



This is a repository copy of *On the existence of conformally coupled scalar field hair for black holes in (anti-)de Sitter space.*

White Rose Research Online URL for this paper:
<http://eprints.whiterose.ac.uk/102578/>

Version: Accepted Version

Article:

Winstanley, E. orcid.org/0000-0001-8964-8142 (2003) On the existence of conformally coupled scalar field hair for black holes in (anti-)de Sitter space. *Foundations of Physics*, 33 (1). pp. 111-143. ISSN 0015-9018

<https://doi.org/10.1023/A:1022871809835>

“The final publication is available at Springer via <http://dx.doi.org/10.1023/A:1022871809835>.”

Reuse

Unless indicated otherwise, fulltext items are protected by copyright with all rights reserved. The copyright exception in section 29 of the Copyright, Designs and Patents Act 1988 allows the making of a single copy solely for the purpose of non-commercial research or private study within the limits of fair dealing. The publisher or other rights-holder may allow further reproduction and re-use of this version - refer to the White Rose Research Online record for this item. Where records identify the publisher as the copyright holder, users can verify any specific terms of use on the publisher's website.

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.



eprints@whiterose.ac.uk
<https://eprints.whiterose.ac.uk/>

On the existence of conformally coupled scalar field hair for black holes in (anti-)de Sitter space

Elizabeth Winstanley (e.winstanley@sheffield.ac.uk)

Department of Applied Mathematics, The University of Sheffield, Hicks Building, Hounsfield Road, Sheffield. S3 7RH. U.K.

Abstract. The Einstein-conformally coupled scalar field system is studied in the presence of a cosmological constant. We consider a massless or massive scalar field with no additional self-interaction, and spherically symmetric black hole geometries. When the cosmological constant is positive, no scalar hair can exist and the only solution is the Schwarzschild-de Sitter black hole. When the cosmological constant is negative, stable scalar field hair exists provided the mass of the scalar field is not too large.

1. INTRODUCTION

For thirty years relativists have been concerned with the question of the uniqueness of black hole solutions to the Einstein equations either in vacuum or coupled to various types of matter. The focus of research during this period has frequently been the proof of uniqueness theorems for various types of black hole geometry (static or axisymmetric) and different types of matter (especially electromagnetic fields). Over the past ten years or so, a plethora of different black hole geometries have been found in many models, so that attention has tended to shift towards finding such solutions (hairy black holes).

The Einstein-scalar field system has long been a favourite during this period. As will be outlined below (see section 2), in asymptotically flat space there are many theorems which rule out scalar field hair for various couplings to the geometry and different types of scalar field potential. In the presence of a cosmological constant, and with minimal coupling to the geometry, non-trivial solutions to the field equations have been found, representing black holes with scalar field hair. In this article we will consider a non-minimally coupled scalar field with a non-zero cosmological constant, so that the geometry approaches (anti-)de Sitter (adS/dS) space at infinity rather than being asymptotically flat.

Space-times which are asymptotically (anti-)de Sitter have become the centre of attention in recent years, for several reasons. Firstly, observations of high-redshift supernovae suggest that the universe may possess a small, positive cosmological constant [1]. Theoretically, there has been much interest in the adS/CFT (conformal field theory) cor-



© 2008 Kluwer Academic Publishers. Printed in the Netherlands.

respondence (see [2] for a review) and its more recent relative the dS/CFT correspondence (see, for example, [3]). Finally, a negative cosmological constant is a key ingredient of “brane world” scenarios (see, for example, [4]). Black holes with these asymptotic geometries play a particularly important role in all these developments.

We shall be primarily concerned with a conformally coupled scalar field since in this case the computations are more straightforward. This is also the coupling which is most interesting from two points of view. Firstly, as outlined in section 2 below, there exists a non-trivial solution to the Einstein-conformally coupled scalar field equations in asymptotically flat space, although the physical interpretation of this solution has been controversial because the scalar field diverges at the event horizon. For other values of the coupling, there are no known non-trivial solutions to the field equations in asymptotically flat space. Secondly, conformal coupling is the most natural quantum scalar field model to study in quantum field theory in curved space.

The paper is structured as follows. In section 2 we briefly review the various no-scalar hair theorems, for both minimally and non-minimally coupled scalar fields. In section 3 we outline the model under consideration and derive the boundary conditions on the scalar field. Next, in section 4, we use a conformal transformation to map the conformally coupled scalar field system to one involving a minimally coupled scalar field. We carefully elucidate the conditions under which this transformation is valid. We are then in a position to study whether or not there are solutions to the latter system. We firstly consider the case in which the cosmological constant is positive (section 5), and show, via a simple proof (following Bekenstein [5]) that does not make use of the conformal transformation, that no solutions are possible. In the situation when the cosmological constant is negative, we find non-trivial solutions in section 6. The stability of these solutions is investigated in section 7. Finally, section 8 includes our conclusions and discussion of our results.

2. STATUS OF THE NO-SCALAR HAIR THEOREMS

In this section we review the status of the no-scalar hair conjecture, for both minimally and non-minimally coupled scalar fields. A more comprehensive review, complete with rigorous theorems, can be found in [6]. A nice summary of the status of the no-hair conjecture in general can be found in [7].

The seminal paper on the subject of no-scalar hair theorems is [5], in which Bekenstein rules out non-trivial hair for a massive scalar field ϕ minimally coupled to gravity. The remarkable thing about [5] is the

simplicity of the argument, which uses only the scalar field equation and the geometric properties of the black hole space-time at the regular event horizon and at infinity. It does not require detailed analysis of possible solutions; nor the assumption of spherical symmetry (although staticity is assumed, the result can be extended to stationary black holes [8]); nor the Einstein equations. The theorem can be readily extended to scalar fields having a convex potential $V(\phi)$, so that

$$\phi \frac{dV}{d\phi} \geq 0,$$

and although Bekenstein's original theorem is concerned only with asymptotically flat space-times, it can be readily extended to spherically symmetric, asymptotically de Sitter geometries [9, 10]. This argument will be exploited in section 5 to show the absence of conformally-coupled scalar field hair for asymptotically de Sitter black holes.

After this early work of Bekenstein, the subject of scalar field hair goes fairly quiet in the literature. The subject of black hole hair received a new lease of life in the early 1990's by the discovery of non-trivial black hole solutions in $\mathfrak{su}(2)$ Einstein-Yang-Mills and related theories (see [11] for a comprehensive review of this topic and an extensive bibliography). In the light of this explosion of interest in hairy black holes, the old question of scalar field hair was re-addressed, and the original theorems of Bekenstein extended. Several authors [12–14] proved the no-scalar hair theorem for a positive semi-definite potential $V(\phi) \geq 0$, although now with the additional assumption of spherical symmetry, but still in asymptotically flat space. These results rule out non-trivial scalar fields having the “double well” Higgs potential. The Sudarsky theorem [14] is particularly simple, but it does not extend to the situation with a cosmological constant [9]. In this paper, we shall not make use of the energy-based arguments of [12, 13], due in part to the subtleties involved in making appropriate definitions of energy in non-asymptotically flat space-times, or where there is non-minimal coupling to the geometry.

Bekenstein himself made an important contribution [15] to this expanding area by proving a no-scalar hair theorem for a very general model involving many scalar fields, again assuming spherical symmetry and minimal coupling to the geometry, but otherwise making no assumptions about the form of the interactions between the scalar fields. The theorem relies on the assumption that the weak energy condition is satisfied by the matter fields, i.e. the stress-energy tensor $T_{\mu\nu}$ satisfies:

$$T_{\mu\nu} u^\mu u^\nu \geq 0 \tag{1}$$

for all time-like vectors u^μ , and the analysis makes use of the conservation equations for the stress tensor and the Einstein equations. The weak energy condition (1) is satisfied by a single scalar field with the usual Lagrangian and a positive semi-definite potential, but it should be stressed that the result in [15] is considerably more general than this and makes much weaker assumptions.

More recently, scalar fields in models with a non-zero cosmological constant have been studied. The inclusion of a positive cosmological constant allows minimally-coupled scalar field hair to exist for positive semi-definite potentials but not convex potentials [9]. However, this hair is unstable. If a negative cosmological constant is included, stable scalar field hair is a possibility [16, 17], in parallel with the situation in Einstein-Yang-Mills theory (see [18], where the existence of stable scalar field hair for asymptotically anti-de Sitter black holes is conjectured). Since the focus of the current work is models containing a cosmological constant, we shall not discuss these results in detail here, but return to them in later sections.

So far we have considered only scalar fields minimally coupled to gravity. Amongst non-minimally coupled scalar field theories, of particular interest is the case of a conformally coupled scalar field, although more general couplings can arise in scalar-tensor theories of gravity. Here, as in the minimally coupled case, seminal contributions to this topic have been made by Bekenstein himself. Bekenstein rediscovered [19, 20] a solution to the conformally coupled scalar field equations previously found by Bocharova et al [21] but unknown in the West. In this paper, following [7], we shall refer to this solution as the BBMB black hole. This solution represents an extremal black hole, but its physical interpretation has been the source of some controversy [22] as the scalar field is divergent on the event horizon, although Bekenstein [20] argued that a particle coupled to the scalar field would experience nothing pathological at the event horizon. Furthermore, this solution is unstable [23]. This solution was revisited in the early 1990's by Zannias and Xanthopoulos [24] who proved that it is the unique static, asymptotically flat, solution of the Einstein-conformal scalar field system. Although the BBMB black hole described above violates the letter of the “no-hair” conjecture, since it is a non-trivial solution of the field equations, no new extra parameters are introduced, the only additional degree of freedom provided by the scalar field being a choice of sign. Therefore this solution has only very limited “hair”, and certainly does not violate the “spirit” of the conjecture. Very recently, Martinez et al [25] have found the generalization of the BBMB black hole in the presence of a positive cosmological constant, with the scalar field having

a quartic potential. In this case the scalar field is regular on and outside the event horizon, with a pole inside the event horizon.

