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SIMULTANEOUS CONFIDENCE BANDS IN NONLINEAR REGRESSION MODELS WITH NONSTATIONARITY

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Abstract: We consider nonparametric estimation of the regression function $g(\cdot)$ in a nonlinear regression model $Y_t = g(X_t) + \sigma(X_t)e_t$, where the regressor (X_t) is a nonstationary unit root process and the error (e_t) is a sequence of independent and identically distributed (i.i.d.) random variables. With proper centering and scaling, the maximum deviation of the local linear estimator of the regression function g is shown to be asymptotically Gumbel. Based on the latter result, we construct simultaneous confidence bands for g, which can be used to test patterns of the regression function. Our results substantially extend existing ones which typically require independent or stationary weakly dependent regressors. Furthermore, we examine the finite sample behavior of the proposed approach via the simulated and real data examples.

Key words and phrases: Gumbel convergence, integrated process, local linear estimation, local time limit theory, maximum deviation, simultaneous confidence bands.

1. Introduction

Following Granger (1981) and Engle and Granger (1987), cointegration has become one of the most important topics in econometrics. For two non-stationary processes (X_t) and (Y_t), we say that they are (linearly) cointegrated if there exists a nonzero constant a such that

$$Y_t - aX_t = \varepsilon_t, \quad t = 1, \ldots, n, \tag{1.1}$$

where (ε_t) is a stationary process. The above cointegration property indicates that the two nonstationary processes (X_t) and (Y_t) have a common stochastic trend over a long time series period. The classic linear cointegration models have been extensively studied in the literature. To account for possible nonlinear relations, in this paper we consider the nonlinear and nonstationary regression model:

$$Y_t - g(X_t) = \sigma(X_t)e_t, \quad t = 1, \ldots, n, \tag{1.2}$$

where (e_t) is a stationary error process, (X_t) is a nonstationary regressor, g and σ are two smooth functions. Model (1.2) generalizes the traditional linear cointegration model (1.1) by allowing possible nonlinear structures on both g and σ. Hence, it provides a flexible tool to analyze nonlinear relationships between two nonstationary processes.

The aim of this paper is to study nonparametric estimates of the function g in model (1.2). Convergence properties of the conventional Nadaraya-Watson kernel-based estimator have been widely discussed under the assumption that (X_t, Y_t) are i.i.d.; see Härdle (1990) and the references therein. In Györfil *et al* (1989), Bosq (1998) and Fan and Yao (2003), the i.i.d. assumption is relaxed and strong mixing stationary processes are allowed. The asymptotic problem becomes quite challenging if (X_t) is non-stationary. Karlsen *et al* (2007, 2010) assumed that (X_t) is a null recurrent Markov chain. Wang and Phillips (2009a, 2009b, 2011) and Cai *et al* (2009) considered the integrated process $X_t = \sum_{j=1}^{t} x_j$, where (x_j) is a stationary linear process. For other recent literature on this research area, we refer to Gao *et al* (2009a, 2009b), Chen *et al* (2010, 2012), Kasparis and Phillips (2012), Wang and Phillips (2012) and Wang (2014).

All the aforementioned papers deal with the point-wise central limit theorem for the Nadaraya-Watson estimator. This type of point-wise distributional result, however, is not useful for testing certain patterns of the regression function g. In recent years, there has been increasing interests on deriving the uniform consistency results for the kernel-based estimator under the nonstationary framework. For instance, Wang and Wang (2013) and Chan and Wang (2014) established the uniform consistency of the kernel-based estimator by using the local time limit theory, Wang and Chan (2014) obtained uniform convergence rates for a class of martingales and studied their application in nonlinear cointegration regression, and Gao *et al* (2015) derived some uniform consistency results using the framework of the null recurrent Markov chains.

Throughout this paper, we estimate the regression function g using the local linear smoothing method in which the estimator of $g(x)$ is defined by

$$\widehat{g}_n(x) = \sum_{t=1}^{n} w_t(x) Y_t \Big/ \sum_{t=1}^{n} w_t(x), \tag{1.3}$$

where the weight function is defined through

$$w_t(x) = K[(X_t - x)/h] V_{n2}(x) - K_1[(X_t - x)/h] V_{n1}(x)$$

with $V_{nj}(x) = \sum_{i=1}^{n} K_j[(X_i - x)/h]$, $K_j(x) = x^j K(x)$, $K(\cdot)$ being a non-negative real function and the bandwidth $h \equiv h_n \to 0$. Under the stationarity assumption on the observations, existing literature such as Fan and Gijbels (1996) has already shown that the local linear estimation method has some advantages over the Nadaraya-Watson kernel estimation method. A recent paper by Chan and Wang (2014) further showed that the performance of $\widehat{g}_n(\cdot)$ is superior to that of the conventional Nadaraya-Watson estimator in uniform asymptotics for nontationary time series.

