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UNIFORM BEHAVIOUR OF THE FROBENIUS CLOSURES OF IDEALS GENERATED BY REGULAR SEQUENCES

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Abstract. This paper is concerned with ideals in a commutative Noetherian ring R of prime characteristic. The main purpose is to show that the Frobenius closures of certain ideals of R generated by regular sequences exhibit a desirable type of 'uniform' behaviour. The principal technical tool used is a result, proved by R. Hartshorne and R. Speiser in the case where R is local and contains its residue field which is perfect, and subsequently extended to all local rings of prime characteristic by G. Lyubeznik, about a left module over the skew polynomial ring $R[x, f]$ (associated to R and the Frobenius homomorphism f, in the indeterminate x) that is both x-torsion and Artinian over R.

0. Introduction

Let R be a commutative Noetherian ring of prime characteristic p, and let $\mathfrak{a}$ be a proper ideal of R. For $n \in \mathbb{N}_0$ (we use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers), the *n-th Frobenius power* $\mathfrak{a}^{[p^n]}$ of $\mathfrak{a}$ is the ideal of R generated by all p^n-th powers of elements of $\mathfrak{a}$. Also R° denotes the complement in R of the union of the minimal prime ideals of R.

An element $r \in R$ belongs to the *tight closure* $\mathfrak{a}^*$ *of* $\mathfrak{a}$ if and only if there exists $c \in R^\circ$ such that $cr^{p^n} \in \mathfrak{a}^{[p^n]}$ for all $n \gg 0$. The theory of tight closure was invented by M. Hochster and C. Huneke [7], and many applications have been found for the theory: see [10].

The *Frobenius closure* $\mathfrak{a}^F$ *of* $\mathfrak{a}$, defined as

$$\mathfrak{a}^F := \left\{ r \in R : \text{there exists } n \in \mathbb{N}_0 \text{ such that } r^{p^n} \in \mathfrak{a}^{[p^n]} \right\},$$

is another ideal relevant to the theory of tight closure. Since $\mathfrak{a}^F$ is finitely generated, there exists $m_0 \in \mathbb{N}_0$ such that $(\mathfrak{a}^F)^{[p^{m_0}]} = \mathfrak{a}^{[p^{m_0}]}$, and we define $Q(\mathfrak{a})$ to be the smallest power of p with this property. An interesting question is whether the set $\{Q(\mathfrak{b}) : \mathfrak{b} \text{ is a proper ideal of } R\}$ of powers of p is bounded. A simpler question is whether, for a given proper ideal $\mathfrak{a}$ of R, the set $\left\{ Q\big(\mathfrak{a}^{[p^n]}\big) : n \in \mathbb{N}_0 \right\}$ is bounded. The main result of this paper shows that the latter question has an affirmative answer when $\mathfrak{a}$ is generated by a regular sequence. The method of proof employs a result for local R, about a left module over the skew polynomial ring $R[x, f]$ that is Artinian as an R-module, that was proved by R. Hartshorne and R. Speiser [6, Proposition 1.11] in the case where the local ring R contains its residue field which is perfect, and subsequently extended to all local rings of characteristic p by G. Lyubeznik [11, Proposition 4.4]. In Section 2 we apply this result, in the case where R is a Cohen–Macaulay local ring, to a top local cohomology module viewed as a left $R[x, f]$-module. This is one more instance where the Frobenius action on such a local cohomology module, as described by K. E. Smith in [19, 3.2], yields valuable insights (for other uses of this technique, see, for example, [2], [3], [5], [6], [11]).

The main result of the paper is presented in the final Section 4, where we use the modules of generalized fractions of the second author and H. Zakeri [16] to construct further modules to which we can apply the Hartshorne–Speiser–Lyubezbik Theorem.

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The uniform behaviour sought in this paper for Frobenius closures has some similarity with the uniform behaviour of tight closures that occurs when there exists a weak test element for R. A p^{m_0}-*weak test element* for R (where $m_0 \in \mathbb{N}_0$) is an element $c' \in R^\circ$ such that, for every ideal $\mathfrak{b}$ of R and for $r \in R$, it is the case that $r \in \mathfrak{b}^*$ if and only if $c'r^{p^n} \in \mathfrak{b}^{[p^n]}$ for all $n \geq m_0$. A p^0-weak test element is called a *test element*. It is a result of Hochster and Huneke [8, Theorem (6.1)(b)] that an algebra of finite type over an excellent local ring of characteristic p has a p^{m_0}-weak test element for some $m_0 \in \mathbb{N}_0$. To illustrate the relevance of the work in this paper to the theory of tight closure, we establish the following lemma.

0.1**. Lemma.** *Let $\mathfrak{a}$ be a proper ideal of the commutative Noetherian ring R of prime characteristic p. Assume that $m_0 \in \mathbb{N}_0$ is such that either*

(i) *there exists a p^{m_0}-weak test element c' for R, or*
(ii) $((\mathfrak{a}^{[p^n]})^F)^{[p^{m_0}]} = (\mathfrak{a}^{[p^n]})^{[p^{m_0}]}$ *for all $n \in \mathbb{N}_0$.*

Then, for $r \in R$, we have $r \in \mathfrak{a}^*$ if and only if there exists $c \in R^\circ$ such that $cr^{p^n} \in (\mathfrak{a}^{[p^n]})^F$ for all $n \gg 0$.

Proof. Only one implication requires proof. Suppose that $r \in R$, $n_0 \in \mathbb{N}_0$, and $c \in R^\circ$ are such that $cr^{p^n} \in (\mathfrak{a}^{[p^n]})^F$ for all $n \geq n_0$. We show that $r \in \mathfrak{a}^*$.

Now $\mathfrak{b}^F \subseteq \mathfrak{b}^*$ for each ideal $\mathfrak{b}$ of R. Therefore, in case (i), we have

$$c'(cr^{p^n})^{p^{m_0}} \in (\mathfrak{a}^{[p^n]})^{[p^{m_0}]} \quad \text{for all } n \geq n_0.$$

In case (ii), we have

$$(cr^{p^n})^{p^{m_0}} \in (\mathfrak{a}^{[p^n]})^{[p^{m_0}]} \quad \text{for all } n \geq n_0.$$

If we let $\widetilde{c} = c'$ in case (i) and $\widetilde{c} = 1$ in case (ii), then we have (in both cases) $(\widetilde{c}c^{p^{m_0}})r^{p^{n+m_0}} \in \mathfrak{a}^{[p^{n+m_0}]}$ for all $n \geq n_0$, and $\widetilde{c}c^{p^{m_0}} \in R^\circ$. Hence $r \in \mathfrak{a}^*$. □

We also draw the reader's attention to the concept of *test exponent* in tight closure theory introduced by Hochster and Huneke in [9, Definition 2.2]. Let c be a test element for a reduced commutative Noetherian ring of prime characteristic p, and let $\mathfrak{a}$ be an ideal of R. A test exponent for c, $\mathfrak{a}$ is a power $q = p^{e_0}$ (where $e_0 \in \mathbb{N}_0$) such that if, for an $r \in R$, we have $cr^{p^e} \in \mathfrak{a}^{[p^e]}$ for *one single* $e \geq e_0$, then $r \in \mathfrak{a}^*$ (so that $cr^{p^n} \in \mathfrak{a}^{[p^n]}$ for all $n \in \mathbb{N}_0$). In [9], it is shown that this concept has strong connections with the major open problem about whether tight closure commutes with localization; indeed, to quote Hochster and Huneke, 'roughly speaking, … test exponents exist, in general, if and only if tight closure commutes with localization'.

