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FROBENIUS MAPS ON INJECTIVE HULLS AND THEIR APPLICATIONS TO TIGHT CLOSURE

MORDECHAI KATZMAN

Abstract. This paper studies Frobenius maps on injective hulls of residue fields of complete local rings with a view toward providing constructive descriptions of objects originating from the theory of tight closure. Specifically, the paper describes algorithms for computing parameter test ideals, and tight closure of certain submodules of the injective hull of residue fields of a class of well-behaved rings which includes all quasi-Gorenstein complete local rings.

1. Introduction

This paper studies problems originating from the theory of tight closure which we now review briefly. Let A be a commutative ring of prime characteristic p; for any positive integers e we define the *iterated Frobenius endomorphism* $f^e : A \to A$ to be the map which raises elements to their p^eth power. This map can be used to endow A with the structure of a A-bimodule. As a left A-module it has the usual A-module structure whereas A acts on itself on the right via the iterated Frobenius map; we denote this bimodule ${}^eA$. Now for all $a \in {}^eA$ and $b \in A$, $b \cdot a = ba$ while $a \cdot b = b^{p^e} a$, where $\cdot$ denotes the action of A. We can extend this construction to obtain the *Frobenius functor* F^e_A sending any A-module M to $F^e_A(M) = {}^eA \otimes_A M$ where A acts on $F^e_A(M)$ via its left-action on ${}^eA$, so for $a \otimes m \in F^e_A(M)$ and $b \in A$ we have $b \cdot (a \otimes m) = ba \otimes m$ and $(a \otimes bm) = a \cdot b \otimes m = b^{p^e} a \otimes m$.

We often find it convenient to think of ${}^eA$ and the associated Frobenius functors as follows. Let Θ be an indeterminate and consider the free A-module $A[\Theta; f^e] = \oplus_{i=0}^{\infty} A\Theta^i$ which we turn into a skew-polynomial ring by defining $\Theta a = a^{p^e}\Theta$ for all $a \in A$. We can now identify ${}^{ej}A$ with $A\Theta^j \subset A[\Theta; f^e]$ and for all A-modules M we may write $F^{ej}_A(M) = A\Theta^j \otimes_A M$.

If M is an A-module and $N \subseteq M$ is an A-submodule we define the tight closure of N in M, denoted N^*_M, to be the set of all $m \in M$ such that for some $c \in A$ not in any minimal prime, $c \otimes m \in F^e_A(M)$ is in the image of the map $F^e_A(N) \to F^e_A(M)$ for all $e \gg 0$.

Among the most interesting and useful results obtained early in the development of the theory of tight closure is the existence of *test-elements* (cf. Chapter 2 in [H]). Notice that the element $c \in A$ occurring in the definition of tight closure could depend on the modules N and M and on the element $m \in M$. Test elements are elements $c \in A$ not in any minimal prime such that for *all* finitely generated modules M and submodules $N \subseteq M$ and *all*

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$m \in M$,

(1) $\quad m \in N^*_M \Leftrightarrow c \otimes m \in F^e_A(M)$ is in the image of $F^e_A(N) \to F^e_A(M)$ for all $e \geq 0$.

A weaker concept, that of a $p^{e'}$*-weak test element* is defined similarly, only that we relax the last condition above and demand that

(2) $\quad m \in N^*_M \Leftrightarrow c \otimes m \in F^e_A(M)$ is in the image of $F^e_A(N) \to F^e_A(M)$ for all $e \geq e'$.

One also defines the *test-ideal* and $p^{e'}$*-weak test ideal* of A to be the ideals generated by all test-elements, and all $p^{e'}$-weak test elements, respectively.

In many applications one restricts one's attention to local rings A and to the tight-closure of ideals generated by systems of parameters. One then naturally considers the notion of *parameter test elements*: these are elements $c \in A$ not in any minimal prime which satisfy (1) with $M = A$ and N being an ideal generated by a system of parameters. Similarly one obtains the notion of $p^{e'}$*-weak parameter test elements*: these are the elements $c \in A$ not in any minimal prime which satisfy (2) with $M = A$ and N being an ideal generated by a system of parameters. One can then define the *parameter test ideal* and $p^{e'}$*-weak parameter test ideal*) of A to be the ideals generated by all parameter test elements, and all $p^{e'}$-weak parameter test elements, respectively. It is worth noting that when S is a Gorenstein ring, the notions of parameter-test-ideals and test-ideals coincide (cf. Chapter 2 in [H]).

We refer the reader to the seminal paper [HH] and to [H] for detailed descriptions of tight closure and its properties.

The main results of this paper produce explicit descriptions of these test-ideals. The first such result is Theorem 3.4 which gives a formula for weak parameter-test-ideals of complete local rings. This is a generalization of Theorem 8.2 in [K] which gave a similar description of the parameter-test-ideals of complete local rings under the assumption that a certain Frobenius map on the the injective hull of the residue field is injective.

Another important result is Theorem 5.5 which gives an explicit description of the tight closure of certain submodules of the injective hull of the residue field of certain complete local rings. In view of the notorious difficulty of computing the tight closure of ideals, the fact that sometimes it is easy to compute the tight closure of submodules of a much larger object seems very interesting. Also, this result has immediate relevance to the study of test-ideals. It is known that test-ideals of local rings are the annihilators of the finitistic tight closure of 0 in the injective hulls of their residue fields (cf. section 8 of [HH]) and it is conjectured that this finitistic tight closure coincides with the regular tight closure (cf. Conjecture 2.6 in [LS] and section 8 of that paper where the conjecture is shown to hold in some cases.) The last section of this paper computes the tight closure of 0 in the injective hulls residue fields of certain complete local rings.

Throughout this paper, we fix $(R, \mathfrak{m})$ to be a complete regular ring of prime characteristic p, we fix $I \subseteq R$ to be an ideal and we write $S = R/I$. We denote with E_R and $E_S = \operatorname{ann}_{E_R} I$ the injective hulls of the residue fields of R and S respectively.