To assess patterns of the regression function g, for example, to test whether g is linear or of other parametric forms, we need to construct a simultaneous confidence band (SCB) for g over a suitable interval. Neither the point-wise central limit theorem nor the uniform convergence of $\widehat{g}_n(\cdot)$ is sufficient for testing whether g has a particular functional form. To deal with the latter problem, we shall obtain in this paper the asymptotic distribution for the normalized maximum absolute deviation

$$\Delta_n = \sup_{|x| \le B_n} \left| V_n(x) \left[\widehat{g}_n(x) - g(x) \right] / \sigma(x) \right|, \tag{1.4}$$

where B_n is a sequence of positive constants which may diverge to infinity and the normalizing term is defined by

$$V_n(x) = V_{n2}^{-1}(x) \sum_{t=1}^{n} w_t(x) \Big/ \Big(\sum_{t=1}^{n} K^2 \big[(X_t - x)/h \big] \Big)^{1/2}. \tag{1.5}$$

Such an asymptotic distributional theory substantially refines the existing uniform consistency results such as those obtained by Chan and Wang (2014), and it further enables one to construct a SCB for the unknown regression function g. In the traditional simultaneous inference theory it is assumed that the regressor process (X_t) is i.i.d. or stationary; see, for example, Bickel and Rosenblatt (1973), Johnston (1982), Hall and Titterington (1988), Xia (1998), Fan and

Zhang (2000), Zhao and Wu (2008), Liu and Wu (2010) and Zhang and Peng (2010). However, in our setting, due to the nonstationarity and the dependence, it is very demanding to establish a limit theory for Δ_n. To this end, we introduce new technical mechanisms and obtain a precise characterization of $V_n(x)$ over an unbounded interval.

The rest of the paper is organized as follows. The assumptions and main theoretical results are stated in Section 2. By using the asymptotic distribution of Δ_n defined in (1.4), in Section 3, we construct SCBs for the regression function g over an expanding interval. In Section 4, we provide both the simulated and real data examples to illustrate the finite sample behavior of the proposed approach. The proofs of the main results are provided in Section 5. Section 6 concludes the paper. Some technical lemmas with the proofs and some supplemental asymptotic theorems are given in a supplemental document.

2. Main results

We start with some regularity conditions which will be used later to establish our main theorem on the asymptotic distribution of Δ_n defined in (1.4). Let C be a positive constant whose value may change from line to line.

(C1) [Regressor process] Let

$$X_t = \sum_{j=1}^{t} x_j, \quad x_j = \sum_{k=0}^{\infty} \phi_k \, \eta_{j-k},$$

where (η_j) is a sequence of i.i.d. random variables with $\mathsf{E}[\eta_1] = 0$, $\mathsf{E}[\eta_1^2] = 1$ and $\mathsf{E}[|\eta_1|^{2+\delta}] < \infty$ for some $\delta > 0$, the characteristic function $\varphi(t)$ of η_1 satisfies $\int_{-\infty}^{\infty}(1+|t|)|\varphi(t)|dt < \infty$, and the coefficients $(\phi_k)_{k\geq 0}$ satisfy $\sum_{k=0}^{\infty}|\phi_k| < \infty$ and $\phi \equiv \sum_{k=0}^{\infty}\phi_k \neq 0$.

(C2) [Regression function] The first derivative of $g(x)$ exists and it is denoted by $g'(x)$. There exists a real positive function $g_0(x)$ such that

$$|g'(y) - g'(x)| \quad \leq \quad g_0(x)|y-x|, \tag{2.1}$$

uniformly for $x \in \mathsf{R}$ and $|y-x|$ sufficiently small.

(C3) [Range and Bandwidth] (i) Let $B_n = M_0 n^{1/2-\epsilon}$, where M_0 is a constant and $\epsilon > 0$ can be arbitrarily small. (ii) For some constant $0 < \delta_0 < 1/4$,

$n^{1/2-\delta_0}h \to \infty$. (iii) $nh^{10}\sup_{|x|\le 2B_n}\left[1+g_0^4(x)\right] = O(\log^{-8} n)$, where $g_0(x)$ is given in (2.1).

(C4) [Errors] (i) The error process (e_t) is i.i.d. with $\mathsf{E}[e_1]=0$, $\mathsf{E}[e_1^2]=1$ and $\mathsf{E}[|e_1|^{2p}]<\infty$, where $p\ge 1+[1/\delta_0]$ and δ_0 is defined as in (**C3**)(ii), and (e_t) is independent of the process (η_t). (ii) For the function $\sigma(\cdot)$, we have

$$\inf_{x\in\mathsf{R}}\sigma(x)>0,\quad \sup_{x\in\mathsf{R}}\frac{|\sigma(x+y)-\sigma(x)|}{\sigma(x)} \quad\le\quad C|y| \tag{2.2}$$

for any $|y|$ sufficiently small.

(C5) [Kernel] The kernel function K is absolutely continuous on a compact support $[-A,A]$ with $A>0$, $\int xK(x)dx=0$ and $|K(x)-K(y)|\le C|x-y|$ for all $x,y\in\mathsf{R}$.

Remark 1. In the literature conditions similar to **(C1)** and **(C2)** have been commonly used; see, for instance, Wang and Phillip (2009b), Wang and Wang (2013) and Chan and Wang (2014). Note that the smoothness condition (2.1) on g' is slightly weaker than the existence of second derivative on g. For instance, if the function g has continuous and bounded second order derivative, we may replace $g_0(x)$ in (2.1) by a positive constant. As a consequence, **(C3)** (iii) can be simplified to $nh^{10}\log^8 n = O(1)$.