In [9, Discussion 5.3], Hochster and Huneke raise the question as to whether there might conceivably exist (when R and c satisfy certain conditions) a 'uniform test exponent' for c, that is, a power of p that is a test exponent for c, $\mathfrak{b}$ for *all* ideals $\mathfrak{b}$ of R simultaneously. There are some similarities between this question and our question (raised in the third paragraph of this Introduction) about whether the set $\{Q(\mathfrak{b}) : \mathfrak{b} \text{ is a proper ideal of } R\}$ is bounded, and so we are hopeful that the work in this paper might give some pointers for the major questions raised by Hochster and Huneke.

1. Notation, terminology and the Hartshorne–Speiser–Lyubeznik Theorem

1.1**. Notation.** Throughout the paper, A will denote a general commutative Noetherian ring and R will denote a commutative Noetherian ring of prime characteristic p. For an ideal $\mathfrak{c}$ of A and an A-module M, we set $\Gamma_{\mathfrak{c}}(M) := \{m \in M : \text{there exists } h \in \mathbb{N} \text{ such that } \mathfrak{c}^h m = 0\}$.

We shall always denote by $f : R \longrightarrow R$ the Frobenius homomorphism, for which $f(r) = r^p$ for all $r \in R$. Throughout, $\mathfrak{a}$ will denote a general proper ideal of R. We shall work with the skew polynomial ring $R[x, f]$ associated to R and f in the indeterminate x over R. Recall that $R[x, f]$ is, as a left R-module, freely generated by $(x^i)_{i \in \mathbb{N}_0}$, and so consists of all polynomials $\sum_{i=0}^n r_i x^i$, where $n \in \mathbb{N}_0$ and $r_0, \ldots, r_n \in R$; however, its multiplication is subject to the rule

$$xr = f(r)x = r^p x \quad \text{for all } r \in R.$$

1.2. **Lemma and Definition.** *Let Z be a left $R[x,f]$-module. Then the set*

$$\Gamma_x(Z) := \left\{ z \in Z : x^j z = 0 \text{ for some } j \in \mathbb{N} \right\}$$

is an $R[x,f]$-submodule of Z, called the x-torsion submodule *of Z.*

Proof. For $j \in \mathbb{N}$, $z \in Z$ and $r \in R$, we have $x^j rz = r^{p^j} x^j z$; the claim follows easily. □

The following lemma enables one to see quickly that, in certain circumstances, an R-module M has a structure as left $R[x,f]$-module extending its R-module structure.

1.3. **Lemma.** *Let G be an R-module and let $\xi : G \longrightarrow G$ be a $\mathbb{Z}$-endomorphism of G such that $\xi(rg) = r^p \xi(g)$ for all $r \in R$ and $g \in G$. Then the R-module structure on G can be extended to a structure of left $R[x,f]$-module in such a way that $xg = \xi(g)$ for all $g \in G$.*

Proof. Note that, if we denote by μ_r, for $r \in R$, the $\mathbb{Z}$-endomorphism of G given by multiplication by r, then $\xi \circ \mu_r = \mu_{r^p} \circ \xi$ for all $r \in R$. We use $\operatorname{End}_{\mathbb{Z}}(G)$ to denote the ring of $\mathbb{Z}$-endomorphisms of the Abelian group G. In view of the universal property of the skew polynomial ring $R[x,f]$, the above shows that there is a ring homomorphism $\phi : R[x,f] \longrightarrow \operatorname{End}_{\mathbb{Z}}(G)$ for which $\phi(x) = \xi$ and $\phi(r) = \mu_r$ for all $r \in R$. The claim is now immediate from the fact that G has a natural structure as a left module over $\operatorname{End}_{\mathbb{Z}}(G)$. □

Crucial to the work in this paper is the following extension, due to G. Lyubeznik, of a result of R. Hartshorne and R. Speiser. It shows that, when R is local, an x-torsion left $R[x,f]$-module which is Artinian (that is, 'cofinite' in the terminology of Hartshorne and Speiser) as an R-module exhibits a certain uniformity of behaviour.

1.4. **Theorem** (G. Lyubeznik [11, Proposition 4.4]). (Compare Hartshorne–Speiser [6, Proposition 1.11].) *Suppose that $(R,\mathfrak{m})$ is local, and let G be a left $R[x,f]$-module which is Artinian as an R-module. Then there exists $e \in \mathbb{N}_0$ such that $x^e \Gamma_x(G) = 0$.*

Hartshorne and Speiser first proved this result in the case where R is local and contains its residue field which is perfect. Lyubeznik applied his theory of F-modules to obtain the result without restriction on the local ring R of characteristic p.

1.5. **Definition.** Suppose that $(R,\mathfrak{m})$ is local, and let G be a left $R[x,f]$-module which is Artinian as an R-module. By the Hartshorne–Speiser–Lyubeznik Theorem 1.4, there exists $e \in \mathbb{N}_0$ such that $x^e \Gamma_x(G) = 0$: we call the smallest such e the *Hartshorne–Speiser–Lyubeznik number*, or *HSL-number* for short, of G.

2. Parameter ideals in Cohen–Macaulay local rings

In this section we give an example of the use the Hartshorne–Speiser–Lyubeznik Theorem to obtain a result, of the type mentioned in the Introduction, about uniform behaviour of Frobenius closures in a Cohen–Macaulay local ring $(R,\mathfrak{m})$ of positive dimension d. The argument is based on the well-known $R[x,f]$-module structure that is carried by the top local cohomology module $H^d_{\mathfrak{m}}(R)$. While this $R[x,f]$-module structure has been used by several authors in the past, the fact that this structure is independent of the choice of a system of parameters for R has not always been transparently clear from the earlier accounts offered; as this independence is crucial for our work, we shall make some comments about it.

2.1. **Reminder.** In this section, we shall sometimes use R' to denote R regarded as an R-module by means of f, at points where such care can be helpful. Let $i \in \mathbb{N}_0$.

(i) With this notation, $f : R \longrightarrow R'$ becomes a homomorphism of R-modules, and so, for the ideal $\mathfrak{a}$ of R, it induces an R-homomorphism $H^i_{\mathfrak{a}}(f) : H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}}(R')$.

(ii) The Independence Theorem for local cohomology (see [1, 4.2.1]) applied to the ring homomorphism $f : R \longrightarrow R$ yields an R-isomorphism $\nu^i_R : H^i_{\mathfrak{a}}(R') \xrightarrow{\cong} H^i_{\mathfrak{a}^{[p]}}(R)$, where $H^i_{\mathfrak{a}^{[p]}}(R)$ is regarded as an R-module via f. Since $\mathfrak{a}$ and $\mathfrak{a}^{[p]}$ have the same radical, $H^i_{\mathfrak{a}}$ and $H^i_{\mathfrak{a}^{[p]}}$ are the same functor.