Definition 1.1. For any S-module M and all $e \geq 0$ we let $\mathcal{F}^e(M)$ denote the set of all additive functions $\phi : M \to M$ with the property that $\phi(sm) = s^{p^e}\phi(m)$ for all $s \in S$ and $m \in M$. Note that each $\mathcal{F}^e(M)$ is naturally an S-module: for all $\phi \in \mathcal{F}^e(M)$ and $s \in S$ the map $s\phi$ defined as $(s\phi)(m) = s\phi(m)$ for all $m \in M$ is in $\mathcal{F}^e(M)$. We also define $\mathcal{F}(M) = \oplus_{e\geq 0}\mathcal{F}^e(M)$.

We call an S-submodule $N \subseteq M$ an $\mathcal{F}^e(M)$*-submodule* if $\phi(N) \subseteq N$ for all $\phi \in \mathcal{F}^e(M)$; if N is an $\mathcal{F}^e(M)$*-submodule* for all $e \geq 0$ we call N an $\mathcal{F}(M)$*-submodule.*

We shall refer to the maps in $\mathcal{F}^e(M)$ defined above as *eth Frobenius maps* (or just *Frobenius maps* when $e = 1$.) The most important Frobenius map is, of course, *the* Frobenius map on $f : S \to S$ given by $f(s) = s^p$.

Notice that given an S-module M, any $\phi \in \mathcal{F}^e(M)$ determines a left $S[\Theta; f^e]$-module structure on M given my $\Theta m = \phi(m)$ for all $m \in M$. Conversely, a left $S[\Theta; f^e]$-module structure on M defines a $\phi \in \mathcal{F}^e(M)$ given by $\phi(m) = \Theta m$ for all $m \in M$.

We shall call an element m of an $S[\Theta; f^e]$-module M *nilpotent* if $\Theta^j m = 0$ for some $j \geq 0$ and we shall denote the set all such elements $\mathrm{Nil}(M)$; this is easily seen to be an $S[\Theta; f^e]$-submodule of M.

In the first part of this paper we will be particularly interested in S-submodules of E_S which are stable under one particular Frobenius map arising from a canonical Frobenius map which we describe next. One of most important examples of modules with Frobenius maps is the top local cohomology module $\mathrm{H}^d_{\mathfrak{m}S}(S)$ which is a left $S[T; f]$-module in the following natural way. $\mathrm{H}^d_{\mathfrak{m}S}(S)$ can be computed as the direct limit of

$$\frac{S}{(x_1, \dots, x_d)S} \xrightarrow{x_1 \cdot \dots \cdot x_d} \frac{S}{(x_1^2, \dots, x_d^2)S} \xrightarrow{x_1 \cdot \dots \cdot x_d} \dots$$

where $x_1, \dots, x_d$ is a system of parameters of S and we can define a Frobenius map $\phi \in \mathcal{F}^e\left(\mathrm{H}^d_{\mathfrak{m}S}(S)\right)$ on this direct limit by mapping the coset $a + (x_1^n, \dots, x_d^n)S$ in the n-th component of the direct limit to the coset $a^{p^e} + (x_1^{np^e}, \dots, x_d^{np^e})S$ in the np^e-th component of the direct limit. When S has a canonical module $\omega \subseteq S$, this $S[T; f]$-module structure induces one in E_S as follows. The inclusion $\omega \subseteq S$ yields a surjection $E_S = \mathrm{H}^d_{\mathfrak{m}S}(\omega) \twoheadrightarrow \mathrm{H}^d_{\mathfrak{m}S}(S)$ which can be made into a surjection of $S[T; f]$-modules by lifting the $S[T; f]$ module structure of $\mathrm{H}^d_{\mathfrak{m}S}(S)$ onto E_S (cf. §7 in [K]). It is this $S[T; f]$-module structure on E_S which, as in [K], will enable us to give a explicit description of the weak parameter test ideals of S.

Recall that as R is a power series ring $\mathbb{K}[\![x_1, \dots, x_n]\!]$ for some field $\mathbb{K}$ of characteristic p, E_R is isomorphic to the module of inverse polynomials $\mathbb{K}[x_1^-, \dots, x_n^-]$ (cf. Example 12.4.1 in [BS]) which has a natural left $R[T; f]$-module structure extending $T x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n} = x_1^{p\alpha_1} \cdot \dots \cdot x_n^{p\alpha_n}$ for all $\alpha_1, \dots, \alpha_n < 0$. One can show that all left $S[\Theta; f^e]$ module structures on $E_S = \mathrm{ann}_{E_S} I$ are given by $\Theta = uT^e$ where $u \in (I^{[p^e]} : I)$ (cf. Proposition 4.1 in [K] and Chapter 3 of [B].) Given a left $S[T; f]$-modules structure on $S[T; f]$, the study of $S[T; f]$-submodules of E_S now translates via Matlis duality to the study of certain ideals of R:

Definition 1.2 (cf. Definition 4.2 in [K])**.** An ideal $J \subseteq S$ is called an E_S-ideal if $\operatorname{ann}_{E_S} J$ is an $S[T;f]$-submodule of E_S. An ideal $J \subseteq R$ is called an E_S-ideal if it contains I and its image in S is an E_S-ideal.

Theorem 4.3 in [K] states that an ideal $J \subseteq R$ containing I is an E_S-ideal if and only if $uJ \subseteq J^{[p]}$ where $u \in (I^{[p]} :_R I)$ determines the $S[T;f]$-module structure of E_S as above. It is this characterization which allows one to transform question regarding submodules of E_S to one regarding ideals of R, and these transformation sometimes renders them tractable.

