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Higher-dimensional solitons and black holes with a non-minimally coupled scalar field

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We study higher-dimensional soliton and hairy black hole solutions of the Einstein equations non-minimally coupled to a scalar field. The scalar field has no self-interaction potential but a cosmological constant is included. Non-trivial solutions exist only when the cosmological constant is negative and the constant governing the coupling of the scalar field to the Ricci scalar curvature is positive. At least some of these solutions are stable when this coupling constant is not too large.

I. INTRODUCTION

The existence and uniqueness of black hole solutions of the Einstein equations with various types of matter has been a rich avenue of research for many years. The classic "no-hair" theorems (see, for example, [1] for a review) proved the uniqueness of the Kerr-Newman family of metrics describing four-dimensional, asymptotically flat, black hole solutions of the Einstein equations with an electromagnetic field or in a vacuum. More recently, there has been an explosion of interest in the generalization of these uniqueness results to higher-dimensional black holes, and particular, in the non-uniqueness of higher-dimensional rotating black holes (see, for example, [2]).

Black hole solutions of the Einstein-scalar field system have been studied for almost as long as those of the Einstein-Maxwell system (see [1] for a detailed review, and [3] for a summary). The case of a minimally coupled scalar field has been the most extensively studied, with a number of "no-hair" results proved, in asymptotically flat space, particularly for static, spherically symmetric black holes [4]. These results depend on some assumptions about the form of the self-interaction potential, typically that it is positive semi-definite. When these assumptions are not satisfied, asymptotically flat black hole and soliton solutions of the field equations can be constructed, some numerically with a particular choice of self-interaction potential [5], and there are also analytic solutions (often with unusual potentials) [6], which can be generated from vacuum solutions in the static case [7].

Including a cosmological constant changes the picture for minimally coupled scalar fields, providing the self-interaction potential is non-zero [8, 9]. When the cosmological constant is positive, numerical [8] and analytic [10] black hole solutions have been found when the self-interaction potential is non-convex [11], although at least some of these solutions are unstable [8]. On the other hand, with a negative cosmological constant, stable black holes with minimally coupled scalar field hair have been found numerically [9], and there are also some analytic solutions [12–14]. Such asymptotically anti-de Sitter (adS) solutions have attracted much recent interest in the literature due to their interpretation via the adS/CFT (conformal field theory) correspondence (some works on this topic include [14, 15]), particularly for black holes with scalar hair in supergravity theories [16], although the latter are unstable [17].

When non-minimal coupling of the scalar field to the Ricci scalar curvature is included in the model, the case of conformal coupling has been of particular interest. The BBMB solution [18] is the unique [19] static, asymptotically flat, black hole solution of the field equations in four space-time dimensions, when the self-interaction potential is zero, but is unstable [20]. Furthermore, the scalar field diverges on the event horizon. An analytic solution has been found in asymptotically flat space when the scalar field self-interaction potential is non-zero [21]. The analogue of the BBMB solution in four dimensions when there is a positive cosmological constant has a quartic self interaction potential [22]. Although the scalar field does not suffer from the divergences of the BBMB black hole, it is still unstable [23]. As for minimal coupling, in the presence of a negative cosmological constant, four-dimensional stable solitons [24] and black holes [25] have been found when the self-interaction potential is either zero or quadratic.

The case of non-minimal coupling which is not conformal coupling has received less attention in the literature. In four-dimensional, asymptotically flat, space-time, various black hole "no-hair" theorems have been proved [26, 27]. With a zero self-interaction potential, four-dimensional black holes are studied in [28]. It is found that non-minimally- and non-conformally-coupled scalar field hair can be supported by the black hole only when there is a negative cosmological constant, in agreement with numerical work [29]. Furthermore, the hair is stable only when the constant governing the coupling between the scalar field and the Ricci scalar curvature is such that the Breitenlohner-Freedman bound [30] on the

arXiv:0907.1487v2 [gr-qc] 11 Jun 2012

"effective mass" is satisfied.

Most of the above work is concerned with static, spherically symmetric black holes in four space-time dimensions. In asymptotically adS space, as well as spherically symmetric black holes, the well-known topological black holes exist (see, for example, [31]). Topological black holes with scalar field hair have not been widely studied, apart from an analytic solution in the minimally coupled case [32], and numerical solutions in the conformally coupled case [24].

Solitons and black holes with scalar hair in more than four space-time dimensions have received more attention in the literature in recent years, particularly in asymptotically adS space [13, 16, 24]. The techniques for generating minimally coupled scalar solutions from solutions of the vacuum Einstein equations have been generalized [33], yielding new analytic solutions. For conformally coupled scalar fields, there is no higher-dimensional analogue of the asymptotically flat BBMB black hole [34], although the corresponding analytic solution in three-dimensional, asymptotically adS space does exist [35], and there are numerical soliton and black hole solutions in adS in higher dimensions [24], all with vanishing self-interaction potential. A set of analytic solutions in various dimensions with a non-zero self-interaction potential and a cosmological constant has also been considered recently in the conformally coupled case [36]. For other non-minimal coupling, Saa [27] has proved a no-hair theorem in the asymptotically flat case, although for certain values of the coupling constant assumptions on the magnitude of the scalar field are required.

Our purpose in this paper is to consider non-minimal, non-conformal coupling, extending the analysis of [28] to solitons, higher-dimensional black holes, and topological black holes in asymptotically adS space. The structure of this paper is as follows. In Sec. II we describe our scalar field model, the field equations and boundary conditions

to be satisfied by soliton and black hole solutions, and the conformal map which transforms our non-minimally coupled scalar field system to one with a minimally coupled scalar field and a non-zero self-interaction potential. For most of the parameter space, we are able to prove that non-trivial solutions cannot exist, and these non-existence results are presented in Sec. III. Non-trivial solutions exist in the remainder of the parameter space, and in Sec. IV we present our numerical solutions and study their thermodynamics and stability. Finally our conclusions are presented in Sec. V.

II. THE MODEL

A. Field equations

We consider the following action, describing a scalar field ϕ with non-minimal coupling to gravity in n space-time dimensions:

$$S = \frac{1}{2} \int d^n x \sqrt{-g} \left[(R - 2\Lambda) - (\nabla\phi)^2 - \xi R \phi^2 \right], \quad (1)$$

where R is the Ricci scalar curvature, and we have included a cosmological constant Λ. Here and throughout this paper, the metric has signature $(-, +, +, +)$ and we use units in which $c = \hbar = 8\pi G = k_B = 1$. The constant ξ governs the coupling between the scalar field and the Ricci scalar curvature. Two values of ξ are particularly important: minimal coupling $\xi = 0$, and conformal coupling $\xi = \xi_c = (n-2)/[4(n-1)]$. In the action (1) we have set any self-interaction potential to zero, for simplicity.

Varying the action (1) with respect to the field variables, we obtain the Einstein equations

$$\left(1 - \xi\phi^2\right) G_{\mu\nu} + g_{\mu\nu}\Lambda = (1 - 2\xi)\nabla_\mu\phi\nabla_\nu\phi + \left(2\xi - \frac{1}{2}\right)g_{\mu\nu}(\nabla\phi)^2 - 2\xi\phi\nabla_\mu\nabla_\nu\phi + 2\xi g_{\mu\nu}\phi\nabla^2\phi; \qquad (2)$$

and scalar field equation

$$\nabla^2\phi - \xi R\phi = 0. \qquad (3)$$

Taking the trace of the Einstein equations (2) gives the Ricci scalar curvature

$$R = -\frac{2(n-1)(\xi - \xi_c)(\nabla\phi)^2 - n\Lambda}{\frac{n}{2} - 1 + 2(n-1)\xi(\xi - \xi_c)\phi^2}. \qquad (4)$$

We are interested in static, spherically symmetric, solitons and black holes for all values of the cosmological constant Λ; and static topological black holes when $\Lambda < 0$.

We therefore use the following ansatz for the metric, in the usual Schwarzschild-like co-ordinates:

$$ds^2 = -H(r)e^{2\delta(r)}\,dt^2 + H(r)^{-1}\,dr^2 + r^2\,d\sigma_{n-2,k}^2, \quad (5)$$

where

$$d\sigma_{n-2,k}^2 = d\psi^2 + f_k^2(\psi)\,d\Omega_{n-3}^2 \qquad (6)$$

denotes the line element of an $(n-2)$-dimensional space Σ_k with constant curvature. The discrete parameter k takes the values $1, 0, -1$, for which we have the following

forms of the function $f_k(\psi)$:

$$f_k(\psi) = \begin{cases} \sin\psi, & \text{for } k = 1, \\ \psi, & \text{for } k = 0, \\ \sinh\psi, & \text{for } k = -1. \end{cases} \qquad (7)$$

The value $k = 1$ gives the usual spherically symmetric metric, with the hypersurface Σ_1 equal to an $(n-2)$-sphere; when $k = -1$, the hypersurface Σ_{-1} has constant negative curvature, and when $k = 0$, the hypersurface Σ_0 is an $(n-2)$-dimensional Euclidean space (see [31] for further details). It is convenient to introduce a new metric function $m(r)$ by

$$H(r) = k - \frac{2m(r)}{r^{n-3}} - \frac{2\Lambda r^2}{(n-2)(n-1)}. \qquad (8)$$

With the ansatz (5), and assuming that the scalar field ϕ depends only on the radial co-ordinate r, the field equations (2, 3) take the form:

$$0 = \frac{n-2}{2r}(1 - \xi\phi^2)\left[H' - \frac{n-3}{r}(k - H)\right] - \left(2\xi - \frac{1}{2}\right)H\phi'^2 + \xi\phi\phi'(H' + 2H\delta') - 2\xi\phi\nabla^2\phi + \Lambda;$$

$$0 = \frac{n-2}{r}(1 - \xi\phi^2)\delta' - (1 - 2\xi)\phi'^2 - 2\xi\phi(\delta'\phi' - \phi'');$$

$$0 = H\phi'' + \phi'\left(H\delta' + H' + H\frac{n-2}{r}\right) - \xi R\phi. \qquad (9)$$

We now summarize the discussion of Sec. I, namely what is known about the existence of soliton and black hole solutions of these field equations, for various values of the cosmological constant Λ and coupling constant ξ. Firstly, we bring together in one table the known results for solitons and black holes in four space-time dimensions:

$n = 4$	$\xi < 0$	$\xi = 0$	$0 < \xi < \xi_c$	$\xi = \xi_c$	$\xi > \xi_c$
$\Lambda > 0$	no black hole hair [28]	no black hole hair [8, 11]	no black hole hair [28]	no black hole hair [25]	no black hole hair [28]
$\Lambda = 0$	no black hole hair [26]	no black hole hair [4]	no black hole hair [26]	unstable black hole hair [18]	no black hole hair [26]
$\Lambda < 0$	no black hole hair [28]	no black hole hair [9]	stable black hole hair [28]	stable solitons and black holes [24, 25]	unstable black hole hair [28]

TABLE I: Summary of existence and non-existence of soliton and black hole solutions in four space-time dimensions.

