Article:

Ye, H and Yang, H (2017) Rational Behavior Adjustment Process with Boundedly Rational User Equilibrium. Transportation Science, 51 (3). pp. 968-980. ISSN: 0041-1655

https://doi.org/10.1287/trsc.2016.0715

Rational Behavior Adjustment Process with Boundedly Rational User Equilibrium

Hongbo Ye[a] and Hai Yang[b]

[a] *Institute for Transport Studies, University of Leeds, Leeds LS2 9JT, United Kingdom*
[b] *Department of Civil and Environmental Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, PR China*

Abstract

This paper extends the framework of “rational behavior adjustment process” (RBAP) to incorporating the “boundedly rational user equilibrium” (BRUE). The proportional-switch adjustment process (PSAP) and the network tatonnement process (NTP) are extended to the BRUE case, and their dynamical equations are shown to be Lipschitz continuous, which guarantees the global uniqueness of the classical solutions. A special group of the BRUE-RBAP is proposed, for which the path flows would increase if the paths are in the acceptable path set, and would decrease otherwise. Classical solutions to this special group of models may not exist. Stability of the BRUE-RBAP with classical solutions is proved with separable link travel cost functions. For non-separable link travel cost functions, the stability of the BRUE-PSAP is proved. Numerical examples are presented to demonstrate the evolution processes of BRUE-PSAP and BRUE-NTP under various bounded rationality thresholds and various initial states. The applicability of BRUE-PSAP in larger networks with asymmetric link travel cost functions is also illustrated.

Keywords: day-to-day dynamics, rational behavior adjustment process, boundedly rational user equilibrium, stability

1. Introduction

The term “bounded rationality” was proposed by Herbert A. Simon in the 1950s to take into account the irrationality on people’s decision-making procedure (Simon, 1955). Different from the classical assumption of utility maximization, a decision maker may pick up a satisfactory choice rather than the utility-maximizing one due to cognition limitation, incomplete information or other restraints (Conlisk, 1996; Gabaix et al., 2006). The concept of bounded rationality was widely implemented and investigated in the transportation area. For example, Chorus and Timmermans (2009) considered travelers' limited awareness and developed the methodology for the ex-ante evaluation of user benefits associated with transport system changes; Gao et al. (2011) considered bounded rationality in a route choice model with cognitive cost in real-time information acquisition; Szeto and Lo (2006) and Han et al. (2015) adopted bounded rationality in the tolerance-based dynamic user equilibrium; Zhao and Huang (2016) introduced the concept of aspiration level and investigated the user equilibrium problem based on it.

A prominent branch of the bounded rationality model is the indifference band model, application of which in the transportation field could trace back to Mahmassani and Chang (1987), who considered bounded rationality in a single bottleneck with departure time choice. The indifference band approach was embedded in simulation frameworks to investigate the transportation system performance with various real-time traffic information strategies (Emmerink et al., 1995a, 1995b; Jayakrishnan et al., 1994; Mahmassani and Jayakrishnan, 1991), as well as incorporating responsive signals (Hu and Mahmassani, 1997) or simultaneous departure time and route choices (Mahamssani and Liu, 1999). Empirical analyses were conducted in Di et al. (2016b) and Jou et al. (2005, 2010).

Among the indifference band models, the boundedly rational user equilibrium (BRUE) is an extension of the classical Wardrop's user equilibrium (UE), allowing travelers to be non-strict utility maximizers. Lou et al. (2010) for the first time examined the mathematical properties of the BRUE traffic assignment problem in general networks. Definition and link/path-based presentation of BRUE were provided, sets of BRUE link/path flow patterns were shown to be non-empty and non-convex, and congestion pricing under BRUE was discussed. Following this study, Di et al. (2013, 2014, 2016a) presented a series of work which respectively discussed the methodology for constructing the BRUE path flow set, the Braess Paradox with regard to BRUE, and the second best toll pricing under bounded rationality. Recently, Di and Liu (2016) provided a comprehensive review on the application of bounded rationality in modelling travelers' route choice behaviors.

Although the behavioral assumption of bounded rationality is widely used in the simulation and empirical studies of dynamic traffic networks, theoretical research on bounded-rationality-based day-to-day dynamics is limited so far. Guo and Liu (2011) developed a BRUE-based day-to-day framework to model the irreversible network change and then applied it to investigate the flow evolution during the collapse and reopening of the I-35W Bridge in the Twin Cities network in Minnesota, without discussing the system stability issues. Di et al. (2015) specialized the model in Guo and Liu (2011) and focused on a parallel-link and single origin-destination (OD) network with separable and linear link time functions. Stability of their special model was examined from the perspective of switched systems. Wu et al. (2013) investigated the path flow evolution in urban railway networks by treating passengers' learning process on perceived travel costs as a boundedly rational behavior.

This study is intended to make a contribution to the modeling of BRUE-based day-to-day dynamics, by introducing BRUE into the conceptual framework of rational behavior adjustment process (RBAP). RBAP is a type of fixed-demand day-to-day dynamics summarized by Zhang et al. (2001) and Yang and Zhang (2009), which includes the day-to-day models in Smith (1984), Friesz et al. (1994) and Nagurney and Zhang (1997) as its special cases. There are also day-to-day models that can be treated as the link-based version of RBAP (Guo et al., 2013, 2015a; Han and Du, 2012; He et al., 2010; Smith and Mounce, 2011) or as the extension of RBAP under elastic demand (Guo et al., 2013, 2015a; Li et al., 2012; Sandholm, 2002, 2005). In this paper, we attempt to establish the BRUE-based RBAP (BRUE-RBAP) and investigate its properties.

The rest of this paper is organized as follows. Section 2 proposes a variational inequality formulation for the BRUE traffic assignment problem and extends the RBAP in Yang and Zhang (2009) to the BRUE-RBAP. The proportional-switch adjustment process (PSAP) (Smith, 1984) and the network tatonnement process (NTP) (Friesz et al., 1994) are extended to be BRUE-based and are shown to be of classical solutions. A special group of BRUE-RBAP is proposed, which incorporates the model in Di et al. (2015). It is further shown by an example that the classical solutions to the model in Di et al. (2015) may not exist. Section 3 discusses the stability of (i) the general BRUE-RBAP with separable link travel cost functions and (ii) the BRUE-PSAP with non-separable link travel cost functions. In Section 4, numerical examples based on the Braess network are presented to compare the evolution processes of the BRUE-PSAP and the BRUE-NTP with respect to different bounded rationality thresholds and different initial states. A larger example based on the

Nguyen-Dupuis network further illustrates the applicability of the BRUE-PSAP with asymmetric link travel cost functions. Section 5 draws the conclusions and provides possible directions for future research.

2. BRUE-RBAP: framework and models

Consider a general network with a set A of links and a set W of OD pairs. Denote by v_a the flow on link $a \in A$ and $v = \left(v_a, a \in A\right)^{\mathrm{T}}$ the link flow vector, where superscript "T" stands for the transpose operation. Each link $a \in A$ is associated with a link travel cost function $c_a\left(v\right)$, while each OD pair $w \in W$ is associated with a fixed demand d_w and a path set R_w. Denote by m_w the total number of paths in R_w, and $m = \sum_{w \in W} m_w$. Let f_{rw} be the flow on path $r \in R_w$, $w \in W$, and $f = \left(f_{rw}, r \in R_w, w \in W\right)^{\mathrm{T}}$ be the path flow vector. Denote by c_{rw} the travel cost on path $r \in R_w$, which is equal to the sum of costs on links constituting this path, and $c = \left(c_{rw}, r \in R_w, w \in W\right)^{\mathrm{T}}$ the vector of path costs. The set $\Omega = \left\{ f \middle| \sum_{r \in R_w} f_{rw} = d_w, f_{rw} \geq 0, r \in R_w, w \in W \right\}$ contains all feasible path flow patterns. The definition of BRUE (Di et al., 2013; Guo and Liu, 2011) is given as follows.

Definition 1. *Define* $\mu_w = \min_{r \in R_w} c_{rw}$, $w \in W$. *A path flow pattern* $f \in \Omega$ *is said to be a boundedly rational user equilibrium (BRUE) path flow pattern if it holds that*

$$\begin{cases} f_{rw} \geq 0, & \text{if } c_{rw} \leq \mu_w + \varepsilon_w \\ f_{rw} = 0, & \text{if } c_{rw} > \mu_w + \varepsilon_w \end{cases}, \quad r \in R_w, \quad w \in W \tag{1}$$

where $\varepsilon_w \geq 0$ *is the bounded rationality threshold of travelers between OD pair* $w \in W$.

Before discussing the day-to-day dynamics, we introduce the following assumption on the path cost functions and present some results on BRUE.

Assumption 1. The path cost functions $c_{rw}\left(f\right)$, $r \in R_w$, $w \in W$, are non-negative, bounded and continuously differentiable on Ω. Thus $c(f)$ is Lipschitz continuous, i.e., there exists $\rho_1 > 0$ such that

$$\|c(f)-c(g)\| \le \rho_1 \|f-g\|, \ \ \forall f, g \in \Omega$$

where $\|\cdot\|$ represents the 2-norm (Euclidean norm).