As in [K] let $\mathcal{C}^e$ be the category of Artinian $S[T;f^e]$-modules and let $\mathcal{D}^e$ be the category of R-linear maps $M \to F_R^e(M)$ where M is a finitely generated S-module and where a morphism between $M \xrightarrow{a} F_R^e(M)$ and $N \xrightarrow{b} F_R^e(N)$ is a commutative diagram of R-linear maps

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "a"] & N \arrow[d, "b"] \\ F_R^e(M) \arrow[r, "F_R^e(\mu)"] & F_R^e(N) \end{tikzcd} .$$

This paper uses the mutually inverse functors $\Delta^e : \mathcal{C}^e \to \mathcal{D}^e$ and $\Psi^e : \mathcal{D}^e \to \mathcal{C}^e$ defined in [K]. We shall also use fragments of the construction of Δ^e, as in the proof of Theorem 2.4. These two functors will enable us to translate problems involving the injective hull E_S to problems involving of ideals in R. The crucial tool in answering the latter will be the ideal operation $I_e(-)$: for an ideal $J \subseteq R$, $I_e(J)$ is defined as the smallest ideal $L \subseteq R$ for which $J \subseteq L^{[p^e]}$. The existence of this operation and its construction are discussed in section 5 of [K]; we shall assume the reader is familiar with the basic properties of this operation described there.

This paper is organized as follows: Section 2 studies basic properties of submodules of E_S and their annihilators which are used throughout this paper. Section 3 generalizes Theorem 8.2 in [K] and gives an explicit description of the weak parameter test ideals of S in the case where S is Cohen-Macaulay with canonical module $\omega \subseteq S$ but where the Frobenius map on E_S induced from the natural Frobenius map on $\mathrm{H}^{\dim S}_{\mathfrak{m}S}(S)$ is not necessarily injective. Section 4 introduces a certain operation on E_S-ideal and applies it to the description of quasimaximal filtrations of E_S. This operation is again used in section 5 which gives fairly explicit descriptions of the tight closure of certain submodules of E_S.

2. Basic properties of graded annihilators and E_S ideals

Throughout this section we consider a fixed $S[T;f]$-module structure of E_S corresponding to a fixed $u \in (I^{[p]} : I)$.

We start by listing some basic properties of E_S-ideals.

Proposition 2.1. (a) *The intersection of E_S-ideals is an E_S-ideal.*

(b) *If $J \subseteq R$ is an E_S-ideal and $A \subset R$ is an ideal, then $(J : A)$ is an E_S-ideal.*

(c) *Assume $J \subseteq R$ is an E_S-ideal with minimal primary decomposition $Q_1 \cap \cdots \cap Q_n$ and write $P_i = \sqrt{Q_i}$ for all $1 \leq i \leq n$. Then $P_1, \ldots, P_n$ are E_S-ideals and, if P_i is not an embedded prime, then Q_i is an E_S-ideal.*

Proof. Let $\{J_\lambda\}_{\lambda \in \Lambda}$ be a set of E_S-ideals. We have

$$u\left(\bigcap_{\lambda \in \Lambda} J_\lambda\right) \subseteq \bigcap_{\lambda \in \Lambda} uJ_\lambda \subseteq \bigcap_{\lambda \in \Lambda} J_\lambda^{[p]} = \left(\bigcap_{\lambda \in \Lambda} J_\lambda\right)^{[p]}$$

where the equality follows from the fact that $R^{1/p}$ is an $\cap$-flat R module (cf. Proposition 5.3 in [K]) and (a) follows.

Since

$$u(J : A)A^{[p]} \subseteq u(J : A)A \subseteq uJ \subseteq J^{[p]}$$

we see that

$$u(J : A) \subseteq (J^{[p]} : A^{[p]}) = (J : A)^{[p]}$$

where the equality follows from the fact that R is regular, and now (b) follows.

To prove (c), first assume that P_i is not an embedded prime, and pick $a \in \cap_{j \neq i} Q_j \setminus P_i$. Now

$$(J : a) = \bigcap_{j=1}^{n} (Q_j : a) = (Q_i : a) = Q_i$$

is an E_S-ideal. Any P_i has the form $(J : a)$ for some $a \in R$, so (b) implies that P_i is an E_S-ideal. □

Definition 2.2. Let H be an $S[T; f]$-module and let $M \subseteq H$ be an S-submodule. For any $e \geq 0$ we write ST^eM for the S-submodule of M generated by $\{T^e m \mid m \in M\}$ and we also write $M^{(e)} = (0 :_R ST^eM)$. We define the *graded annihilator of* M, denoted $\operatorname{gr-ann} M$, to be the ideal $\oplus_{e \geq 0} M^{(e)} ST^e \subseteq S[T; f]$.

We shall call an ideal $L \subseteq S$ *H-special,* if there exists an $S[T; f]$-submodule $N \subseteq H$ for which $\operatorname{gr-ann} N = LS[T; f]$. When $H = E_S$ and $\operatorname{Nil}(E_S) = 0$ the notions of E_S-special ideals and E_S ideals coincide (cf. §6 in [K]).

Note that whenever $M \subseteq H$ is an $S[T; f]$-submodule, $\left\{M^{(e)}\right\}_{e \geq 0}$ is an ascending chain of ideals. When $\operatorname{Nil}(M) = 0$ that ascending chain is constant and that constant value is a radical M-special ideal, whose minimal primes are themselves M-special ideals. (cf. Corollary 3.7 in [S1]). In general the ascending chain $\left\{M^{(e)}\right\}_{e \geq 0}$ need not be constant (e.g., while $\operatorname{Nil}(E_S)^{(e)} = S$ for all large e, $\operatorname{Nil}(E_S)^{(0)} \neq S$ whenever $\operatorname{Nil}(E_S) \neq 0$), and the ideals there may be non-radical. We next study the properties of these chains of ideals.