It can be seen that the existence of hair for spherically symmetric black holes has been studied for all values of the cosmological constant Λ and coupling constant ξ. However, to date solitons and topological black holes have been considered only when the scalar field is conformally coupled, $\xi = \xi_c$.

A similar table for $n > 4$ reveals that the parameter space has been much less widely explored:

$n > 4$	$\xi < 0$	$\xi = 0$	$0 < \xi < \xi_c$	$\xi = \xi_c$	$\xi > \xi_c$
$\Lambda > 0$					
$\Lambda = 0$	no black hole hair [27]		no black hole hair [27]	no black hole hair [34]	no black hole hair [27]
$\Lambda < 0$				stable solitons and black holes [24]	

TABLE II: Summary of existence and non-existence of soliton and black hole solutions in more than four space-time dimensions.

It should be emphasized that the results in Tabs. I and II are only for *zero* self-interaction potential, as many of the solutions which have been found in more than four space-time dimensions have a non-zero potential V [13, 16, 36]. The aim of the present paper is to complete table II.

B. Boundary conditions

Before we can investigate the existence (or otherwise) of black hole and soliton solutions of the field equations (2, 3) we need to specify the boundary conditions at the origin (for solitons), black hole event horizon (if there is one) and at infinity.

Firstly, regular soliton solutions can exist only when

$k = 1$. Near the origin, the field variables have the expansions [24]:

$$H = 1 + \frac{2\left(\Lambda - 2\xi^2\phi_0^2 R_0\right)}{(n-1)(n-2)(\xi\phi_0^2-1)}r^2 + O(r^4);$$

$$\delta = \delta_0 + \frac{\xi^2\phi_0^2 R_0}{(n-1)(n-2)(\xi\phi_0^2-1)}r^2 + O(r^4);$$

$$\phi = \phi_0 + \frac{\xi\phi_0 R_0}{2(n-1)}r^2 + O(r^4); \quad (10)$$

where

$$R_0 = \frac{n\Lambda}{\frac{n}{2}-1+2(n-1)\xi(\xi-\xi_c)\phi_0^2} \quad (11)$$

is the value of the Ricci scalar curvature at the origin. We note that the constants δ_0 and ϕ_0 are arbitrary; the value of δ_0 will be fixed by the boundary conditions at infinity.

For black holes with a regular, non-extremal event horizon at $r = r_h$, we assume that all the variables have regular Taylor series expansions in a neighbourhood of the event horizon:

$$H(r) = H'(r_h)(r-r_h) + O\left(r-r_h\right)^2;$$

$$\delta(r) = \delta_h + \delta'(r_h)(r-r_h) + O\left(r-r_h\right)^2;$$

$$\phi(r) = \phi_h + \phi'(r_h)(r-r_h) + O\left(r-r_h\right)^2; \quad (12)$$

where [24]

$$H'(r_h) = (n-3)\frac{k}{r_h} + \frac{2r_h\left(\Lambda-\xi^2\phi_h^2 R_h\right)}{(n-2)(\xi\phi_h^2-1)};$$

$$\phi'(r_h) = \frac{\xi R_h \phi_h}{H'(r_h)}; \quad (13)$$

and

$$R_h = \frac{n\Lambda}{n/2-1+2(n-1)\xi(\xi-\xi_c)\phi_h^2} \quad (14)$$

is the value of the Ricci scalar curvature on the event horizon. We computed $\delta'(r_h)$ directly (without using $\phi''(r_h)$ as in [24]), but the expression so obtained is too lengthy to reproduce here. Since the field equations do not involve δ, but only its derivatives, the constant δ_h is arbitrary, and will be fixed by the boundary conditions at infinity. The constant ϕ_h is also arbitrary. Similar expressions to (12, 13) hold at the cosmological horizon in the case $\Lambda > 0$.

We also require the behaviour of the field variables as $r \to \infty$. To ensure consistency between the Einstein equations (9) and the form of the Ricci scalar (4), since the self-interaction potential is zero, we will assume that the scalar field vanishes as $r \to \infty$.

Considering firstly the asymptotically flat case $\Lambda = 0$, the field variables have the following behaviour:

$$\phi(r) = \frac{c_1}{r^{n-3}} + O\left(r^{-n+2}\right);$$

$$H(r) = 1 + O(r^{-n+3});$$

$$\delta(r) = O\left(r^{-2(n-3)}\right); \quad (15)$$

and the Ricci scalar curvature is $R \sim O(r^{-2(n-2)})$. The asymptotic form (15) means that the metric function $m(r)$ (8) tends to a constant as $r \to \infty$, namely the mass of the solution.

For asymptotically anti-de Sitter solutions with $\Lambda < 0$, the boundary conditions near infinity for a massive, minimally coupled, scalar field have been studied in depth [37]. In our non-minimally coupled case, with $\Lambda \neq 0$, we define a constant p such that the behaviour of the scalar field as $r \to \infty$ is

$$\phi(r) = \frac{c_1}{r^p} + O(r^{-(p+1)}), \quad (16)$$

then from the scalar field equation (9) we find

$$p = \left(\frac{n-1}{2}\right)\left(1 \pm \sqrt{1 - \frac{4n}{n-1}\xi}\right). \quad (17)$$

The value of p in (17) has the appropriate limits when $\xi = \xi_c$ [24] or $n = 4$ [28]. If $\xi < 0$, then one of the values of p is negative, leading to a scalar field which diverges as $r \to \infty$. We wish to rule this out and therefore choose the positive sign in (17) in this case. If $0 \leq \xi < (n-1)/(4n)$, both roots for p are positive, and the dominant behaviour of ϕ will be given by the smaller root, which has the negative sign in (17). If $\xi > (n-1)/(4n)$, the constant p has a non-zero imaginary part, which leads us to expect oscillatory behaviour in the function ϕ as $r \to \infty$, as was observed in the four-dimensional case [28].

The behaviour of the metric function $\delta(r)$ as $r \to \infty$ is readily found, from the second equation in (9), to be

$$\delta(r) = \frac{\delta_1}{r^{2p}} + O(r^{-2p-1}), \quad (18)$$

while the leading order behaviour of the metric function $H(r)$ is found, from the first equation in (9):

$$H(r) = k - \frac{2\Lambda r^2}{(n-2)(n-1)} + O(r^{-2(p-1)}). \quad (19)$$

The behaviour (18) and (19) ensures that the metric (5) approaches adS or de Sitter space as $r \to \infty$.

The behaviour of the metric function $H(r)$ (19) has interesting consequences for the metric function $m(r)$ (8). We find

$$m(r) = M + \frac{M_1}{r^q} + O(r^{-q-1}), \quad (20)$$

where

$$q = 2p + 1 - n = \pm(n-1)\sqrt{1 - \frac{4n}{n-1}\xi}, \quad (21)$$

and M and M_1 are constants. When $\xi < 0$, the plus sign in (21) is relevant and q is real and positive, so that $m(r)$ converges as $r \to \infty$. When $0 < \xi < (n-1)/(4n)$, the negative sign in (21) will be the dominant behaviour, so q is real and $q < 0$. When $\xi > (n-1)/(4n)$, we have that

q is purely imaginary. Thus the function $m(r)$ diverges as $r \to \infty$ for $0 < \xi < (n-1)/(4n)$ and is oscillatory (with no decay in the oscillations) for $\xi > (n-1)/(4n)$. Therefore, although the metric (5) approaches pure adS or de Sitter as $r \to \infty$, the function $m(r)$ has no finite limit. We expect that the definition of the total mass of the soliton or black hole, using the conserved charges approach of [37], will have a contribution from the scalar field as well as from the metric function $m(r)$. We will not examine this further in this paper, leaving open the question of how a finite mass for the configurations is defined.

However, it is not immediately apparent from (20, 21) how a finite limit for $m(r)$ arises in the case of conformal coupling [24], $\xi = \xi_c = (n-2)/[4(n-1)]$, when $q = -1$ (21). To see this, we need to consider the constant M_1. By comparing the form of the Ricci scalar expressed in terms of the scalar field (4) with that computed directly from the metric (5), we find the following equation for M_1:

$$
\begin{aligned}
&- q(q-1)\frac{3}{c_1^2 \Lambda} M_1 \\
&= n(n-1)^2 (\xi - \xi_c) \left[\frac{2\xi}{n-2} - \frac{2p^2}{(n-1)(n-2)n} \right] \\
&\quad - (2p-n)\frac{p}{n-1} \left[p(1-4\xi) - 2\xi \right].
\end{aligned}
\tag{22}
$$

From the expression for p it is straightforward to show that the right-hand-side of (22) vanishes for all n when $\xi = \xi_c$, while the coefficients multiplying M_1 are non-zero in this case. Therefore it must be the case that $M_1 = 0$ for $\xi = \xi_c$, which means that the leading order term in $m(r)$ as $r \to \infty$ becomes the constant M, in agreement with [24].

C. Conformal transformation

As in four space-time dimensions [28], we will find it useful at various points in our analysis to employ a conformal transformation [18, 38] which maps the Einstein-non-minimally-coupled-scalar field system to the mathematically simpler Einstein-minimally-coupled-scalar field system (see [39] for related works on the conformal transformation).