In the later 1990's efforts were made to extend the uniqueness theorem of Xanthopoulos and Zannias to other non-minimally coupled scalar field theories, not just conformal coupling. Again, Bekenstein led the way, in a paper with Mayo [26]. There the no-scalar hair theorem is proved when the parameter ξ which couples a neutral scalar field to the curvature (for a precise definition see equation (2) in section 3 below) is either negative or greater than one half. This paper leaves open the question of $0 < \xi < 1/2$, which includes the conformally coupled case $\xi = 1/6$. Interestingly, the corresponding proof for a charged, non-minimally coupled scalar field is valid for all ξ . Related theorems for the neutral case and all values of ξ can be found in two papers by Saa [27, 28], which however make various restrictions on the value of the scalar field, whilst the paper of Bekenstein and Mayo has stronger results in that the restrictions on the value of the scalar field required to prove the theorems are themselves proved, using energy arguments. It is striking that energy arguments can be applied to this system, as the non-minimal coupling means that the weak energy condition (1), so crucial in the minimally coupled case, is no longer valid. The application of energy arguments to this non-minimally coupled system has not been without controversy [29], although there is additional numerical evidence [29] that the theorems of Bekenstein and Mayo are in fact true. We will examine whether or not the energy conditions of [26] are satisfied for the solutions we find in section 6.

We are now in a position to test the no-scalar hair conjecture for non-minimally coupled scalar fields in space-times which are no longer asymptotically flat. Our model will contain a (positive or negative) cosmological constant and so we are interested in black holes in (a)dS space.

3. THE MODEL AND BOUNDARY CONDITIONS

In this section we will outline the model to be considered in this paper, derive the corresponding field equations and use these to elucidate the boundary conditions on the fields.

Consider the following action, which describes a self-interacting scalar field ϕ with non-minimal coupling to gravity:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} (R - 2\Lambda) - \frac{1}{2} (\nabla\phi)^2 - \frac{1}{2} \xi R\phi^2 - V(\phi) \right], \quad (2)$$

where R is the Ricci scalar, Λ the cosmological constant (which may be either positive or negative), ξ is the coupling constant, $V(\phi)$ the scalar

field self-interaction potential and $(\nabla\phi)^2 = \nabla_\mu\phi\nabla^\mu\phi$. For a minimally coupled scalar field, $\xi = 0$ and for conformal coupling (which is the main focus of this paper), $\xi = 1/6$ in four dimensions. We are particularly interested in the cases where the potential V is either zero (massless scalar field, no self-interaction) or $V(\phi) = \frac{1}{2}\mu^2\phi^2$ which describes a scalar field of mass μ . Here and throughout this paper, the metric has signature $(-+++)$, and we will use units in which the gravitational coupling constant $\kappa^2 = 8\pi G$ (with G being Newton's constant) is set equal to unity and $c = 1$.

Variation of the action (2) yields the following field equations:

$$\begin{aligned} \left[1 - \xi\phi^2\right] G_{\mu\nu} + g_{\mu\nu}\Lambda &= (1 - 2\xi) \nabla_\mu\phi\nabla_\nu\phi + \left(2\xi - \frac{1}{2}\right) g_{\mu\nu} (\nabla\phi)^2 \\ &\quad - 2\xi\phi\nabla_\mu\nabla_\nu\phi + 2\xi g_{\mu\nu}\phi\nabla^\rho\nabla_\rho\phi \\ &\quad - g_{\mu\nu}V(\phi); \end{aligned} \quad (3a)$$

$$\nabla_\mu\nabla^\mu\phi = \xi R\phi + \frac{dV}{d\phi}. \quad (3b)$$

It is useful to take the trace of the Einstein equations (3a) to give the Ricci scalar:

$$R = \frac{(1 - 6\xi) (\nabla\phi)^2 - 6\xi\phi\frac{dV}{d\phi} + 4V(\phi) + 4\Lambda}{1 - \xi(1 - 6\xi)\phi^2}, \quad (4)$$

where we have made use of the scalar field equation (3b) to substitute for $\nabla_\mu\nabla^\mu\phi$ arising in the expression for R . This equation enables one to eliminate higher-order derivatives of the metric from the scalar field equation (3b). In this paper we shall focus primarily on a conformally coupled scalar field, in which case $\xi = 1/6$ and (4) takes a particularly simple form:

$$R = 4\Lambda + 4V(\phi) - \phi\frac{dV}{d\phi};$$

and the scalar field equation (3b) can be written as:

$$\nabla_\mu\nabla^\mu\phi = \frac{2}{3}(\Lambda + V(\phi))\phi + \left(1 - \frac{1}{6}\phi^2\right)\frac{dV}{d\phi}. \quad (5)$$

We consider a spherically symmetric black hole geometry with line element

$$\begin{aligned} ds^2 &= - \left(1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}\right) \exp(2\delta(r)) dt^2 \\ &\quad + \left(1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2, \end{aligned}$$

and we assume that the scalar field ϕ depends only on the radial coordinate r . For later convenience, we define the quantity $N(r)$ as

$$N(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}.$$

The field equations for this system are (a prime $'$ denotes d/dr):

$$\begin{aligned} \frac{2}{r^2} (1 - \xi\phi^2) m' &= \xi\Lambda\phi^2 + \left(\frac{1}{2} - 2\xi\right) N\phi'^2 - \xi\phi\phi'N' - 2\xi N\phi\phi'' \\ &\quad - \frac{4N}{r}\xi\phi\phi' + V(\phi); \end{aligned} \quad (6a)$$

$$\frac{2}{r} (1 - \xi\phi^2) \delta' = (1 - 2\xi) \phi'^2 - 2\xi\phi\phi'' + 2\xi\phi\phi'\delta'; \quad (6b)$$

$$0 = N\phi'' + \left(N\delta' + N' + \frac{2N}{r}\right) \phi' - \xi R\phi - \frac{dV}{d\phi}. \quad (6c)$$

It is now possible, using (4), to eliminate the Ricci scalar curvature from the scalar field equation (6c) and then also eliminate ϕ'' from the right-hand-side of the Einstein equations (6a,6b) above. Writing the equations in this form would make them more amenable to numerical integration.

We are interested in black hole solutions possessing a regular, non-extremal event horizon at $r = r_h$, close to which the field variables have the form

$$\begin{aligned} N(r) &= N'(r_h)(r - r_h) + O(r - r_h)^2; \\ \delta(r) &= \delta(r_h) + O(r - r_h); \\ \phi(r) &= \phi(r_h) + O(r - r_h). \end{aligned}$$

There may also be a cosmological event horizon at $r = r_c \neq r_h$ (depending upon the sign of the cosmological constant), with similar expansions of the fields nearby.

At infinity, we will assume that the scalar field ϕ approaches a constant, ϕ_∞ , as $r \rightarrow \infty$, and that the geometry approaches (a)dS, so that $\delta \rightarrow 0$ at infinity. We do not necessarily, at this stage, assume that ϕ has an analytic form near infinity, but rather, as observed for minimally coupled scalar fields [9, 16],

$$\phi = \phi_\infty + O(r^{-k}) \quad (7)$$

for some $k > 0$, whose value we shall ascertain shortly. This means that

$$m(r) \rightarrow \tilde{\Lambda}r^3 + M + O(r^{-1}),$$

so that

$$N(r) \rightarrow 1 - \frac{2M}{r} - \frac{\Lambda_{eff}r^2}{3} + O(r^{-2}),$$

where we have an “effective” cosmological constant

$$\Lambda_{eff} = \Lambda + 6\tilde{\Lambda}.$$

We will assume that both Λ and Λ_{eff} are non-zero. At infinity,

$$R \rightarrow 4\Lambda_{eff} + O(r^{-1}),$$

and therefore the scalar field equation gives, to leading order,

$$4\xi\Lambda_{eff}\phi_\infty = -\frac{dV}{d\phi}(\phi_\infty).$$

Therefore, in the particular case in which $V(\phi) \equiv 0$, (and $\Lambda_{eff} \neq 0$), it must be the case that

$$\phi \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

In the other case of particular interest to us in this paper, $V(\phi) = \frac{1}{2}\mu^2\phi^2$, we have

$$4\xi\Lambda_{eff}\phi_\infty = -\mu^2\phi_\infty,$$

which means that either $\phi_\infty = 0$ or, if

$$4\xi\Lambda_{eff} = -\mu^2, \tag{8}$$

then ϕ_∞ is arbitrary. The condition (8) is applicable only when $\Lambda_{eff} < 0$, i.e. when the geometry is asymptotically anti-de Sitter.

The first Einstein equation now gives, to leading order,

$$6\tilde{\Lambda} (1 - \xi\phi_\infty^2) = \xi\Lambda\phi_\infty^2 + V(\phi_\infty).$$

Hence, in the particular case $\phi_\infty = 0$, $\tilde{\Lambda} = 0$ and $\Lambda_{eff} = \Lambda$. On the other hand, for a massive scalar field such that (8) holds,

$$\Lambda_{eff} = \Lambda \left(1 + \xi\phi_\infty^2\right)^{-1}.$$

This equation, combined with (8), fixes the value of ϕ_∞ in this case to be such that

$$\phi_\infty^2 = -\frac{1}{\xi} \left(1 + \frac{4\xi\Lambda}{\mu^2}\right),$$

which can hold only if $\mu^2 \leq -4\xi\Lambda$. Therefore, for conformally coupled scalar fields for which $\xi = 1/6$ (the primary interest in this paper), we have, for $\mu^2 \leq -2\Lambda/3$, either $\phi_\infty = 0$ or

$$\phi_\infty^2 = -6 \left(1 + \frac{2\Lambda}{3\mu^2}\right). \tag{9}$$

In section 6 we will rule out solutions in which $\phi_\infty \neq 0$, and therefore we will not consider this possibility further.

Our final task in this section is to determine the rate at which ϕ vanishes as we approach infinity, i.e. the power k in (7). For the minimally coupled scalar field, it is not necessarily the case that k is an integer (or even real) [16]. We will assume that $\phi_\infty = 0$ since this is the case for the solutions we find in later sections. Substituting the asymptotic form (7) into the scalar field equation (6c) gives the following equation for k :

$$0 = k^2 - 3k + 12\xi + 3\frac{\mu^2}{\Lambda},$$

where we have used $V(\phi) = \frac{1}{2}\mu^2\phi^2$. This has roots, for the conformally coupled ($\xi = 1/6$) case:

$$k = \frac{3}{2} \pm \sqrt{\frac{1}{4} - \frac{3\mu^2}{\Lambda}}. \quad (10)$$

If the cosmological constant Λ is negative, then k is always real, whereas k can be complex if Λ is positive and $\mu^2 > \Lambda/12$. With $\Lambda < 0$, both values of k are positive only if $\mu^2 < -2\Lambda/3$; otherwise there is a negative value of k , which represents a scalar field ϕ which does not converge to zero at infinity. Therefore, when we come to consider solutions in anti-de Sitter space in section 6, we shall restrict our attention only to those fields having mass μ such that $\mu^2 < -2\Lambda/3$. The leading order behaviour of ϕ at infinity is given by the smaller value of k , i.e.

$$k = \frac{3}{2} - \sqrt{\frac{1}{4} - \frac{3\mu^2}{\Lambda}}.$$

If $\Lambda > 0$, both values of k (10) are positive when k is real, and if k is complex, then it always has positive real part. In this latter situation complex k represents a scalar field solution which oscillates about zero as $r \rightarrow \infty$, with the amplitude of the oscillations decreasing like $r^{-\Re k}$. Similar behaviour is observed for the minimally coupled scalar field in anti-de Sitter space [16].