(iii) It is important for an understanding of this paper to note that the R-isomorphism ν_R^i does not depend on any choice of generators for $\mathfrak{a}$ or, for that matter, for any ideal having the same radical as $\mathfrak{a}$. Indeed, ν_R^i is a constituent isomorphism in a natural equivalence of functors ν^i that forms part of an isomorphism of connected sequences of functors that is uniquely determined by the identity natural equivalence from $\Gamma_{\mathfrak{a}}$ to $\Gamma_{\mathfrak{a}^{[p]}}$: see [1, 4.2.1] for details.

(iv) Composition yields a $\mathbb{Z}$-endomorphism $\xi := \nu_R^i \circ H^i_{\mathfrak{a}}(f) : H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}}(R)$ which is such that $\xi(r\gamma) = r^p\xi(\gamma)$ for all $\gamma \in H^i_{\mathfrak{a}}(R)$ and $r \in R$. It therefore follows from Lemma 1.3 that $H^i_{\mathfrak{a}}(R)$ has a natural structure as left $R[x, f]$-module in which $x\gamma = \xi(\gamma)$ for all $\gamma \in H^i_{\mathfrak{a}}(R)$.

We emphasize that this $R[x, f]$-module structure on $H^i_{\mathfrak{a}}(R)$ does not depend on any choice of generators for an ideal having the same radical as $\mathfrak{a}$.

We can now define an invariant of a local ring R of characteristic p.

2.2. **Definition.** Suppose that $(R, \mathfrak{m})$ is a d-dimensional local ring. By 2.1, the top local cohomology module $H^d_{\mathfrak{m}}(R)$, which is well known to be Artinian as an R-module, has a natural structure as a left $R[x, f]$-module. We define $\eta(R)$ to be the HSL-number (see 1.5) of $H^d_{\mathfrak{m}}(R)$.

To exploit the properties of this invariant, we are going to use a concrete description of a typical element of $H^d_{\mathfrak{m}}(R)$ and the way in which the indeterminate x acts on such an element.

2.3. **Discussion.** If one chooses generators $b_1, \ldots, b_s$ for an ideal $\mathfrak{b}$ having the same radical as $\mathfrak{a}$, then one can represent the local cohomology module $H^i_{\mathfrak{a}}(R)$ quite concretely, either as a direct limit of homology modules of Koszul complexes, as in Grothendieck [4, Theorem 2.3] or [1, §5.2], or as a cohomology module of a Čech complex, as in [1, §5.1]. These representations can lead to an explicit formula for the effect of ξ (of 2.1(iv)) on an element of $H^i_{\mathfrak{a}}(R)$. The following illustration for the top local cohomology module $H^s_{\mathfrak{a}}(R)$ with respect to $\mathfrak{a}$ is relevant to the work in this paper.

(i) Let M be an R-module. We are going to use the description of the local cohomology module $H^s_{\mathfrak{a}}(M)$ as the s-th cohomology module of the Čech complex of M with respect to $b_1, \ldots, b_s$. Thus $H^s_{\mathfrak{a}}(M)$ can be represented as the residue class module of $M_{b_1\ldots b_s}$ modulo the image, under the Čech 'differentiation' map, of $\bigoplus_{i=1}^s M_{b_1\ldots b_{i-1}b_{i+1}\ldots b_s}$. See [1, §5.1]. We use '[ ]' to denote natural images of elements of $M_{b_1\ldots b_s}$ in this residue class module.

Denote the product $b_1 \ldots b_s$ by b. A typical element of $H^s_{\mathfrak{a}}(M)$ can be represented as $[m/b^n]$ for some $m \in M$ and $n \in \mathbb{N}_0$; moreover, for $m, m_1 \in M$ and $n, n_1 \in \mathbb{N}_0$, we have $[m/b^n] = [m_1/b^{n_1}]$ if and only if there exists $k \in \mathbb{N}_0$ such that $k \geq \max\{n, n_1\}$ and $b^{k-n}m - b^{k-n_1}m_1 \in (b_1^k, \ldots, b_s^k)M$. In particular, it should be noted that, if $b_1, \ldots, b_s$ form an M-sequence, then $[m/b^n] = 0$ if and only if $m \in (b_1^n, \ldots, b_s^n)M$, by [14, Theorem 3.2], for example.

(ii) For an element $v \in R$, there is an R-isomorphism $\omega : R'_v \xrightarrow{\cong} R_{v^p}$, where R_{v^p} is regarded as an R-module via f, for which $\omega(r/v^n) = r/v^{np}$ for all $r \in R'$ and $n \in \mathbb{N}_0$. It is straightforward to use such isomorphisms (and the uniqueness aspect of [1, Theorem 4.2.1]) to see that, with the notation of (i) above and 2.1(ii),

$$\nu_R^i\left(\left[\frac{r}{(b_1 \ldots b_s)^n}\right]\right) = \left[\frac{r}{(b_1 \ldots b_s)^{np}}\right] \quad \text{for all } r \in R'.$$

(iii) It follows that the left $R[x, f]$-module structure on $H^s_{\mathfrak{a}}(R)$ is such that

$$x\left[\frac{r}{(b_1 \ldots b_s)^n}\right] = \left[\frac{r^p}{(b_1 \ldots b_s)^{np}}\right] \quad \text{for all } r \in R \text{ and } n \in \mathbb{N}_0.$$

In this section, we shall use the above formula in the special case in which $(R, \mathfrak{m})$ is local, $\mathfrak{a} = \mathfrak{m}$, and $b_1, \ldots, b_s$ is a system of parameters for R.

2.4. **Theorem.** *Suppose that $(R, \mathfrak{m})$ (as in 1.1) is a d-dimensional Cohen–Macaulay local ring, where $d > 0$. The invariant $\eta(R)$ of R is defined in 2.2.*

Let $e \in \mathbb{N}_0$. Then the following statements are equivalent:

(i) *$e \geq \eta(R)$;*

(ii) *$(\mathfrak{q}^F)^{[p^e]} = \mathfrak{q}^{[p^e]}$ for each ideal $\mathfrak{q}$ of R that can be generated by a full system of parameters for R;*

(iii) *for one system of parameters $a_1, \ldots, a_d$ for R, we have*
$$((a_1^{n_1}, \ldots, a_d^{n_d})^F)^{[p^e]} = (a_1^{n_1}, \ldots, a_d^{n_d})^{[p^e]} \quad \textit{for all } n_1, \ldots, n_d \in \mathbb{N};$$
(iv) *for one ideal $\mathfrak{s}$ of R that can be generated by a full system of parameters for R, we have*
$$((\mathfrak{s}^{[p^n]})^F)^{[p^e]} = (\mathfrak{s}^{[p^n]})^{[p^e]} \quad \textit{for all } n \in \mathbb{N}_0.$$

Proof. Since R is Cohen–Macaulay, every system of parameters for R forms an R-sequence.