The conformal transformation is defined by the following [38],

$$
\overline{g}_{\mu\nu} = \Omega^{\frac{2}{n-2}} g_{\mu\nu},
\tag{23}
$$

where $\Omega = 1 - \xi\phi^2$. This is only valid for $\Omega \neq 0$, which is always true for $\xi < 0$ but bounds the scalar field when $\xi > 0$. With this transformation, the action (1) becomes

$$
\overline{S} = \int d^n x \sqrt{-\overline{g}} \left[\frac{1}{2} \left(\overline{R} - 2\Lambda \right) - \frac{1}{2} \left(\overline{\nabla}\Phi \right)^2 - U(\Phi) \right],
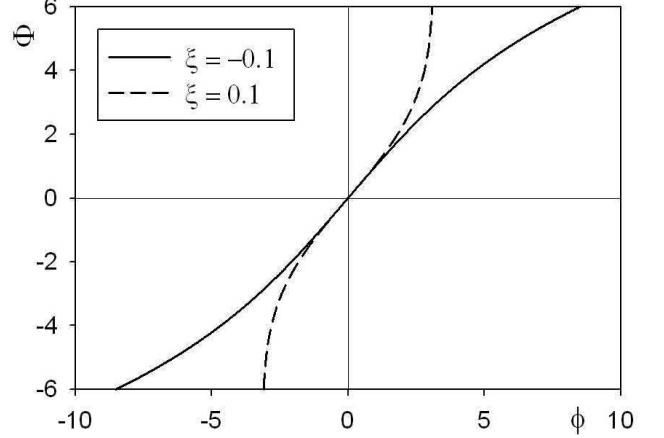\tag{24}
$$

FIG. 1: The minimally coupled scalar field Φ (25) as a function of the non-minimally coupled scalar field ϕ for $n = 6$ and two values of the coupling constant ξ. Similar behaviour is observed for different values of n and ξ.

where a bar denotes quantities calculated using the transformed metric (23), and the cosmological constant Λ is unchanged by the transformation. As in [38], we define a new scalar field Φ by

$$
\Phi = \int d\phi \left[\frac{(n-2)\Omega + (n-1)(4\xi^2\phi^2)}{(n-2)\Omega^2} \right]^{\frac{1}{2}},
\tag{25}
$$

choosing the constant of integration so that $\Phi = 0$ when $\phi = 0$. The integral (25) can be performed analytically, however the expression is very long except in the conformally coupled case [24]. The new scalar field Φ (25) is a single-valued function of ϕ as long as the quantity under the square root is positive, which places a further constraint on the values of ϕ. The scalar field Φ is plotted as a function of ϕ in Fig. 1 for $n = 6$ and two particular values of the coupling constant ξ. When $\xi < 0$, the function Φ is defined for all values of ϕ, but when $\xi > 0$, the function Φ is defined only for those values of ϕ such that the integrand in (25) is real, which gives a finite range of values of ϕ centred on the origin.

The transformed potential $U(\Phi)$ can be written implicitly in terms of ϕ [38]:

$$
U = \Lambda \left[\Omega^{-\frac{n}{n-2}} - 1 \right].
\tag{26}
$$

As in [24, 28], the presence of a cosmological constant means that the potential $U(\Phi)$ for the minimally coupled scalar field Φ is non-zero although the potential for the non-minimally coupled scalar field ϕ vanishes. It should be emphasized that the potential $U(\Phi)$ is not physical, which means that we can circumvent the conditions on the potential in the "no-hair" theorems in the minimally coupled case [4] and have non-trivial solutions. Examples of the shape of the potential $U(\Phi)$ can be seen in Fig. 2. For $\xi < 0$, the potential is defined for all values of Φ and we find that $U(\Phi)/\Lambda$ is negative for all Φ, with a maximum at $\Phi = 0$. For $\xi > 0$, the range of values of
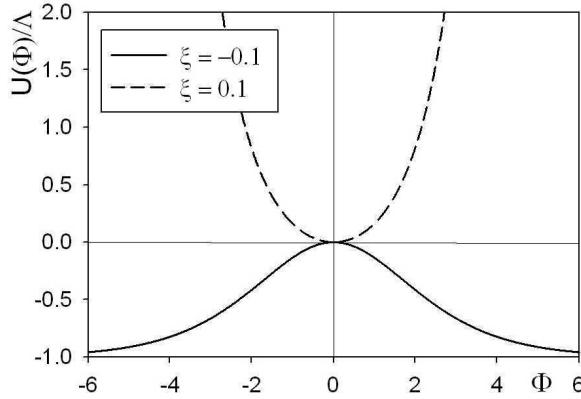
FIG. 2: The potential $U(\Phi)/\Lambda$ (26) as a function of the minimally coupled scalar field Φ for $n = 6$ and two values of the coupling constant ξ. Similar behaviour is observed for other values of n and ξ.

Φ is finite and we find that $U(\Phi)/\Lambda$ is positive, with a minimum at $\Phi = 0$.

The main advantage of the conformal transformation is that it simplifies the field equations (9) considerably. If we assume that the transformed metric (23) is spherically symmetric, we may write it in the form

$$d\overline{s}^2 = -\overline{H}(\overline{r})e^{2\overline{\delta}(\overline{r})}\,dt^2 + \overline{H}(\overline{r})^{-1}\,d\overline{r}^2 + \overline{r}^2\,d\sigma_{n-2,k}^2, \quad (27)$$

where we have introduced a new radial co-ordinate $\overline{r}$ by

$$\overline{r} = \Omega^{\frac{1}{n-2}}r, \quad (28)$$

and the new metric functions $\overline{H}(\overline{r})$ and $\overline{\delta}(\overline{r})$ are given by:

$$\overline{H}(\overline{r}) = H(r)\left[1 - \frac{2\xi\phi\phi'r}{(n-2)\Omega}\right]^2;$$

$$\overline{\delta}(\overline{r}) = \delta(r) + \ln\left[\Omega^{\frac{1}{n-2}}\left(1 - \frac{2\xi\phi\phi'r}{(n-2)\Omega}\right)^{-1}\right]. \quad (29)$$

We note that $\overline{H} = 0$ whenever $H = 0$ so the horizon structure of the space-time is preserved by the conformal transformation. With the metric ansatz (27), the field equations for the transformed system take the form

$$0 = \frac{n-2}{2\overline{r}}\left[\frac{d\overline{H}}{d\overline{r}} - \frac{n-3}{\overline{r}}(k - \overline{H})\right] - \frac{1}{2}\overline{H}\left(\frac{d\Phi}{d\overline{r}}\right)^2$$
$$+ U(\Phi) + \Lambda;$$

$$0 = \frac{n-2}{\overline{r}}\frac{d\overline{\delta}}{d\overline{r}} - \left(\frac{d\Phi}{d\overline{r}}\right)^2;$$

$$0 = \overline{H}\frac{d^2\Phi}{d\overline{r}^2} + \frac{d\Phi}{d\overline{r}}\left(\overline{H}\frac{d\overline{\delta}}{d\overline{r}} + \frac{d\overline{H}}{d\overline{r}} + \overline{H}\frac{n-2}{\overline{r}}\right)$$
$$- \frac{dU(\Phi)}{d\Phi}. \quad (30)$$

III. NON-EXISTENCE OF SOLUTIONS

In this section we generalize the simple results in [28] to higher dimensions and solitons. As in [28], we emphasize that our method can only be used to show non-existence of solutions for which the scalar field remains regular everywhere, including at the origin, infinity and on any event or cosmological horizon. Therefore our results are evaded by the BBMB black hole [18] as in that case the scalar field diverges on the event horizon. We begin with non-existence results which can be proven simply from the scalar field equation (3) before turning to results where we need to use the conformal transformation of section II C.

A. Simple non-existence results

Starting with the scalar field equation (9), we multiply both sides by $\phi r^{n-2}e^\delta$ and integrate from $r = x$ to $r = y$, where $x = 0$ for soliton solutions and $x = r_h$, the event horizon, for black hole solutions; and where $y = r_c$, the radius of the cosmological horizon, for $\Lambda > 0$ and $y = \infty$ for $\Lambda \leq 0$. This gives the equation

$$0 = \int_x^y dr\left[\xi R\phi^2 r^{n-2}e^\delta - \phi\left(H\phi'r^{n-2}e^\delta\right)'\right]$$
$$= -\left[\phi H\phi'r^{n-2}e^\delta\right]_x^y$$
$$+ \int_x^y dr\, r^{n-2}e^\delta\left(\xi R\phi^2 + H\phi'^2\right), \quad (31)$$

where we have integrated by parts in the second line.

We now need to consider carefully the behaviour of the boundary term in (31) at each of the four possible boundary points ($r = 0$, $r = r_h$, $r = r_c$ and $r = \infty$), using the boundary conditions derived in Sec. II B:

$r = 0$: At the origin, from (10), we have H, ϕ and δ all $O(1)$, while $\phi' = O(r)$. Therefore the boundary term is $O(r^{n-1})$ and vanishes as $r \to 0$.

$r = r_h$ and $r = r_c$: Since $H = 0$ at either an event or cosmological horizon, the boundary term vanishes providing that all the other variables remain finite there.

$r = \infty$: These are the most complicated boundary conditions and we need to consider the asymptotically flat and asymptotically adS cases separately.

Firstly, if $\Lambda = 0$, from (15) we have $\phi = O(r^{-(n-3)})$ and $\phi' = O(r^{-(n-2)})$, with $H, e^\delta \to 1$, so that the boundary term is $O(r^{-(n-3)})$ and vanishes as $r \to \infty$.

Secondly, if $\Lambda < 0$, we have $H = O(r^2)$, and, from (16), we have $\phi = O(r^{-p})$, $\phi' = O(r^{-p-1})$, so the boundary term is $O(r^{-2p+n-1}) = O(r^{-q})$ where q is given by (21). Following the discussion in Sec. II B, the boundary term therefore vanishes only when $\xi < 0$ in this case.

Assuming that the boundary term vanishes (as will be the case in the rest of this section), substituting in for the form of the Ricci scalar using (4) we can write (31) as:

$$0 = \int_x^y dr \; r^{n-2} e^\delta \left[\frac{\mathcal{F}}{\mathcal{G}} \right], \qquad (32)$$

where

$$\begin{aligned} \mathcal{F} &= H\phi'^2 \left(\frac{n}{2} - 1 \right) + n\Lambda\xi\phi^2; \\ \mathcal{G} &= 2\xi\phi^2 (n-1)(\xi - \xi_c) + \left(\frac{n}{2} - 1 \right). \end{aligned} \qquad (33)$$

We now apply (32) to various cases where we have empty spaces in Tab. II. In many of these cases the argument is a simple generalization of the corresponding result in [28].

1. $\Lambda < 0$ and $\xi < 0$

In this case the boundary term at infinity vanishes, as well as at the origin or black hole event horizon. For these parameter values the function $\mathcal{G} > 0$ and finite everywhere inside the integration range. It is straightforward to see that $\mathcal{F}$ is a sum of positive terms, and therefore the only way the integral (32) can vanish is if $\mathcal{F} \equiv 0$ everywhere inside the integration range. This is only possible if $\phi \equiv 0$ everywhere.