4. CONFORMAL TRANSFORMATION

The system described by the action (2) above can be transformed under a conformal transformation as follows [30]:

$$\bar{g}_{\mu\nu} = \Omega g_{\mu\nu} \quad (11)$$

where

$$\Omega = 1 - \xi\phi^2. \quad (12)$$

This transformation is valid only for those solutions for which Ω does not vanish. In asymptotically flat space, there is strong numerical [29] and analytic [26] evidence that solutions in which $\Omega = 0$ for some value of r do not exist. For the solutions to be exhibited in section 6, it will be automatic that Ω has no zeros. We leave open the question of whether there are in fact solutions in which $\Omega = 0$ at some point.

Under the transformation (11) the action becomes that of a minimally coupled scalar field:

$$S = \int d^4x \sqrt{-\bar{g}} \left[\frac{1}{2} (\bar{R} - 2\Lambda) - \frac{1}{2} (\bar{\nabla}\Phi)^2 - U(\Phi) \right] \quad (13)$$

where a bar denotes quantities calculated using the transformed metric $\bar{g}_{\mu\nu}$ and we have defined a new scalar field Φ as [30]

$$\Phi = \int d\phi \left[\frac{(1 - \xi\phi^2) + 6\xi^2\phi^2}{(1 - \xi\phi^2)^2} \right]^{\frac{1}{2}}. \quad (14)$$

Note that, although the cosmological constant sets a length scale for the theory, it is unchanged by this conformal transformation.

When $\xi = 1/6$, so the scalar field is conformally coupled, the equation above simplifies to

$$\begin{aligned} \Phi &= \int d\phi \frac{1}{(1 - \phi^2/6)} \\ &= \sqrt{6} \tanh^{-1} \left(\frac{\phi}{\sqrt{6}} \right). \end{aligned} \quad (15)$$

For all values of ξ , we will choose the constant of integration so that $\Phi = 0$ when $\phi = 0$. For values of ξ not equal to $1/6$, the field Φ (14) can be written in terms of inverse sinh and tanh functions of ϕ , which means that it is not so readily inverted as in the conformally coupled case. The transformed potential $U(\Phi)$ takes the form [30]

$$U(\Phi) = \frac{V(\phi) + \Lambda\xi\phi^2(2 - \xi\phi^2)}{(1 - \xi\phi^2)^2}.$$

For the remainder of this paper, we shall focus only on the conformally coupled case, since the conformal transformation is straightforward to implement and invert in this case. The conformally coupled scalar field is also of most interest physically, as outlined in the introduction.

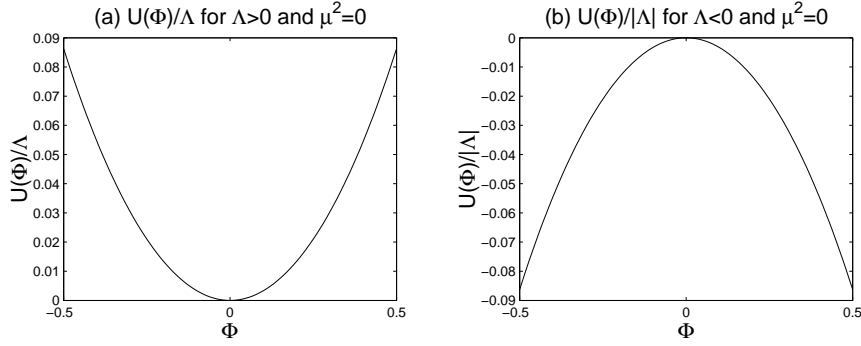


Figure 1. (a) Graph of $U(\Phi)/\Lambda$ (vertical axis) against Φ (horizontal axis) for positive Λ and vanishing mass. For a massive conformally coupled scalar field and $\Lambda > 0$, the potential has the same overall shape. (b) Graph of $U(\Phi)/|\Lambda|$ (vertical axis) against Φ (horizontal axis) for negative Λ and vanishing mass.

Suppose now that the field is conformally coupled, and $V(\phi) \equiv 0$, then

$$U(\Phi) = \frac{\Lambda\phi^2(12 - \phi^2)}{(6 - \phi^2)^2}.$$

If we use the relation

$$\phi = \sqrt{6} \tanh\left(\frac{\Phi}{\sqrt{6}}\right), \quad (16)$$

then

$$U(\Phi) = \Lambda \left[1 + \cosh^2\left(\frac{\Phi}{\sqrt{6}}\right) \right] \sinh^2\left(\frac{\Phi}{\sqrt{6}}\right). \quad (17)$$

For a massive conformally coupled scalar field, $V(\phi) = \frac{1}{2}\mu^2\phi^2$, the transformed potential is

$$U(\Phi) = \Lambda \left[1 + \cosh^2\left(\frac{\Phi}{\sqrt{6}}\right) \right] \sinh^2\left(\frac{\Phi}{\sqrt{6}}\right) + 3\mu^2 \sinh^2\left(\frac{\Phi}{\sqrt{6}}\right) \cosh^2\left(\frac{\Phi}{\sqrt{6}}\right). \quad (18)$$

Notice that the sign of the transformed potential U depends on that of the cosmological constant Λ . If $\Lambda > 0$, then both the massless (17) and massive (18) potentials are positive, and have the form sketched in figure 1 (a). However, for negative Λ , the potential corresponding to a massless conformally coupled scalar field (17) is always strictly negative and has the form of that for positive Λ , but inverted (see figure 1 (b)). This is rather unphysical, but it should be borne in mind that this potential is for the transformed scalar field, which is not intended to be

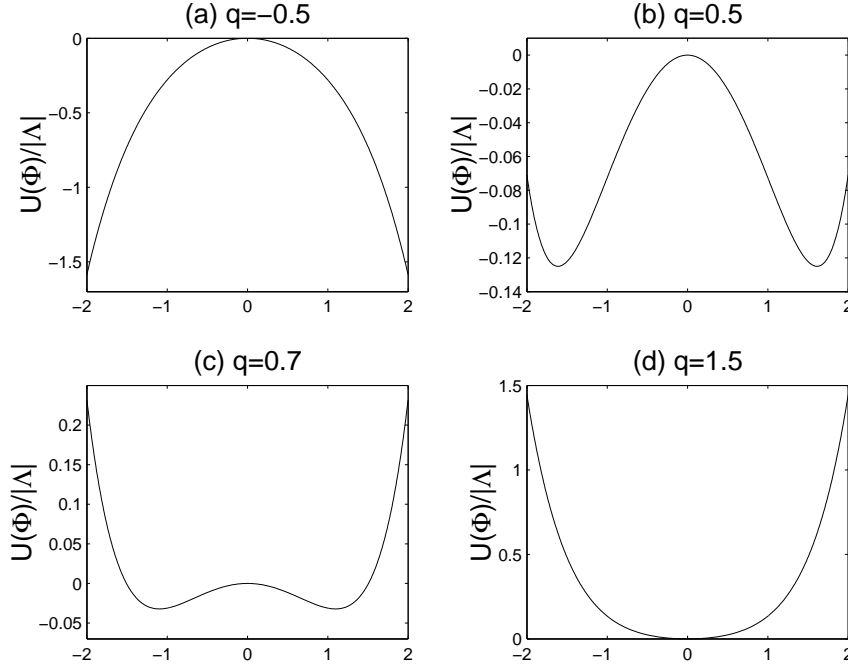


Figure 2. Graphs of $U(\Phi)/|\Lambda|$ (vertical axis in each case) against Φ (horizontal axis in each case) for negative Λ and various values of $q = -1 + \frac{3\mu^2}{|\Lambda|}$. For $q \leq 0$, the potential is negative everywhere as in the massless case (graph (a)). For $0 < q < 1$ (graphs (b) and (c)), the potential is negative for small Φ but becomes positive for sufficiently large Φ . If $q \geq 1$ (graph (d)), then the potential is positive for all Φ .

physically interpreted. Rather, the transformation serves to simplify the mathematics, and we should not be too concerned at this stage that we have an unphysical transformed potential. For a massive conformally coupled scalar field, the sign of the potential (18) depends on the mass μ as well as Λ . If $0 \leq 3\mu^2 \leq -\Lambda$, then the potential is negative for all Φ , as for the massless case. If $-2\Lambda > 3\mu^2 > -\Lambda$, then the potential is positive for sufficiently large Φ , but negative for small Φ , whilst for $3\mu^2 \geq -2\Lambda$, the potential is positive for all Φ . The possibilities are sketched in figure 2.