In condition **(C3)**(i), we allow B_n to diverge to infinity, which has an essential difference from the literature that investigates the SCB for stationary regressor where B_n is usually assumed to be fixed (c.f., Liu and Wu, 2010). Our main theorem is established for diverging B_n, which indicates that the SCB for unknown g with $I(1)$ regressor may be constructed under an expanding interval whose length is allowed to diverge to infinity; see Remark 4 in Section 3 for more details. **(C3)**(ii) is close to the necessary condition of $\sqrt{n}h\to\infty$ by letting δ_0 be sufficiently close to zero. **(C3)**(iii) implies that there is a trade-off between the function g, the bandwidth h and the range B_n. If B_n is assumed to be a positive constant B, **(C3)**(iii) is satisfied when $\sup_{|x|\le 2B} g_0(x) = O(1)$ and $nh^{10}\log^8 n = O(1)$. In particular, the condition **(C3)**(iii) ensures that the bias term of the local linear estimator (1.3) with nonstationarity is asymptotically negligible, which can simplify the presentation of our limit theory in Theorem 1 and the subsequent construction of the SCBs in Section 3.

In condition **(C4)**, $\mathsf{E}[e_1^2]=1$ is not necessary. If $\mathsf{E}[e_1^2]=a^2\ne 1$, it suffices to standardize the model (1.2) with $e_t' = e_t/a$ and replace $\sigma(x)$ by $a\sigma(x)$. The moment condition of e_1 depends on the δ_0 in **(C3)**(ii), which is reasonable. Weaker bandwidth restriction (smaller δ_0) would lead to stronger moment condition on e_t (larger p). We further require the independence between the regressor (X_t) and the error (e_t) by noting that (e_t) is independent of (η_t) in **(C4)**(i). This condition is restrictive, but seems difficult to be relaxed, even with the stationary regressor, due to the technical reasons. The condition imposed on $\sigma(x)$ is mild and is satisfied by a large class of functions. Typical examples include $\sigma(x) = 1 + |x|^k$ with $k \geq 0$ and $\sigma(x) = e^x/(1+e^x)$.

The kernel condition **(C5)** implies that the derivative $K'(x)$ exists almost everywhere and $\int_{-A}^{A} [K'(x)]^2 dx < \infty$. Define $\lambda_1 = \int_{-A}^{A} K(x)dx$, $\lambda_2 = \int_{-A}^{A} K^2(x)dx$, and $\kappa_2 = \int_{-A}^{A} (K'(x))^2 dx/(2\lambda_2)$. We further have

$$r(s) \equiv \int_{-A}^{A} K(x)K(x+s)dx/\lambda_2 = 1 - \kappa_2 s^2 + o(s^2), \quad \text{as } s \to 0. \tag{2.3}$$

This follows from the Taylor's expansion of $r(s)$ and Theorems B1–B2 of Bickel and Rosenblatt (1973). As in Bickel and Rosenblatt (1973), $r(s)$ is used as the covariance function of a certain Gaussian process, which will appear in the proof of Theorem 1. Note that the kernel condition **(C5)** is mild and it can be satisfied by many commonly-used functions such as the Epanechnikov kernel $K(u) = \max\{3(1-u^2)/4, 0\}$.

We now introduce our main theoretical result: asymptotic distribution for the normalized maximum absolute deviation Δ_n defined in (1.4).

Theorem 1. *Let Conditions (C1)-(C5) be satisfied. Then, for $z \in R$,*

$$P\left\{(2\log \bar{h}^{-1})^{1/2}\,(\Delta_n - d_n) \leq z\right\} \to e^{-2e^{-z}}, \tag{2.4}$$

where $\bar{h} = h/(2B_n)$ and

$$d_n = (2\log \bar{h}^{-1})^{1/2} + \frac{1}{(2\log \bar{h}^{-1})^{1/2}} \log \frac{\kappa_2^{1/2}}{\pi 2^{1/2}}.$$

Remark 2. Letting $\mathcal{G}_n = (2\log \bar{h}^{-1})^{1/2}\,(\Delta_n - d_n)$ and $\mathcal{G}$ be the standard Gumbel distribution with the cumulative distribution function defined as the limit in

(2.4), we may reformulate the above limit result as $\mathcal{G}_n \xrightarrow{d} \mathcal{G}$. Unlike the existing results with stationary regressors, an important and useful feature of Theorem 1 is that we allow that B_n is divergent. Using Theorem 1, we can construct SCBs for the unknown function g on expanding intervals which will be given in Section 3 below. The requirement of intervals being expanding is crucial in our functional estimation since the regressor process (X_t) is integrated and thus stochastically unbounded, behaving like random walk. In fact, in the proof of Theorem 1 in Section 5, we obtain the asymptotic Gumbel distribution theory for $B_n = O(\sqrt{n}\log^{-c_0} n)$ for some $c_0 > 0$, which is optimal up to a logarithmic multiplicative factor, which cannot be extended, for example, to $B_n \geq \sqrt{n}\log n$. To see this, consider the simplest case in which $X_t = \sum_{j=1}^{t} x_j$ and x_j are i.i.d. standard normal random variables. Then $\mathsf{P}(\max_{t\leq n}|X_t| \geq \sqrt{n}\log n) \to 0$ and there is almost no data point for estimating $g(x)$ with $x \geq \sqrt{n}\log n$. Liu and Wu (2010) obtained Gumbel convergence for the nonlinear regression models with stationary regressors. In the latter paper, however, it is assumed that the involved interval is bounded and non-expanding. The result in Liu and Wu (2010) generally fails if the interval is unbounded. This explains the significant difference between models with stationary and nonstationary regressors. The model with nonstationary $I(1)$ regressor presently allows expanding intervals, however, the theoretical derivation is much more challenging and it uses local time limit theory together with multivariate Gaussian approximation.