Lemma 2.3. *Let $J_1 \subseteq J_2 \subseteq R$ be any ideals.*

$$\left(\frac{J_2}{J_1}\right)^{\vee} \cong \frac{\operatorname{ann}_{E_R} J_1}{\operatorname{ann}_{E_R} J_2}.$$

Proof. Apply $(-)^\vee$ to the short exact sequence

$$0 \to J_2/J_1 \to R/J_1 \to R/J_2 \to 0$$

to obtain the short exact sequence

$$0 \to \operatorname{ann}_{E_R} J_2 \to \operatorname{ann}_{E_R} J_1 \to \left(\frac{J_2}{J_1}\right)^\vee \to 0.$$

□

For any $e \geq 1$ write $\nu_e = 1 + \cdots + p^{e-1}$.

Theorem 2.4. *Let M be an S-submodule of E_S and write $M = \operatorname{ann}_{E_S} L$ for some ideal $L \subseteq R$.*

(a) *For all $e \geq 0$,*
$$ST^e M \cong \frac{\operatorname{ann}_{E_S} L^{[p^e]}}{\operatorname{ann}_{E_S}(u^{\nu_e}R + L^{[p^e]})}.$$
(b) *For all $e \geq 0$, $M^{(e)} = (L^{[p^e]} : u^{\nu_e})$.*
(c) *For all $e \geq 0$, $uM^{(e)} \subseteq M^{(e-1)^{[p]}}$.*
(d) *Assume further that M is an $S[T; f]$-submodule of E_S. Then $\operatorname{ann}_{E_S} M^{(e)}$ is an $S[T; f]$ submodule of E_S and if for some $e \geq 0$ we have $M^{(e)} = M^{(e+1)}$, then $M^{(j)} = M^{(e)}$ for all $j \geq e$.*

Proof. Fix any $e \geq 0$ and consider the map of R-modules $\psi_e : RT^e \otimes_R E_S \to E_S$ given by $\psi_e(rT^e \otimes m) = rT^e m$; notice that $\psi_e(RT^e \otimes_R M) = ST^e M$. Since R is regular, we have an injection $RT^e \otimes_R M \hookrightarrow RT^e \otimes_R E_S$; let $\overline{\psi}_e$ be the restriction of ψ_e to $RT^e \otimes_R M$ and consider the following commutative diagram

$$\begin{tikzcd} RT^e \otimes E_S \arrow[rr, "\psi_e"] & & E_S \arrow[d, equal] \\ RT^e \otimes M \arrow[u, hook] \arrow[r, two heads, "\overline{\psi}_e"] & ST^e M \arrow[r, hook] & E_S \end{tikzcd} .$$

An application of Matlis duality together with the fact that $(RT^e \otimes_R M)^\vee \cong RT^e \otimes_R M^\vee$ (cf. Lemma 4.1 in [L]) yields the commutative diagram

$$\begin{tikzcd} R/I^{[p^e]} \arrow[d, two heads] & & R/I \arrow[ll, "u^{\nu_e}"'] \arrow[d, equal] \\ R/L^{[p^e]} & (ST^e M)^\vee \arrow[l, hook'] & R/I \arrow[l, two heads] \end{tikzcd} .$$

The image of the composition of the top and left maps is $u^{\nu_e}R + L^{[p^e]}/L^{[p^e]}$ and this coincides with the image of $(ST^e M)^\vee$ in $R/L^{[p^e]}$. We deduce that $(ST^e M)^\vee$ is isomorphic to $u^{\nu_e}R + L^{[p^e]}/L^{[p^e]}$. Now

$$(ST^e M) \cong (ST^e M)^{\vee\vee} = \left(\frac{u^{\nu_e}R + L^{[p^e]}}{L^{[p^e]}}\right)^\vee$$

and an application of Lemma 2.3 gives (a).

We now compute

$$\begin{aligned}
M^{(e)} &= (0 :_R ST^e M)\\
&= (0 :_R (ST^e M)^\vee)\\
&= \left(0 :_R \frac{u^{\nu_e}R + L^{[p^e]}}{L^{[p^e]}}\right)\\
&= (L^{[p^e]} :_R u^{\nu_e} R)
\end{aligned}$$

and obtain (b). Next we notice that, for all $e \geq 0$, $\nu_e - 1 = p\nu_{e-1}$ and

$$uM^{(e)} = u(L^{[p^e]} : u^{\nu_e}) \subseteq (L^{[p^e]} : u^{p\nu_{e-1}}) = (L^{[p^{e-1}]} : u^{\nu_{e-1}})^{[p]} = M^{(e-1)^{[p]}}$$

and (c) follows.

If M is an $S[T;f]$-submodule of E_S then $\left\{M^{(e)}\right\}_{e\geq 0}$ is an ascending chain of ideals and we deduce from (c) that $uM^{(e)} \subseteq M^{(e)^{[p]}}$, i.e., that $M^{(e)}$ is an E_S ideal and hence $\operatorname{ann}_{E_S} M^{(e)}$ is an $S[T;f]$ submodule of E_S.

Consider the maps $\beta_i : R/L \to F^i_R(R/L) = R/L^{[p^i]}$ given by the composition

$$R/L \xrightarrow{u} R/L^{[p]} \xrightarrow{u^p} \dots \xrightarrow{u^{p^{i-1}}} R/L^{[p^i]},$$

i.e., by multiplication by u^{ν_i}. For each $i \geq 1$, the kernel of β_i is the image of $M^{(i)} = (L^{[p^i]} :_R u^{\nu_i})$ in R/L; (d) now follows from Proposition 2.3(b) in [L]. □

We can now prove the following generalization of Proposition 3.3 in [S1].