2. $\Lambda > 0$ and $\xi > \xi_c$

The non-existence proof in this case follows the argument in Sec. III A 1, except that here we consider only the region inside the cosmological horizon. Again $\mathcal{G} > 0$ and finite everywhere within the integration range. Also, the function $\mathcal{F}$ is again a sum of positive terms so the only solution is again $\phi \equiv 0$.

3. $\Lambda > 0$ and $0 < \xi < \xi_c$

Here the argument is slightly more complicated as $\mathcal{G}$ could have a zero somewhere in the integration range. However, at such a point the Ricci scalar curvature would be infinite unless $\mathcal{F}$ also has a zero at that point. Now $\mathcal{F}$ is a sum of positive terms, so if $\mathcal{F} = 0$ at a point, each term must be zero at that point, meaning that $\phi = 0$ at that point. Substituting $\phi = 0$ into $\mathcal{G}$ gives $\mathcal{G} \neq 0$, which is a contradiction, so $\mathcal{G}$ cannot have any zeros in the region of integration. As $\mathcal{F}$ is always positive and $\mathcal{G}$ is of constant sign, the only way the integral (32) can be zero is if $\mathcal{F} \equiv 0$ everywhere, which again gives us the trivial solution $\phi \equiv 0$.

4. $\Lambda = 0$, $\xi < 0$ and $\xi > \xi_c$

When $\xi < 0$ or $\xi > \xi_c$ it is the case that $\mathcal{G} > 0$. With $\Lambda = 0$, the function $\mathcal{F}$ simplifies to give a single positive term. Therefore, as in previous subsections, $\phi' \equiv 0$, and the condition that $\phi \to 0$ as $r \to \infty$ means that $\phi \equiv 0$ everywhere. These results are in agreement with those derived in [27] by using the conformal transformation of Sec. II C.

5. $\xi = 0$

The minimally coupled case is very straightforward: the function $\mathcal{G} = \frac{n}{2} - 1 > 0$ everywhere. Once again $\mathcal{F}$ reduces to a single positive term, and as in Sec. III A 4, it must be the case that $\phi' \equiv 0$, and hence $\phi \equiv 0$.

6. $\xi = \xi_c$ and $\Lambda \geq 0$

In the conformally coupled case, the function $\mathcal{G} = \frac{n}{2} - 1 > 0$, exactly as in Sec. III A 5. The same argument holds about $\mathcal{F}$ being sum of positive terms so only possible solution is the trivial one where $\phi \equiv 0$.

7. Summary

At this stage it is useful to update Tab. II to incorporate our new results. In Tab. III, we have highlighted new

$n > 4$	$\xi < 0$	$\xi = 0$	$0 < \xi < \xi_c$	$\xi = \xi_c$	$\xi > \xi_c$
$\Lambda > 0$	see Sec. III B	**no solutions**	**no solutions**	**no solutions**	**no solutions**
$\Lambda = 0$	**no solutions**	**no solutions**	no solutions [27]	**no solutions**	**no solutions**
$\Lambda < 0$	**no solutions**	**no solutions**		stable solutions [24]	

TABLE III: Summary of existence and non-existence of soliton and black hole solutions in more than four space-time dimensions.

results in bold, and emphasize that our results rule out both soliton and black hole solutions, including topological black holes. We are not able to use our simple approach in this section to say anything about the case $\Lambda = 0$, $0 < \xi < \xi_c$, which was considered in [27] using the conformal transformation of Sec. II C. We are so far unable to say anything about the cases $\Lambda > 0, \xi < 0$; $\Lambda < 0, 0 < \xi < \xi_c$ and $\Lambda < 0, \xi > \xi_c$. The first of these will be dealt with in

the next section.

B. Non-existence result for $\Lambda > 0$ and $\xi < 0$

The non-existence of black hole solutions in this case was shown in four space-time dimensions in [28]. That result extends trivially to higher-dimensions and solitons, so we only provide a very brief outline.

Since $\xi < 0$, the conformal transformation (23) is valid providing all field variables remain finite. The transformed scalar field equation (30) gives, at the *cosmological* horizon $r = r_c$,

$$\frac{d\Phi}{d\overline{r}} \frac{d\overline{H}}{d\overline{r}} = \frac{dU(\Phi)}{d\Phi}. \tag{34}$$

Now $\overline{H} > 0$ inside the cosmological horizon, and $\overline{H} < 0$ outside the cosmological horizon, so $\frac{d\overline{H}}{d\overline{r}} < 0$ on the cosmological horizon. We then have two cases: $\frac{d\Phi}{d\overline{r}} > 0$, $\frac{dU(\Phi)}{d\Phi} < 0$ and vice versa.

The argument is the same for both cases so we only consider $\frac{d\Phi}{d\overline{r}} > 0$, $\frac{dU(\Phi)}{d\Phi} < 0$. With $\Lambda > 0$ and $\xi < 0$, the potential $U(\Phi)$ is everywhere negative and has a maximum at $\Phi = 0$ (see Fig. 2). Therefore Φ is positive and increasing on the horizon. In order to satisfy the boundary condition on the original scalar field, namely $\phi \to 0$ at infinity, it must be the case that $\Phi \to 0$ at infinity, and hence Φ must have a maximum somewhere outside the cosmological horizon. At such a maximum $\frac{d\Phi}{d\overline{r}} = 0$ and the field equation (30) gives:

$$\overline{H} \frac{d^2\Phi}{d\overline{r}^2} = \frac{dU(\Phi)}{d\Phi} < 0. \tag{35}$$

However, since we are outside the cosmological horizon, it is the case that $\overline{H} < 0$ and so equation (35) gives $\frac{d^2\Phi}{d\overline{r}^2} > 0$ which is a contraction. Therefore there can be no non-trivial solutions in this case.

The conformal transformation can also be applied to the case $\Lambda = 0$, $0 < \xi < \xi_c$, when the transformed potential $U(\Phi) = 0$ (26) [27].

IV. NON-TRIVIAL SOLUTIONS

In the previous section, we were able to rule out non-trivial higher-dimensional soliton or black hole solutions of the field equations (9) except when $\Lambda < 0$ and $0 < \xi < \xi_c$ or $\xi > \xi_c$, solutions with $\Lambda < 0$ and $\xi = \xi_c$ having already been found [24]. It is no surprise that we find non-trivial solutions for these other values of ξ. In four dimensions, spherically symmetric black hole solutions were found in [28]. We find solutions which generalize these to higher dimensions, as well as solitons and topological black holes in four and more space-time dimensions. In this section we first discuss the properties of the numerical solutions, before studying their thermodynamics and stability.
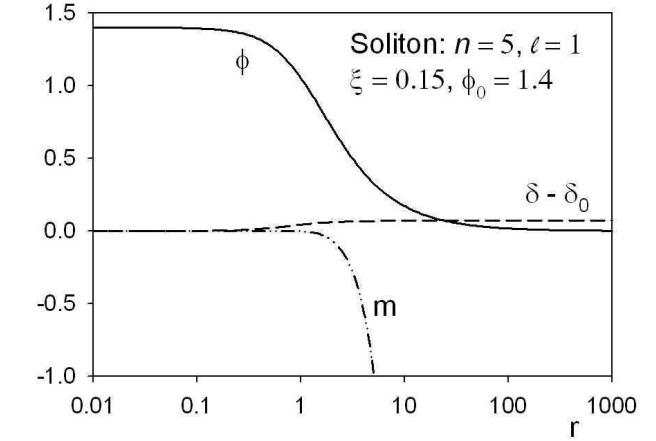

FIG. 3: Example of a five-dimensional soliton solution with $\ell = 1$, $\xi = 0.15$ and $\phi_0 = 1.4$. In this case the scalar field ϕ is monotonic and has no zeros.

A. Numerical solutions

Using a standard ordinary differential equation solver, we integrate the field equations (9), starting close to either the origin or black hole event horizon, as applicable, and integrating towards $r \to \infty$. Since the field equations (9) are invariant under the transformation $\phi \to -\phi$, we only consider positive values of ϕ at the origin or event horizon. Without loss of generality, we set $\Lambda = -(n-1)(n-2)/2$, so that the length scale set by the cosmological constant, $\ell^2 = -(n-2)(n-1)/(2\Lambda) = 1$ (this can be achieved by a rescaling of the co-ordinates and metric function m).

We find soliton, spherically symmetric ($k = 1$) black holes and topological black holes ($k \neq 1$) for any number of space-time dimensions, $n \geq 4$, and any value of the coupling constant $\xi > 0$. Our numerical work indicates that solutions exist for any value of ϕ at the origin or event horizon such that $0 < \phi < \xi^{-1/2}$.

Two typical solutions are illustrated in Figs. 3 and 4. In Fig. 3, a typical five-dimensional soliton solution is shown, for $\xi = 0.15$. This value of ξ is less than $(n-1)/(4n) = 0.20$, and, as predicted in Sec. II B, the scalar field ϕ is monotonic and has no zeros. In Fig. 4 we plot a typical six-dimensional black hole solution with $\xi = 0.2501$. Since this value of ξ is greater than $(n-1)/(4n) = 5/24$, in this case we find that the scalar field ϕ oscillates as it tends to zero as $r \to \infty$, as predicted in Sec. II B, although the magnitude of the oscillations is too small to see in Fig. 4.