(i) $\Rightarrow$ (ii) Let $b_1, \ldots, b_d$ be a system of parameters for R, and let $\mathfrak{q} = (b_1, \ldots, b_d)R$. Let $r \in \mathfrak{q}^F$. Thus there exists $n \in \mathbb{N}_0$ such that $r^{p^n} \in \mathfrak{q}^{[p^n]} = (b_1^{p^n}, \ldots, b_d^{p^n})R$. Use $b_1, \ldots, b_d$ in the notation of 2.3(i) for $H^d_{\mathfrak{m}}(R)$, and write $b := b_1 \ldots b_d$. We have $x^n\, [r/b] = \left[r^{p^n}/b^{p^n}\right] = 0$ in $H^d_{\mathfrak{m}}(R)$. It therefore follows from the definition of $\eta(R)$ as the HSL-number of $H^d_{\mathfrak{m}}(R)$ that $x^e\, [r/b] = x^{e-\eta(R)} x^{\eta(R)}\, [r/b] = 0$, so that $\left[r^{p^e}/b^{p^e}\right] = 0$ and $r^{p^e} \in \mathfrak{q}^{[p^e]}$ because $b_1, \ldots, b_d$ form an R-sequence. Therefore $(\mathfrak{q}^F)^{[p^e]} = \mathfrak{q}^{[p^e]}$.

(ii) $\Rightarrow$ (iii) This is immediate from the fact that, if $a_1, \ldots, a_d$ is a system of parameters for R, then so too is $a_1^{n_1}, \ldots, a_d^{n_d}$ for all $n_1, \ldots, n_d \in \mathbb{N}$.

(iii) $\Rightarrow$ (iv) This is clear.

(iv) $\Rightarrow$ (i) Suppose that $\mathfrak{s}$ is generated by the system of parameters $a_1, \ldots, a_d$ for R. Use $a_1, \ldots, a_d$ in the notation of 2.3(i) for $H^d_{\mathfrak{m}}(R)$, and write $a := a_1 \ldots a_d$. Let $\zeta \in H^d_{\mathfrak{m}}(R)$. There exist $r \in R$ and $n \in \mathbb{N}_0$ such that $\zeta = \left[r/a^{p^n}\right]$; furthermore, $\zeta \in \Gamma_x(H^d_{\mathfrak{m}}(R))$ if and only if there exists $j \in \mathbb{N}_0$ such that $\left[r^{p^j}/(a^{p^n})^{p^j}\right] = x^j\zeta = 0$, that is (since $a_1, \ldots, a_d$ form an R-sequence), if and only if there exists $j \in \mathbb{N}_0$ such that $r^{p^j} \in (((a_1, \ldots, a_d)R)^{[p^n]})^{[p^j]}$. Therefore, $\zeta \in \Gamma_x(H^d_{\mathfrak{m}}(R))$ if and only if $r \in (\mathfrak{s}^{[p^n]})^F$. Since $((\mathfrak{s}^{[p^n]})^F)^{[p^e]} = (\mathfrak{s}^{[p^n]})^{[p^e]}$ for all $n \in \mathbb{N}_0$, it follows that $x^e(\Gamma_x(H^d_{\mathfrak{m}}(R))) = 0$, so that $e \geq \eta(R)$. □

As a special case of Theorem 2.4, we recover the result of R. Fedder (see [2, Proposition 1.4]) that $\mathfrak{q}^F = \mathfrak{q}$ for each ideal $\mathfrak{q}$ of R (as in 2.4) that can be generated by a full system of parameters (that is, R is F-contracted in the sense of [2, p. 49]) if and only if $\eta(R) = 0$ (that is, R is F-injective in the sense of [3, Definition 1.7]). We point out, however, that Fedder also proved that, in order for these conditions to be satisfied, it is sufficient that $(a_1, \ldots, a_d)^F = (a_1, \ldots, a_d)$ for *one single* system of parameters $a_1, \ldots, a_d$ for R.

Our final result in this section shows that the set of ideals $\mathfrak{a}$ in a Cohen–Macaulay local ring $(R, \mathfrak{m})$ (as in 1.1) of positive dimension for which $(\mathfrak{a}^F)^{[p^{\eta(R)}]} = \mathfrak{a}^{[p^{\eta(R)}]}$ is larger than the set of all ideals generated by full systems of parameters, as it contains all parameter ideals. (We use the term 'parameter ideal' in the sense of Smith [19, Definition 2.8]; however, a proper ideal in a Cohen–Macaulay local ring is a parameter ideal if and only if it can be generated by part of a system of parameters.)

2.5**. Theorem.** *Suppose that $(R, \mathfrak{m})$ (as in 1.1) is a d-dimensional Cohen–Macaulay local ring, where $d > 0$. The invariant $\eta(R)$ of R is defined in 2.2.*

Let $\mathfrak{a}$ be an ideal of R generated by part of a system of parameters. Then $(\mathfrak{a}^F)^{[p^{\eta(R)}]} = \mathfrak{a}^{[p^{\eta(R)}]}$.

Proof. In view of Theorem 2.4, we can, and do, assume that $\operatorname{ht}\mathfrak{a} < d$. There exist a system of parameters $a_1, \ldots, a_d$ for R and an integer $i \in \{0, \ldots, d-1\}$ such that $\mathfrak{a} = (a_1, \ldots, a_i)R$. Let $r \in \mathfrak{a}^F$. Then, for each $n \in \mathbb{N}$, we have $r \in ((a_1, \ldots, a_i, a_{i+1}^n, \ldots, a_d^n)R)^F$, and, since $a_1, \ldots, a_i, a_{i+1}^n, \ldots, a_d^n$ is a system of parameters for R, it follows from Theorem 2.4 that $r^{p^{\eta(R)}} \in ((a_1, \ldots, a_i, a_{i+1}^n, \ldots, a_d^n)R)^{[p^{\eta(R)}]}$. Therefore
$$\begin{aligned} r^{p^{\eta(R)}} &\in \bigcap_{n\in\mathbb{N}} \left(a_1^{p^{\eta(R)}}, \ldots, a_i^{p^{\eta(R)}}, a_{i+1}^{np^{\eta(R)}}, \ldots, a_d^{np^{\eta(R)}}\right)R \\ &\subseteq \bigcap_{n\in\mathbb{N}} \left(\mathfrak{a}^{[p^{\eta(R)}]} + \mathfrak{m}^{np^{\eta(R)}}\right) = \mathfrak{a}^{[p^{\eta(R)}]} \end{aligned}$$
by Krull's Intersection Theorem. The result follows. □

3. **Preparatory results about modules of generalized fractions**

For terminology and notation concerning modules of generalized fractions, the reader is referred to Sharp–Zakeri [16]. Our first few results in this section concern the general commutative Noetherian ring A. For an A-module M, we say that a sequence $a_1, \ldots, a_t$ of elements of A is a *poor M-sequence* precisely when $((a_1, \ldots, a_i)M :_M a_{i+1}) = (a_1, \ldots, a_i)M$ for all $i = 0, \ldots, t-1$. (Thus a poor A-sequence is just a *possibly improper regular sequence* in the sense of [7, Discussion (7.3)].)