Theorem 2.5. *Let $A \subseteq B$ be $S[T;f]$-submodules of E_S. Write $A = \operatorname{ann}_{E_S} K$ and $B = \operatorname{ann}_{E_S} J$ for some ideals $J \subseteq K \subseteq R$. Also write* $\operatorname{gr-ann} B = \oplus_{e\geq 0} b_e T^e$ *and* $\operatorname{gr-ann} B/A = \oplus_{e\geq 0} \overline{b}_e T^e$ *where b_e and $\overline{b}_e$ are ideals of R for all $e \geq 0$. For all $e \geq 0$,*

$$\overline{b}_e = \left((J^{[p^e]} :_R u_{\nu_e}) :_R K\right) = (b_e :_R K).$$

Proof. As in the proof of Theorem 2.4, fix any $e \geq 0$ and consider the map of R-modules $\psi_e : RT^e \otimes_R B/A \to B/A$ given by $\psi_e(rT^e \otimes m) = rT^e m$; we notice that the image of ψ_e is $(ST^e B + A)/A$. An application of Matlis duality to the maps

$$RT^e \otimes_R B/A \twoheadrightarrow (ST^e B + A)/A \hookrightarrow B/A$$

yields the R-linear maps

$$K/J \twoheadrightarrow ((ST^e B + A)/A)^\vee \hookrightarrow K^{[p^e]}/J^{[p^e]}$$

whose composition is given by multiplication by u^{ν_e}. We deduce that $((ST^e B + A)/A)^\vee$ is isomorphic to the image of the map $R/J \to R/J^{[p^e]}$ given by multiplication by u^{ν_e}, i.e., to

$u^{\nu_e}K + J^{[p^e]}/J^{[p^e]}$. Now

$$\begin{aligned}\overline{b}_e &= (0 :_R (ST^eB + A)/A)\\ &= (0 :_R ((ST^eB + A)/A)^\vee)\\ &= (J^{[p^e]} :_R u^{\nu_e}K)\\ &= \Big((J^{[p^e]} :_R u^{\nu_e}) :_R K\Big)\\ &= (b_e :_R K).\end{aligned}$$

□

Recall that there exists an integer $\eta \geq 0$ such that $T^\eta \operatorname{Nil}(E_S) = 0$ (cf. Proposition 4.4. in [L]) which we shall refer to as the *index of nilpotency of* E_S and that $\operatorname{Nil}(E_S) = \operatorname{ann}_{E_S} I_\eta(u^{\nu_\eta}R) + I$ where for all ideals $L \subseteq R$ and positive integers e, $I_e(L)$ is defined as the smallest ideal J for which $L \subseteq J^{[p^e]}$ (cf. Theorem 4.6 and section 5 in [K].)

Corollary 2.6. *Let B be an $S[T; f]$-submodule of E_S and write $B = \operatorname{ann}_{E_S} J$ for some ideal $J \subseteq R$. Let η be the index of nilpotency of E_S and $K = I_\eta(u^{\nu_\eta}R) + I$. We have $\big((J^{[p^e]} :_R u^{\nu_e}) :_R K\big) = ((J :_R u) :_R K)$ for all $e \geq 0$ and these are radical ideals.*

We conclude this section by recording some additional properties of the associated primes of the ideals occurring in graded annihilators of $S[T; f]$-submodules of E_S. These properties will not be used elsewhere in this paper.

Proposition 2.7. *Let M be an $S[T; f]$-submodule of E_S, write $\overline{M} = M + \operatorname{Nil}(E_S)/\operatorname{Nil}(E_S)$ and write $\operatorname{Nil}(E_S) = \operatorname{ann}_{E_S} K$.*

(a) *For all $e \geq 0$, $\operatorname{Ass} M^{(e)} \supseteq \operatorname{Ass} M^{(e+1)}$.*
(b) *For all $e \geq 0$, if $P \in \operatorname{Ass} M^{(0)} \setminus \operatorname{Ass} M^{(e)}$ and P is not an embedded prime of $M^{(0)}$ then $P \supseteq K$.*
(c) *Assume that $\operatorname{ht} KS > 0$. If $\operatorname{ht} M^{(e)}S > 0$ for some $e \geq 0$ then $\operatorname{ht} M^{(e)}S > 0$ for all $e \geq 0$.*

Proof. Write $M = \operatorname{ann}_{E_S} J$ for some E_S-ideal J. For all $e \geq 0$ let $Q_1^{(e)} \cap \cdots \cap Q_{n_e}^{(e)}$ be a minimal primary decomposition of $M^{(e)}$ and write $P_i^{(e)} = \sqrt{Q_1^{(e)}}$ for all $1 \leq i \leq n_e$.

Theorem 2.4(b) shows that

$$M^{(e+1)} = \Big(J^{[p^{e+1}]} :_R u^{p\nu_e+1}\Big) = \Big(\Big(J^{[p^e]} :_R u^{\nu_e}\Big)^{[p]} :_R u\Big) = \Big({M^{(e)}}^{[p]} :_R u\Big).$$

Now

$$\bigcap \Big\{ \Big({Q_i^{(e)}}^{[p]} :_R u\Big) \mid 1 \leq i \leq n_e,\ u \notin {Q_i^{(e)}}^{[p]} \Big\}$$

is a primary decomposition of $M^{(e+1)}$ and (a) follows.

Theorem 2.5 implies that $\overline{M}^{(e)} = (M^{(e)} :_R K)$ for all $e \geq 0$ and $\overline{M}^{(e)} = \overline{M}^{(0)}$ is a radical ideal (cf. Lemma 1.9 in [S1]). Now we obtain primary decompositions

$$\bigcap \Big\{ \Big(Q_i^{(e)} :_R K\Big) \mid 1 \leq i \leq n_e,\ K \nsubseteq Q_i^{(e)} \Big\}$$

where the primary components associated with minimal primes are irredundant, and since these primary components occur for all $e \geq 0$, (b) follows.