The transformed metric $\bar{g}_{\mu\nu}$ (11) will also be spherically symmetric, and we take it to have the form:

$$d\bar{s}^2 = -\bar{N}(\bar{r})e^{2\bar{\delta}(\bar{r})}dt^2 + \bar{N}(\bar{r})^{-1}d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where we have defined a transformed radial co-ordinate r by

$$\bar{r} = \left(1 - \xi\phi^2\right)^{\frac{1}{2}} r, \quad (19)$$

and the transformed metric quantities \bar{N} and $\bar{\delta}$ are given by

$$\begin{aligned}\bar{N} &= N \left(1 - \xi\phi^2 - \xi r\phi\phi'\right)^2 \left(1 - \xi\phi^2\right)^{-2}; \\ e^{\bar{\delta}} &= e^{\delta} \left(1 - \xi\phi^2 - \xi r\phi\phi'\right)^{-1} \left(1 - \xi\phi^2\right)^{\frac{3}{2}}.\end{aligned}\quad (20)$$

In addition to our earlier requirement that $(1 - \xi\phi^2) \neq 0$, the transformed radial co-ordinate r ceases to be a good co-ordinate if

$$1 - \xi\phi^2 - \xi r\phi\phi' = 0,$$

so we are also restricting our attention to those solutions for which the left-hand-side of the above equation does not vanish. Using the relationships (16,19), we see that, in the conformally coupled case,

$$\begin{aligned}1 - \frac{1}{6}\phi^2 &= \operatorname{sech}^2\left(\frac{\Phi}{\sqrt{6}}\right); \\ 1 - \frac{1}{6}\phi^2 - \frac{1}{6}r\phi\phi' &= \left[1 + \frac{1}{\sqrt{6}}\bar{r}\frac{d\Phi}{d\bar{r}}\tanh\left(\frac{\Phi}{\sqrt{6}}\right)\right]^{-1} \operatorname{sech}^2\left(\frac{\Phi}{\sqrt{6}}\right)\end{aligned}\quad (21)$$

so that both conditions are automatically satisfied if Φ is well-defined, in other words if we can find a solution of the transformed, minimally coupled system, then we automatically have a solution of the conformally coupled system, provided that

$$\mathcal{A} = 1 + \frac{1}{\sqrt{6}}\bar{r}\frac{d\Phi}{d\bar{r}}\tanh\left(\frac{\Phi}{\sqrt{6}}\right) \neq 0.\quad (22)$$

This condition will be explicitly checked for the solutions we find in section 6. The converse, however, is not true, and a solution to the conformally coupled system does not necessarily lead to a well-defined solution of the minimally coupled system.

We may use the equations (21) to write (20) in the alternative form

$$\begin{aligned}\bar{N} &= N \left[1 + \frac{1}{\sqrt{6}}\bar{r}\frac{d\Phi}{d\bar{r}}\tanh\left(\frac{\Phi}{\sqrt{6}}\right)\right]^{-2}; \\ e^{\bar{\delta}} &= e^{\delta} \left[1 + \frac{1}{\sqrt{6}}\bar{r}\frac{d\Phi}{d\bar{r}}\tanh\left(\frac{\Phi}{\sqrt{6}}\right)\right] \operatorname{sech}\left(\frac{\Phi}{\sqrt{6}}\right).\end{aligned}\quad (23)$$

From (20,23), if there is a horizon at for some value of the untransformed co-ordinate r , so that $N(r) = 0$, then at the corresponding \bar{r} we will have $\bar{N} = 0$, so that the transformation (providing it is regular) does not affect the horizon structure of the geometry. At infinity, we have $\phi \rightarrow 0$, and so $\Phi \rightarrow 0$ as well, from our earlier choice of constant of integration in the definition of Φ . This means that $\bar{r} \sim r$ as $r \rightarrow \infty$,

and the metric function \bar{N} and $\bar{\delta}$ behave in exactly the same way as their untransformed counterparts.

We now define a new quantity $\bar{m}(\bar{r})$ by

$$\bar{N}(\bar{r}) = 1 - \frac{2\bar{m}(\bar{r})}{\bar{r}} - \frac{\Lambda\bar{r}^2}{3};$$

in terms of which the field equations derived from the transformed action (13) are:

$$\frac{d\bar{m}}{d\bar{r}} = \frac{\bar{r}^2}{4}\bar{N}\left(\frac{d\Phi}{d\bar{r}}\right)^2 + \frac{\bar{r}^2}{2}U(\Phi); \quad (24a)$$

$$\frac{d\bar{\delta}}{d\bar{r}} = \frac{\bar{r}}{2}\left(\frac{d\Phi}{d\bar{r}}\right)^2; \quad (24b)$$

$$0 = \bar{N}\frac{d^2\Phi}{d\bar{r}^2} + \left(\bar{N}\frac{d\bar{\delta}}{d\bar{r}} + \frac{d\bar{N}}{d\bar{r}} + \frac{2\bar{N}}{\bar{r}}\right)\frac{d\Phi}{d\bar{r}} - \frac{dU}{d\Phi}. \quad (24c)$$

From these equations it is clear that the main advantage of using the conformal transformation is that the minimally coupled scalar field equations are considerably less complicated than those for a scalar field with non-minimal coupling, as only first derivatives of the scalar field now appear on the right-hand-side of the Einstein equations, and only first derivatives of the metric functions appear in the scalar field equation. This means that the minimally coupled scalar field equations are much easier to solve numerically, although we are restricting our attention to those solutions for which the conformal transformation is valid. The other advantage is that the minimally coupled scalar field system has been extensively studied [9, 16] and so we can exploit known results to give us insight into the more complex non-minimally coupled case.

5. NON-EXISTENCE OF SOLUTIONS IN DE SITTER SPACE

We consider first the case where the cosmological constant Λ is positive, so that the geometry is asymptotically de Sitter.

In this situation, the potential in the transformed system (17,18) is positive semi-definite for both a massless and massive conformally coupled scalar field (see figure 1 (a)). Furthermore, it is convex, since

$$\frac{dU}{d\Phi} = \frac{1}{2\sqrt{6}}\left(\Lambda + 3\mu^2\right)\sinh\left(\frac{4\Phi}{\sqrt{6}}\right) + \frac{\Lambda}{\sqrt{6}}\sinh\left(\frac{2\Phi}{\sqrt{6}}\right),$$

which has the same sign as Φ . The fact that the potential is convex means that it is straightforward to prove that there are no non-trivial solutions to the transformed, minimally coupled system, using Bekenstein's original technique [5]. This argument has been presented in [9]. However, this above argument is clearly only valid if the conformal transformation is well-defined everywhere between the event and cosmological horizons. In other words, we have assumed that $\phi^2 \neq 6$ and that $6 - \phi^2 - r\phi\phi'$ does not vanish anywhere.

In fact, the same technique can be applied directly to the untransformed system, in the conformally coupled case, to show that the only solution is $\phi \equiv 0$. Firstly, we consider the scalar field equation in the simple form (5), for a spherically symmetric geometry in which

$$\nabla_\mu \nabla^\mu \phi = r^{-2} e^{-\delta} \left(N r^2 e^\delta \phi' \right)'.$$

Multiply both sides of the scalar field equation by $\phi r^2 e^\delta$ and integrate from the black hole event horizon $r = r_h$ to the cosmological event horizon $r = r_c$:

$$\begin{aligned} 0 &= \int_{r_h}^{r_c} dr \left\{ r^2 e^\delta \left[\frac{2}{3} (\Lambda + V) \phi^2 + \left(1 - \frac{1}{6} \phi^2 \right) \phi \frac{dV}{d\phi} \right] \right. \\ &\quad \left. - \phi \left(N r^2 e^\delta \phi' \right)' \right\} \\ &= \int_{r_h}^{r_c} dr r^2 e^\delta \left[\frac{2}{3} (\Lambda + V) \phi^2 + \left(1 - \frac{1}{6} \phi^2 \right) \phi \frac{dV}{d\phi} + N \phi'^2 \right] \\ &\quad - \left[N r^2 e^\delta \phi \phi' \right]_{r_h}^{r_c}, \end{aligned} \quad (25)$$

where we have integrated by parts in the second line. Using the boundary conditions of regular event and cosmological horizons, the boundary term vanishes. If the integrand is positive definite, the only possibility is then that $\phi' \equiv 0$, in which case $\phi \equiv 0$ (using the boundary conditions at infinity), giving only the trivial solution.

The integrand in (25) will be positive if the potential V is positive semi-definite and convex, and if $1 - \frac{1}{6}\phi^2$ is non-negative everywhere between the event and cosmological horizons (which is tantamount to assuming the validity of the conformal transformation). However, we are interested in the potentials $V(\phi) = 0$ and $V(\phi) = \frac{1}{2}\mu^2\phi^2$, in which case (25) becomes

$$0 = \int_{r_h}^{r_c} dr r^2 e^\delta \left[\left(\frac{2}{3} \Lambda + \mu^2 + \frac{1}{6} \mu^2 \phi^2 \right) \phi^2 + N \phi'^2 \right], \quad (26)$$

so that the integrand is manifestly positive definite and the only possible solution is the trivial one.

We note that the fact that we are considering only a conformally coupled scalar field is crucial in this proof, and for other, non-minimal couplings, the proof is considerably more complicated. Furthermore, we are only considering a very restricted class of potentials (corresponding to massless or massive scalar fields with no self-interactions) and the proof will not readily extend to more general potentials (as is the case with the original Bekenstein proof in asymptotically flat space [5, 8]). The proof given here works equally well for conformal coupling (and suitably restricted potentials) in asymptotically flat space. For example, in asymptotically flat space, if the potential $V \equiv 0$, then the scalar field ϕ vanishes like r^{-1} at infinity, whilst if $V(\phi) = \frac{1}{2}\mu^2\phi^2$ then $\phi \sim e^{-\mu r}$ as $r \rightarrow \infty$, so that in each case the boundary term at infinity in (25) vanishes. We are therefore left with (26), with $\Lambda = 0$, so that the integrand is manifestly positive definite and the only possibility is the trivial solution.

Note also that our proof does not rule out the BBMB black hole [19–21] because the divergence of the scalar field on the (extremal) event horizon means that the boundary terms in (25) do not vanish. For the extension of the BBMB black hole in the presence of a positive cosmological constant [25], the boundary terms in (25) do vanish because the scalar field is now regular on the event horizon (and the event horizon is no longer extremal). However, in this case, it is straightforward to check that the integrand in (25) is no longer positive definite.

We note, to end this section, that the proof also does not extend to the case where $\Lambda < 0$. In this situation, in the absence of a cosmological event horizon, we would integrate from the black hole event horizon out to infinity. From the analysis of section 3, we are interested only in values of μ satisfying $\mu^2 < -2\Lambda/3$, in which case the behaviour of the scalar field ϕ at infinity is such that the boundary term in (25) does not vanish, although it is negative and so makes a positive contribution to the right-hand-side of (25). However, since $\mu^2 < -2\Lambda/3$, we can no longer say that the integrand in the first term of (25), i.e. the integrand in (26), is positive definite and so the proof fails. This leaves open the possibility of non-trivial solutions in anti-de Sitter space, and in the following section we shall show that such solutions do in fact exist.

6. EXISTENCE OF SOLUTIONS IN ANTI-DE SITTER SPACE

Having shown that there are no non-trivial solutions to the conformally coupled scalar field equations in the presence of a positive cosmological constant, we shall, in this section, exhibit numerical solutions when

the cosmological constant is negative, so that the geometry approaches anti-de Sitter space at infinity.