Denoting $\hat{c}_{rw} = c_{rw} - \mu_w - \varepsilon_w$ and $\hat{c} = \left(\hat{c}_{rw}, r \in R_w, w \in W\right)^{\mathrm{T}}$, the BRUE condition (1) can be rewritten as

$$\begin{cases} f_{rw} \ge 0, & \text{if } \left[\hat{c}_{rw}\right]_+ = 0 \\ f_{rw} = 0, & \text{if } \left[\hat{c}_{rw}\right]_+ > 0 \end{cases}, \ \ r \in R_w, \ \ w \in W \tag{2}$$

where $[x]_+ = \max\{x, 0\}$. Referring to Smith (1979), any path flow pattern $f \in \Omega$ satisfying condition (2) must satisfy the following variational inequality (VI) condition (3), and vice versa:

$$(f' - f)^{\mathrm{T}} \left[\hat{c}(f)\right]_+ \ge 0, \ \ \forall \ f' \in \Omega \tag{3}$$

Equation (3) provides the VI formulation for BRUE traffic assignment problems. It is worth pointing out that, different from the classical VI form for traffic assignment (Dafermos, 1980), the path-based VI (3) has no equivalent link-based form, and $\left[\hat{c}(f)\right]_+$ is Lipschitz continuous but neither differentiable nor pseudo-monotone. Furthermore, since Ω is compact and convex, then the solution set to VI problem (3), i.e. the set of the BRUE path flow patterns, denoted by Ω^*, is nonempty and compact (Corollary 2.2.5, Facchinei and Pang, 2003). Also, we have known that Ω^* is usually nonconvex (Lou et al., 2010), and it is connected under very special settings (with affine linear and strictly monotone link cost functions, see Di et al., 2015). However, its connectedness under general nonlinear and non-separable link cost functions remains an open question.

Regarding the day-to-day flow dynamics, denote by $f(t)$ the path flow vector at calendar time t. Let $\dot{f}(t) = \mathrm{d}f(t)/\mathrm{d}t$ and consider the path flow dynamics

$$\dot{f}(t) = F(f(t)), \ \ t \ge 0, \ \ f(0) = f_0 \in \Omega \tag{4}$$

where $F(f(t)) = \left(F_{rw}(f(t)), r \in R_w, w \in W\right)^{\mathrm{T}}$, under which the nonnegative path flows and fixed demands are assumed to be guaranteed. Also, assume system (4) has classical solutions, that is to say, for every $f_0 \in \Omega$, there exists a unique solution $f(t)$ of (4) defined on $[0,\infty)$ satisfying $f(0) = f_0$. The day-to-day dynamics with no classical solutions would be illustrated in Example 1 in Section 2.3.

Referring to the rational behavior adjustment process (RBAP) in Yang and Zhang (2009), we define the BRUE-RBAP in Definition 2. It is clear that RBAP is a special case of BRUE-RBAP with $\varepsilon_w = 0$ for all $w \in W$.

Definition 2. *The flow dynamics (4) is BRUE-RBAP if* $c(f)^{\mathrm{T}} \dot{f} \leq 0$, *with the equality holding if and only if* f *is a BRUE path flow pattern.*

The abstract concept of BRUE-RBAP in Definition 2 is not applicable until specific day-to-day models are built based on this framework, and to do it the most direct way would be extending existing models in the RBAP category into BRUE-RBAP.

2.1. PSAP under BRUE

Based on Guo (2013), the PSAP in Smith (1984) can be extended to BRUE-PSAP as follows,

$$F_{rw}(f) = \sum_{s \in R_w} \left(f_{sw} \left[c_{sw} - c_{rw} - \varepsilon_w \right]_+ - f_{rw} \left[c_{rw} - c_{sw} - \varepsilon_w \right]_+ \right), \quad r \in R_w, \quad w \in W \tag{5}$$

The right hand side of Eq. (5) can always be multiplied by a positive scalar to form a more general flow dynamics. Since adding this scaler would not affect the analysis in this paper, it is omitted for simplicity.

The global existence and uniqueness of the solution to BRUE-PSAP is assured by Theorem 1 shown below, and the following lemma is necessitated.

Lemma 1. *Consider* $(x_1, \cdots, x_m)^{\mathrm{T}} \in \mathbb{R}^m$. *The following relations hold:*

$$\max\left(|x_1|, \cdots, |x_m|\right) \leq \left(\sum_{i=1}^{m} x_i^2 \right)^{1/2} \leq \sqrt{m} \max\left(|x_1|, \cdots, |x_m|\right) \tag{6}$$

Proof. The proof is obvious and thus omitted. □

Theorem 1. *Function* $F(f)$ *in Eq. (5) is Lipschitz continuous on* Ω*, therefore the solution to BRUE-PSAP is globally unique.*

Proof. First, by Eq. (6), we have

$$\left\|F(f)-F(g)\right\| \le \sqrt{m}\max_{w\in W}\max_{r\in R_w}\left|F_{rw}(f)-F_{rw}(g)\right| \tag{7}$$

Letting $\Delta c_{rsw}=c_{rw}-c_{sw}-\varepsilon_w$ and substituting Eq. (5) into Eq. (7), we have that, for each OD pair $w\in W$,

$$\begin{aligned}
&\max_{r\in R_w}\left|F_{rw}(f)-F_{rw}(g)\right| \\
&\le 2m_w\max_{r\in R_w}\max_{s\in R_w}\left|f_{sw}\left[\Delta c_{srw}(f)\right]_+ - g_{sw}\left[\Delta c_{srw}(g)\right]_+\right| \\
&\le 2m_w\max_{r\in R_w}\max_{s\in R_w}\left|f_{sw}\Delta c_{srw}(f)-g_{sw}\Delta c_{srw}(g)\right| \\
&\le 2m_w\max_{r\in R_w}\max_{s\in R_w}\left|f_{sw}\left(c_{sw}(f)-c_{rw}(f)\right)-g_{sw}\left(c_{sw}(g)-c_{rw}(g)\right)\right| \\
&\quad +2\varepsilon_w m_w\max_{s\in R_w}\left|f_{sw}-g_{sw}\right|
\end{aligned} \tag{8}$$

Further substituting Eq. (8) into Eq. (7) reads

$$\begin{aligned}
&\left\|F(f)-F(g)\right\| \\
&\le 2\sqrt{m}\max_{w\in W}(m_w)\max_{w\in W}\max_{r\in R_w}\max_{s\in R_w}\left|f_{sw}\left(c_{sw}(f)-c_{rw}(f)\right)-g_{sw}\left(c_{sw}(g)-c_{rw}(g)\right)\right| \\
&\quad +2\sqrt{m}\max_{w\in W}(\varepsilon_w m_w)\max_{w\in W}\max_{s\in R_w}\left|f_{sw}-g_{sw}\right| \\
&\le 2\sqrt{m}\max_{w\in W}(m_w)\left(\sum_{w\in W}\sum_{r\in R_w}\sum_{s\in R_w}\left(f_{sw}\left(c_{sw}(f)-c_{rw}(f)\right)-g_{sw}\left(c_{sw}(g)-c_{rw}(g)\right)\right)^2\right)^{1/2} \\
&\quad +2\sqrt{m}\max_{w\in W}(\varepsilon_w m_w)\left\|f-g\right\|
\end{aligned} \tag{9}$$

Since $c_{rw}(\cdot)$, $r\in R_w$, $w\in W$, are all differentiable, then there exists $\rho_2>0$ such that

$$\left(\sum_{w\in W}\sum_{r\in R_w}\sum_{s\in R_w}\left(f_{sw}\left(c_{sw}(f)-c_{rw}(f)\right)-g_{sw}\left(c_{sw}(g)-c_{rw}(g)\right)\right)^2\right)^{1/2} \le \rho_2\left\|f-g\right\| \tag{10}$$

Substituting Eq. (10) into Eq. (9) yields

$$\left\|F(f)-F(g)\right\| \le 2\sqrt{m}\left(\rho_2\max_{w\in W}(m_w)+\max_{w\in W}(\varepsilon_w m_w)\right)\left\|f-g\right\|$$

That is to say, $F(f)$ in Eq. (5) is Lipschitz continuous. Therefore BRUE-PSAP has a classical solution which is globally unique (Theorem 3.2, Khalil, 2002). □

We show that BRUE-PSAP is BRUE-RBAP in the following theorem.