Assume now that that $\operatorname{ht} KS > 0$ and that $\operatorname{ht} M^{(e)} S > 0$ for some $e \geq 0$. If $\operatorname{ht} M^{(0)} S = 0$ then there exists an associated prime P of $M^{(0)}$ such that $\operatorname{ht} PS = 0$. But P is not an associated prime of $M^{(e)}$, so $K \subseteq P$ and $\operatorname{ht} KS \leq \operatorname{ht} PS = 0$, a contradiction. □

All the results in this section have natural analogues when working with a $S[\Theta; f^e]$-module structure on E_S− these were omitted for the sake of simplicity. The proofs of this analogous results consist of straightforward modifications of the proofs given above. In what follows we shall assume the more general results.

3. Weak parameter test ideals

In this section we describe an algorithm for computing the p^e-weak parameter test ideal of complete local Cohen-Macaulay rings. This extends the main result in [K] where this was done under the assumption that a certain Frobenius map on E_S is injective.

Throughout this section we assume S to be Cohen-Macaulay with canonical module $\omega \subseteq S$ and we write $H = \mathrm{H}^{\dim S}_{\mathfrak{m}S}(S)$. Now $\dim S/\omega < \dim S$ and the short exact sequence

$$0 \to \omega \to S \to S/\omega \to 0$$

yields a surjection $\Upsilon : E_S \twoheadrightarrow H$. We can now endow E_S with a structure of an $S[T; f]$-module which makes this surjection into a map of $S[T; f]$-modules (cf. section 7 in [K]). We fix this $S[T; f]$-module structure throughout this section.

Let $J \subseteq R$ be henceforth in this section the ideal for which $\ker \Upsilon = \operatorname{ann}_{E_S} J$. This ideal can be computed effectively as follows. The map Υ is obtained from the long exact sequence of local cohomology modules arising from the the short exact sequence $0 \to \omega \to S \to S/\omega \to 0$, i.e., from

$$0 \to \mathrm{H}^{\dim S-1}_{\mathfrak{m}S}(S/\omega) \to \mathrm{H}^{\dim S}_{\mathfrak{m}S}(\omega) \xrightarrow{\Upsilon} \mathrm{H}^{\dim S}_{\mathfrak{m}S}(S) \to 0.$$

We may rewrite this short exact sequence using local duality to obtain

$$0 \to \operatorname{Ext}^{\dim R - \dim S + 1}_R(S/\omega, R)^\vee \to \operatorname{Ext}^{\dim R - \dim S}_R(\omega, R)^\vee \xrightarrow{\Upsilon} \operatorname{Ext}^{\dim R - \dim S}_R(S, R)^\vee \to 0$$

which yields

$$0 \to \operatorname{Ext}^{\dim R - \dim S}_R(S, R) \to \operatorname{Ext}^{\dim R - \dim S}_R(\omega, R) \to (\ker \Upsilon)^\vee \to 0$$

Now $\operatorname{Ext}^{\dim R - \dim S}_R(\omega, R) \cong S$ and we identify $\omega' = \operatorname{Ext}^{\dim R - \dim S}_R(S, R)$, which is a canonical module for S, with its image in S. We have $(\ker \Upsilon)^\vee \cong S/\omega'$ and another application of $(-)^\vee$ gives $(S/JS)^\vee \cong \ker \Upsilon \cong (S/\omega')^\vee$ and so $S/JS \cong S/\omega'$, and, therefore, $JS = \omega'$.

Definition 3.1. For all $e \geq 0$ we define

$$\mathcal{I}_e = \left\{ M^{(e)} \mid M \subseteq H \text{ is an } S[T; f]\text{-submodule} \right\}.$$

Notice that this extends the definition of the set of H-special ideals given in [S1] for the case where H is T-torsion-free.

Fix a system of parameters $x_1, \dots, x_d$ of S and think of H as the direct limit

$$\frac{S}{(x_1, \dots, x_d)S} \xrightarrow{x_1 \cdot \ldots \cdot x_d} \frac{S}{(x_1^2, \dots, x_d^2)S} \xrightarrow{x_1 \cdot \ldots \cdot x_d} \dots$$

with its standard Frobenius described in the introduction and notice that as we assume S to be Cohen-Macaulay, the maps in this direct limit are injective. Pick some element $a + (x_1^i, \dots, x_d^i)S$. In what follows we will tacitly use the fact that $\overline{c}T^e(a + (x_1^i, \dots, x_d^i)S) = 0$ in the direct limit for some $\overline{c} \in S$ not in any minimal prime if and only if $a \in ((x_1^i, \dots, x_d^i)S)^*$ (cf. Remark 4.2 in [S1]).

Theorem 3.2. *Assume that S has a parameter-test-element. For all $e \geq 0$, the p^e-weak parameter test ideal of S is the image of*

$$\cap \{K \mid K \in \mathcal{I}_e, \operatorname{ht} KS > 0\}$$

is S.

Proof. Let τ be the intersection in the statement of the theorem. Assume that d is a p^e-weak parameter test element. If $M \subseteq H$ is an $S[T; f]$-submodule for which $\operatorname{ht} M^{(e)} > 0$, we can find a $c \in M^{(e)}$ whose image in S is not in a minimal prime such that $cT^{e'}M = 0$ for all $e' \geq e$, and hence $dT^{e'}M = 0$ for all $e' \geq e$, and in particular $d \in M^{(e)}$. We deduce that $d \in \tau$. We next show that all elements in τ are p^e-weak parameter test elements.