3.1. *Remark.* A helpful tool for working with modules of generalized fractions, particularly in the context of this paper, is the Exactness Theorem, which provides an exactness criterion for complexes of modules of generalized fractions.

Let $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ be a chain of triangular subsets on A in the sense of [14, page 420], and let M be an A-module. By [18, 3.3], the complex of modules of generalized fractions $C(\mathcal{U}, M)$ is exact if and only if, for each $i \in \mathbb{N}$, each member of U_i is a poor M-sequence. (Although the present second author and H. Zakeri first proved this result, a shorter proof, which applies also in the case in which the underlying commutative ring is not necessarily Noetherian, was later provided by O'Carroll [14, 3.1].)

For each $i \in \mathbb{N}$, we shall let $W_i = W_i(A)$ denote the set of all poor A-sequences of length i. It is an easy consequence of [16, Example 3.10] that W_i is a triangular subset of A^i. In fact, $\mathcal{W} = \mathcal{W}(A) := (W_i(A))_{i\in\mathbb{N}}$ is a chain of triangular subsets on A in the sense of [14, p. 420], and so the above-mentioned Exactness Theorem yields that the complex $C(\mathcal{W}, A)$ is exact.

The following consequence of the Exactness Theorem for generalized fractions is very useful.

3.2. **Proposition** (Sharp–Zakeri [18, Corollary 3.15]). *Let $n \in \mathbb{N}$, let M be an A-module, and let U be a triangular subset of A^n that consists entirely of poor M-sequences. Then, for $m \in M$ and $(u_1, \ldots, u_n) \in U$, it is the case that*

$$\frac{m}{(u_1, \ldots, u_n)} = 0 \text{ in } U^{-n}M \quad \text{if and only if} \quad m \in \sum_{i=1}^{n-1} u_i M.$$

3.3. **Theorem.** *Suppose that $(A, \mathfrak{m})$ is a local ring of depth $t > 0$. Then*

$$\operatorname{Hom}_A(A/\mathfrak{m}, (W_t \times \{1\})^{-(t+1)}A)$$

is a finitely generated A-module.

Proof. We consider the complex $C(\mathcal{W}(A), A)$ of modules of generalized fractions. Note that this is exact. It is also a consequence of the Exactness Theorem for modules of generalized fractions that there are exact sequences

$$0 \longrightarrow A \xrightarrow{d^0} W_1^{-1}A \xrightarrow{\pi_1} (W_1 \times \{1\})^{-2}A \longrightarrow 0$$

and (for each $i \in \mathbb{N}$ with $i > 1$)

$$0 \longrightarrow (W_{i-1} \times \{1\})^{-i}A \xrightarrow{e^{i-1}} W_i^{-i}A \xrightarrow{\pi_i} (W_i \times \{1\})^{-(i+1)}A \longrightarrow 0,$$

where $d^0(a) = a/(1)$ for all $a \in A$, $\pi_1(a/(r_1)) = a/(r_1, 1)$ for all $a \in A$, $(r_1) \in W_1$,

$$e^{i-1}\left(\frac{a}{(r_1, \ldots, r_{i-1}, 1)}\right) = \frac{a}{(r_1, \ldots, r_{i-1}, 1)} \qquad \text{for all } a \in A, (r_1, \ldots, r_{i-1}) \in W_{i-1},$$

and

$$\pi_i\left(\frac{a}{(r_1, \ldots, r_i)}\right) = \frac{a}{(r_1, \ldots, r_i, 1)} \qquad \text{for all } a \in A, (r_1, \ldots, r_i) \in W_i.$$

(It follows from Proposition 3.2 that d^0 and the e^j $(j \in \mathbb{N}_0)$ are monomorphisms.)

We next show that, for each $i \in \mathbb{N}$ with $i \leq t$ and all $j \in \mathbb{N}_0$, we have $\operatorname{Ext}^j_A(A/\mathfrak{m}, W_i^{-i}A) = 0$. This is easy when $i = 1$, and so we suppose that $i > 1$. For each $\mathbf{r} := (r_1, \ldots, r_i) \in W_i$, let

$$U_{\mathbf{r}} := \{(r_1^{n_1}, \ldots, r_i^{n_i}) : n_k \in \mathbb{N} \text{ for all } k = 1, \ldots, i\}.$$

This is a triangular subset of A^i, and it follows from [16, page 39] that $\left(U_{\mathbf{r}}^{-i}A\right)_{\mathbf{r}\in W_i}$ can be turned into a direct system in such a way that

$$\varinjlim_{\mathbf{r}\in W_i} U_{\mathbf{r}}^{-i}A \cong W_i^{-i}A.$$

Since $A/\mathfrak{m}$ is a finitely generated A-module, we have

$$\operatorname{Ext}^j_A(A/\mathfrak{m}, W_i^{-i}A) \cong \varinjlim_{\mathbf{r}\in W_i} \operatorname{Ext}^j_A(A/\mathfrak{m}, U_{\mathbf{r}}^{-i}A).$$

We note next that $\mathrm{Ext}^j_A(A/\mathfrak{m}, U_{\mathbf{r}}^{-i}A) = 0$ for each A-sequence $\mathbf{r} := (r_1, \ldots, r_i) \in W_i$, because multiplication by r_i on $U_{\mathbf{r}}^{-i}A$ provides an automorphism, by [17, Lemma 2.1], so that, since $r_i \in \mathfrak{m}$, multiplication by r_i on $\mathrm{Ext}^j_A(A/\mathfrak{m}, U_{\mathbf{r}}^{-i}A)$ provides an endomorphism that is both zero and an automorphism.

Next, if $\mathbf{r}' := (r'_1, \ldots, r'_i) \in W_i$ is such that $U_{\mathbf{r}'}^{-i}A \neq 0$, then $r'_1, \ldots, r'_{i-1} \in \mathfrak{m}$ (by 3.2), so that $(r'_1, \ldots, r'_{i-1})$ is actually an A-sequence. If $r'_i \notin \mathfrak{m}$, then (since $i-1 < t$) there exists $s_i \in \mathfrak{m}$ such that $(r'_1, \ldots, r'_{i-1}, s_i)$ is an A-sequence; of course, $s_i \in Rr'_i = R$, and so

$$(r'_1, \ldots, r'_{i-1}, r'_i) \leq (r'_1, \ldots, r'_{i-1}, s_i)$$

in the partial order used to form the direct limit in [16, Proposition 3.5].

These considerations show that $\mathrm{Ext}^j_A(A/\mathfrak{m}, W_i^{-i}A) = 0$, for all $i \in \mathbb{N}$ with $i \leq t$ and all $j \in \mathbb{N}_0$.