The asymptotic Gumbel distribution result (2.4) also holds when we replace $\widehat{g}_n(\cdot)$ by the Nadaraya-Watson kernel estimation $\widetilde{g}_n(\cdot)$ in the definition of Δ_n. To save the space of the main document, we provide some relevant results in Appendix C of the supplemental document.

3. Construction of SCBs

This section constructs the SCBs for the regression function g. Since $\sigma(x)$ in the definition of Δ_n given in (1.4) is unknown, Theorem 1 is not directly usable. The consistent estimate of $\sigma(x)$ satisfying certain rates are required over the set $\{x : |x| \leq B_n\}$. Using the similar arguments as in Wang and Wang (2013), we construct the kernel estimation:

$$\widehat{\sigma}_n^2(x) = \frac{\sum_{t=1}^{n}[Y_t - \widehat{g}_n(X_t)]^2 K[(X_t - x)/b]}{\sum_{t=1}^{n} K[(X_t - x)/b]}, \tag{3.1}$$

where $\widehat{g}_n(x)$ is the local linear estimate defined in (1.3), and b is a bandwidth. Let $a_n \asymp b_n$ denote that a_n and b_n have the same asymptotic order.

Proposition 1. *Under the conditions of Theorem 1 and that $b \asymp h$, we have*

$$\sup_{|x|\le B_n}\Big|\frac{\sigma(x)}{\widehat{\sigma}_n(x)}-1\Big| = O_P\big[h+(nh^2)^{-1/4}\log^2 n\big]. \tag{3.2}$$

Remark 3. The proof of Proposition 1 will be given in Section 5. Proposition 1 can be seen as an extension of Wang and Wang (2013)'s uniform consistency results from the case of bounded range to the case of diverging range. The uniform convergence rate in (3.2) is close to be optimal. Using the bandwidth conditions in **(C3)**, we can see that the rate in (3.2) is sufficient for us to replace $\sigma(\cdot)$ by $\widehat{\sigma}_n(\cdot)$ when constructing the SCBs of the regression function g.

Define

$$\widehat{\Delta}_n = \sup_{|x|\le B_n}\left|V_n(x)\left[\widehat{g}_n(x)-g(x)\right]/\widehat{\sigma}_n(x)\right|.$$

Due to (3.2) and Theorem 1, $\widehat{\Delta}_n$ and Δ_n have the same limit distribution. Consequently, for given α, the $(1-\alpha)$-SCB for g over the set $\{x : |x| \le B_n\}$ can be constructed by

$$\big[\widehat{g}_n(x)-l_\alpha(x),\quad \widehat{g}_n(x)+l_\alpha(x)\big], \tag{3.3}$$

where

$$\begin{aligned}
l_\alpha(x) &= \Big[z_\alpha(2\log \bar{h}^{-1})^{-1/2}+d_n\Big]\ \widehat{\sigma}_n(x)\, V_n^{-1}(x),\\
z_\alpha &= -\log\left(-\frac{1}{2}\log(1-\alpha)\right),\quad \bar{h}=h/(2B_n),\\
d_n &= (2\log\bar{h}^{-1})^{1/2}+\frac{1}{(2\log\bar{h}^{-1})^{1/2}}\log\frac{\kappa_2^{1/2}}{\sqrt{2}\pi}.
\end{aligned}$$

Remark 4. As Theorem 1 implies that the asymptotic bias term is negligible due to the bandwidth condition **(C3)**, we do not need to correct the bias of the local linear estimation when constructing the SCBs in (3.3). The simulation study in Section 4 shows that such construction of SCBs works reasonably well in the finite sample case. When the regression function $g(x)$ has a thin tail such as

$g(x) = (\alpha + \beta e^x)/(1 + e^x)$, some routine calculations show that $\sup_x |g_0(x)| < \infty$. As a consequence, the SCB result (3.3) holds over the set $\{x : |x| \le M_0 n^{1/2-\epsilon}\}$, whenever $n^{1/2-\delta_0} h \to \infty$ and $nh^{10} \log^8 n \to 0$, where ϵ can be chosen sufficiently small. As pointed out in Section 2, this is fundamentally different from the existing literature with stationary regressor, which generally fails if the interval is unbounded.

4. Numerical studies

In this section, we provide both the simulated and real data examples to illustrate the finite sample behavior of our SCBs. For the construction, we need to choose an appropriate cut-off value so that the pre-assigned nominal confidence level can be achieved. However, the Gumbel convergence in Theorem 1 can be quite slow, which implies that the SCB in (3.3) by using the asymptotically cut-off value may not have a good finite sample performance. To circumvent such a problem, as in Wu and Zhao (2007) and Liu and Wu (2010), we next introduce a simulation-based method.

1. Choose an appropriate bandwidth h and a kernel function K satisfying **(C5)**. Then using (1.3), we compute $\widehat{g}_n(x)$ and the estimated residuals $\widehat{e}_t = Y_t - \widehat{g}_n(X_t)$. Based on the latter, with another bandwidth b, we estimate $\sigma^2(\cdot)$ by using (3.1) and denote the estimate by $\widehat{\sigma}_n^2(\cdot)$.