In this section we shall use the conformal transformation approach outlined in section 4 and are therefore tacitly assuming that this transformation is valid. This method depends on the quantities $1 - \phi^2/6$ and $1 - \phi^2/6 - r\phi\phi'/6$ being well-behaved and non-zero everywhere outside the event horizon. For the transformation of the fields given in (15), both these conditions automatically hold, so that if we can find a solution in the minimally coupled, transformed system, then we automatically have a well-behaved solution of the conformally coupled system. The converse is not however true.

We firstly investigate the range of masses for which solutions of this nature are possible, since the mass alters the form of the minimally coupled scalar field potential (see figure 2). From the analysis of section 3, if $\mu^2 \geq -2\Lambda/3$, then $\phi_\infty = 0$, and the scalar field does not have convergent behaviour at infinity. Therefore, for the remainder of this section, we consider only the case $\mu^2 \leq -2\Lambda/3$.

If $\mu^2 \leq -2\Lambda/3$, there are two possible behaviours at infinity: either $\phi_\infty = 0$ or $\phi_\infty \neq 0$ and is given by (9). In the case that $\phi_\infty \neq 0$, the minimally coupled scalar field Φ does not tend to zero at infinity (since in section 4 we chose our constant of integration so that $\Phi = 0$ if and only if $\phi = 0$). Working with the minimally coupled scalar field equations means that we can immediately apply the results of [16]. From [16, 17], it must be the case that Φ approaches a stationary point of the potential $U(\Phi)$ at infinity. For the minimally coupled scalar field potential $U(\Phi)$ given by (18), we have

$$\begin{aligned} \frac{dU}{d\Phi} &= \frac{1}{2\sqrt{6}} \left[\Lambda + 3\mu^2 \right] \sinh\left(\frac{4\Phi}{\sqrt{6}}\right) + \frac{\Lambda}{\sqrt{6}} \sinh\left(\frac{2\Phi}{\sqrt{6}}\right); \\ \frac{d^2U}{d\Phi^2} &= \frac{1}{3} \left[\Lambda + 3\mu^2 \right] \cosh\left(\frac{4\Phi}{\sqrt{6}}\right) + \frac{\Lambda}{3} \cosh\left(\frac{2\Phi}{\sqrt{6}}\right). \end{aligned} \quad (27)$$

Therefore the potential has stationary points at $\Phi = 0$ and $\Phi = \pm\Phi_s$, where $\Phi_s > 0$ and

$$\cosh\left(\frac{2\Phi_s}{\sqrt{6}}\right) = -\frac{\Lambda}{\Lambda + 3\mu^2};$$

which is only possible if $-\Lambda/3 \leq \mu^2 \leq -2\Lambda/3$. When $\Phi = 0$, we have

$$\frac{d^2U}{d\Phi^2} = \frac{2\Lambda}{3} + \mu^2; \quad (28)$$

and therefore this is a local maximum of the potential (since we are considering only the masses $\mu^2 < -2\Lambda/3$), while the values $\Phi = \pm\Phi_s$

correspond to local minima. These points can be seen in figure 2. In [16], it was found that a local maximum of the potential acted as an “attractor” for the scalar field at infinity, and that, while solutions in which the scalar field tended to a local minimum at infinity were possible, they necessitated a “fine tuning” in the parameters of the theory and were unstable. (See also [17] for a simple explanation of why the scalar field must approach a local maximum of the potential at infinity.) In view of this, we shall from now on only consider solutions in which Φ and ϕ both tend to zero at infinity.

The minimally coupled field equations of the transformed system (24a-24c), are considerably easier to integrate numerically than the conformally coupled scalar field equations (6a-6c). Our approach is to numerically integrate the minimally coupled scalar field equations outwards from the event horizon for a particular initial value of the scalar field. However, not all initial values of the scalar field at the event horizon, $\Phi_h = \Phi(\bar{r}_h)$, give solutions. It must be the case that $\Phi_h \frac{d\Phi}{d\bar{r}}(\bar{r}_h) < 0$, which from the scalar field equation (24c) implies that

$$\Phi_h \frac{dU}{d\Phi}(\Phi_h) < 0. \quad (29)$$

If not, then if $\Phi_h > 0$, initially the scalar field is increasing and it will remain in a region for which $dU/d\Phi > 0$ (see figure 2). If, subsequently, Φ has a stationary point at $\bar{r} = \bar{r}_0$, then from (24c),

$$\bar{N} \frac{d^2\Phi}{d\bar{r}^2}(\bar{r}_0) = \frac{dU}{d\Phi}(\Phi(\bar{r}_0)) > 0,$$

and so the scalar field can only have a minimum in this region. Therefore the scalar field is unable to have a maximum, which is necessary if it is to tend to zero at infinity. A similar line of reasoning applies if $\Phi_h < 0$. Therefore, using (27) the condition (29) becomes

$$\cosh\left(\frac{2\Phi_h}{\sqrt{6}}\right) > -\frac{\Lambda}{\Lambda + 3\mu^2}, \quad (30)$$

if $\Lambda + 3\mu^2 < 0$, and

$$\cosh\left(\frac{2\Phi_h}{\sqrt{6}}\right) < -\frac{\Lambda}{\Lambda + 3\mu^2}, \quad (31)$$

if $\Lambda + 3\mu^2 > 0$. In the former case, then the right-hand-side of the inequality in (30) is negative and the condition is automatically satisfied. However, in the latter case, (31) places a restriction on the value Φ_h of the minimally coupled scalar field at the event horizon, and the condition can only be satisfied for some Φ_h if $3\mu^2 < -2\Lambda$ since

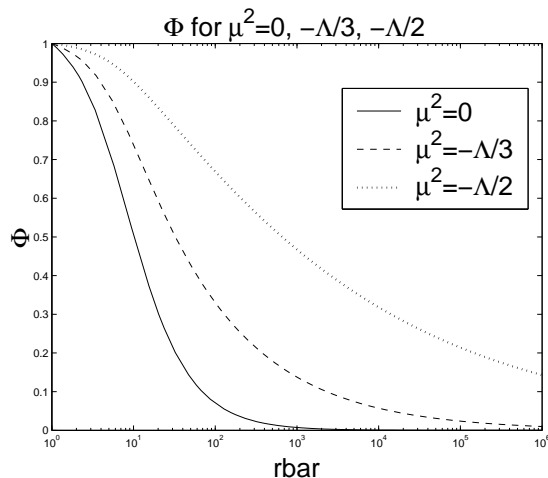


Figure 3. Graph of Φ (vertical axis) against \bar{r} (horizontal axis) for $\mu^2 = 0, -\Lambda/3$ and $-\Lambda/2$, with $\Lambda = -0.1$, $\bar{r}_h = 1$ and $\Phi(\bar{r}_h) = 1$. For all values of the mass, Φ has no zeros and tends to zero at infinity.

$\cosh(x) \geq 1$ for all x . This is not a problem since, as discussed in section 3 the conformally coupled scalar field has the desired behaviour at infinity only if $3\mu^2 < -2\Lambda$. However, for increasing mass, there is a decreasing interval of initial values Φ_h which will give us solutions.

With suitable initial conditions, it is then straightforward to numerically integrate the minimally coupled scalar field equations (24a-24c). Typical results are shown in figure 3. Here we have chosen the particular values $\Lambda = -0.1$, $\bar{r}_h = 1$, $\Phi(\bar{r}_h) = 1$, and three values of the mass: $\mu = 0$, $\mu^2 = -\Lambda/3$ and $\mu^2 = -\Lambda/2$, similar behaviour being found for other values of the parameters. We plot the scalar field Φ in figure 3. We find that the scalar field has no zeros, and simply decays to zero at infinity, the rate of decay to zero being slower for larger mass.

Next, we check that the additional condition (22) required for the conformal transformation to be well-behaved. As shown in figure 4 this condition is easily satisfied for the solutions we have found.

We are thus in a position to transform, via (16,19,20), to give the conformally coupled quantities. In figures 5-7 we plot the corresponding solutions of the conformally coupled scalar field equations for the parameter values given above. In common with the minimally coupled scalar field Φ , the conformally coupled scalar field ϕ is monotonic and vanishes asymptotically at infinity, as can be seen in figure 5. This is exactly as is anticipated since ϕ has the same sign as Φ from (16). The field decays like r^{-k} , where the power k is given by the smaller of the

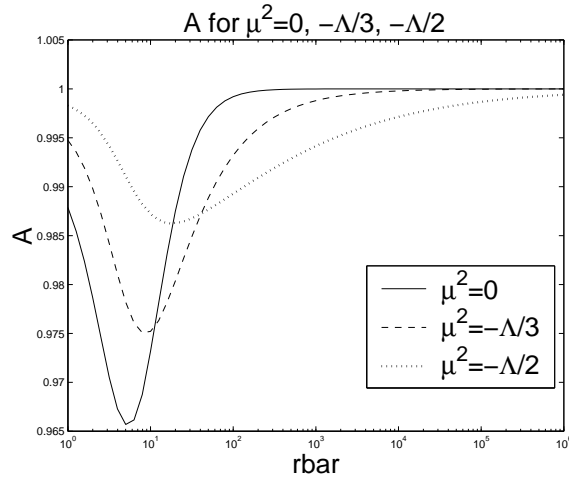


Figure 4. Graph of \mathcal{A} (given by (22)) (vertical axis) against \bar{r} (horizontal axis) for $\mu^2 = 0, -\Lambda/3$ and $-\Lambda/2$, with the other parameters as in figure 3. The graph shows clearly that this quantity never vanishes, so that condition (22) holds and the conformal transformation remains valid for these solutions. Similar results are found for other values of the parameters.

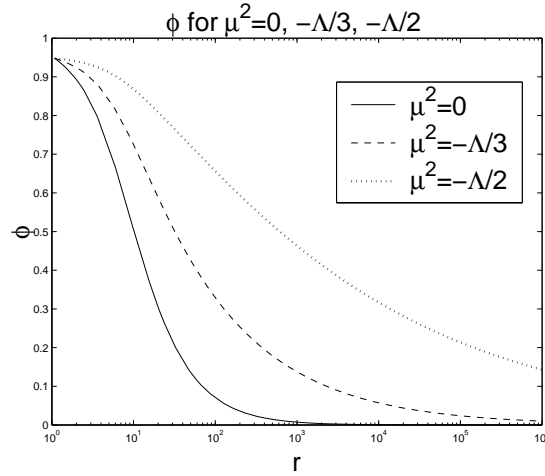


Figure 5. Graph of ϕ (vertical axis) against r (horizontal axis), for the parameter values given previously. The scalar field function is monotonic for all values of the mass.