It now follows from the long exact sequences that result from application of the functor $\mathrm{Hom}_A(A/\mathfrak{m}, \bullet)$ to the short exact sequences displayed in the first paragraph of this proof that

$$\begin{aligned} \mathrm{Hom}_A(A/\mathfrak{m}, (W_t \times \{1\})^{-(t+1)}A) &\cong \mathrm{Ext}^1_A(A/\mathfrak{m}, (W_{t-1} \times \{1\})^{-t}A) \cong \cdots \\ &\cong \mathrm{Ext}^{t-1}_A(A/\mathfrak{m}, (W_1 \times \{1\})^{-2}A) \\ &\cong \mathrm{Ext}^t_A(A/\mathfrak{m}, A), \end{aligned}$$

and, since the last module is finitely generated, the result is proved. □

3.4. **Corollary.** *Suppose that $(A, \mathfrak{m})$ is a local ring of positive depth t. Then $\Gamma_{\mathfrak{m}}\left((W_t \times \{1\})^{-(t+1)}A\right)$ is an Artinian A-module.*

Proof. Note that

$$\mathrm{Hom}_A\left(A/\mathfrak{m}, \Gamma_{\mathfrak{m}}\big((W_t \times \{1\})^{-(t+1)}A\big)\right) \cong \mathrm{Hom}_A(A/\mathfrak{m}, (W_t \times \{1\})^{-(t+1)}A);$$

the latter module is finitely generated, by Theorem 3.3. Thus $G := \Gamma_{\mathfrak{m}}\left((W_t \times \{1\})^{-(t+1)}A\right)$ has finitely generated socle; since each element of G is annihilated by some power of $\mathfrak{m}$, it follows that G is Artinian (by E. Matlis [12, Proposition 3] or L. Melkersson [13, Theorem 1.3], for example). □

So far, the work in this section has concerned the general commutative Noetherian ring A. We now specialize to the commutative Noetherian ring R of characteristic p.

3.5. **Lemma.** *Let $n \in \mathbb{N}$ and let U be a triangular subset of R^n. Then the module of generalized fractions $U^{-n}R$ has a structure as left $R[x, f]$-module with*

$$x\left(\frac{r}{(u_1, \ldots, u_n)}\right) = \frac{r^p}{(u_1^p, \ldots, u_n^p)} \quad \textit{for all } r \in R \textit{ and } (u_1, \ldots, u_n) \in U.$$

Proof. Suppose that $\mathbf{u} = (u_1, \ldots, u_n)$, $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ are three elements of U for which there exist lower triangular matrices $\mathbf{H}, \mathbf{K}$ with entries in R such that $\mathbf{H}\mathbf{u}^T = \mathbf{w}^T = \mathbf{K}\mathbf{v}^T$ and that $r, s \in R$ are such that $|\mathbf{H}|r - |\mathbf{K}|s \in (w_1, \ldots, w_{n-1})R$. (As in [16], we use $|\mathbf{H}|$ to denote the determinant of $\mathbf{H}$, *etcetera*.) If $\mathbf{G}$ is a matrix with entries from R, we shall denote by $\mathbf{G}^p$ the matrix having the same size as $\mathbf{G}$ obtained by raising each entry of $\mathbf{G}$ to the p-th power. Application of the Frobenius homomorphism yields that

$$\mathbf{H}^p(\mathbf{u}^p)^T = (\mathbf{w}^p)^T = \mathbf{K}^p(\mathbf{v}^p)^T$$

and $|\mathbf{H}^p|r^p - |\mathbf{K}^p|s^p \in (w_1^p, \ldots, w_{n-1}^p)R$. It follows that $r^p/(u_1^p, \ldots, u_n^p) = s^p/(v_1^p, \ldots, v_n^p)$ in $U^{-n}R$, and that there is a $\mathbb{Z}$-endomorphism $\xi : U^{-n}R \longrightarrow U^{-n}R$ which is such that

$$\xi\left(\frac{r}{(u_1, \ldots, u_n)}\right) = \frac{r^p}{(u_1^p, \ldots, u_n^p)} \quad \text{for all } r \in R \text{ and } (u_1, \ldots, u_n) \in U.$$

Note that $\xi(s\alpha) = s^p\xi(\alpha)$ for all $s \in R$ and $\alpha \in U^{-n}R$. The claim therefore follows from Lemma 1.3. □

3.6. **Proposition.** *Suppose that $(R, \mathfrak{m})$ is a local ring of positive depth t. Then*

$$\Gamma_{\mathfrak{m}}\Big((W_t(R) \times \{1\})^{-(t+1)}R\Big)$$

is an $R[x, f]$-submodule of $(W_t(R) \times \{1\})^{-(t+1)}R$ which is Artinian as an R-module.

Proof. If $\alpha \in (W_t(R)\times\{1\})^{-(t+1)}R$ is annihilated by $\mathfrak{m}^h$ for a positive integer h, then $(\mathfrak{m}^h)^{[p]}x\alpha = 0$, and so $\Gamma_{\mathfrak{m}}\left((W_t(R)\times\{1\})^{-(t+1)}R\right)$ is an $R[x,f]$-submodule of $(W_t(R)\times\{1\})^{-(t+1)}R$. The result therefore follows from Corollary 3.4. □

4. **Families of ideals generated by regular sequences**

The purpose of this final section of the paper is to establish a theorem which yields, as a corollary, the promised result that $\{Q(\mathfrak{a}^{[p^n]}) : n \in \mathbb{N}_0\}$ is bounded when $\mathfrak{a}$ is generated by a regular sequence. Our main theorem concerns a family $(\mathfrak{b}_\lambda)_{\lambda\in\Lambda}$ of ideals of R that can be generated by R-sequences with the property that $\mathcal{P} := \bigcup_{\lambda\in\Lambda} \operatorname{ass}\mathfrak{b}_\lambda$ is a finite set (we use $\operatorname{ass}\mathfrak{b}_\lambda$ or $\operatorname{ass}(\mathfrak{b}_\lambda)$ to denote $\operatorname{Ass}(R/\mathfrak{b}_\lambda)$). We believe that Proposition 4.1, which shows that there is a good supply of examples of such families, is well known, but as we have been unable to locate a precise reference for it, we give a brief indication of proof.

4.1**. Proposition.** *Let $\mathbf{r} := (r_1,\dots,r_t)$ and $\mathbf{s} := (s_1,\dots,s_t)$ be poor A-sequences.*

(i) *If $\mathbf{s}A \subseteq \mathbf{r}A$, then $\operatorname{ass}(\mathbf{r}A) \subseteq \operatorname{ass}(\mathbf{s}A)$.*
(ii) *We have $\operatorname{ass}(r_1^{n_1},\dots,r_t^{n_t})A = \operatorname{ass}(\mathbf{r}A)$ for all $n_1,\dots,n_t \in \mathbb{N}$.*
(iii) *If $\sqrt{\mathbf{r}A} = \sqrt{\mathbf{s}A}$, then $\operatorname{ass}(\mathbf{r}A) = \operatorname{ass}(\mathbf{s}A)$.*

Proof. (i) There is a $t\times t$ matrix $\mathbf{M}$ with entries in R such that $\mathbf{s}^T = \mathbf{M}\mathbf{r}^T$. By O'Carroll [15, Theorem 3.7], the map $A/\mathbf{r}A \longrightarrow A/\mathbf{s}A$ induced by multiplication by the determinant of $\mathbf{M}$ is an A-monomorphism. Therefore $\operatorname{ass}(\mathbf{r}A) \subseteq \operatorname{ass}(\mathbf{s}A)$.