2. Generate i.i.d. standard normal random variables $e_1^*, \ldots, e_n^*$ which are independent of $x_1, \ldots, x_n$; compute $Y_t^* = \widehat{\sigma}_n(X_t) e_t^*$; perform a local linear regression of (Y_t^*) on (X_t) with bandwidth h and kernel K and let $\hat{g}_n^*(\cdot)$ be the estimated regression function. Then compute the normalized maximum deviation
$$\Delta_n^* = \max_{|x| \le B_n} \left| V_n(x) \hat{g}_n^*(x) / \widehat{\sigma}_n(x) \right|. \tag{4.1}$$

3. Repeat the preceding Step 2 for N times, and then compute the $(1-\alpha)$th sample quantile, denoted by $\Delta_{n,1-\alpha}^*$.

4. Construct the $(1-\alpha)$th SCB for $g(x)$ over the interval $x \in [-B_n, B_n]$ as
$$\left[\widehat{g}_n(x) - \widehat{\sigma}_n(x) \Delta_{n,1-\alpha}^* / V_n(x), \quad \widehat{g}_n(x) + \widehat{\sigma}_n(x) \Delta_{n,1-\alpha}^* / V_n(x)\right].$$

From Theorem 1, with the normalization in (2.4), Δ_n^* and Δ_n have the same asymptotic Gumbel distribution. Then the empirical quantile of the former can approximate that of the latter. In comparison with the asymptotic Gumbel distribution, the distribution of Δ_n^* better approximates that of Δ_n. Consequently it is expected that the latter has a better finite sample performance. Our simulation-based construction has an important practical convenience: the procedure is the same as when data are stationary. Therefore we do not have to be concerned with whether the true data generating process is $I(0)$ or $I(1)$ for constructing SCBs with asymptotically correct coverage probabilities.

Example 1. Consider the nonlinear regression model

$$Y_t = g(X_t) + \sigma(X_t)e_t, \quad X_t = X_{t-1} + x_t, \quad t = 1, 2, \cdots, n, \tag{4.2}$$

where $g(x) = \log(10 + 0.25x^{1.2})$, $\sigma(x) = (6 + 0.2x^2)^{0.5}$, (x_t) is generated by the AR(1) process $x_t = 0.4x_{t-1} + v_t$, $v_t, t \in \mathsf{Z}$, are i.i.d. $\mathsf{N}(0, 1)$, e_t, $t \in \mathsf{Z}$, are i.i.d. uniform($-3^{1/2}$, $3^{1/2}$), and $(v_t)_{t\in\mathsf{Z}}$ and $(e_t)_{t\in\mathsf{Z}}$ are independent. We choose three levels of n: $n = 100, 200, 500$ and consider SCBs over the interval $x \in [B_l, B_u]$, where B_l (resp. B_u) is the 0.1th (resp. 0.9th) sample quantile of the $I(1)$ regressor (X_t).

Table 1. Coverage Probabilities for SCBs

	True $\sigma^2(\cdot)$		Estimated $\sigma^2(\cdot)$	
$(\alpha; n)$	Simulation	Theoretical	Simulation	Theoretical
$0.05; 100$	0.9411	0.9568	0.8859	0.9069
$0.01; 100$	0.9874	0.9971	0.9564	0.9805
$0.05; 200$	0.9376	0.9530	0.9098	0.9271
$0.01; 200$	0.9875	0.9960	0.9723	0.9878
$0.05; 500$	0.9495	0.9622	0.9447	0.9573
$0.01; 500$	0.9901	0.9974	0.9880	0.9963

In our simulation, for convenience we use the Gaussian kernel function $K(u) = (2\pi)^{-1/2}\exp(-u^2/2)$, since it has a very thin tail and is the default choice in R package KernSmooth. We also use the function dpill (Ruppert *et al*, 1995) in the package KernSmooth for choosing the bandwidth and the function locpoly for performing the local linear regression. Table 1 shows the coverage probabilities which are the proportions of the SCBs that cover the true function $g(x)$ over $x \in [B_l, B_u]$ based on 10^5 repetitions. Two levels are considered: $\alpha = 0.05$ and $\alpha = 0.01$. Coverage probabilities based on theoretical cut-off values computed from Theorem 1 are also shown. In the left panel we use the true variance function $\sigma^2(\cdot)$, while in the right one the estimated $\widehat{\sigma}_n^2(\cdot)$ is used. Using the true variance function $\sigma^2(\cdot)$, with $\alpha = 5\%$ and $n = 500$, the coverage probability for the simulation-based method is 0.9495 which is very close to the nominal level 0.95 with the relative accuracy level $(1 - 0.9495)/(1 - 0.95) = 1.01$. However, for the theoretical cut-off value based on Theorem 1, the coverage probability is 0.9622, which is quite different from 0.95 and has a relative accuracy level $(1 - 0.9622)/(1 - 0.95) = 0.76$. As expected, larger n leads to more accurate coverage probabilities. A similar claim can be made for the SCBs with $\alpha = 1\%$. Hence the simulation-based method has a more accurate finite sample performance in particular when $n = 500$. The accuracy of both coverage probabilities can be slightly affected if the estimated variance function $\widehat{\sigma}_n^2(\cdot)$ is used.