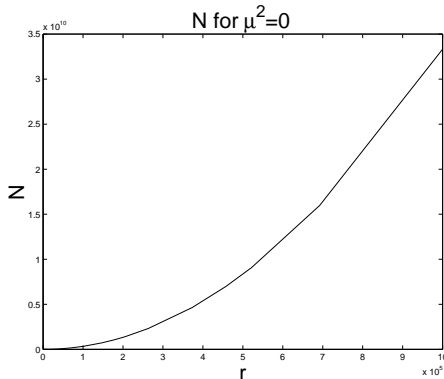


Figure 6. Graph of N (vertical axis) against r (horizontal axis), for the parameter values given previously, just in the specific case $\mu^2 = 0$. The functions for other values of the mass behave similarly.

values (10):

$$k = \frac{3}{2} - \sqrt{\frac{1}{4} - \frac{3\mu^2}{\Lambda}}$$

It is straightforward to check that our numerical solutions have precisely this power-law fall-off at infinity.

The corresponding metric functions $N(r)$ and $e^{\delta(r)}$ are plotted in figures 6 and 7 respectively. Their behaviour is exactly as anticipated, the geometry approaching anti-de Sitter space as $r \rightarrow \infty$. Note that, using (19), the values of the event horizon radius r_h is 1.0845 for each solution, since we have used the same value of Φ at the event horizon. Similar behaviour is found for other values of the event horizon radius.

The monotonicity of the scalar field ϕ is in contrast to some of the minimally coupled scalar field equations found in [16], where for certain values of the parameters in their double-well potential, they find solutions in which the scalar field oscillates as it decays to zero. Although our potential has, for some values of the parameters, similar qualitative features as the double-well potential (see figure 2) for all values of the parameters of interest in our case, the scalar field is monotonic. For hairy black holes in the Einstein-Yang-Mills (EYM) system in adS, it is found that those solutions in which the single function describing the gauge field has no zeros are linearly stable [18]. Stable solutions are also found in the minimally coupled system when the scalar field has no zeros [16]. One may therefore conjecture that, since in the solutions here the scalar field has no zeros, these configurations may too be stable.

Before we address this question in the next section, we firstly consider the stress-energy tensor for the conformally coupled scalar field.

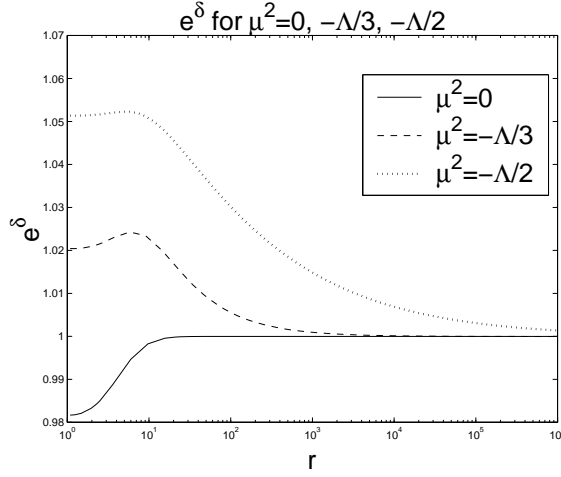


Figure 7. Graph of e^δ (vertical axis) against r (horizontal axis), for the parameter values given previously. In each case $\delta \rightarrow 0$ at infinity, so that the geometry is asymptotically anti-de Sitter.

Its components are given by the following:

$$\begin{aligned}
 \left(1 - \frac{1}{6}\phi^2\right) T_t^t &= -\frac{1}{6}N\phi'^2 - \frac{1}{6}e^{-2\delta} (Ne^{2\delta})' \phi\phi' + \frac{1}{3}\phi\nabla_\mu\nabla^\mu\phi \\
 &\quad -V(\phi) - \frac{1}{6}\Lambda\phi^2; \\
 \left(1 - \frac{1}{6}\phi^2\right) (T_t^t - T_r^r) &= -\frac{2}{3}N\phi'^2 + \frac{1}{3}N\phi\phi'' - \frac{1}{3}\phi\phi'\delta'; \\
 \left(1 - \frac{1}{6}\phi^2\right) (T_t^t - T_\theta^\theta) &= \frac{1}{6} \left[\frac{2N}{r} - e^{-2\delta} (Ne^{2\delta})' \right] \phi\phi'. \quad (32)
 \end{aligned}$$

The above formulae are almost identical to those given in [26] for the conformally coupled scalar field in asymptotically flat space. The cosmological constant simply adds an extra term to T_t^t . One of the key assumptions of [26] is an energy condition that

$$\text{sign}(T_t^t) = \text{sign}(T_t^t - T_r^r) = \text{sign}(T_t^t - T_\theta^\theta). \quad (33)$$

This is equivalent to the consensus amongst all observers on time-like paths as to the sign of the energy density. This is a considerably weaker statement than the weak energy condition (1), which states that the three quantities given above are all negative. It is by no means clear whether this is a reasonable expectation for our system, given not only the non-minimal coupling of matter to the geometry, but also the presence of a negative cosmological constant. We plot, in figures

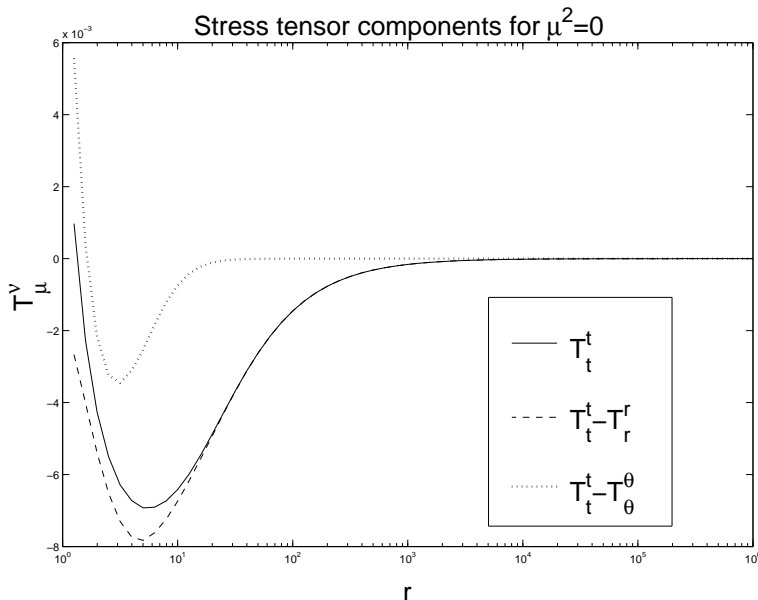


Figure 8. Graph of the stress tensor components for the $\mu^2 = 0$ solution presented above. Note that all three components plotted are negative except in a region very close to the event horizon.

8-10, the three quantities (32) for the solutions presented earlier in this section. Figures 8-10 show that (33) holds, apart from in a small region close to the event horizon. In consequence, the weak energy condition is also violated close to the event horizon. Violation of the weak energy condition close to the event horizon has been observed to be a key feature of other models (albeit in asymptotically flat space) possessing hairy black hole solutions [31, 32]. For all values of the mass, we find that $T_t^t - T_\theta^\theta$ is positive near the event horizon, and that $T_t^t - T_r^r$ is negative everywhere outside the event horizon. For sufficiently small mass, T_t^t is positive close to the event horizon, but for larger values of the mass it is negative everywhere. We compare this behaviour with that for the Gauss-Bonnet-dilaton black holes in [31], where $T_t^t - T_r^r$ is always negative, $T_t^t - T_\theta^\theta$ is positive close to the event horizon, and T_t^t is positive close to the event horizon. In both these models there is non-minimal coupling of the scalar field to the geometry, which seems to be the key factor which results in the violation of the very weak energy condition (33) near the event horizon. This means that, close to the event horizon, there is no consensus amongst observers as to the sign of the energy density, and some observers at fixed r (who are highly accelerated) measure a negative energy density.

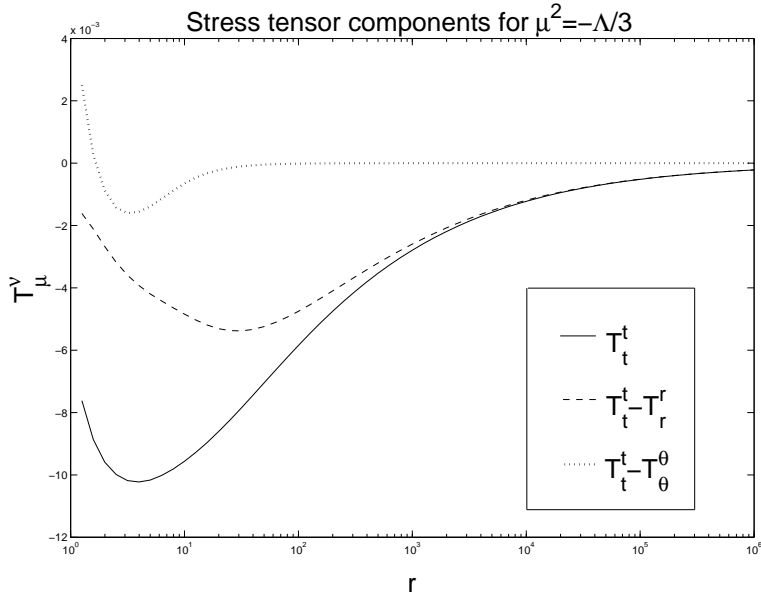


Figure 9. As in figure 8, but with $\mu^2 = -\Lambda/3$.