Theorem 2. *BRUE-PSAP in (5) is BRUE-RBAP.*

Proof. We first write

$$\begin{aligned} c^{\mathrm{T}} \dot{f} &= \sum_{w \in W} \sum_{r \in R_w} c_{rw} \sum_{s \in R_w} \left(f_{sw} \left[\Delta c_{srw} \right]_+ - f_{rw} \left[\Delta c_{rsw} \right]_+ \right) \\ &= \sum_{w \in W} \sum_{r \in R_w} \sum_{s \in R_w} c_{rw} f_{sw} \left[\Delta c_{srw} \right]_+ - \sum_{w \in W} \sum_{r \in R_w} \sum_{s \in R_w} c_{rw} f_{rw} \left[\Delta c_{rsw} \right]_+ \end{aligned} \tag{11}$$

For the second term on the right hand side of the last equality in Eq. (11), interchanging the indices of r and s, we have

$$c^{\mathrm{T}} \dot{f} = \sum_{w \in W} \sum_{r \in R_w} \sum_{s \in R_w} \left(c_{rw} - c_{sw} \right) f_{sw} \left[\Delta c_{srw} \right]_+ \leq 0$$

with the equality holding if and only if the following condition holds:

$$f_{sw} = 0 \ \text{ if } \ \Delta c_{srw} > 0, \ \ \forall \ r, s \in R_w, \ \ w \in W \tag{12}$$

which indicates BRUE. Therefore, the flow dynamics (5) is BRUE-RBAP. □

2.2. NTP under BRUE

The NTP (Friesz et al., 1994; Yang and Zhang, 2009) is given as

$$F(f) = P_\Omega(f - \beta c) - f, \ \ \beta > 0 \tag{13}$$

where

$$P_\Omega(x) = \arg\min_{y \in \Omega} \| y - x \| \tag{14}$$

Based on the VI formulation (3), we can extend the NTP to the following BRUE-NTP:

$$F(f) = P_\Omega\left(f - \beta[\hat{c}]_+\right) - f, \ \ \beta > 0 \tag{15}$$

The existence and uniqueness of the classical solution to BRUE-NTP is assured by the following theorem.

Theorem 3. *$F(f)$ in Eq. (13) is Lipschitz continuous on Ω, therefore, the classical solution to BRUE-NTP is globally unique.*

Proof. From Eq. (15),

$$
\begin{aligned}
\left\|F(f)-F(g)\right\| &= \left\|\left(P_\Omega\left(f-\beta\left[\hat{c}(f)\right]_+\right)-f\right)-\left(P_\Omega\left(g-\beta\left[\hat{c}(g)\right]_+\right)-g\right)\right\| \\
&\le \left\|P_\Omega\left(f-\beta\left[\hat{c}(f)\right]_+\right)-P_\Omega\left(g-\beta\left[\hat{c}(g)\right]_+\right)\right\|+\left\|f-g\right\| \\
&\le \left\|\left(f-\beta\left[\hat{c}(f)\right]_+\right)-\left(g-\beta\left[\hat{c}(g)\right]_+\right)\right\|+\left\|f-g\right\| \\
&\le \beta\left\|\left[\hat{c}(f)\right]_+-\left[\hat{c}(g)\right]_+\right\|+2\left\|f-g\right\| \\
&\le \beta\left\|\hat{c}(f)-\hat{c}(g)\right\|+2\left\|f-g\right\| \\
&\le \beta\left\|c(f)-c(g)\right\|+2\left\|f-g\right\|+\left(\sum_{w\in W} m_w\left(\mu_w(f)-\mu_w(g)\right)^2\right)^{1/2}
\end{aligned}
\tag{16}
$$

where the second inequality is due to $\left\|P_\Omega(x)-P_\Omega(y)\right\|\le\left\|x-y\right\|$ (Theorem 1.5.5(d), Facchinei and Pang, 2003). For the third term on the right hand side of the last inequality in Eq. (16), without loss of generality, assuming that, for OD pair $w\in W$, we have $\mu_w(f)\ge\mu_w(g)\ge 0$ and $s'\in\arg\min_{s\in R_w} c_{sw}(g)$, then by Eq. (6),

$$
\begin{aligned}
\left|\mu_w(f)-\mu_w(g)\right| &= \left|\min_{r\in R_w} c_{rw}(f)-\min_{s\in R_w} c_{sw}(g)\right| \\
&= \min_{r\in R_w} c_{rw}(f)-c_{s'w}(g) \\
&\le c_{s'w}(f)-c_{s'w}(g) \\
&\le \max_{r\in R_w}\left|c_{rw}(f)-c_{rw}(g)\right| \\
&\le \left(\sum_{r\in R_w}\left(c_{rw}(f)-c_{rw}(g)\right)^2\right)^{1/2}
\end{aligned}
\tag{17}
$$

and thus

$$
\sum_{w\in W} m_w\left(\mu_w(f)-\mu_w(g)\right)^2 \le \sum_{w\in W} m_w\sum_{r\in R_w}\left(c_{rw}(f)-c_{rw}(g)\right)^2 \le \max_{w\in W} m_w\left\|c(f)-c(g)\right\|^2 \tag{18}
$$

Substituting Eq. (18) into Eq. (16) leads to

$$
\begin{aligned}
\left\|F(f)-F(g)\right\| &\le \beta\left\|c(f)-c(g)\right\|+2\left\|f-g\right\|+\max_{w\in W}\sqrt{m_w}\left\|c(f)-c(g)\right\| \\
&\le \left(2+\rho_1\beta+\rho_1\max_{w\in W}\sqrt{m_w}\right)\left\|f-g\right\|
\end{aligned}
\tag{19}
$$

Therefore $F(f)$ is Lipschitz continuous and the BRUE-NTP admits globally unique solution given any feasible initial state. □

To show that BRUE-NTP is BRUE-RBAP, we first introduce the following lemma.

Lemma 2. *Under BRUE-NTP in (15),* $\hat{c}^{\mathrm{T}}\dot{f} \le [\hat{c}]_+^{\mathrm{T}}\dot{f}$.

Proof. Define the acceptable path set of OD pair $w \in W$ with respect to path costs $c(f)$ as

$$S_w = \{r | c_{rw} \le \mu_w + \varepsilon_w, r \in R_w\} = \{r | \hat{c}_{rw} \le 0, r \in R_w\}, \quad w \in W \tag{20}$$

and $S_w^c = R_w \setminus S_w$ the complement of S_w with respect to R_w. Referring to the definition of $P_\Omega(f - \beta[\hat{c}]_+)$ in Eq. (14), given f, $\dot{f}$ is the unique solution to the following minimization problem (21)-(23):

$$\min_y \sum_{w \in W} \sum_{r \in R_w} \left(y_{rw} + \beta[\hat{c}_{rw}]_+ \right)^2 \tag{21}$$

$$\text{s.t.} \ \sum_{r \in R_w} y_{rw} = 0, \quad w \in W \tag{22}$$

$$y_{rw} \ge -f_{rw}, \quad r \in R_w, \quad w \in W \tag{23}$$

Now we prove that $\dot{f}_{rw} \ge 0$ for all $r \in S_w$, $w \in W$, by contradiction. Suppose $\dot{f}_{iw} < 0$ for some path $i \in S_w$, then $[\hat{c}_{iw}]_+ = 0$. By $\sum_{r \in R_w} \dot{f}_{rw} = 0$, there must exist another path $j \in R_w$ such that $\dot{f}_{jw} > 0$. By choosing $0 < \delta < \min\{\dot{f}_{jw}, -\dot{f}_{iw}\}$, we will have

$$\dot{f}_{jw} > \dot{f}_{jw} - \delta > 0 > -f_{jw}, \quad 0 > \dot{f}_{iw} + \delta > \dot{f}_{iw} \ge -f_{iw}$$

and

$$\left(\dot{f}_{jw} - \delta + \beta[\hat{c}_{jw}]_+\right)^2 + \left(\dot{f}_{iw} + \delta + [\hat{c}_{iw}]_+\right)^2 < \left(\dot{f}_{jw} + \beta[\hat{c}_{jw}]_+\right)^2 + \left(\dot{f}_{iw} + [\hat{c}_{iw}]_+\right)^2$$

which violates the optimality of $\dot{f}$. Therefore, we must have $\dot{f}_{rw} \ge 0$ for all $r \in S_w$, $w \in W$, which further leads to $\sum_{r \in S_w} \hat{c}_{rw}\dot{f}_{rw} \le 0 = \sum_{r \in S_w} [\hat{c}_{rw}]_+ \dot{f}_{rw}$. Since $\sum_{r \in S_w^c} \hat{c}_{rw}\dot{f}_{rw} = \sum_{r \in S_w^c} [\hat{c}_{rw}]_+ \dot{f}_{rw}$, we can conclude that $\hat{c}^{\mathrm{T}}\dot{f} \le [\hat{c}]_+^{\mathrm{T}}\dot{f}$. □

Now we have the following theorem.

Theorem 4. *The BRUE-NTP is BRUE-RBAP.*

Proof. From Lemma 2, we have

$$c^{\mathrm{T}}\dot{f}=\hat{c}^{\mathrm{T}}\dot{f}\le[\hat{c}]_{+}^{\mathrm{T}}\dot{f}=[\hat{c}]_{+}^{\mathrm{T}}\left(P_{\Omega}\left(f-\beta[\hat{c}]_{+}\right)-f\right) \tag{24}$$

By Theorem 1.5.5(b) in Facchinei and Pang (2003),

$$\begin{aligned}&\left(P_{\Omega}\left(f-\beta[\hat{c}]_{+}\right)-f\right)^{\mathrm{T}}\left(f-\beta[\hat{c}]_{+}-P_{\Omega}\left(f-\beta[\hat{c}]_{+}\right)\right)\\&=-\left\|P_{\Omega}\left(f-\beta[\hat{c}]_{+}\right)-f\right\|^{2}-\beta[\hat{c}]_{+}^{\mathrm{T}}\left(P_{\Omega}\left(f-\beta[\hat{c}]_{+}\right)-f\right)\\&\ge 0\end{aligned} \tag{25}$$

Combining Eqs. (24) and (25) yields $c^{\mathrm{T}}\dot{f}\le-\left\|P_{\Omega}\left(f-\beta[\hat{c}]_{+}\right)-f\right\|^{2}\big/\beta\le 0$, and $c^{\mathrm{T}}\dot{f}=0$ if and only if

$$P_{\Omega}\left(f-\beta[\hat{c}]_{+}\right)-f=0 \tag{26}$$

Referring to Proposition 1.5.8 in Facchinei and Pang (2003), the solution to Eq. (26) is equivalent to that of the VI problem (3) and thus is BRUE. Therefore BRUE-NTP is BRUE-RBAP. □

2.3. A special group of BRUE-RBAP models

We refer to Assumption 2 in Zhang et al. (2001) to establish a special group of BRUE-RBAP models in Eq. (27). The readers may note that the original assumption in Zhang et al. (2001) violated the flow conservation, which assumed that under a fixed-demand day-to-day dynamics, traffic flows on paths with costs higher than the minimum cost would decrease, while flows on the minimum-cost paths would keep unchanged.