Example 2. We next consider the monthly US share price indices and treasury bill rates for the period January/1957–December/2009, which are downloaded from International Monetary Fund's website and were also used by Chen *et al* (2012). The upper two plots in Figure 1 give the two series with the log transformation applied to the share price data. The augmented Dickey-Fuller test indicates that the treasury bill rates are the $I(1)$ process, and the log-transformed share price indices are the $I(1)$ process with drift. To ensure that our methodology and theory are applicable to the data, we remove the drift from the share price indices and the adjusted share price indices are plotted in the lower-left plot of Figure 1. The lower-right plot in Figure 1 gives the scatter plot of the adjusted share price indices against the treasury bill rates, which indicates the existence of heteroskedasticity. The aim of this example is to analyze the relationship between the share price series and the treasury bill rates. Let (Y_t) be the adjusted share price series, and (X_t) be the treasury bill series. We first fit the data with the linear regression model defined by $Y_t = a + bX_t + e_t$, and obtain the least squares estimation $\widehat{a}$ and $\widehat{b}$ of the two parameters: -0.1582 and -0.0514, respectively. However, the cointegration test shows that the null hypothesis of the $I(1)$ error process cannot be rejected. This indicates that the linear cointegration model is not appropriate for the data.

We next consider the nonlinear regression model:

$$Y_t = g(X_t) + \varepsilon_t^*, \quad \varepsilon_t^* = \sigma(X_t)e_t, \quad t = 1, \cdots, 636, \tag{4.3}$$

where (e_t) is assumed to be independent of (X_t), g and σ are two nonlinear functions as in model (1.2). The structure of $\varepsilon_t^* = \sigma(X_t)e_t$ allows the existence of nonstationarity and heteroskedasticity for the model error term. The local linear estimation method is used to estimate the regression function g and the resulting estimate is denoted by $\widehat{g}_n(\cdot)$, where the bandwidth is chosen by the commonly-used cross-validation method. Let $\widehat{\varepsilon}_t^* = Y_t - \widehat{g}_n(X_t)$. Figure 2 plots the residual $(\widehat{\varepsilon}_t^*)$ against the regressor (X_t), which shows that our nonlinear model (4.3) with heteroskedasticity on errors is appropriate for the data.

The SCB for the regression function is plotted in Figure 3 with the cut-off value chosen by using the simulation-based method. The upper and lower dashed lines are the 95%-SCB for the regression function g. The solid line is the local linear estimated regression function and the dotted line is the estimated linear regression function. From the figure, we have to reject the hypothesis that g is linear at 5% level as part of the dotted line lies outside the SCB. This again shows that (X_t) and (Y_t) are not linearly cointegrated and the traditional linear model is not suitable for the data.

It is well-known that it is difficult to derive the local time limit theory for local linear estimation when (X_t) is multivariate $I(1)$, which limits the empirical applicability of the proposed SCB construction. While Figures 3 suggests a nonlinear relationship between the adjusted share price series and the treasury bill series, the true relationship may also depend on some other macroeconomic variables such as the long-term bond yields (Chen *et al*, 2012). To address the latter problem, we may extend the nonlinear regression model (4.3) to the nonlinear varying-coefficient models (c.f., Cai *et al*, 2009) and include some other macroeconomic time series in the empirical analysis. However, this is beyond the scope of this paper, and we will leave this in our future study.

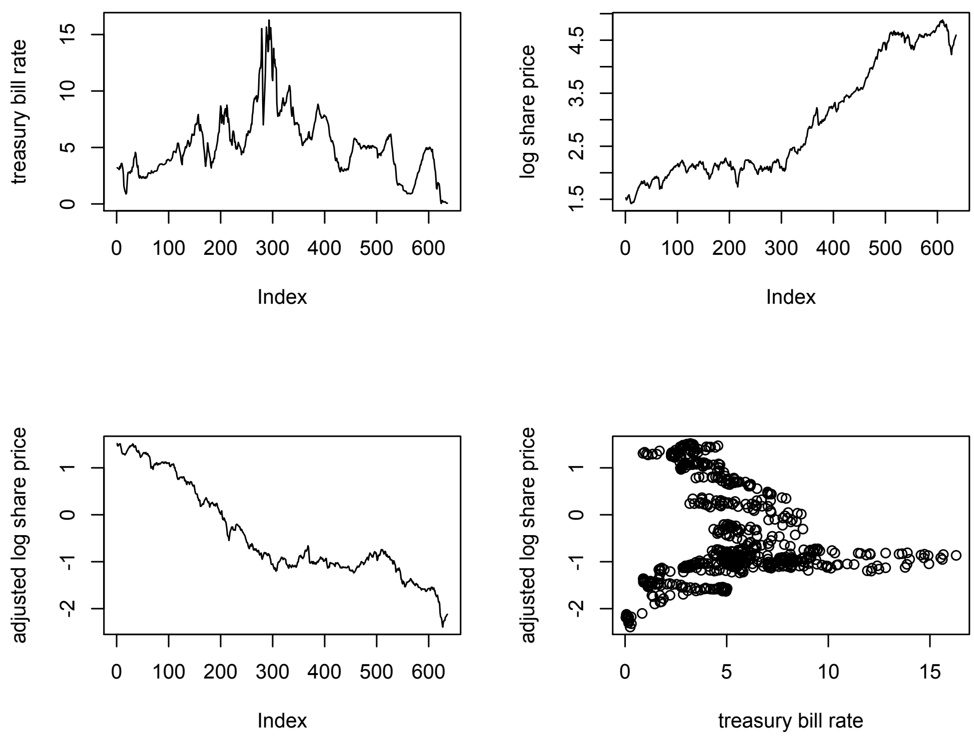

Figure 1: The data plots over period of January/1959–December/2009.