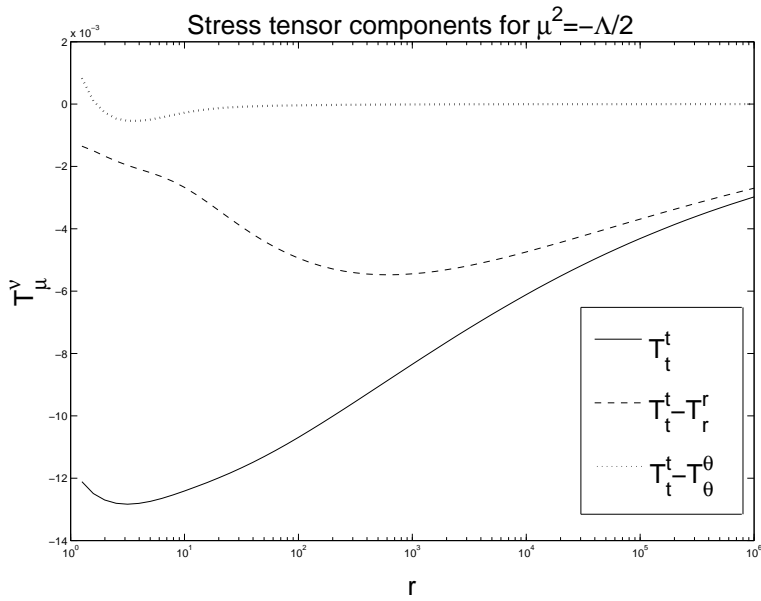


Figure 10. As in figure 8, but with $\mu^2 = -\Lambda/2$.

The solutions found in this section certainly violate the “letter” of the no-hair conjectures, as they represent non-trivial solutions of the Einstein-conformally coupled scalar field equations. However, it remains to be seen whether they violate the “spirit” of the conjecture, i.e. whether or not they are stable. That is the subject of the next section.

7. STABILITY ANALYSIS

We now turn to the question of whether the solutions of the conformally coupled scalar field equations, exhibited in the previous section, are stable.

It is algebraically easiest to begin with the transformed, minimally coupled scalar field system, since we are interested in equilibrium solutions for which the transformation is well-behaved. Therefore we consider time-dependent perturbations of the equilibrium solutions, keeping the geometry and scalar field configuration spherically symmetric:

$$\begin{aligned}\Phi(t, r) &= \Phi(r) + \Phi_1(t, r); \\ \bar{m}(t, r) &= \bar{m}(r) + \bar{m}_1(t, r); \\ \bar{\delta}(t, r) &= \bar{\delta}(r) + \bar{\delta}_1(t, r);\end{aligned}$$

where in each line the first term is the static, equilibrium solution as discussed in the previous section, and the second term (with the subscript 1) denotes a small perturbation. We follow the standard procedure and substitute the above values in the time-dependent field equations, keeping only terms linear in the perturbations. It is straightforward to eliminate the metric perturbations, which are given in terms of the scalar field perturbation Φ_1 by

$$\begin{aligned}\bar{m}_1 &= \frac{1}{2} \bar{N} \bar{r}^2 \frac{d\Phi}{d\bar{r}} \Phi_1; \\ \bar{\delta}_1 &= \bar{r} \frac{d\Phi}{d\bar{r}} \frac{d\Phi_1}{d\bar{r}}.\end{aligned}$$

We consider the usual “tortoise” co-ordinate \bar{r}_* given by

$$\frac{d\bar{r}_*}{d\bar{r}} = \frac{e^{-\bar{\delta}}}{\bar{N}}, \quad (34)$$

and end up with a single perturbation equation for $\psi = r\Phi_1$,

$$-\frac{\partial^2 \psi}{\partial t^2} = -\frac{\partial^2 \psi}{\partial \bar{r}_*^2} + \mathcal{U}\psi; \quad (35)$$

where the perturbation potential \mathcal{U} is given by (cf. [16]):

$$\begin{aligned} \mathcal{U} = \frac{\bar{N}e^{2\bar{\delta}}}{\bar{r}^2} & \left[1 - \bar{N} - \Lambda\bar{r}^2 - U\bar{r}^2 + 2\bar{r}^3 \frac{dU}{d\Phi} \frac{d\Phi}{d\bar{r}} + \bar{r}^2 \frac{d^2U}{d\Phi^2} \right. \\ & \left. + \bar{r}^4 (U + \Lambda) \left(\frac{d\Phi}{d\bar{r}} \right)^2 - \bar{r}^2 \left(\frac{d\Phi}{d\bar{r}} \right)^2 \right]. \end{aligned} \quad (36)$$

It is possible to recast the perturbation equations in terms of quantities in the untransformed, conformally coupled system. Using (23), the perturbations of the conformally coupled system are given in terms of those in the minimally coupled system by:

$$\begin{aligned} \phi_1 &= \Phi_1 \operatorname{sech}^2 \left(\frac{\Phi}{\sqrt{6}} \right); \\ \delta_1 &= \bar{\delta}_1 - \frac{1}{\sqrt{6}} \Phi_1 \tanh \left(\frac{\Phi}{\sqrt{6}} \right) \\ &\quad - \frac{1}{\sqrt{6}} \bar{r} \mathcal{A}^{-1} \left[\frac{d\Phi_1}{d\bar{r}} \tanh \left(\frac{\Phi}{\sqrt{6}} \right) + \frac{1}{\sqrt{6}} \Phi_1 \frac{d\Phi}{d\bar{r}} \operatorname{sech}^2 \left(\frac{\Phi}{\sqrt{6}} \right) \right]; \\ N_1 &= \bar{N}_1 \mathcal{A}^2 + 2\bar{N} \mathcal{A} \frac{1}{\sqrt{6}} \bar{r} \left[\frac{d\Phi_1}{d\bar{r}} \tanh \left(\frac{\Phi}{\sqrt{6}} \right) + \frac{1}{\sqrt{6}} \Phi_1 \frac{d\Phi}{d\bar{r}} \operatorname{sech}^2 \left(\frac{\Phi}{\sqrt{6}} \right) \right], \end{aligned}$$

where \mathcal{A} is as given in equation (22):

$$\mathcal{A} = \left[1 + \frac{1}{\sqrt{6}} \bar{r} \frac{d\Phi}{d\bar{r}} \tanh \left(\frac{\Phi}{\sqrt{6}} \right) \right].$$

In the above equations we have considered the perturbations of the metric function N rather than m as the relationship is simpler to write down. In addition, because the radial co-ordinates r and \bar{r} are related by (19), it is no longer the case that r is unchanged by the perturbations, because we have kept \bar{r} fixed during the perturbations of the minimally coupled system. This corresponds simply to a choice of gauge in the conformally coupled system, which, although not the usual choice, does not affect our conclusions about stability. Note, however, that in transforming back to the conformally coupled system, the definition of the ‘‘tortoise’’ co-ordinate is unchanged, i.e. if

$$\frac{dr_*}{dr} = \frac{e^{-\delta}}{N},$$

then $r_* = \bar{r}_*$ as a consequence of the transformation relations (20,34). Therefore, one could follow the procedure of [23], and rewrite (35) in terms of quantities in the conformally coupled system. However, the numerical values of the potential will be unchanged by this procedure

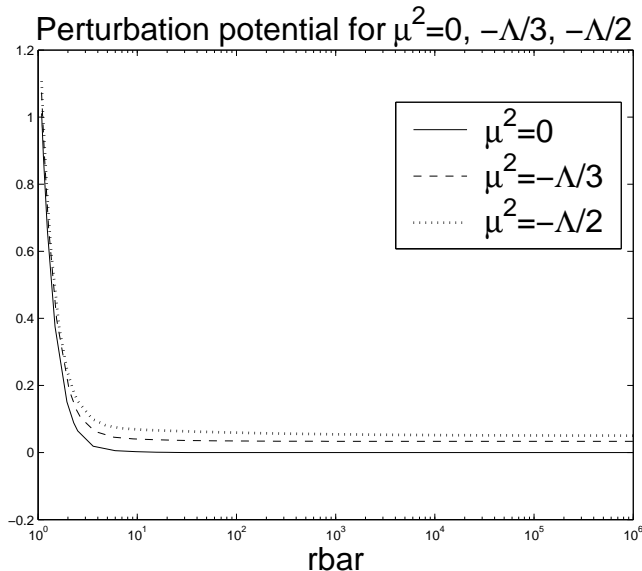


Figure 11. Graph of the stability potential $\bar{r}^2 \mathcal{U} e^{-2\bar{\delta}} \bar{N}^{-1}$ given by (36) for the solutions of section 6. Note that in each case the potential \mathcal{U} vanishes at the event horizon because we have divided by a factor of \bar{N} and tends to a constant at infinity.

(although the algebraic form of the potential will become rather more complicated), and so it is simpler to consider the potential in the form (36).

The potential (36) vanishes at the event horizon ($\bar{N} = 0$) and its form at infinity can be calculated using (28) and the asymptotic form of the scalar field Φ to be

$$\mathcal{U} \rightarrow \bar{N} \mu^2,$$

which diverges like \bar{r}^2 as $\bar{r} \rightarrow \infty$ due to the factor of \bar{N} . For all (non-zero) values of the mass, therefore, the potential is positive at infinity but we cannot immediately say whether the solutions are stable or not. For all the solutions we found, it was the case that the potential is positive everywhere outside the event horizon. Typical examples of such a potential are shown in figure 11, where we have plotted $\bar{r}^2 \mathcal{U} e^{-2\bar{\delta}} \bar{N}^{-1}$ (i.e. the terms in square brackets in (36)) for the solutions exhibited in section 6. The potential \mathcal{U} will vanish at the event horizon because the functions given in figure 11 will be multiplied by a factor of \bar{N} . The functions given in figure 11 are multiplied by a positive constant at infinity and so the potential \mathcal{U} approaches a positive constant at infinity. We therefore conclude that the scalar field solutions in this mass range are linearly stable to spherically symmetric perturbations.

In other words, those solutions in which the scalar field has no zeros are linearly stable, in common with the EYM system [18]. It may be, of course, that there is some instability due to non-spherically symmetric perturbations, but we consider this unlikely. A very detailed investigation of the EYM system [33, 34] has revealed no instabilities for arbitrary linear perturbations, provided the cosmological constant is sufficiently large and negative, and it is reasonable to expect that the same is true here.

8. DISCUSSION AND CONCLUSIONS

In this paper we have investigated the possibility of giving a black hole conformally coupled scalar field hair when we remove the restriction that the geometry be asymptotically flat. We considered a massless or massive scalar field with potential given by $V(\phi) = \frac{1}{2}\mu^2\phi^2$, where μ is the mass of the field, which may be zero. For simplicity, we assumed that the scalar field had no other self-interactions, and that the black hole geometry is spherically symmetric. If the model contains a positive cosmological constant, so that the geometry approaches de Sitter space asymptotically at infinity, we are able to show using a simple method due to Bekenstein [5] that there are no non-trivial solutions of the field equations. With a negative cosmological constant, the geometry is asymptotically anti-de Sitter and we used a conformal transformation to map the conformally coupled scalar field system to a minimally coupled scalar field system with an unusual potential. We numerically integrated the minimally coupled scalar field equations (which are simpler than those for the conformally coupled scalar field) and found non-trivial black hole solutions with scalar field hair, provided that the mass μ satisfies $\mu^2 \leq -2\Lambda/3$. These solutions are found to be stable to spherically symmetric linear perturbations. If $\mu^2 \geq -2\Lambda/3$, then there are no solutions of the minimally coupled system with the desired boundary conditions.