Theorem 5. *The flow dynamics is BRUE-RBAP if the following condition holds:*

$$\begin{cases}\dot{f}_{rw}\ge 0,\ \ r\in S_{w},w\in W\\ \dot{f}_{rw}\le 0,\ \ r\in S_{w}^{c},w\in W\end{cases}\text{, with all equalities holding simultaneously iff } f\in\Omega^{*} \tag{27}$$

Proof. From condition (27),

$$
\begin{aligned}
c^{\mathrm{T}}\dot{f} &= \sum_{w\in W}\left(\sum_{r\in S_w} c_{rw}\dot{f}_{rw} + \sum_{r\in S_w^c} c_{rw}\dot{f}_{rw}\right)\\
&\le \sum_{w\in W}\left(\sum_{r\in S_w}\left(\mu_w+\varepsilon_w\right)\dot{f}_{rw} + \sum_{r\in S_w^c}\left(\mu_w+\varepsilon_w\right)\dot{f}_{rw}\right)\\
&= \sum_{w\in W}\left(\mu_w+\varepsilon_w\right)\left(\sum_{r\in S_w}\dot{f}_{rw} + \sum_{r\in S_w^c}\dot{f}_{rw}\right)\\
&= 0
\end{aligned}
\tag{28}
$$

hence $c^{\mathrm{T}}\dot{f} \le 0$, with the equality holding if and only if $f \in \Omega^*$. Therefore dynamics (27) is BRUE-RBAP. □

It is easy to verify that BRUE-PSAP and BRUE-NTP do not necessarily satisfy condition (27), although they are BRUE-RBAP. Further, it is not easy, if not impossible, to build a dynamics of this type that has classical solutions. One of the models in the literature satisfying condition (27) is the one in Di et al. (2015), with the formulation given as follows:

$$
F_{rw}\left(f\right) = \begin{cases} \dfrac{1}{\left|S_w\right|}\displaystyle\sum_{s\in S_w^c} f_{sw}, & r\in S_w \\ -f_{rw}, & r\in S_w^c \end{cases},\quad w\in W \tag{29}
$$

where $F_{rw}\left(f\right)$ could be discontinuous because the set S_w could change with f. In some usual cases, the classical solution under rule (29) may not exist, which is illustrated by the following example.

Example 1. In Figure 1, consider a network consisting of a single OD and three parallel links (thus link is equivalent to path). The demand is fixed to be 16. The travel cost functions of the three links are respectively $c_1 = v_1 + 5$, $c_2 = 2v_2 + 1$, $c_3 = 5v_3 + 10$. The bounded rationality threshold is $\varepsilon = 2$. Suppose that the flow pattern at some time moment is $v = \left(6,6,4\right)$, and hence $c\left(v\right) = \left(11,13,30\right)$. Then path 1 is of the minimum cost, while path 2 is also acceptable. By rule (29), $\dot{v} = \left(2,2,-4\right)$, with which path 2 will immediately become unacceptable since the marginal cost of link 2 is greater than that of link 1. However, once path 2 becomes unacceptable, its flow will immediately decrease and path 2 will rejoin the acceptable path set. As a result, the flow on link 2 cannot increase exactly at the rate given by $\dot{v} = \left(2,2,-4\right)$, which means that the classical solution does not exist at $v = \left(6,6,4\right)$. □

3. Stability analysis

We begin with LaSalle's theorem to conduct the stability analysis on BRUE-RBAP.

Definition 3. *Consider the dynamical system*

$$\dot{f}(t) = F(f(t)), \quad f(0) \in \Phi, \quad t \geq 0 \tag{30}$$

where $F:\Phi \to \mathbb{R}^m$ *and* $\Phi \subseteq \mathbb{R}^m$. *A set* $M \subset \Phi$ *is said to be invariant if each solution starting in* M *remains in* M *for all* t.

Theorem 6. (Theorem 1, LaSalle, 1960) *Consider that the dynamical system (30) has classical solutions, and the classical solutions are continuous functions of the initial conditions. Let* Φ *be a bounded closed (compact) set with the property that every solution of (30) which begins in* Φ *remains in* Φ *for all future time. Suppose there is a scalar function* $V(x)$ *which has continuous first partials in* Φ *and is such* $\dot{V}(x) \leq 0$. *Let* E *be the set of all points in* Φ *where* $\dot{V}(x) = 0$. *Let* M *be the largest invariant set in E. Then every solution starting in* Φ *approaches M as* $t \to \infty$.

Based on Theorem 6, we prove the stability of BRUE-RBAP under separable link travel cost functions in the following theorem.

Theorem 7. *Assume the BRUE-RBAP has classical solutions that are continuous functions of the initial conditions, and the link travel cost functions are separable. If* $f(0) \in \Omega$, *then* $f(t) \to \Omega^*$ *as* $t \to \infty$.

Proof. Obviously, Ω is a compact, and if $f(0) \in \Omega$ then $f(t) \in \Omega$ for all $t \geq 0$. Define

$$V(f) = \sum_{a \in A} \int_0^{v_a(f)} c_a(\omega) \mathrm{d}\omega. \tag{31}$$

whose partial derivatives are continuous according to Assumption 1. By the definition of BRUE-RBAP, $\dot{V}(f) = c^{\mathrm{T}} \dot{f} \leq 0$, while $\dot{V}(f) = 0$ if and only if $f \in \Omega^*$, i.e.,

$\{f | f \in \Omega, \dot{V}(f) = 0\} = \Omega^*$. Since the largest invariant set contained in Ω^* is Ω^* itself, we have that, if $f(0) \in \Omega$, then $f(t) \to \Omega^*$ as $t \to \infty$. □

Remark 1. The proof for Theorem 7 does not require a positive definite $V(f)$, i.e., $V(f) \geq 0$ for all $f \in \Omega$ and $V(f) = 0$ if and only if $f \in \Omega^*$, which is actually not satisfied by Eq. (31). □

Remark 2. The global uniqueness of classical solutions and their continuity to the initial conditions are guaranteed when $F(f)$ is Lipschitz continuous (see Remark 2.2 in Chellaboina et al., 1999), such as BRUE-PSAP and BRUE-NTP. The situation becomes more complicated when $F(f)$ is discontinuous, which is not unusual for day-to-day dynamics, for example, the projected dynamical system (Nagurney and Zhang, 1997). □

Theorem 7 establishes the stability of the general BRUE-RBAP with separable link travel cost functions. When the link travel cost functions are non-separable, the stability analysis is difficult to conduct for the general case. Nonetheless, specifically, we show the stability of BRUE-PSAP with general link travel cost functions as follows.

Lemma 3. *For BRUE-PSAP, if the path travel cost function* $c(f)$ *is monotone, i.e.,*

$$(f-g)^{\mathrm{T}}(c(f)-c(g)) \geq 0, \ \forall f, g \in \Omega$$

then

$$F(f)^{\mathrm{T}} J_c(f) F(f) \geq 0, \ \forall f \in \Omega \tag{32}$$

where $J_c(f)$ *is the Jacobian of function* $c(f)$ *evaluated at* f.