This change in the behaviour of the scalar field as $r \to \infty$ on varying the coupling constant ξ is the same as found for the four-dimensional solutions [28], although typically the oscillations in ϕ for $\xi > (n-1)/(4n)$ have
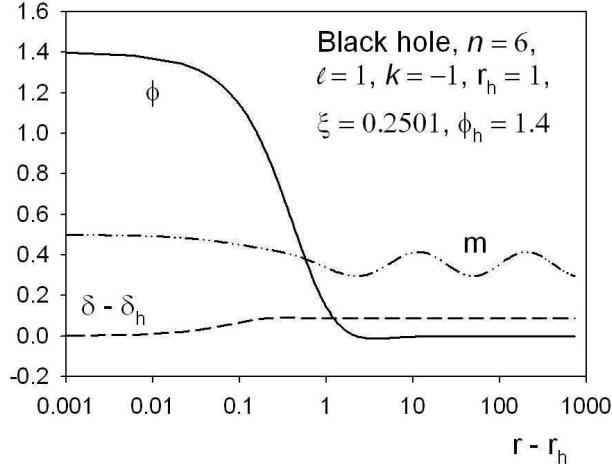
FIG. 4: Example of a six-dimensional topological black hole solution with $k = -1$, $\ell = 1$, $\xi = 0.2501$ and $\phi_0 = 1.4$. In this case we find that the scalar field ϕ oscillates as it tends to zero as $r \to \infty$, although the magnitude of the oscillations is too small to see on the graph.
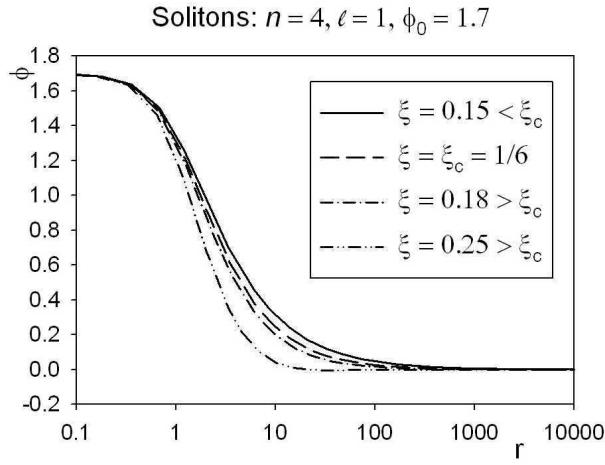

FIG. 5: The effect of varying ξ is shown for four-dimensional soliton solutions with $\ell = 1$ and $\phi_0 = 1.7$. As discussed in Sec. II B, for $\xi < (n-1)/(4n)$, the scalar field ϕ monotonically decreases to zero as $r \to \infty$ and has no zeros, but for $\xi > (n-1)/(4n)$, the scalar field oscillates as it tends to zero, although the magnitude of the oscillations is very small. Apart from this, varying ξ does not have a great effect on the form of the scalar field ϕ. For higher dimensions, the effect on ϕ of varying ξ is even less.

only a small magnitude. Apart from these oscillations, the values of the function ϕ do not seem to change very much as ξ varies, as shown in Fig. 5.

Varying the coupling constant ξ has a much more significant effect on the metric functions δ and m, as shown in Figs. 6–8. The effect on the metric function δ is shown in Fig. 6 for some four-dimensional soliton solutions. In four dimensions, the value of ξ at which the oscillatory behaviour of the scalar field begins is $\xi = 3/16 = 0.1875$.
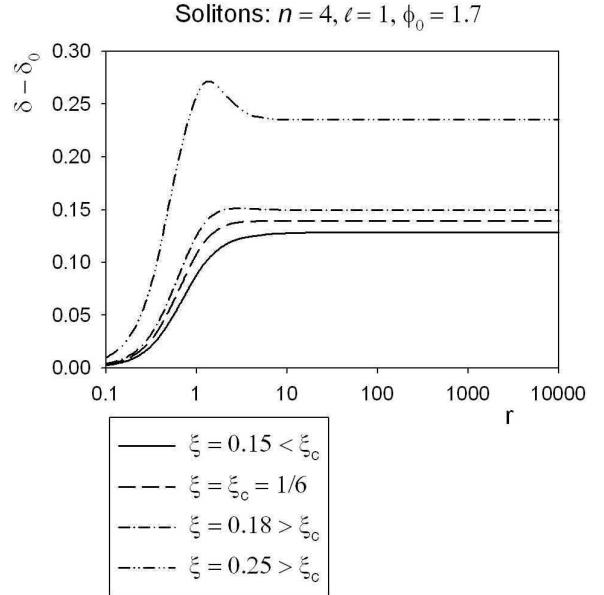

FIG. 6: The effect on the metric function δ of varying ξ is shown for four-dimensional soliton solutions with $\ell = 1$ and $\phi_0 = 1.7$. The function δ has the limit $\delta \to 0$ as $r \to \infty$, so we have plotted $\delta - \delta_0$ in order to make the behaviour of the functions easier to visualize.
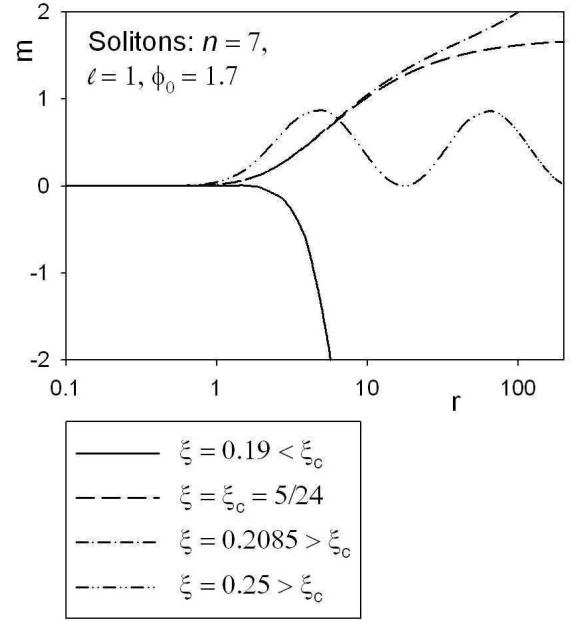

FIG. 7: The effect on the metric function m of varying ξ is shown for seven-dimensional soliton solutions with $\ell = 1$ and $\phi_0 = 1.7$.
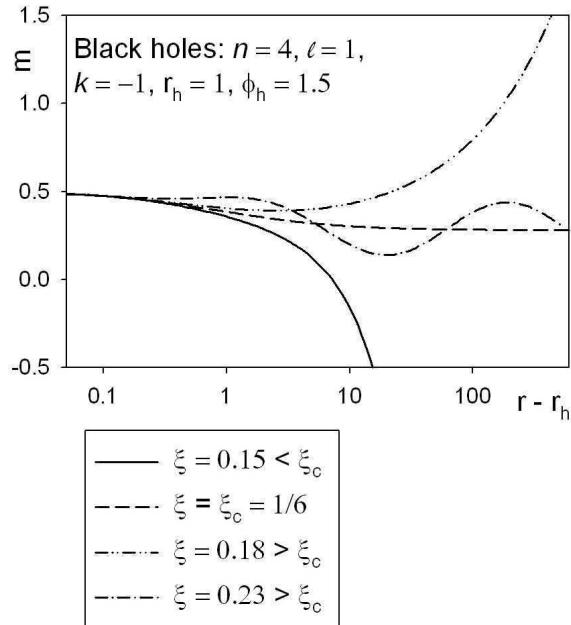
FIG. 8: The effect on the metric function m of varying ξ is shown for four-dimensional black hole solutions with $\ell = 1$, $r_h = 1$, $k = -1$ and $\phi_h = 1.5$.

It can be seen from Fig. 6 that for values of ξ below 0.1875, the function δ is monotonically increasing with r, although its value at the origin is decreasing as ξ increases (recall that $\delta \to 0$ as $r \to \infty$, although in Fig. 6 we have plotted $\delta - \delta_0$ rather than δ to make the graph easier to read). However, for values of ξ greater than 0.1875, the function δ has a maximum before decreasing monotonically to its value at infinity.

As discussed in Sec. II B, the effect of ξ on the behaviour of the metric function m is even more dramatic, and shown in Figs. 7 and 8. When $\xi < \xi_c = (n-2)/[4(n-1)]$, the metric function m diverges to $-\infty$ as $r \to \infty$. Only for $\xi = \xi_c$ does the metric function m converge as $r \to \infty$, as observed in [24]. The function m diverges to $+\infty$ as $r \to \infty$ for values of ξ between $(n-2)/[4(n-1)]$ and $(n-1)/(4n)$, a shrinking interval as n increases. For $\xi > (n-1)/(4n)$, the metric function m oscillates about a non-zero value as $r \to \infty$.

For black hole solutions, there is an additional parameter to vary, namely the constant $k \in \{0, 1, -1\}$, which governs the topology of the event horizon. The effect on the scalar field ϕ and metric function δ of varying k is shown in Figs. 9 and 10 respectively. Varying the constant k does not significantly change the scalar field ϕ, as seen in Fig. 9, in agreement with the observations in the conformally coupled case [24]. However, we find more significant differences in the magnitude of the function δ in the non-conformally coupled case than in the conformally coupled case (compare Fig. 10 and [24]).
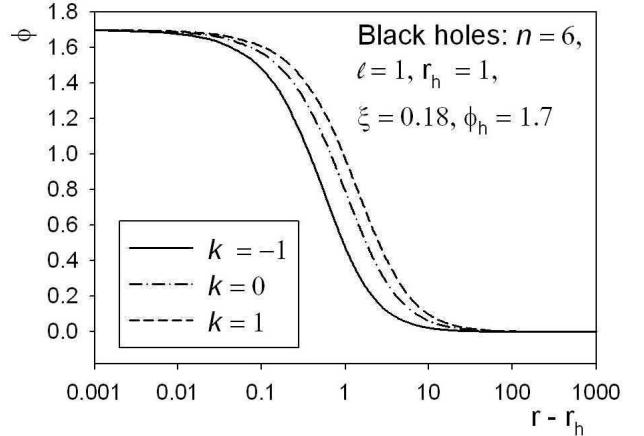
When the scalar field is conformally coupled and the

FIG. 9: The effect on the scalar field ϕ of varying the constant k is shown for some six-dimensional black holes with $\ell = 1$, $r_h = 1$, $\xi = 0.18$ and $\phi_h = 1.7$. The functions ϕ for the different values of k do not differ significantly.
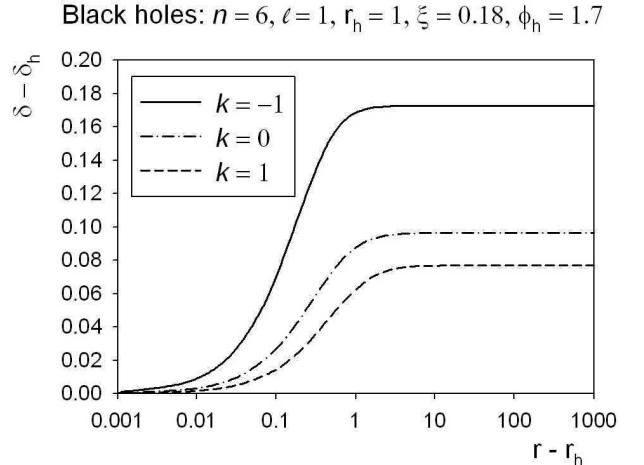

FIG. 10: The effect on the metric function δ of varying the constant k is shown for the same black hole solutions as in Fig. 9. The function δ has the limit $\delta \to 0$ as $r \to \infty$, so we have plotted $\delta - \delta_h$ in order to make the behaviour of the functions easier to visualize.