(ii) It is enough to establish this result in the case where A is local and $\mathbf{r}A$ is a proper ideal. In a local ring A, every permutation of an A-sequence is again an A-sequence; it is therefore sufficient for us to establish the result in the case where $n_1 = \cdots = n_{t-1} = 1$. This can be achieved easily by use of part (i) in conjunction with an inductive argument based on the exact sequence

$$0 \longrightarrow A/(r_1,\dots,r_{t-1},r_t)A \longrightarrow A/(r_1,\dots,r_{t-1},r_t^{n_t+1})A \longrightarrow A/(r_1,\dots,r_{t-1},r_t^{n_t})A \longrightarrow 0,$$

in which the second homomorphism is induced by multiplication by $r_t^{n_t}$ and the third map is the canonical epimorphism.

(iii) There exists $n\in\mathbb{N}$ such that $(r_1^n,\dots,r_t^n)A \subseteq \mathbf{s}A$. By parts (i) and (ii), we have

$$\operatorname{ass}(\mathbf{s}A) \subseteq \operatorname{ass}(r_1^n,\dots,r_t^n)A = \operatorname{ass}(\mathbf{r}A).$$

Now reverse the roles of $\mathbf{r}$ and $\mathbf{s}$ to complete the proof. □

We are now ready to present the main theorem of the paper.

4.2**. Theorem.** *Let Λ be an indexing set and let $(\mathfrak{b}_\lambda)_{\lambda\in\Lambda}$ be a family of ideals of R that can be generated by R-sequences with the property that $\mathcal{P} := \bigcup_{\lambda\in\Lambda}\operatorname{ass}\mathfrak{b}_\lambda$ is a finite set. Then there exists $e\in\mathbb{N}_0$ such that*

$$(\mathfrak{b}_\lambda^F)^{[p^e]} = \mathfrak{b}_\lambda^{[p^e]} \quad \textit{for all } \lambda\in\Lambda.$$

Proof. We can assume that no $\mathfrak{b}_\lambda$ is 0. Let $\mathfrak{p}\in\mathcal{P}$, and let $t_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}}$. By Proposition 3.6,

$$\Gamma_{\mathfrak{p}R_{\mathfrak{p}}}\left(\left(W_{t_{\mathfrak{p}}}(R_{\mathfrak{p}})\times\{1/1\}\right)^{-(t_{\mathfrak{p}}+1)}R_{\mathfrak{p}}\right)$$

is an $R_{\mathfrak{p}}[x,f]$-submodule of $\left(W_{t_{\mathfrak{p}}}\times\{1/1\}\right)^{-(t_{\mathfrak{p}}+1)}R_{\mathfrak{p}}$ which is Artinian as an $R_{\mathfrak{p}}$-module. Let $e_{\mathfrak{p}}$ be its HSL-number (see 1.5). Let

$$\{\operatorname{ht}\mathfrak{p} : \mathfrak{p}\in\mathcal{P}\} = \{h_1,\dots,h_w\}, \qquad \text{where } h_1 < h_2 < \cdots < h_w.$$

Noting that $\mathcal{P} := \bigcup_{\lambda\in\Lambda}\operatorname{ass}\mathfrak{b}_\lambda$ is a finite set, we define (for each $i = 1,\dots,w$)

$$e_i := \max\{e_{\mathfrak{p}} : \mathfrak{p}\in\mathcal{P} \text{ and } \operatorname{ht}\mathfrak{p} = h_i\},$$

and we claim that $e = \sum_{i=1}^w e_i$ has the desired property. However, before we embark on the proof of this claim, we make the following observations, in which $\mathfrak{b},\mathfrak{p}$ denote ideals of R with $\mathfrak{p}$ prime, and $i\in\mathbb{N}_0$.

(i) We have $(\mathfrak{b}R_{\mathfrak{p}})^F = (\mathfrak{b}^F)R_{\mathfrak{p}}$ and $(\mathfrak{b}R_{\mathfrak{p}})^{[p^i]} = (\mathfrak{b}^{[p^i]})R_{\mathfrak{p}}$.
(ii) If $(\mathfrak{b}^F)^{[p^i]} = \mathfrak{b}^{[p^i]}$, then $(\mathfrak{b}^F)^{[p^{i+j}]} = (\mathfrak{b}^F)^{[p^{i+j}]}$ for all $j\in\mathbb{N}$.

(iii) If $(\mathfrak{b}^F)^{[p^i]} \neq \mathfrak{b}^{[p^i]}$, then the R-module $(\mathfrak{b}^F)^{[p^i]}/\mathfrak{b}^{[p^i]}$ has an associated prime ideal $\mathfrak{q}$. This $\mathfrak{q}$ will be such that $((\mathfrak{b}^F)^{[p^i]})R_\mathfrak{q} \neq (\mathfrak{b}^{[p^i]})R_\mathfrak{q}$, that is (in view of (i) above) such that $((\mathfrak{b}R_\mathfrak{q})^F)^{[p^i]} \neq (\mathfrak{b}R_\mathfrak{q})^{[p^i]}$. Note that such a $\mathfrak{q}$ has to be an associated prime of $\mathfrak{b}^{[p^i]}$; consequently, if $\mathfrak{b}$ can be generated by an R-sequence, then it follows from Proposition 4.1(ii) that such a $\mathfrak{q}$ has to be an associated prime of $\mathfrak{b}$.

We can now use these observations (i)–(iii) to see that, in order to establish our claim that $e = \sum_{i=1}^{w} e_i$ has the desired property, it is enough for us to show that, for each $\mathfrak{p} \in \mathcal{P}$, it is the case that

$$((\mathfrak{b}_\lambda R_\mathfrak{p})^F)^{[p^{e_1+\cdots+e_i}]}/(\mathfrak{b}_\lambda R_\mathfrak{p})^{[p^{e_1+\cdots+e_i}]} = 0 \quad \text{for all } \lambda \in \Lambda, \text{ where } \operatorname{ht}\mathfrak{p} = h_i.$$

We suppose that this is not the case, and we let $\mathfrak{p}$ be a minimal counterexample. Set $\operatorname{ht}\mathfrak{p} = h_j$; there must exist $\mu \in \Lambda$ such that

$$((\mathfrak{b}_\mu R_\mathfrak{p})^F)^{[p^{e_1+\cdots+e_j}]}/(\mathfrak{b}_\mu R_\mathfrak{p})^{[p^{e_1+\cdots+e_j}]} \neq 0.$$

Set $e' := \sum_{\gamma=1}^{j-1} e_\gamma$ (interpreted as 0 if $j = 1$). By choice of $\mathfrak{p}$, each of the $R_\mathfrak{p}$-modules

$$((\mathfrak{b}_\mu R_\mathfrak{p})^F)^{[p^{e'}]}/(\mathfrak{b}_\mu R_\mathfrak{p})^{[p^{e'}]} \quad \text{and} \quad ((\mathfrak{b}_\mu R_\mathfrak{p})^F)^{[p^{e'+e_j}]}/(\mathfrak{b}_\mu R_\mathfrak{p})^{[p^{e'+e_j}]}$$

has $\mathfrak{p}R_\mathfrak{p}$ as its only possible associated prime, because a smaller associated prime would lead to a contradiction to the minimality of $\mathfrak{p}$. Therefore, both of the $R_\mathfrak{p}$-modules in this last display have finite length.