Fix a $c \in R$ whose image $\overline{c}$ in S is a parameter-test-element. Let $h \in H$ be such that such that $cT^{e'}h = 0$ for all $e' \geq 0$. Define $L = \oplus_{e' \geq 0} S\overline{c}T^{e'}$ and $M = \operatorname{ann}_H L$; notice that $h \in M$. Now $c \in M^{(0)} \subseteq M^{(e)}$ and so $\operatorname{ht} M^{(e)}S > 0$. Also $\tau \subseteq M^{(e)}$ so $\tau T^e M \subseteq M^{(e)}T^e M = 0$, and in particular $\tau T^e h = 0$. $\square$

Lemma 3.3. *Assume that S has a parameter test element. Let M be a $S[T; f]$-submodule of H. If $\operatorname{ht} M^{(e)}S > 0$ for some $e \geq 0$, then $\operatorname{ht} M^{(e')}S > 0$ for all $e' \geq 0$.*

Proof. We assume that $\operatorname{ht} M^{(e)}S > 0$ for some $e \geq 0$ and show that $\operatorname{ht} M^{(0)}S > 0$. Since $M^{(e')} \supseteq M^{(0)}$ for all $e' \geq 0$, we will then have $\operatorname{ht} M^{(e')}S > 0$ for all $e' \geq 0$.

Pick any element $d \in M^{(e)}$ whose image in S is not in any minimal prime and notice that $d \in M^{(j)}$ for all $j \geq e$, i.e., $dST^jM = 0$ for all $j \geq e$.

Let $\mathbf{x} = (x_1, \dots, x_{\dim S})$ be a full system of parameters of S and write $\mathbf{x}^n S$ for the ideal of S generated by $x_1^n, \dots, x_{\dim S}^n$. Now think of H as the direct limit of

$$\frac{S}{\mathbf{x}S} \to \frac{S}{\mathbf{x}^2 S} \to \frac{S}{\mathbf{x}^3 S} \to \dots$$

where the (injective) maps are given by multiplication by $x_1 \cdot \ldots \cdot x_{\dim S}$.

Any element $m \in M$ can be identified with an element represented by $s + \mathbf{x}^i S$ in the direct limit system above, and the fact that $dST^jm = 0$ for all $j \geq e$ shows that $ds^{[p^j]} \in (\mathbf{x}^i S)^{[p^j]}$ for all $j \geq e$ and hence $s \in (\mathbf{x}^i S)^*$. Now for any parameter-test-element c, $c(\mathbf{x}^i S)^* \subseteq \mathbf{x}^i S$, and we deduce that $cs \in \mathbf{x}^i S$. We now see that any parameter-test-element kills M and so is in $M^{(0)}S$, hence $M^{(0)}S$ has positive height.

$\square$

We are now ready to give an explicit description of weak parameter-test-ideals and to do so we need to recall the following notion (cf. section 5 in [K]). For any ideal $L \subseteq R$ and $u \in R$ we define $L^{\star u}$ to be the smallest ideal A containing L with the property that $uA \subseteq A^{[p]}$.

Theorem 3.4. *Let $c \in R$ be such that its image in S is a test element. For all $e \geq 0$, the p^e-weak parameter test ideal $\overline{\tau}_e$ of S is given by*

$$\left(\left((cJ+I)^{\star u}\right)^{[p^e]} :_R u^{\nu_e} J\right) S.$$

Proof. Write $L = (cJ+I)^{\star u}$. Notice that L is an E_S-ideal and that since $c \in ((cJ+I)^{\star u} : J)$, we have $\operatorname{ht} LS > 0$. Now

$$\begin{aligned} \overline{\tau}_e &= \cap\{(M^{(e)}S \,|\, M \subseteq H \text{ is an } S[T;f] \text{ submodule, } \operatorname{ht} M^{(e)} > 0\} \\ &= \cap\{\left(A^{[p^e]} :_R u^{\nu_e} J\right) S \,|\, A \subseteq J \text{ is an } E_S\text{-ideal, } \operatorname{ht}\left(A^{[p^e]} :_R u^{\nu_e} J\right) S > 0\} \end{aligned}$$

and $\left(L^{[p^e]} :_R u^{\nu_e} J\right) S$ is one of the ideals in this intersection, hence $\overline{\tau}_e \subseteq \left(L^{[p^e]} :_R u^{\nu_e} J\right) S$.

Now let $A \subseteq J$ be any E_S-ideal for which $\operatorname{ht}\left(A^{[p^e]} :_R u^{\nu_e} J\right) S > 0$. Lemma 3.3 implies that $\operatorname{ht}(\operatorname{ann}_{E_S} A)^{(0)} = \operatorname{ht} AS > 0$ and since the image of c in S is in $\overline{\tau}_0 \subseteq (A : J)S$, we have $cJ \subseteq A$. Proposition 5.5 in [K] now implies that $L \subseteq A$ and hence that $\left(L^{[p^e]} :_R u^{\nu_e} J\right) \subseteq \left(A^{[p^e]} :_R u^{\nu_e} J\right)$. We conclude that $\left(L^{[p^e]} :_R u^{\nu_e} J\right) S \subseteq \overline{\tau}_e$. $\square$

Corollary 3.5. *Let $\overline{\tau}$ be the union of the ascending chain $\{\overline{\tau}_e \,|\, e \geq 0\}$. If $\overline{\tau}_i = \overline{\tau}_{i+1}$ for some $i \geq 0$ then $\overline{\tau} = \overline{\tau}_i$.*

Proof. Write $L = (cJ+I)^{\star u}$ and let $M = \operatorname{ann}_{E_S} L$. Theorem 3.4 together with Theorem 2.4(b) imply that $\operatorname{gr-ann} M = \oplus_{e\geq 0} \overline{\tau}_e T^e$ and the result follows from Theorem 2.4(d). $\square$

We can translate Theorem 3.4 above to an algorithm as follows.

(1) Given R and I, compute the $u \in (I^{[p]} :_R I)$ corresponding to the Frobenius map on E_S which makes Υ into an homomorphism of $S[T;f]$ modules and also find $J = \ker \Upsilon$ (cf. §7 in [K]).
(2) Find a single parameter test element c (e.g., by inspecting the Jacobian of I (cf. Chapter 2 in [H])).
(3) Compute $(cJ+I)^{\star u}$ (cf. §5 in [K]).
(4) Output the p^e-weak parameter test ideal $\left(\left((cJ+I)^{\star u}\right)^{[p^e]} :_R u^{\nu_e} J\right) S$.