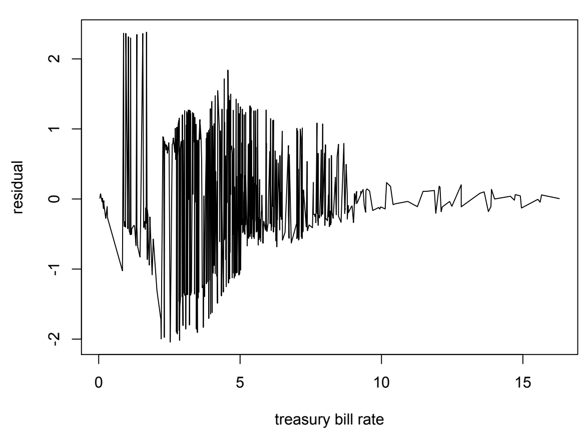

Figure 2: The residuals from the nonlinear regression model against the treasury bill rates.

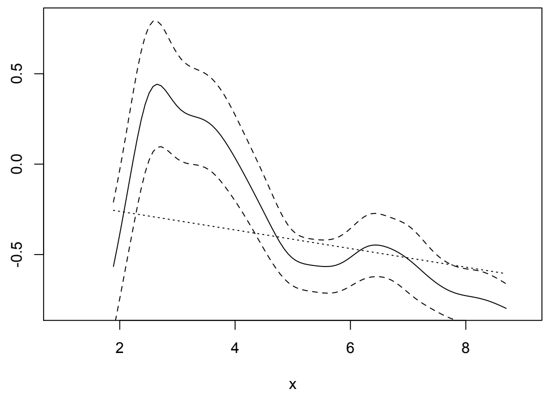

Figure 3: The SCB for the regression function g.

5. Proofs of the main results

In this section, we give the proofs of Theorem 1 in Section 2 and Proposition 1 in Section 3. We shall prove Theorem 1 for $B_n = \sqrt{n}/(\log^{c_0} n)$ with c_0 being a positive constant, which is asymptotically wider than $B_n = M_0 n^{1/2-\epsilon}$. Some of the arguments used in this section and the supplemental material are similar to those in Bickel and Rosenblatt (1973) and Liu and Wu (2010). However, due to the presence of non-stationary series, our proofs are quite challenging, using uniform asymptotics for martingale and functionals of non-stationary time series over some expanding intervals, which are more complicated than the uniform asymptotic results over a bounded interval used in the stationary time series literature.

Proof of Theorem 1. Let

$$\begin{aligned}
\Gamma_{1n}(x) &= \frac{1}{S_n(x)V_{n2}(x)} \cdot \sum_{t=1}^{n} w_t(x)\big[g(X_t) - g(x)\big], \\
\Gamma_{2n}(x) &= \frac{V_{n1}(x)}{S_n(x)V_{n2}(x)} \cdot \sum_{t=1}^{n} K_1[(X_t - x)/h]e_t, \\
\Gamma_{3n}(x) &= \frac{1}{S_n(x)V_{n2}(x)} \cdot \sum_{t=1}^{n} w_t(x)\big[\sigma(X_t) - \sigma(x)\big]e_t/\sigma(x),
\end{aligned}$$

where $S_n^2(x) = \sum_{t=1}^n K^2\big[(X_t - x)/h\big]$, $V_{nj}(\cdot)$, $w_t(\cdot)$ and $K_1(\cdot)$ are defined as in

Section 1. Similarly to Liu and Wu (2010), we may split the $V_n(x)\big[\widehat{g}_n(x) - g(x)\big]/\sigma(x)$ as follows:

$$\begin{aligned} & V_n(x)\big[\widehat{g}_n(x) - g(x)\big]/\sigma(x) \\ = \quad & V_n(x)\Big\{\sum_{t=1}^n w_t(x)\sigma(X_t)e_t + \sum_{t=1}^n w_t(x)\big[g(X_t) - g(x)\big]\Big\}/\big[\sigma(x)\sum_{t=1}^n w_t(x)\big] \\ = \quad & \frac{1}{S_n(x)} \cdot \sum_{t=1}^n K[(X_t - x)/h]e_t + \Gamma_{1n}(x)/\sigma(x) + \Gamma_{2n}(x) + \Gamma_{3n}(x). \qquad (5.1) \end{aligned}$$

Furthermore, we apply the truncation technique to deal with the first term on the right hand side of (5.1). Let $\mathcal{I}_n = [-h^{-1}B_n, h^{-1}B_n]$ and $e'_t = \tilde{e}_t/(\mathsf{E}\tilde{e}_t^2)^{1/2}$, where

$$\tilde{e}_t = e_t I\{|e_t| \le \log n\} - \mathsf{E}\left[e_t I\{|e_t| \le \log n\}\right], \quad 1 \le t \le n. \qquad (5.2)$$

Define $Z_t(x) = K(X_t/h - x)/S_n(xh)$ and

$$\begin{aligned} M_n(x) &= \sum_{t=1}^n Z_t(x)e_t, \quad M_n = \sup_{x\in\mathcal{I}_n} |M_n(x)|; \\ \widetilde{M}_n(x) &= \sum_{t=1}^n Z_t(x)e'_t, \quad \widetilde{M}_n = \sup_{x\in\mathcal{I}_n} |\widetilde{M}_n(x)|. \end{aligned}$$