Proof. Since $c(f)$ is bounded on Ω, so is $F(f)$ in Eq. (5). Also we know that $\sum_{r \in R_w} F_{rw} = 0$ for all $w \in W$, and $F_{rw} \geq 0$ if $f_{rw} = 0$. Then for each $f \in \Omega$, there exists a sufficiently small scalar $\alpha > 0$ (where α is indeed a function of f) such that

$f+\alpha F(f)\in\Omega$. Therefore by treating $F(f)^{\mathrm{T}}c\left(f+\alpha F(f)\right)$ as a function of α and applying the *Mean Value Theorem*, we have

$$\alpha F(f)^{\mathrm{T}}\left(c\left(f+\alpha F(f)\right)-c(f)\right)=\alpha F(f)^{\mathrm{T}}J_c\left(f+\gamma F(f)\right)F(f)\geq 0,\ \ \gamma\in(0,\alpha)$$

where the inequality holds due to the monotonicity of $c(f)$, and thus

$$F(f)^{\mathrm{T}}J_c\left(f+\gamma F(f)\right)F(f)\geq 0,\ \ \gamma\in(0,\alpha)$$

Since α can be arbitrarily small, then by the continuity of $J_c(\cdot)$ on Ω, we have $F(f)^{\mathrm{T}}J_c(f)F(f)\geq 0$ for all $f\in\Omega$. □

Remark 3. An alternative proof for Lemma 3 can be found in 5.4.3 in Ortega and Rheinboldt (2000). However, to use the monotonicity of $c(f)$ in their proof, $F(f)$ in Eq. (32) cannot be any arbitrarily vector in $\mathbb{R}^m$, therefore, one cannot conclude that $J_c(f)$ is positive semidefinite from Eq. (32); in other words, the monotonicity of $c(f)$ doesn't necessary mean that its Jacobian is positive semidefinite. □

Theorem 8. *If the path travel cost function $c(f)$ is monotone, then under BRUE-PSAP in Eq. (5), if $f(0)\in\Omega$, then $f(t)\to\Omega^*$ as $t\to\infty$.*

Proof. Similar to Smith (1984), defining

$$V(f)=\sum_{w\in W}\sum_{k\in R_w}\sum_{s\in R_w}f_{kw}\left[c_{kw}-c_{sw}-\varepsilon_w\right]_+^2 \tag{33}$$

then we have

$$\frac{\partial V(f)}{\partial f_{rw}}=\sum_{s\in R_w}\left[\Delta c_{rsw}\right]_+^2+2\sum_{u\in W}\sum_{k\in R_u}\sum_{s\in R_u}f_{ku}\left[\Delta c_{ksu}\right]_+\frac{\partial c_{ku}}{\partial f_{rw}}-2\sum_{u\in W}\sum_{k\in R_u}\sum_{s\in R_u}f_{ku}\left[\Delta c_{ksu}\right]_+\frac{\partial c_{su}}{\partial f_{rw}} \tag{34}$$

For the second term on the right hand side of Eq. (34), interchanging the indices of k and s yields

$$\frac{\partial V(f)}{\partial f_{rw}} = \sum_{s\in R_w}[\Delta c_{rsw}]_+^2 + 2\sum_{u\in W}\sum_{k\in R_u}\sum_{s\in R_u}\frac{\partial c_{su}}{\partial f_{rw}}\left(f_{su}[\Delta c_{sku}]_+ - f_{ku}[\Delta c_{ksu}]_+\right)$$
$$= \sum_{s\in R_w}[\Delta c_{rsw}]_+^2 - 2\sum_{u\in W}\sum_{s\in R_u}\frac{\partial c_{su}}{\partial f_{rw}}\dot{f}_{su}$$

and thus

$$\begin{aligned}\dot{V}(f) &= \sum_{w\in W}\sum_{r\in R_w}\left(\sum_{s\in R_w}[\Delta c_{rsw}]_+^2 - 2\sum_{u\in W}\sum_{s\in R_u}\frac{\partial c_{su}}{\partial f_{rw}}\dot{f}_{su}\right)\dot{f}_{rw} \\ &= \sum_{w\in W}\sum_{r\in R_w}\sum_{s\in R_w}[\Delta c_{rsw}]_+^2\dot{f}_{rw} - 2\sum_{w\in W}\sum_{r\in R_w}\sum_{u\in W}\sum_{s\in R_u}\frac{\partial c_{su}}{\partial f_{rw}}\dot{f}_{su}\dot{f}_{rw} \\ &\le \sum_{w\in W}\sum_{r\in R_w}\sum_{s\in R_w}[\Delta c_{rsw}]_+^2\dot{f}_{rw}\end{aligned} \tag{35}$$

where the inequality holds due to Lemma 3. For the right hand side of the inequality in Eq. (35), considering a specific OD pair $w\in W$, we have

$$\begin{aligned}&\sum_r\sum_s[\Delta c_{rsw}]_+^2\dot{f}_{rw} \\ &= \sum_r\sum_s[\Delta c_{rsw}]_+^2\sum_k\left(f_{kw}[\Delta c_{krw}]_+ - f_{rw}[\Delta c_{rkw}]_+\right) \\ &= \sum_r\sum_s[\Delta c_{rsw}]_+^2\sum_k f_{kw}[\Delta c_{krw}]_+ - \sum_r\sum_s[\Delta c_{rsw}]_+^2\sum_k f_{rw}[\Delta c_{rkw}]_+ \qquad (36) \\ &= \sum_k f_{kw}\sum_r[\Delta c_{krw}]_+\sum_s\left([\Delta c_{rsw}]_+^2 - [\Delta c_{ksw}]_+^2\right) \qquad (37) \\ &\le \sum_k f_{kw}\sum_r[\Delta c_{krw}]_+\sum_{s=r}\left([\Delta c_{rsw}]_+^2 - [\Delta c_{ksw}]_+^2\right) \\ &= -\sum_k\sum_r f_{kw}[\Delta c_{krw}]_+^3 \\ &\le 0\end{aligned}$$

where Eq. (37) is obtained by interchanging the indices of r and k in the second term of Eq. (36) and then rearranging the summation orders in both terms; the first inequality is due to the fact that $[\Delta c_{krw}]_+\left([\Delta c_{rsw}]_+^2 - [\Delta c_{ksw}]_+^2\right)\le 0$ for all $k,r,s\in R_w$, $w\in W$. Therefore, if $\dot{V}(f)=0$, then we have

$$f_{kw}[\Delta c_{krw}]_+^3 = 0,\ \ \forall\ k,r\in R_w,\ \ w\in W \tag{38}$$

Since for each path $k\in S_w^c$, there must exist a path $j\in S_w$ such that $\Delta c_{kjw}>0$, so condition (38) requires $f_{kw}=0$ if $k\in S_w^c$, which indicates BRUE. It means that, if $\dot{V}(f)=0$, then f is BRUE. Conversely, referring to Eq. (35), if f is BRUE, then $\dot{V}(f)=0$. That is to say, $\dot{V}(f)=0$ if and only if $f \in \Omega^*$. With reference to Theorem 6, we have $f(t) \to \Omega^*$ as $t \to \infty$. □

4. Numerical examples

In this section, we present numerical examples to explore the properties of BRUE-RBAP that are not discussed in the theoretical analyses so far. In the first example, the well-known Braess network (Braess, 1968; Braess et al., 2005) is adopted. With this simple network, we compare the evolution trajectories of BRUE-PSAP and BRUE-NTP with different bounded rationality thresholds and different initial states. In the second example, we conduct simulation on the Nguyen-Dupuis network (Nguyen and Dupuis, 1984) to show the applicability of BRUE-PSAP in larger networks with general link travel cost functions. For BRUE-NTP, we set $\beta = 1$.

4.1. Braess network

The Braess network is shown in Figure 2. The travel cost function of each link takes the widely-used BPR (Bureau of Public Roads, 1964) form, $c_a(v_a) = c_a^0\left(1+0.15\left(v_a/Y_a\right)^4\right)$, where c_a^0 is the free flow cost and Y_a the capacity of link $a \in A$, and their values are listed in Table 1. There is only one OD pair with a fixed demand of 10 served by three paths (Path 1, $O \to 1 \to 3 \to D$; Path 2, $O \to 2 \to 4 \to D$; Path 3, $O \to 2 \to 5 \to 3 \to D$). The Wardrop's UE path flow pattern is uniquely $(3,3,4)^{\mathrm{T}}$, which is also the equilibrium point of RBAP. Given the bounded rationality threshold, the complete BRUE set could be obtained using the method proposed by Di et al. (2013). Alternatively, due to the small network size in this example, we can plot the approximate BRUE set by enumeration.

Since the feasible path flow patterns have to meet the flow conservation constraint, a two-dimensional figure is enough to display the flow trajectories. The ordinary differential equation sets representing the day-to-day dynamics, including BRUE-PSAP in Eq. (5) and BRUE-NTP in Eq. (15), are solved numerically in *Matlab* by the built-in function *ode45* using the fourth-order Runge-Kutta method. It is worth mentioning that, the observations we made in this example are based on the specific network structure and parameter settings.

Generalization of these observations requires further elaboration under general conditions.

We set the bounded rationality thresholds to be 0, 0.3, 0.5 and 0.8, respectively, and see how the evolution trajectories change in both BRUE-PSAP and BRUE-NTP. The results are shown in Figure 3 and Figure 4, respectively. First of all, when $\varepsilon = 0$, both models converge to the unique Wardrop's UE as predicted. Second, as the threshold expands, trajectories starting from the same initial state shift for both BRUE-PSAP and BRUE-NTP, but less for the former, which may indicate that the trajectories of BRUE-PSAP are more "robust" with respect to the threshold change. Moreover, given a same threshold, it happens in both models that different trajectories converge to the same equilibrium point, but less frequently for BRUE-PSAP. For example, when $\varepsilon = 0.3$, starting from 30 different initial states, the numbers of final equilibria of BRUE-PSAP and BRUE-NTP are respectively 22 and 10. In this sense, BRUE-PSAP has a much larger diversity regarding the final equilibria than BRUE-NTP.