metric function m converges to a finite limit as $r \to \infty$, it was observed in [24] that varying k had a significant impact on the properties of the metric function m (see, Fig. 3 in [24]). For non-conformal coupling, when the metric function m either diverges or oscillates as $r \to \infty$, we find that varying k does not alter these dominant behaviours very much, as illustrated in Figs. 11 and 12. When the metric function m diverges, as seen in Fig. 11, the rate of divergence of the function m as $r \to \infty$ does not change as k varies, but the coefficient M_1 multiplying the dominant divergent term in the behaviour of m (20) does vary (see Eq. (22) - the constant M_1 depends on the constant c_1 in the expansion of the scalar field ϕ at infinity (16), which varies as k varies). When the metric
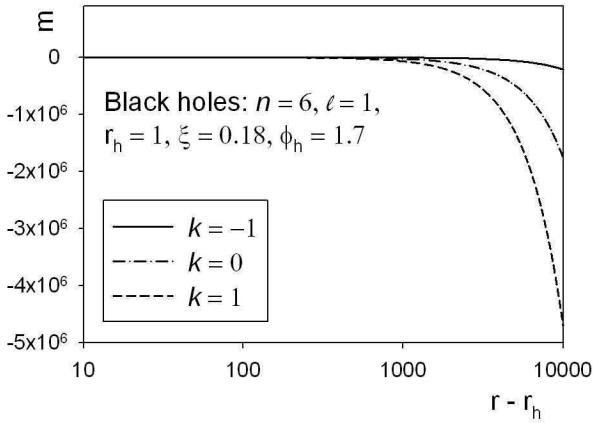
FIG. 11: The effect on the metric function m of varying the constant k is shown for the same black hole solutions as in Fig. 9.
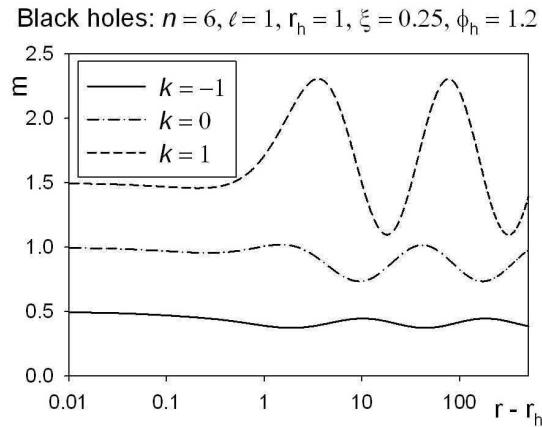

FIG. 12: The effect on the metric function m of varying the constant k is shown for the some six-dimensional black hole solutions with $\ell = 1$, $r_h = 1$, $\xi = 0.25$ and $\phi_h = 1.2$.

function m oscillates as $r \to \infty$, the effect of varying k can be seen in Fig. 12. In accordance with the analysis of Sec. II B, it can be seen that changing k does not affect the period of the oscillations, but does affect their magnitude and phase. In particular, the magnitude of the oscillations increases as k increases.

B. Thermodynamics

The thermodynamics of black holes with minimally coupled scalar field hair has recently attracted attention in relation to the adS/CFT correspondence [14, 15, 17], and a number of authors have studied the thermodynam-

ics of black holes with a conformally coupled scalar field [36, 40]. Given the difficulties, outlined in Sec. II B, of defining a mass for our non-minimally coupled solutions, we do not attempt a full thermodynamical analysis here. Instead we just make some brief remarks about the temperature and entropy of the black holes.

The temperature is given by the usual Hawking formula:

$$T = \frac{1}{4\pi} H'(r_h) e^{\delta_h}; \qquad (36)$$

while the entropy is modified by the non-minimally coupled scalar field [41]:

$$S = 2\pi V r_h^{n-2} \left[1 - \xi \phi_h^2 \right], \qquad (37)$$

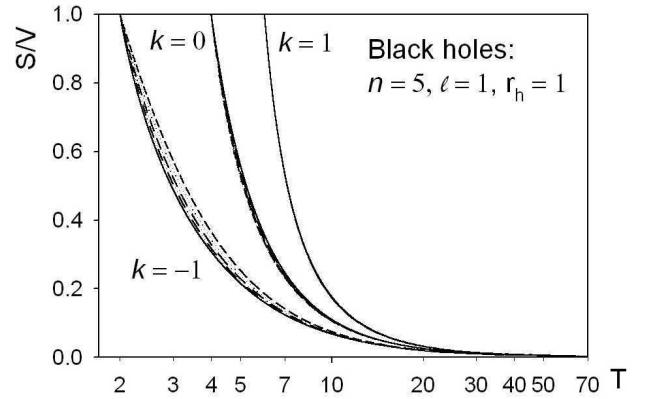where V is the volume of the maximally symmetric space with metric $d\sigma_{n-2,k}$ (6).

FIG. 13: Temperature-entropy relation for some five-dimensional black holes with fixed horizon radius $r_h = 1$, and $\ell = 1$. For each value of k, we have plotted curves for five values of ξ: 0.15, 0.1875, 0.225, 0.2625 and 0.3, and varied the value of the scalar field on the horizon ϕ_h such that both S and T are positive.

In Fig. 13, we have plotted the relationship between temperature T and entropy S/V for some five-dimensional black holes with $r_h = 1$, varying k and, for each value of k, considering five values of the coupling constant ξ. For each curve, we computed the temperature and entropy for all values of ϕ_h such that $0 < \phi_h < 1/\sqrt{\xi}$. It can be seen from Fig. 13 that varying k and ξ makes little difference to the relationship between temperature and entropy, particularly for $k = 0$ and $k = 1$ (in the latter case the curves for different ξ are indistinguishable).

In Fig. 14 we fix $k = 1$ and $\xi = 1/6$ and consider the effect of changing the event horizon radius r_h on the temperature and entropy of some five-dimensional black holes. From Fig. 14, it can be seen that for small values of
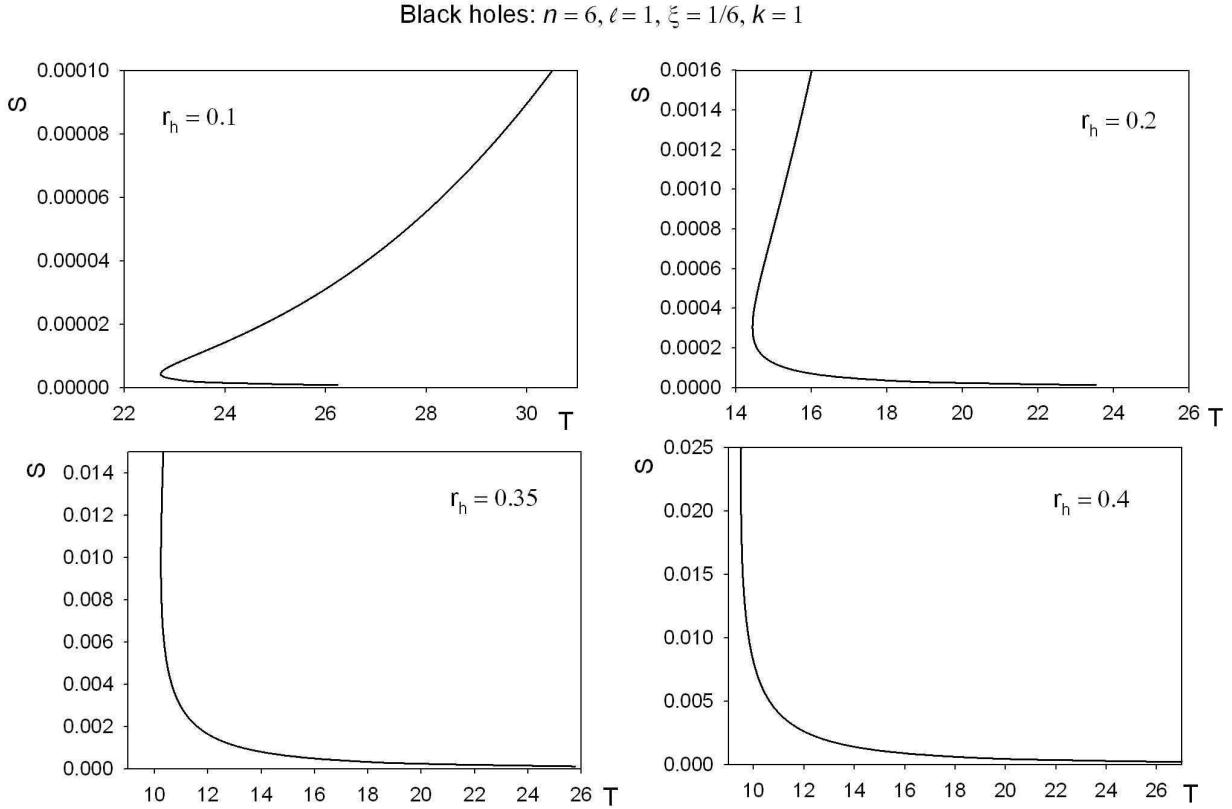
Black holes: $n = 6$, $\ell = 1$, $\xi = 1/6$, $k = 1$

FIG. 14: Temperature-entropy relations for some values of the event horizon radius r_h, for five-dimensional black holes with $k = 1$, $\ell = 1$ and $\xi = 1/6$. For each value of r_h considered, we vary ϕ_h.

the event horizon radius r_h, we have two branches of solutions, one with a low entropy and one with a much higher entropy. Above about $r_h \sim 0.35$, we have just one branch of solutions.

C. Stability of solutions

We now investigate the key question of whether our solitons and black holes with a non-minimally coupled scalar field are stable. We consider spherically symmetric perturbations of the metric and scalar field. The algebra rapidly becomes somewhat unwieldy, even with a computer algebra package, so to simplify matters we follow the approach of [24, 28] and use the conformal transformation, described in Sec. II C, to derive the perturbation equations with a minimally coupled scalar field, then transforming back to the frame with a non-minimally coupled scalar field.