Let $r_1, \ldots, r_t$ be an R-sequence that generates $\mathfrak{b}_\mu$. Note that $\mathfrak{p} \in \operatorname{ass}\mathfrak{b}_\mu$, by point (iii) above; it follows that $t = \operatorname{depth} R_\mathfrak{p} = t_\mathfrak{p}$ and $\mathfrak{b}_\mu R_\mathfrak{p}$ is generated by the maximal $R_\mathfrak{p}$-sequence $r_1/1, \ldots, r_t/1$.

There exists $\rho \in (\mathfrak{b}_\mu R_\mathfrak{p})^F$ such that $\rho^{p^{e'+e_j}} \notin (\mathfrak{b}_\mu R_\mathfrak{p})^{[p^{e'+e_j}]}$. Consider

$$\alpha := \frac{\rho}{\left(\frac{r_1}{1}, \ldots, \frac{r_t}{1}, \frac{1}{1}\right)} \in \left(W_{t_\mathfrak{p}}(R_\mathfrak{p}) \times \{1/1\}\right)^{-(t_\mathfrak{p}+1)} R_\mathfrak{p}.$$

We have

$$x^{e'}\alpha = x^{e'} \frac{\rho}{\left(\frac{r_1}{1}, \ldots, \frac{r_t}{1}, \frac{1}{1}\right)} = \frac{\rho^{p^{e'}}}{\left(\frac{r_1^{p^{e'}}}{1}, \ldots, \frac{r_t^{p^{e'}}}{1}, \frac{1}{1}\right)} \in \Gamma_{\mathfrak{p}R_\mathfrak{p}}\left(\left(W_{t_\mathfrak{p}}(R_\mathfrak{p}) \times \{1/1\}\right)^{-(t_\mathfrak{p}+1)} R_\mathfrak{p}\right),$$

by Proposition 3.2 (because $\rho^{p^{e'}} \in ((\mathfrak{b}_\mu R_\mathfrak{p})^F)^{[p^{e'}]}$ and $((\mathfrak{b}_\mu R_\mathfrak{p})^F)^{[p^{e'}]}/(\mathfrak{b}_\mu R_\mathfrak{p})^{[p^{e'}]}$ has finite length). Also, $\alpha \in \Gamma_x\left(\left(W_{t_\mathfrak{p}}(R_\mathfrak{p}) \times \{1/1\}\right)^{-(t_\mathfrak{p}+1)} R_\mathfrak{p}\right)$ because $\rho \in (\mathfrak{b}_\mu R_\mathfrak{p})^F$. However, it follows from Proposition 3.2 that $x^{e_j}(x^{e'}\alpha) \neq 0$, because $\rho^{p^{e'+e_j}} \notin (\mathfrak{b}_\mu R_\mathfrak{p})^{[p^{e'+e_j}]}$. Hence

$$x^{e'}\alpha \in \Gamma_x\left(\Gamma_{\mathfrak{p}R_\mathfrak{p}}\left(\left(W_{t_\mathfrak{p}}(R_\mathfrak{p}) \times \{1/1\}\right)^{-(t_\mathfrak{p}+1)} R_\mathfrak{p}\right)\right) \quad \text{but} \quad x^{e_j}(x^{e'}\alpha) \neq 0.$$

This is a contradiction, and so the theorem is proved. □

Corollary 4.3 below is a special case of Theorem 4.2; it presents one of the results mentioned in the Introduction.

4.3. **Corollary.** *Suppose that the ideal $\mathfrak{a}$ of R can be generated by an R-sequence $r_1, \ldots, r_t$. Then there exists $e \in \mathbb{N}_0$ such that $((r_1^{n_1}, \ldots, r_t^{n_t})^F)^{[p^e]} = (r_1^{n_1}, \ldots, r_t^{n_t})^{[p^e]}$ for all $n_1, \ldots, n_t \in \mathbb{N}$. In particular, $((\mathfrak{a}^{[p^n]})^F)^{[p^e]} = (\mathfrak{a}^{[p^n]})^{[p^e]}$ for all $n \in \mathbb{N}_0$.*

Proof. For all $n_1, \ldots, n_t \in \mathbb{N}$, the sequence $r_1^{n_1}, \ldots, r_t^{n_t}$ is an R-sequence, and $\operatorname{ass}(r_1^{n_1}, \ldots, r_t^{n_t})R = \operatorname{ass}\mathfrak{a}$ by Proposition 4.1(ii). The result is therefore immediate from Theorem 4.2. □

4.4. **Corollary.** *Let Λ be an indexing set and let $(\mathfrak{b}_\lambda)_{\lambda \in \Lambda}$ be a family of ideals of R that can be generated by R-sequences with the property that $\mathcal{Q} := \left\{\sqrt{\mathfrak{b}_\lambda} : \lambda \in \Lambda\right\}$ is a finite set. Then there exists $e \in \mathbb{N}_0$ such that*

$$(\mathfrak{b}_\lambda^F)^{[p^e]} = \mathfrak{b}_\lambda^{[p^e]} \quad \textit{for all } \lambda \in \Lambda.$$

Proof. For each $\mathfrak{q} \in \mathcal{Q}$, we can select a $\lambda(\mathfrak{q}) \in \Lambda$ such that $\sqrt{\mathfrak{b}_{\lambda(\mathfrak{q})}} = \mathfrak{q}$. Since $\mathcal{Q}$ is finite, the set $\mathcal{P}' := \bigcup_{\mathfrak{q} \in \mathcal{Q}} \operatorname{ass}(\mathfrak{b}_{\lambda(\mathfrak{q})})$ is finite. Let $\mu \in \Lambda$. Then $\sqrt{\mathfrak{b}_\mu}$ is equal to some $\mathfrak{q} \in \mathcal{Q}$, and so $\sqrt{\mathfrak{b}_\mu} = \sqrt{\mathfrak{b}_{\lambda(\mathfrak{q})}}$. By Proposition 4.1(iii), $\operatorname{ass}(\mathfrak{b}_\mu) = \operatorname{ass}(\mathfrak{b}_{\lambda(\mathfrak{q})}) \subseteq \mathcal{P}'$. Hence $\bigcup_{\mu \in \Lambda} \operatorname{ass} \mathfrak{b}_\mu$ is a subset of the finite set $\mathcal{P}'$, and so the result follows from Theorem 4.2. □

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