4.2. Nguyen-Dupuis network

In this example, we simulate BRUE-PSAP in a larger network with non-separable and asymmetric link travel cost functions. The Nguyen-Dupuis network is illustrated in Figure 5, with paths and other OD-related information being listed in Table 2. The travel cost on link a is given by $c_a = y_a + 10^{-3}\sum_{b=1}^{19} x_{ab} v_b$, where $X = \left[x_{ab}\right]_{19\times 19}$ and $y = \left[y_a\right]_{1\times 19}$ are given in Table 3. Two different initial states are chosen, whose values and the corresponding equilibrium path flows and costs are listed in Table 4. Flow evolution on representative paths is drawn on Figure 6. In both cases, the flow patterns eventually approach the corresponding equilibria, which is consistent with Theorem 8.

5. Conclusions

This paper proposed an extended RBAP model on the basis of BRUE. The path-based VI formulation for BRUE traffic assignment was established. The existing RBAP models (PSAP and NTP) were extended to the BRUE case. A special group of BRUE-RBAP was built, with which coming along some discussion on the non-existence of the classical solutions to some models belonging to this special group. Stability of the general BRUE-RBAP with separable link travel cost functions was proved; while considering non-separable link travel cost functions, the stability of BRUE-PSAP was proved. The numerical examples verified the stability of BRUE-RBAP and further uncovered some properties of BRUE-PSAP and BRUE-NTP.

While making some progresses in the static and dynamic modeling related to BRUE, this paper raises more questions which remain unsolved.

First, regarding the BRUE solution set, its topology (e.g. connectedness) is still not fully investigated, and although the VI formulation is established, it is unclear how to obtain (at least part of) the BRUE solutions based on this formulation in an efficient way. For this topic, some discussion can be found in Di et al. (2013) and Han et al. (2015).

Second, although the existence and uniqueness of the solutions to the day-to-day dynamics is assured in most of the previous literature, it is not a trivial problem as shown in Sections 2.1 and 2.2. In some cases, the classical solution doesn't always exist, as illustrated in Example 1. Under this circumstance, other notions of solutions such as Caratheodory solutions and Filippov solutions (Cortes, 2008) can be defined, the discussion of which is beyond the scope of the current paper but could be an interesting extension in the future.

Third, the stability analyses under BRUE-RBAP are much more complex compared with that under RBAP. It is very difficult to construct the Lyapunov functions under general non-separable link cost functions, even for the RBAP based on Wardrop's UE (Guo et al., 2015a), while the non-convexity of the BRUE link/path flow set could make the problem even more challenging for BRUE-RBAP. Furthermore, even the LaSalle's theorem can only be used for day-to-day formulations with classical solutions that are continuous w.r.t. the initial states. The discontinuous day-to-day formulations may lead to discontinuous Lyapunov functions, and the non-existence of the classical solutions further requires more advanced mathematical methods.

Besides the above-mentioned problems, there are broader directions for future research, such as the link-based BRUE-RBAP, the BRUE-RBAP under flexible demand and the associated congestion pricing scheme (Guo et al., 2015b; Ye and Yang, 2015), as well as validation of these models with real data.

Acknowledgement

The authors express their thanks to the Associate Editor and two anonymous reviewers, whose useful comments have improved the exposition of this paper. The research described in this paper was substantially supported by Grants from the Hong Kong Grant Council [Grant HKUST16211114] and the National Natural Science Foundation of China [Grant NSFC 71371020].

References

Braess D (1968) Über ein Paradoxon aus der Verkehrsplanung. *Unternehmensforschung* 12(1):258-268.

Braess D, Nagurney A, Wakolbinger T (2005) On a paradox of traffic planning. *Transportation Science* 39(4):446-450.

Bureau of Public Roads (1964) *Traffic Assignment Manual* (US Department of Commerce).

Chorus CG, Timmermans HJP (2009) Measuring user benefits of changes in the transport system when traveler awareness is limited. *Transportation Research Part A* 43(5):536-547.

Conlisk J (1996) Why bounded rationality? *Journal of Economic Literature* 34(2):669-700.

Cortes J (2008) Discontinuous dynamical systems. *IEEE Control Systems* 28(3):36-73.

Dafermos S (1980) Traffic equilibrium and variational inequalities. *Transportation Science* 14(1):42-54.

Di X, Liu HX (2016) Boundedly rational route choice behavior: A review of models and methodologies. *Transportation Research Part B* 85:142-179.

Di X, He X, Guo X, Liu HX (2014) Braess paradox under the boundedly rational user equilibria. *Transportation Research Part B* 67:86-108.

Di X, Liu HX, Ban X (2016a) Second best toll pricing within the framework of bounded rationality. *Transportation Research Part B* 83:74-90.

Di X, Liu H, Ban X, Yu J (2015) Submission to the DTA 2012 Special issue: on the stability of a boundedly rational day-to-day dynamic. *Networks and Spatial Economics* 15(3):537-557.

Di X, Liu HX, Pang J, Ban X (2013) Boundedly rational user equilibria (BRUE): Mathematical formulation and solution sets. *Transportation Research Part B* 57:300-313.

Di X, Liu HX, Zhu S, Levinson DM (2016b) Indifference bands for boundedly rational route switching. *Transportation* (in press)

Dupuis P, Nagurney A (1993) Dynamical systems and variational inequalities. Annals *of Operations Research* 44(1):7-42.

Emmerink RHM, Axhausen KW, Nijkamp P, Rietveld P (1995a) The potential of information provision in a simulated road transport network with non-recurrent congestion. *Transportation Research Part C* 3(5):293-309.

Emmerink RM, Axhausen K, Nijkamp P, Rietveld P (1995b) Effects of information in road transport networks with recurrent congestion. *Transportation* 22(1):21-53.

Facchinei F, Pang J (2003) *Finite-Dimensional Variational Inequalities and Complementarity Problems* (Springer-Verlag, New York).

Friesz TL, Bernstein D, Mehta NJ, Tobin RL, Ganjalizadeh S (1994) Day-to-day dynamic network disequilibria and idealized traveler information systems. *Operations Research* 42(6):1120-1136.

Gabaix X, Laibson DI, Moloche G, Weinberg S (2006) Costly information acquisition: experimental analysis of a boundedly rational model. *American Economic Review* 96(4):1043-1068.

Gao S, Frejinger E, Ben-Akiva M (2011) Cognitive cost in route choice with real-time information: An exploratory analysis. *Transportation Research Part A* 45(9):916-926.

Guo RY, Yang H, Huang HJ (2013) A discrete rational adjustment process of link flows in traffic networks. *Transportation Research Part C* 34:121-137.

Guo RY, Yang H, Huang HJ, Tan ZJ (2015a) Link-based day-to-day network traffic dynamics and equilibria. *Transportation Research Part B* 71:248-260.

Guo RY, Yang H, Huang HJ, Tan ZJ (2015b) Day-to-day flow dynamics and congestion control. *Transportation Science* (in press)

Guo X (2013) Toll sequence operation to realize target flow pattern under bounded rationality. *Transportation Research Part B* 56:203-216.

Guo X, Liu HX (2011) Bounded rationality and irreversible network change. *Transportation Research Part B* 45(10):1606-1618.

Han K, Szeto WY, Friesz TL (2015) Formulation, existence, and computation of boundedly rational dynamic user equilibrium with fixed or endogenous user tolerance. *Transportation Research Part B* 79:16-49.

Han L, Du L (2012) On a link-based day-to-day traffic assignment model. *Transportation Research Part B* 46(1):72-84.

He X, Guo X, Liu HX (2010) A link-based day-to-day traffic assignment model. *Transportation Research Part B* 44(4):597-608.

Hu T, Mahmassani HS (1997) Day-to-day evolution of network flows under real-time information and reactive signal control. *Transportation Research Part C* 5(1):51-69.

Jayakrishnan R, Mahmassani HS, Hu T (1994) An evaluation tool for advanced traffic information and management systems in urban networks. *Transportation Research Part C* 2(3):129-147.

Jou R, Hensher DA, Liu Y, Chiu C (2010) Urban commuters' mode-switching behaviour in Taipai, with an application of the bounded rationality principle. *Urban Studies* 47(3):650-665.

Jou R, Lam S, Liu Y, Chen K (2005) Route switching behavior on freeways with the provision of different types of real-time traffic information. *Transportation Research Part A* 39(5):445-461.

Khalil HK (2002). *Nonlinear Systems, third edition* (Prentice Hall, Upper Saddle River, NJ).

LaSalle JP (1960) Some extensions of Liapunov's second method. *IRE Transactions on Circuit Theory* 7(4):520-527.

Li Y, Tan Z, Chen Q (2012) Dynamics of a transportation network model with homogeneous and heterogeneous users. *Discrete Dynamics in Nature and Society* 2012:1-16.

Lou Y, Yin Y, Lawphongpanich S (2010) Robust congestion pricing under boundedly rational user equilibrium. *Transportation Research Part B* 44(1):15-28.

Mahmassani HS, Chang G (1987) On boundedly rational user equilibrium in transportation systems. *Transportation Science* 21(2):89-99.

Mahmassani HS, Jayakrishnan R (1991) System performance and user response under real-time information in a congested traffic corridor. *Transportation Research Part A* 25(5):293-307.

Mahmassani HS, Liu Y (1999) Dynamics of commuting decision behaviour under advanced traveller information systems. *Transportation Research Part C* 7(2-3):91-107.

Nagurney A, Zhang D (1997) Projected dynamical systems in the formulation, stability analysis, and computation of fixed-demand traffic network equilibria. *Transportation Science* 31(2):147-158.