As usual, the metric perturbations can be eliminated to yield a single perturbation equation for the quantity Ψ, which is defined in terms of the scalar field perturbation $\delta\phi$ as follows:

$$\Psi = r^{\frac{1}{2}(n-2)} \Omega^{-\frac{1}{2}} \mathcal{B}^{\frac{1}{2}} \delta\phi, \tag{38}$$

where $\Omega = 1 - \xi\phi^2$ as in Sec. II C and

$$\mathcal{B} = \Omega + \frac{4(n-1)}{n-2} \xi^2 \phi^2. \tag{39}$$

The perturbation equation for time-periodic perturbations $\Psi(t, r) = e^{i\sigma t} \Psi(r)$ takes the standard Schrödinger form

$$\sigma^2 \Psi = -\frac{d^2}{dr_*^2} \Psi + \mathcal{U}\Psi, \tag{40}$$

where we have introduced the usual 'tortoise' co-ordinate r_* by

$$\frac{dr_*}{dr} = \frac{1}{He^\delta}, \tag{41}$$

and the perturbation potential $\mathcal{U}$ takes the lengthy form

$$\mathcal{U} = -\frac{He^{2\delta}}{r^2}\left\{ \frac{4r^2}{(n-2)^2}\Omega^{-2}\mathcal{A}^2 H - (n-2)(n-3)\frac{k}{2} + r^2\Omega^{-1}\Lambda - \mathcal{B}^{-1}r^2\left[\frac{2\xi n\Lambda}{n-2} + \frac{8\xi^2 n(n-1)}{(n-2)^2}\phi^2\Omega^{-1}\Lambda\right]\right.$$
$$+\frac{2\mathcal{B}^{-2}r^2\xi n\phi\Lambda\Omega^{-1}}{n-2}\left[\Omega\xi\phi + \frac{4(n-1)}{n-2}\xi^2\phi(1+\xi\phi^2)\right] + 2\xi n\phi\Lambda\Omega^{-1}r^2\phi'\mathcal{A}^{-1}$$
$$\left.+\frac{(n-2)^3(n-3)k}{8}\phi'^2\mathcal{A}^{-2}\mathcal{B} - \frac{(n-2)^2}{4}r^2\Omega^{-1}\Lambda\phi'^2\mathcal{A}^{-2}\mathcal{B}\right\}, \tag{42}$$

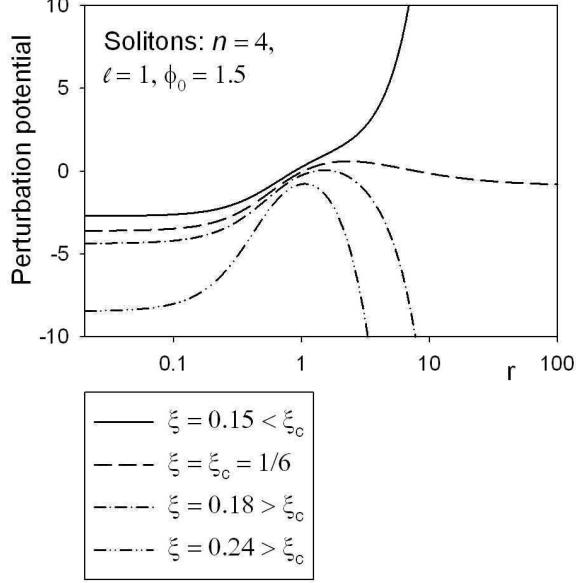

FIG. 15: Perturbation potential $\mathcal{U}$ (42) plotted as a function of r for some four-dimensional soliton solutions with $\ell = 1$, $\phi_0 = 1.5$ and four different values of the coupling constant ξ. In this case the potential is finite at the origin. The behaviour of the potential at infinity is in accordance with (45).

with Ω and $\mathcal{B}$ defined above and

$$\mathcal{A} = \xi\phi\phi' - \frac{(n-2)}{2r}\Omega. \tag{43}$$

When $n = 4$, the perturbation potential (42) reduces to that in [28], as expected.

It is helpful to derive the behaviour of the perturbation potential at $r = 0$, $r = r_h$ and as $r \to \infty$. Near the origin, using the boundary conditions (10), we have

$$\mathcal{U} = \frac{e^{2\delta_0}}{r^2}\left[\frac{(n-2)(n-3)}{2} - 1\right] + O(1). \tag{44}$$

This means that the perturbation potential $\mathcal{U}$ diverges to $+\infty$ at the origin unless $n = 4$, as can be seen by comparing the potentials in Figs. 15 and 16.

At the event horizon, $H(r_h) = 0$ and all other quantities are finite, so it is straightforward to see that the potential $\mathcal{U}$ vanishes there, as illustrated in Fig. 17.
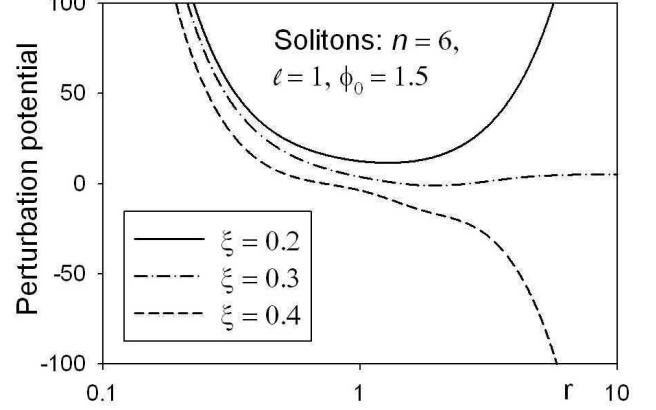

FIG. 16: Perturbation potential $\mathcal{U}$ (42) plotted as a function of r for some six-dimensional soliton solutions with $\ell = 1$, $\phi_0 = 1.5$ and three different values of the coupling constant ξ. In this case the potential diverges at the origin, as expected from equation (44). The behaviour of the potential at infinity is as predicted in (45).

As $r \to \infty$, using the boundary conditions (16, 18, 19), we find

$$\mathcal{U} = \frac{2n\Lambda^2 r^2}{(n-2)^2(n-1)^2}\left[(n-3) - 2\xi(n-1)\right]$$
$$+\frac{2k\Lambda}{(n-2)(n-1)}\left[\xi n(n-1) - n^2 + 4n - 2\right]$$
$$+O\left(r^{-2}\right). \tag{45}$$

Therefore the potential $\mathcal{U}$ diverges as $r \to \infty$ unless $\xi = (n-3)/[2(n-1)]$, which corresponds to $\xi = \xi_c = 1/6$ when $n = 4$. This is confirmed by the example potentials plotted in Figs. 15–17.

As in [24], we find that in general the perturbation potential has a complex dependence on n, ϕ_0 or ϕ_h as applicable, r_h, k and ξ (see particularly Fig. 17). In some cases, the perturbation potential $\mathcal{U}$ is positive everywhere and we can immediately deduce that the solutions are stable. From (45), this is only possible for $\xi \le (n-3)/[2(n-1)]$.

To study the stability of the solutions for which the perturbation potential $\mathcal{U}$ is not everywhere positive, we follow the approach of [24, 28], and consider the zero mode perturbation Ψ_0, that is, the solution of the per-
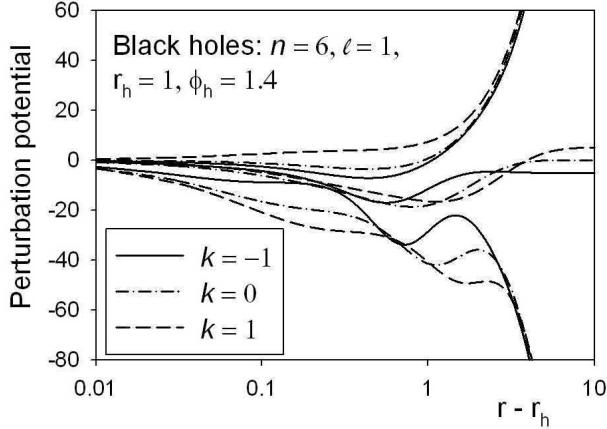
FIG. 17: Perturbation potential $\mathcal{U}$ (42) plotted as a function of r for some six-dimensional black hole solutions with $\ell = 1$, $r_h = 1$ and $\phi_h = 1.4$. For each value of k, we plot the potential for $\xi = 0.2$, 0.3 and 0.4. The potential vanishes at the event horizon, and the behaviour at infinity is governed by (45). The potentials for $\xi = 0.2$ diverge to ∞ as $r \to \infty$, those for $\xi = 0.3$ converge to a finite limit as $r \to \infty$, while those for $\xi = 0.4$ diverge to $-\infty$ as $r \to \infty$.

turbation equation (40) with $\sigma = 0$. Near $r = 0$, from the differential equation (40), we find that $\Psi_0 = O(r^\alpha)$, where

$$\alpha = \frac{1}{2} \pm \frac{1}{2}\sqrt{2n^2 - 10n + 9}, \qquad (46)$$

and we chose the positive sign so that Ψ_0 is regular at the origin. Near $r = r_h$, we simply require $\Psi = O(r - r_h)$, and as $r \to \infty$, we have $\Psi_0 = O(r^{-\beta})$, where

$$\beta = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 2n(n-3) - 4n(n-1)\xi}. \qquad (47)$$

The behaviour of Ψ_0 as $r \to \infty$ is therefore complicated. For $\xi < [1 + 2n(n-3)]/[4n(n-1)]$, the constant β is real, and the dominant behaviour of Ψ_0 will be from taking the negative sign in (47). This value of β will be positive (corresponding to $\Psi_0 \to 0$ as $r \to \infty$) as long as $\xi > (n-3)/[2(n-1)]$, but is negative (corresponding to Ψ_0 diverging as $r \to \infty$) for $\xi < (n-3)/[2(n-1)]$. For $\xi > [1 + 2n(n-3)]/[4n(n-1)]$, the constant β is complex, with positive real part. In this case $\Psi_0 \to 0$ as $r \to \infty$, but Ψ_0 is oscillating as $r \to \infty$. Integrating the perturbation equation (40) over the original radial co-ordinate r, from $r = 0$ or $r = r_h$, as applicable, gives examples of the typical behaviour of the zero modes, which are shown in Figs. 18–20.