The main idea of proving Theorem 1 is to show that $\Gamma_{jn}(x)$, $j = 1, 2, 3$, are asymptotically dominated by $M_n(x)$ uniformly over $|x| \le B_n$, and $M_n(x)$ is asymptotically equivalent to its truncated version $\widetilde{M}_n(x)$. Due to (5.1) and $\inf_{x\in\mathsf{R}} \sigma(x) > 0$, it is clearly seen that Theorem 1 follows from the following three propositions, which are proved in the supplemental materials. □

Proposition 2. *Under the conditions of Theorem 1, we have*

$$\sup_{|x|\le B_n} |\Gamma_{jn}(x)| = O_P(\log^{-2} n), \quad j = 1, 2, 3.$$

Proposition 3. *Under the conditions of Theorem 1, we have*

$$\sup_{x\in\mathcal{I}_n} |M_n(x) - \widetilde{M}_n(x)| = O_P(\log^{-2} n).$$

Proposition 4. *Under the conditions of Theorem 1, we have for any $z \in \mathsf{R}$,*

$$\mathsf{P}\Big\{(2\log \bar{h}^{-1})^{1/2}(\widetilde{M}_n - d_n) \le z\Big\} \to e^{-2e^{-z}},$$

where d_n is defined as in Theorem 1.

Proof of Proposition 1. Note that, due to the condition (C4)(ii) and $K(s) = 0$ if $|s| \ge A$,

$$\frac{\big|\sigma^i(X_t) - \sigma^i(x)\big|}{\sigma^i(x)} \cdot K\big[(X_t - x)/b\big] \quad \le \quad CbK\big[(X_t - x)/b\big],$$

for $i = 1$ and 2, all $x \in \mathsf{R}$ and a sufficiently small b. Similarly, we have

$$|\widehat{g}_n(X_k) - g(X_k)|^i\, K\big[(X_k - x)/b\big] \le CK\big[(X_k - x)/b\big] \cdot \sup_{|x| \le 2B_n} |\widehat{g}_n(x) - g(x)|^i\,,$$

for $i = 1$ and 2, $|x| \le B_n$ and a sufficiently small b. Then, we have

$$\begin{aligned}\frac{\big|\widehat{\sigma}_n^2(x) - \sigma^2(x)\big|}{\sigma^2(x)} \quad &\le \quad C\left[b + \sigma^{-2}(x) \sup_{|z| \le 2B_n} |\widehat{g}_n(z) - g(z)|^2\right] + \\ &\qquad C\left|\frac{\sum_{t=1}^n K\big[(X_t - x)/b\big](e_t^2 - 1)}{\sum_{t=1}^n K\big[(X_t - x)/b\big]}\right| + \\ &\qquad C\left[\frac{\sum_{t=1}^n K\big[(X_t - x)/b\big]|e_t|}{\sigma(x)\sum_{t=1}^n K\big[(X_t - x)/b\big]} \cdot \sup_{|z| \le 2B_n} |\widehat{g}_n(z) - g(z)|\right],\end{aligned}$$

for $|x| \le B_n$ and a sufficiently small b. Furthermore, by Theorem 4.1 of Chan and Wang (2014) and (C3)(iii), we have

$$\begin{aligned}\sup_{|x| \le 2B_n} |\widehat{g}_n(x) - g(x)| \quad &= \quad O_P\big[(nh^2)^{-1/4}\log^{1/2} n + h^2 \sup_{|x| \le 2B_n} |g_0(x)|\big] \\ &= \quad O_P\big[(nh^2)^{-1/4}\log^2 n\big].\end{aligned}$$

The above arguments, together with Lemma A.4 given in the supplemental document, $b \asymp h$ and $\inf_{x \in \mathsf{R}} \sigma(x) > 0$, lead to

$$\sup_{|x| \le B_n} \frac{\big|\widehat{\sigma}_n^2(x) - \sigma^2(x)\big|}{\sigma^2(x)} \quad = \quad O_P\big[h + (nh^2)^{-1/4}\log^2 n\big].$$

Hence, we have

$$\sup_{|x| \le B_n}\left|\frac{\sigma(x)}{\widehat{\sigma}(x)} - 1\right| \quad \le \quad \sup_{|x| \le B_n} \frac{\sigma(x)}{\widehat{\sigma}(x)}\, \frac{\big|\widehat{\sigma}^2(x) - \sigma^2(x)\big|}{\sigma^2(x)} = O_P\big[h + (nh^2)^{-1/4}\log^2 n\big].$$

This completes the proof of Proposition 1. □

6. Conclusion

In this paper we study a local linear estimator of the regression function in a nonlinear regression model, where the univariate regressor is generated by a nonstationary $I(1)$ process. Under some regularity conditions and with a proper normalization, we derive the asymptotic Gumbel distribution for the maximum deviation of the developed local linear estimator, and then use this result to construct the SCBs of the regression function. Furthermore, we also propose a nonparametric kernel-based method to estimate the variance function, and introduced a simulation-based procedure to choose the cut-off value in the construction of SCBs which can circumvent the slow convergence issue of the asymptotic Gumbel distribution. The simulation study and the empirical application show that the proposed approach of constructing SCBs performs reasonably well in finite samples.

Supplementary materials

The supplementary materials contain the proofs of Propositions 2–4, some technical lemmas with proofs, and the discussion on the asymptotic Gumbel distribution for the Nadaraya-Watson kernel estimation.

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