One might be surprised that we have found a mass bound for conformally coupled scalar field hair, such that hair exists only for sufficiently small mass. However this is in accord with studies of scalar field perturbations of anti-de Sitter space [35, 36]. The scalar field equation on this background can be written as:

$$\nabla_\mu \nabla^\mu \phi - \alpha \phi = 0,$$

where

$$\alpha = \mu^2 + 4\xi\Lambda$$

is a constant, for arbitrary coupling to the scalar curvature and arbitrary mass. It is found that stable fluctuations having positive energy are possible only for values of α which are bounded above. In this situation there is some ambiguity because several different models (i.e. with different values of mass μ^2 and coupling constant ξ) give the same value of α . However, in our situation in which we have fixed ξ , the conclusion is that μ^2 should be bounded above, as we have found.

Our results should be compared with those for minimally coupled scalar field hair. For minimally coupled scalar fields, there are no non-trivial solutions in asymptotically flat space. In the presence of a positive cosmological constant, hairy black holes are possible for non-convex potentials [9] but these configurations are always unstable. With a negative cosmological constant, stable scalar field hair is possible [16]. For conformally coupled scalar fields, the only solution in asymptotically flat space is the BBMB black hole [19–21], which has a divergent scalar field on the event horizon. For the simple potentials considered in this paper, we have shown that there is no conformally coupled scalar field hair if the cosmological constant is positive, while stable hair is possible in the presence of a negative cosmological constant. Very recently, a hairy black hole has been found if the potential is quartic [25], generalizing the BBMB black hole (except that now the scalar field is regular on and outside the event horizon). The stability of this new solution is not yet known, although we think it is reasonable to conjecture that it is unstable. These results are summarised in the table below:

	$\xi = 0$	$\xi = 1/6$
$\Lambda = 0$	No hair	No regular hair
$\Lambda > 0$	Unstable hair	(Unstable?) hair
$\Lambda < 0$	Stable hair	Stable hair

Our main result is that, even within the restricted class of potentials considered in this paper, we have found stable hairy black holes with conformally coupled scalar field hair if the cosmological constant is negative.

All the solutions we have found are related to solutions of the minimally coupled scalar field system, via the conformal transformation (11). In asymptotically flat space, there is strong analytic [26] and numerical [29] evidence that there are no solutions for which the conformal transformation is not valid. In asymptotically de Sitter space, the new solution of [25] is such that the conformal transformation breaks down at a point outside the event horizon. However, it remains an open question as to whether, in asymptotically anti-de Sitter space, all

solutions of the conformally coupled scalar field system can be obtained from the minimally coupled system, or whether there are conformally coupled solutions for which the conformal transformation is not valid.

In this paper we have concentrated on a conformally coupled scalar field because it is the most physically relevant coupling and also the conformal transformation has its simplest form for this coupling. The question naturally arises as to whether our results generalise to other (non-minimal) couplings to the scalar curvature. In asymptotically flat space, conformal coupling is special in that only for this coupling is there known to be a non-trivial solution of the field equations, namely the Bekenstein black hole, even though the scalar field diverges at the event horizon. The theorem of Bekenstein and Mayo [26] leaves open the possibility of solutions for $0 < \xi < 1/2$, although we do not expect that solutions will be found for this range of couplings. We would conjecture that the results we have presented here are, qualitatively at least, the same as for more general values of ξ , namely we anticipate that there is stable hair in adS and any hair that exists in dS is unstable. These questions are left for future work.

Acknowledgements

We would like to thank Olivier Sarbach for helpful discussions, and Ricardo Troncoso and Cristian Martinez for their helpful comments. This work was supported by a grant from the Nuffield Foundation, reference NUF-NAL 02.

References

1. Supernova Cosmology Project (S. Perlmutter et al.), “Discovery of a supernova explosion at half the age of the universe”. *Nature* **391**, 51–54 (1998).
2. O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, “Large N field theories, string theory and gravity”. *Phys. Rep.* **323**, 184–386 (2000).
3. A. Strominger, “The dS/CFT correspondence”. *J. High Energy Physics* **0110**, 034 (2001).
4. L. Randall and R. Sundrum, “An alternative to compactification”. *Phys. Rev. Lett.* **83**, 4690–4693 (1999).
5. J. D. Bekenstein, “Nonexistence of baryon number for static black holes”. *Phys. Rev.* **D5**, 1239–1246 (1972).
6. M. Heusler, *Black Hole Uniqueness Theorems*. (Cambridge University Press, Cambridge, 1996).
7. J. D. Bekenstein, “Black hole hair: twenty-five years after”. In: I. M. Dremin and A. M. Semikhatov (eds.): *Proceedings of the Second International Sakharov*

- Conference on Physics, Moscow, Russia, 20-23 May 1996*. pp. 216–219 (World Scientific, Singapore, 1997).
8. J. D. Bekenstein, “Nonexistence of baryon number for black holes. II”. *Phys. Rev. D* **5**, 2403–2412 (1972).
 9. T. Torii, K. Maeda, and M. Narita, “Toward the no-scalar hair conjecture in asymptotically de-Sitter space-time”. *Phys. Rev. D* **59**, 064027 (1999).
 10. R-G. Cai and J-Y. Ji, “Hairs on the cosmological horizon”. *Phys. Rev. D* **58**, 024002 (1998).
 11. M. S. Volkov and D. V. Gal'tsov, “Gravitating non-Abelian solitons and black holes with Yang-Mills fields”. *Phys. Rep.* **319**, 2–83 (1999).
 12. M. Heusler, “A mass bound for spherically symmetrical black hole spacetimes”. *Class. Quantum Gravity* **12**, 779–789 (1995).
 13. M. Heusler and N. Straumann, “Scaling arguments for the existence of static, spherically symmetric solutions of self-gravitating systems”. *Class. Quantum Gravity* **9**, 2177–2189 (1992).
 14. D. Sudarsky, “A simple proof of a no-hair theorem in Einstein-Higgs theory”. *Class. Quantum Gravity* **12**, 579–584 (1995).
 15. J. D. Bekenstein, “Novel ‘no-scalar-hair’ theorem for black holes”. *Phys. Rev. D* **51**, R6608–R6611 (1995).
 16. T. Torii, K. Maeda, and M. Narita, “Scalar hair on the black hole in asymptotically anti-de Sitter space-time”. *Phys. Rev. D* **64**, 044007 (2001).
 17. D. Sudarsky and J. A. Gonzalez, “On black hole scalar hair in asymptotically anti de Sitter spacetimes”. Preprint *gr-qc/0207069* (2002).
 18. E. Winstanley, “Existence of stable hairy black holes in $\mathfrak{su}(2)$ Einstein-Yang-Mills theory with a negative cosmological constant”. *Class. Quantum Gravity* **16**, 1963–1978 (1999).
 19. J. D. Bekenstein, “Exact solutions of Einstein-conformal scalar equations”. *Ann. Phys.* **82**, 535–547 (1974).
 20. J. D. Bekenstein, “Black holes with scalar charge”. *Ann. Phys.* **91**, 75–82 (1975).
 21. N. M. Bocharova, K. A. Bronnikov, and V. N. Mel'nikov, “An exact solution of the system of Einstein equations and mass-free scalar field”. *Vestnik Moskov. Univ. Fizika* **25**, 706–709 (1970).
 22. D. Sudarsky and T. Zannias, “Spherical black holes cannot support scalar hair”. *Phys. Rev. D* **58**, 087502 (1998).
 23. K. A. Bronnikov and Y. N. Kireyev, “Instability of black holes with scalar charge”. *Phys. Lett.* **67A**, 95–96 (1978).
 24. B. C. Xanthopoulos and T. Zannias, “The uniqueness of the Bekenstein black hole”. *J. Math. Phys.* **32**, 1875–1880 (1991).
 25. C. Martinez, R. Troncoso, and J. Zanelli, “De Sitter black hole with a conformally coupled scalar field in four dimensions”. Preprint *hep-th/0205319* (2002).
 26. A. E. Mayo and J. D. Bekenstein, “No hair for spherical black holes: Charged and non-minimally coupled scalar field with self-interaction”. *Phys. Rev. D* **54**, 5059–5069 (1996).
 27. A. Saa, “New no-scalar-hair theorem for black holes”. *J. Math. Phys.* **37**, 2346–2351 (1996).
 28. A. Saa, “Searching for non-minimally coupled scalar hairs”. *Phys. Rev. D* **53**, 7377–7380 (1996).
 29. I. Peña and D. Sudarsky, “Numerical evidence against black holes with non-minimally coupled scalar hair”. *Class. Quantum Gravity* **18**, 1461–1474 (2001).

30. K-I. Maeda, “Towards the Einstein-Hilbert action via conformal transformation”. *Phys. Rev.* **D39**, 3159–3162 (1989).
31. P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis, and E. Winstanley, “Dilatonic black holes in higher curvature string gravity”. *Phys. Rev.* **D54**, 5049–5058 (1996).
32. N. E. Mavromatos and E. Winstanley, “Aspects of hairy black holes in spontaneously-broken Einstein-Yang-Mills systems: Stability analysis and entropy considerations”. *Phys. Rev.* **D53**, 3190–3214 (1996).
33. O. Sarbach and E. Winstanley, “On the linear stability of solitons and hairy black holes with a negative cosmological constant: the odd-parity sector”. *Class. Quantum Gravity* **18**, 2125–2146 (2001).
34. E. Winstanley and O. Sarbach, “On the linear stability of solitons and hairy black holes with a negative cosmological constant: the even-parity sector”. *Class. Quantum Gravity* **19**, 689–723 (2002).
35. P. Breitenlohner and D. Z. Freedman, “Positive energy in anti-de Sitter backgrounds and gauged extended super-gravity”. *Phys. Lett.* **B115**, 197–201 (1982).
36. P. Breitenlohner and D. Z. Freedman, “Stability in gauged extended super-gravity”. *Ann. Phys.* **144**, 249–281 (1982).