For $0 < \xi < \xi_c$, we find that the zero mode Ψ_0 has no zeros, and from this we can deduce that there can be no negative eigenvalues $\sigma^2 < 0$ of the perturbation equation (40), in other words, the solutions are stable. For $\xi > [1 + 2n(n-3)]/[4n(n-1)]$, we find, as expected, that the zero mode Ψ_0 oscillates many times as $r \to \infty$, leading us to conclude that there is at least one negative
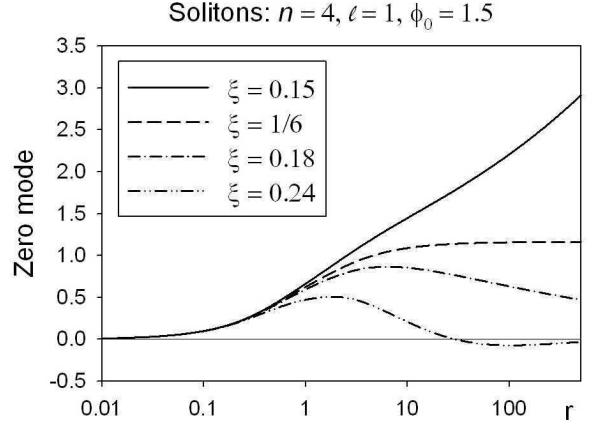

FIG. 18: Zero mode Ψ_0 plotted as a function of r for some four-dimensional soliton solutions with $\ell = 1$, $\phi_0 = 1.5$ and four different values of the coupling constant ξ.
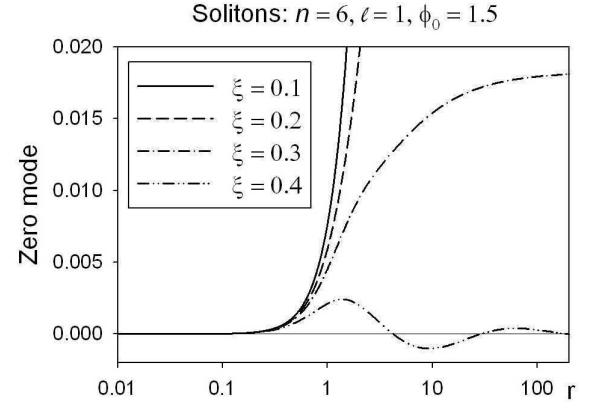

FIG. 19: Zero mode Ψ_0 plotted as a function of r for some six-dimensional soliton solutions with $\ell = 1$, $\phi_0 = 1.5$ and four different values of the coupling constant ξ.

eigenvalue $\sigma^2 < 0$ of the perturbation equation (40), and the solutions are unstable. For $\xi_c < \xi < [1 + 2n(n-3)]/[4n(n-1)]$, the situation is more complicated. We find for some solutions that the zero mode Ψ_0 has no zeros, indicating the stability of the solutions; but for others it has at least one zero and we conclude that the solutions are unstable.

As discussed in Sec. II B, the equilibrium scalar field oscillates as $r \to \infty$ if $\xi > (n-1)/(4n)$ (17). Interestingly, this coincides with the value of ξ coming from the Breitenlohner-Freedman bound in n dimensions [30], which states that scalar fields in pure anti-de Sitter space are stable if their mass satisfies the inequality

$$m_{BF}^2 > \frac{\Lambda(n-1)}{2(n-2)}. \qquad (48)$$

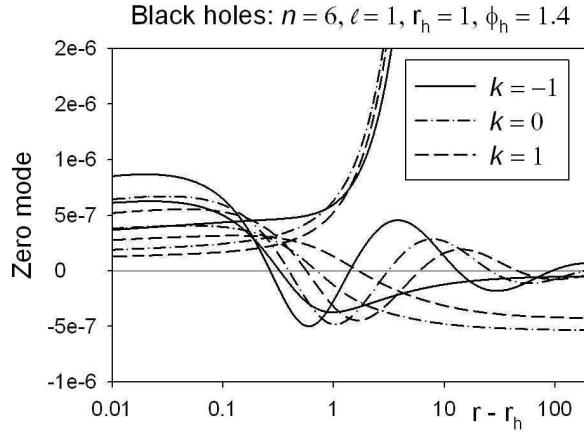In our case the "effective" mass of the scalar field is given by ξ multiplied by the value of the Ricci scalar

FIG. 20: Zero mode Ψ_0 plotted as a function of r for some six-dimensional black hole solutions with $\ell = 1$, $r_h = 1$ and $\phi_h = 1.4$. For each value of k, we plot the zero mode for $\xi = 0.1$, 0.3 and 0.4. The zero modes for $\xi = 0.1$ diverge to $+\infty$ as $r \to \infty$; those for $\xi = 0.3$ have a single zero; and those for $\xi = 0.4$ oscillate about 0 as $r \to \infty$.
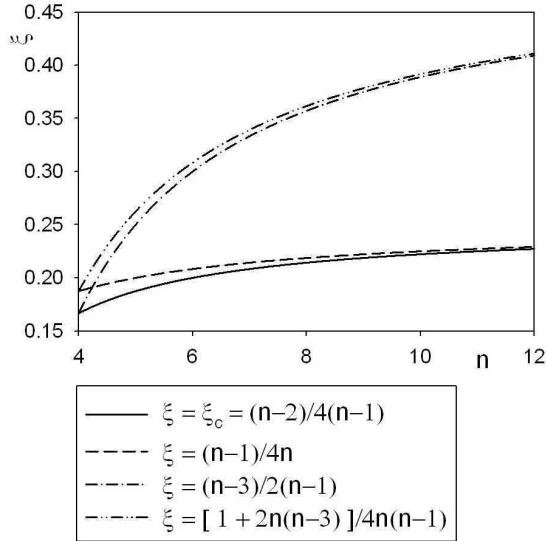

FIG. 21: The key values of the coupling constant ξ discussed in the text, as functions of the number of dimensions n.

curvature at infinity, which is $2\Lambda n/(n-2)$. Bearing in mind that Λ is negative, the inequality (48) then becomes $\xi < (n-1)/(4n)$. For any $n > 4$, this value of ξ is always greater than ξ_c, as can be seen in Fig. 21, but the gap between ξ_c and the value of ξ coming from the Breitenlohner-Freedman bound narrows as n gets large. From the Breitenlohner-Freedman bound, we therefore might expect that any non-trivial solutions with $\xi > (n-1)/(4n)$ are unstable. We find that for $\xi > (n-3)/[2(n-1)]$, the perturbation potential $\mathcal{U}$ diverges to $-\infty$ as $r \to \infty$. Analysis of the zero mode Ψ_0 reveals that it oscillates as $r \to \infty$ only

if $\xi > [1 + 2n(n-3)]/[4n(n-1)]$. These latter two values of ξ are also shown in Fig. 21. The value of ξ at which the potential diverges to $-\infty$ at infinity is equal to ξ_c only when $n = 4$, and for $n > 4$ it is considerably larger than both ξ_c and the value of ξ coming from the Breitenlohner-Freedman bound. The value of ξ at which the zero mode oscillates equals the value of ξ coming from the Breitenlohner-Freedman bound when $n = 4$, but for all $n \geq 4$ it is greater than the value of ξ at which the perturbation potential diverges to $-\infty$.

For $n > 4$, we therefore have a complicated picture. For all the solutions we have studied, we find that they are stable when $0 < \xi < \xi_c$, in accordance with the results in [28] for $n = 4$. All the solutions we have studied are unstable for $\xi > [1 + 2n(n-3)]/[4n(n-1)]$. However, while the Breitenlohner-Freedman bound would lead us to expect that all the solutions with $\xi > (n-1)/(4n)$ would be unstable, in practice we are unable to reach any general conclusions about the stability of solutions when $(n-1)/(4n) < \xi < [1 + 2n(n-3)]/[4n(n-1)]$, with some solutions being stable and some unstable.

V. CONCLUSIONS

In this paper we have studied the existence of soliton and black hole solutions of the Einstein equations in the presence of a cosmological constant and a non-minimally coupled scalar field with zero self-interaction potential. We have considered all space-time dimensions greater than or equal to four, and topological black holes as well as the usual spherically symmetric solitons and black holes, covering all the possibilities not previously considered in the literature [24, 28].

Since we have a zero self-interaction potential, we are able to use elementary arguments to show the non-existence of soliton or black hole solutions, except when the cosmological constant is negative and the coupling constant ξ (which governs the coupling between the scalar field and the Ricci scalar curvature) is positive, when we find non-trivial soliton and hairy black hole solutions. The field equations are highly complex and so we have only considered numerical solutions in this paper. It is likely that solution generating techniques [33] could be used to give analytic solutions, but only with non-zero self-interaction potentials.

The behaviour of the solutions as ξ varies is rather more complex in more than four dimensions than it was in four space-time dimensions [28]. If $\xi > 0$ but less than the value for conformal coupling, for all the numerical solutions we studied, the scalar field was monotonically decreasing to zero from its value at the origin or the black hole event horizon, as applicable, and, furthermore, all such solutions that we studied were stable. We should emphasize that we have only considered spherically symmetric perturbations in this paper, and that while these solutions are stable under such perturbations, it is possible that there exist non-spherically symmetric unstable

modes.

For values of ξ greater than that for conformal coupling, the picture is less straightforward. The scalar field is oscillating for $\xi > (n-1)/(4n)$, in accordance with the Breitenlohner-Freedman [30] bound for scalar fields in pure anti-de Sitter space. This would lead us to expect that all solutions for $\xi > (n-1)/(4n)$ would be unstable. However, while all the solutions we have studied with $\xi > [1 + 2n(n-3)]/[4n(n-1)]$ have been unstable, we have found conflicting results for $(n-1)/(4n) < \xi < [1 + 2n(n-3)]/[4n(n-1)]$, with

some solutions being stable and some unstable.

The oscillatory nature of the scalar field for $\xi > (n-1)/(4n)$ also leads to oscillations in the metric function $m(r)$, whose limit as $r \to \infty$ one would normally take as a measure of the mass of the solution. We leave the question of how to define the mass when $m(r)$ is oscillating for future work. In the absence of an appropriate definition of mass, we have been unable to perform a complete thermodynamic analysis of our solutions, which would perhaps help to resolve the stability issue outlined above. We hope to return to these questions.

Our results are summarized in the updated Tab. IV, where the new results in this paper are highlighted in bold.

$n > 4$	$\xi < 0$	$\xi = 0$	$0 < \xi < \xi_c$	$\xi = \xi_c$	$\xi > \xi_c$
$\Lambda > 0$	**no solutions**	**no solutions**	**no solutions**	**no solutions**	**no solutions**
$\Lambda = 0$	**no solutions**	**no solutions**	no solutions [27]	**no solutions**	**no solutions**
$\Lambda < 0$	**no solutions**	**no solutions**	**stable solutions**	stable solutions [24]	**mostly unstable solutions**

TABLE IV: Summary of existence and non-existence of soliton and black hole solutions in more than four space-time dimensions.

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