Nguyen S, Dupuis C (1984) An efficient method for computing traffic equilibria in networks with asymmetric transportation costs. *Transportation Science* 18(2):185-202.

Ortega J, Rheinboldt W (2000) *Iterative Solution of Nonlinear Equations in Several Variables* (Society for Industrial and Applied Mathematics).

Sandholm WH (2002) Evolutionary implementation and congestion pricing. *The Review of Economic Studies* 69(3):667-689.

Sandholm WH (2005) Excess payoff dynamics and other well-behaved evolutionary dynamics. *Journal of Economic Theory* 124(2):149-170.

Simon H (1955) A behavioral model of rational choice. *The Quarterly* Journal *of Economics* 69(1):99-118.

Smith MJ (1979) The existence, uniqueness and stability of traffic equilibria. *Transportation Research Part B* 13(4):295-304.

Smith MJ (1984) The stability of a dynamic model of traffic assignment - An application of a method of Lyapunov. *Transportation Science* 18(3):245-252.

Smith M, Mounce R (2011) A splitting rate model of traffic re-routing and traffic control. *Transportation Research Part B* 45(9):1389-1409.

Szeto WY, Lo HK (2006) Dynamic traffic assignment: properties and extensions. *Transportmetrica* 2(1):31-52.

Wu J, Sun H, Wang DZW, Zhong M, Han L, Gao Z (2013) Bounded-rationality based day-to-day evolution model for travel behavior analysis of urban railway network. *Transportation Research Part C* 31:73-82.

Yang F, Zhang D (2009) Day-to-day stationary link flow pattern. *Transportation Research Part B* 43(1):119-126.

Ye H, Yang H, Tan Z (2015) Learning marginal-cost pricing via a trial-and-error procedure with day-to-day flow dynamics. *Transportation Research Part B* 81:794-807.

Zhang D, Nagurney A (1995) On the stability of projected dynamical systems. Journal *of Optimization Theory and Applications* 85(1):97-124.

Zhang D, Nagurney A, Wu J (2001) On the equivalence between stationary link flow patterns and traffic network equilibria. *Transportation Research Part B* 35(8):731-748.

Zhao C, Huang H (2016) Experiment of boundedly rational route choice behavior and the model under satisficing rule. *Transportation Research Part C* 68:22-37.

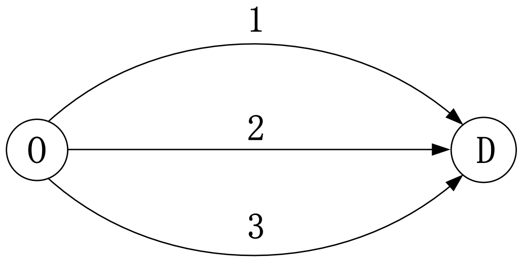


Figure 1. A single-OD and parallel-link network.

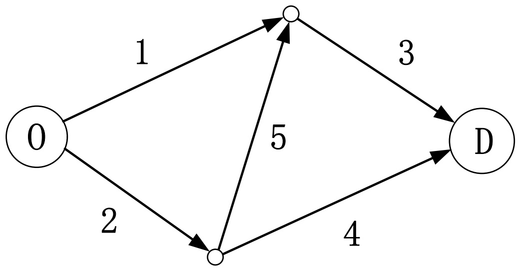


Figure 2. Braess network.

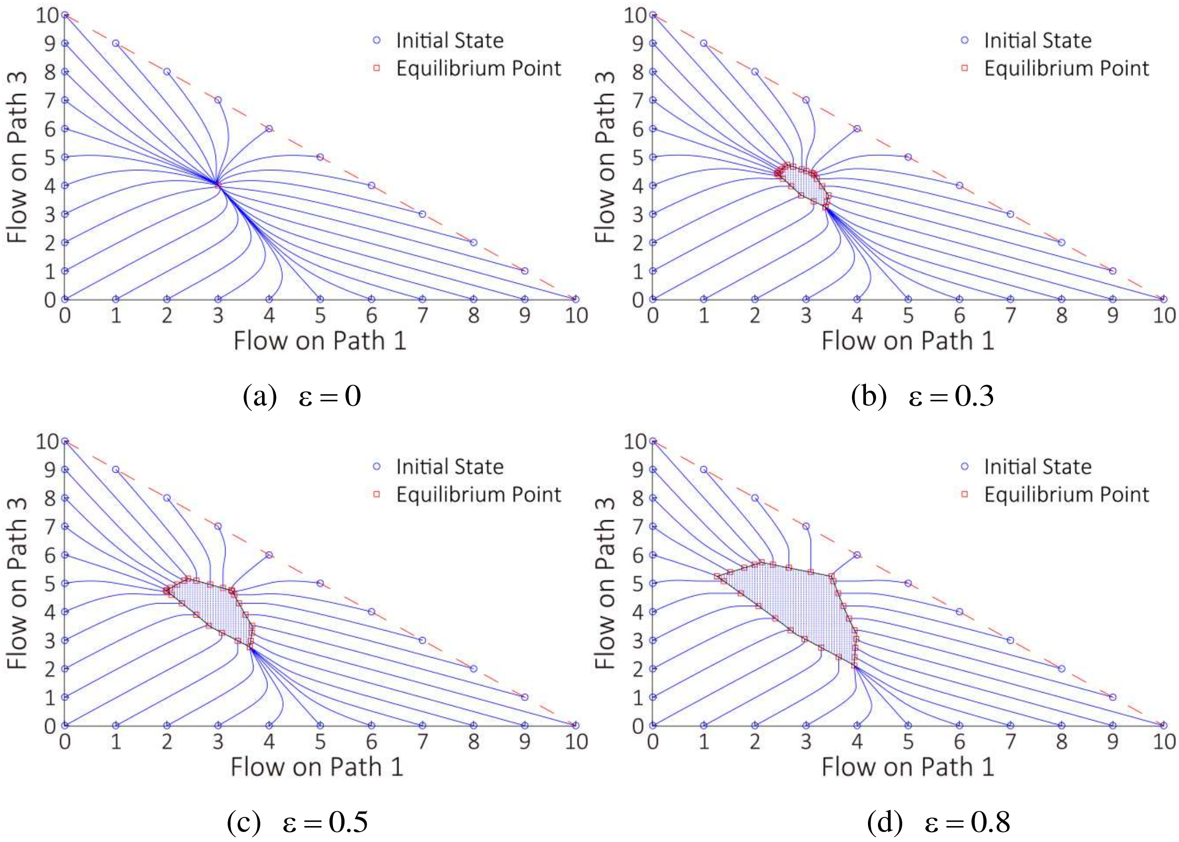
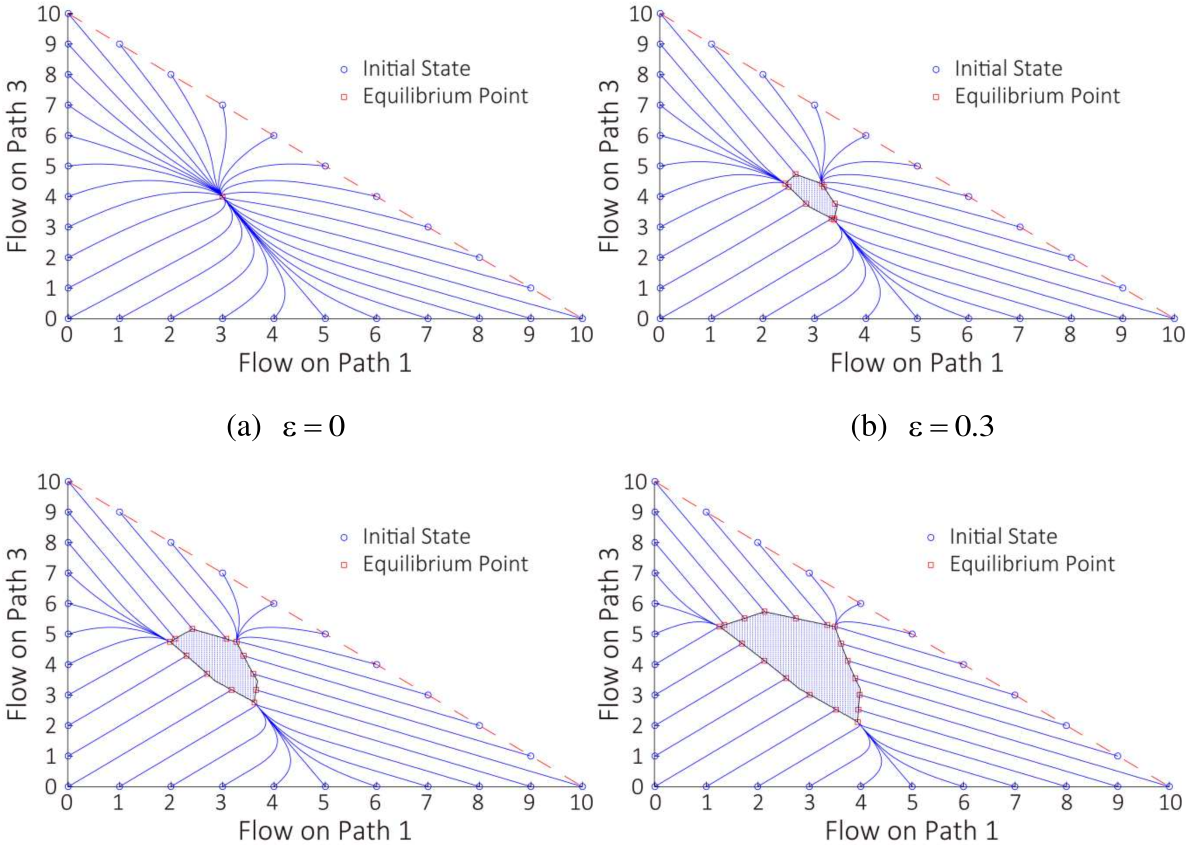


Figure 3. Trajectories of BRUE-PSAP with different thresholds.



(a) $\varepsilon = 0$ (b) $\varepsilon = 0.3$

(c) $\varepsilon = 0.5$ (d) $\varepsilon = 0.8$

Figure 4. Trajectories of BRUE-NTP with different thresholds.

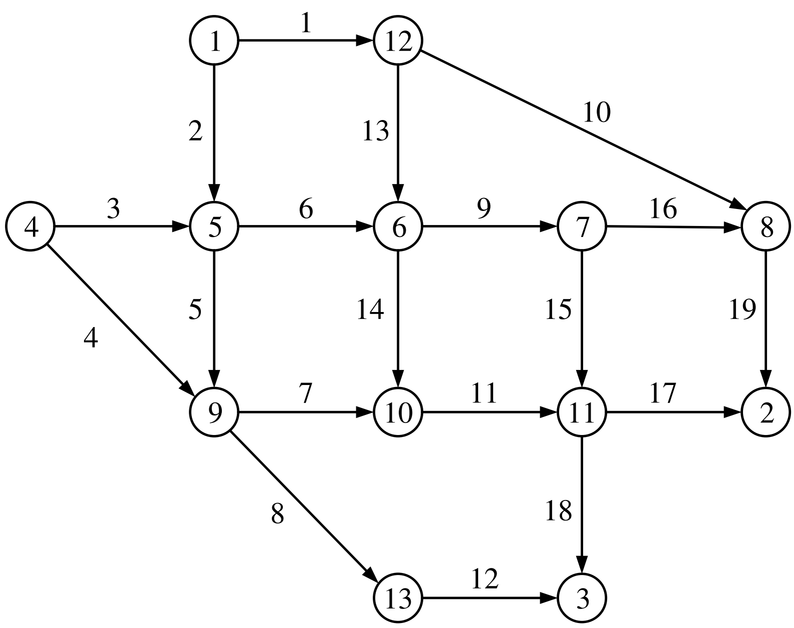
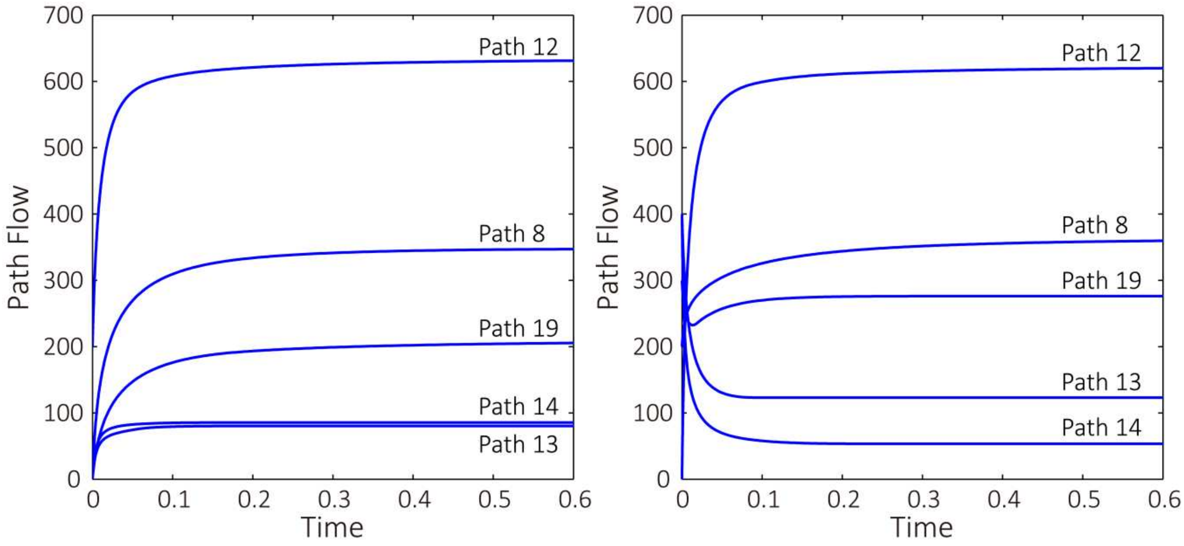


Figure 5. Nguyen-Dupuis network.



(a) Pattern #1 (b) Pattern #2

Figure 6. Evolution of path flows in the asymmetric Nguyen-Dupuis network.

Table 1. Parameters of the link performance functions in Braess network

Link no. a	1	2	3	4	5
Free flow cost, c_a^0	2	1	1	2	1
Link capacity, Y_a	3	7	7	3	4

Table 2. OD demands, thresholds and paths in Nguyen-Dupuis network

Origin	Destination	Demand	Threshold	Path #	Consisting of links
1	2	400	20	1	1, 5, 7, 9, 11
				2	1, 5, 7, 10, 15
				3	1, 5, 8, 14, 15
				4	1, 6, 12, 14, 15
				5	2, 7, 9, 11, 17
				6	2, 7, 10, 15, 17
				7	2, 8, 14, 15, 17
				8	2, 11, 18,
1	3	800	10	9	1, 5, 7, 10, 16
				10	1, 5, 8, 14, 16
				11	1, 6, 12, 14, 16
				12	1, 6, 13, 19
				13	2, 7, 10, 16, 17
				14	2, 8, 14, 16, 17
4	2	600	15	15	3, 5, 7, 9, 11
				16	3, 5, 7, 10, 15
				17	3, 5, 8, 14, 15
				18	3, 6, 12, 14, 15
				19	4, 12, 14, 15
4	3	200	20	20	3, 5, 7, 10, 16
				21	3, 5, 8, 14, 16
				22	3, 6, 12, 14, 16
				23	3, 6, 13, 19
				24	4, 12, 14, 16
				25	4, 13, 19

Table 3. Parameters of the link performance functions in Nguyen-Dupuis network

	18	10	5		1	1.2													
	1	19															0.2	0.1	
	2		15	9	2	1.1													
			5	11		2						0.3	0.1						
	3		2		13.5	6	0.4	1									1		
	2		1	0.9	2	17.5						0.4	0.2						
					0.6		12.5	0.1	4	4.4							0.3		
					0.5		2	5.5				1.1		0.1			0.4		
							0.5		13.5	1	0.6							5	
X							0.2		5	33.3				2	13	12			
									0.2		12.5				3			2	
				0.5		4		2				9.8	2	0.3					
				0.1		2						1	5						1
								0.3		10		3		37	0.1	14			
										0.1	11			0.2	20	0.5			
										0.2				0.1	2	50			6
		4			3		0.3	2									20	1	
		4							4		0.3						2	11	
													2			5			10
y	7	9	9	12	3	9	5	13	5	9	9	10	9	6	9	8	7	14	11

Table 4. Initial flows and equilibrium flows/costs for the Nguyen-Dupuis example

Path #	Pattern #1			Pattern #2		
	Initial flow	BRUE flow	BRUE cost	Initial flow	BRUE flow	BRUE cost
1	200.0	0.0	87.8	0.0	0.0	85.8
2	200.0	0.0	100.8	0.0	0.0	99.0
3	0.0	0.0	102.7	0.0	0.0	101.4
4	0.0	0.0	112.3	0.0	0.0	112.6
5	0.0	51.9	81.3	0.0	37.0	80.9
6	0.0	0.0	94.4	0.0	0.0	94.1
7	0.0	0.0	96.3	200.0	0.0	96.5
8	0.0	348.1	62.1	200.0	363.0	62.5
9	200.0	0.0	100.0	0.0	0.0	98.6
10	200.0	0.0	101.9	0.0	0.0	101.0
11	200.0	0.0	111.4	0.0	0.0	112.2
12	200.0	634.0	87.7	0.0	623.3	86.8
13	0.0	80.4	93.5	400.0	123.3	93.7

14	0.0	85.6	95.4	400.0	53.5	96.1
15	200.0	304.1	81.9	0.0	304.9	79.8
16	200.0	59.8	94.9	0.0	18.8	93.1
17	200.0	24.9	96.8	0.0	0.0	95.5
18	0.0	0.0	106.4	300.0	0.0	106.6
19	0.0	211.3	81.8	300.0	276.3	83.4
20	50.0	0.0	94.1	0.0	0.0	92.7
21	50.0	0.0	96.0	0.0	0.0	95.1
22	50.0	0.0	105.5	0.0	0.0	106.2
23	50.0	0.0	81.8	0.0	0.0	80.8
24	0.0	0.0	81.0	100.0	0.0	83.0
25	0.0	200.0	57.2	100.0	200.0	57.5