4. Quasimaximal filtrations

Throughout this section we consider a fixed $S[T;f]$-module structure of E_S corresponding to a fixed $u \in (I^{[p]} : I)$.

As in section 4 of [L], for any $S[T;f]$ module M we write M_{red} for $M/\operatorname{Nil}(M)$ and M^* for the $S[T;f]$-submodule $\cap_{e\geq 0} ST^e M$ of M. We note that if M is Artinian as an S-module, there exists an $\alpha \gg 0$ such that

$$\begin{aligned}
(M_{\mathrm{red}})^* &= \frac{(\cap_{e\geq 0} ST^e M) + \operatorname{Nil}(M)}{\operatorname{Nil}(M)} \\
&= \frac{ST^\alpha M + \operatorname{Nil}(M)}{\operatorname{Nil}(M)} \\
&\cong \frac{ST^\alpha M}{\operatorname{Nil}(M) \cap (ST^\alpha M)} \\
&= \frac{\cap_{e\geq 0} ST^e M}{\operatorname{Nil}(M) \cap (\cap_{e\geq 0} ST^e M)} \\
&= (M^*)_{\mathrm{red}}
\end{aligned}$$

and denote both of these M^*_{red}. We also recall the following:

Definition 4.1. A filtration $0 = M_0 \subset \cdots \subset M_s = M$ of an $S[T;f]$ module M is called *quasimaximal* if for all $1 \leq i \leq s$ the modules $(M_i/M_{i-1})^*_{\mathrm{red}}$ are non-zero simple $S[T;f]$ modules.

Artinian $S[T;f]$ modules have quasimaximal filtrations; their lengths and simple factors are invariants of the module (cf. section 4 in [L]).

In this section we study quasimaximal filtrations of E_S and in doing so we introduce an operation on E_S-ideals which will be a key ingredient for obtaining the results of the next section. We start with a description of such filtrations in general.

Definition 4.2. Let M be an $S[T;f]$ module. We define $\mathcal{A}(M)$ to be the set of all $S[T;f]$-submodules $N \subseteq M$ with the property that $\operatorname{Nil}(M/N) = 0$.

Theorem 4.3. *Let M be an $S[T;f]$ module which is Artinian as an S-module. Let $0 \subseteq N_1 \subsetneq \cdots \subsetneq N_s = M$ be a chain with $N_1, \ldots, N_s \in \mathcal{A}(M)$ which is saturated in the sense that for all $1 \leq i \leq s-1$, there is no element in $\mathcal{A}(M)$ strictly between N_i and N_{i+1} and there is no element in $\mathcal{A}(M)$ strictly contained in N_1. Then $0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_s = M$ is a quasimaximal filtration of M whenever $N_1 \neq 0$ and $N_1 \subsetneq \cdots \subsetneq N_s = M$ is a quasimaximal filtration of M whenever $N_1 = 0$.*

Proof. Fix any $1 \leq i \leq s-1$. We have $\operatorname{Nil}(N_{i+1}/N_i) \subseteq \operatorname{Nil}(M/N_i) = 0$ hence $(N_{i+1}/N_i)^*_{\mathrm{red}} = (N_{i+1}/N_i)^*$.

Pick any $S[T;f]$-submodule $A \subseteq M$ such that $N_i \subseteq A \subseteq N_{i+1}$ and let B be the $S[T;f]$ submodule of M for which $\operatorname{Nil}(M/A) = B/A$. We have

$$\operatorname{Nil}(M/B) = \operatorname{Nil}\left(\frac{M/A}{B/A}\right) = \operatorname{Nil}\left(\frac{M/A}{\operatorname{Nil}(M/A)}\right) = 0$$

so $B \in \mathcal{A}(M)$. Also, the natural surjection $M/A \twoheadrightarrow M/N_{i+1}$ maps $\operatorname{Nil}(M/A) = B/A$ into $\operatorname{Nil}(M/N_{i+1}) = 0$ hence $B \subseteq N_{i+1}$. Now $N_i \subseteq B \subseteq N_{i+1}$ and the saturation of our chain implies that either $B = N_i$ (in which case $A = N_i$) or $B = N_{i+1}$.

We now show that $(N_{i+1}/N_i)^* = (N_{i+1}^* + N_i)/N_i$ is simple. Pick any sub-$S[T;f]$-module A/N_i of $(N_{i+1}/N_i)^*$ where $A \subseteq M$ is an $S[T;f]$ submodule of M containing N_i for which $A/N_i \subseteq (N_{i+1}/N_i)^*$. Now $(A/N_i)^* \subseteq (N_{i+1}/N_i)^*$; if $(A/N_i)^* = (A^* + N_i)/N_i = 0$, then $A^* \subseteq N_i$ and $A/N_i \subseteq \mathrm{Nil}(M/N_i) = 0$. Assume that $(A/N_i)^* \neq 0$ and let B be as in the previous paragraph, i.e., $\mathrm{Nil}(M/A) = B/A$. Since $ST^eB \subseteq A$ for all $e \gg 0$, we have $B^* = A^*$, hence $(A/N_i)^* = (B/N_i)^* = (N_{i+1}/N_i)^*$. Now $(N_{i+1}/N_i)^* = (A/N_i)^* \subseteq A/N_i \subseteq (N_{i+1}/N_i)^*$ so $A/N_i = (N_{i+1}/N_i)^*$.

It remains to show that, if $N_1 \neq 0$,

$$(N_1)^*_{\mathrm{red}} = \left(\frac{N_1}{N_1 \cap \mathrm{Nil}(M)}\right)^*$$

is simple. To simplify notation, write $N = N_1$. Pick any $S[T;f]$-submodule A of M for which $N \cap \mathrm{Nil}(M) \subseteq A \subseteq N$, and, as before, write $\mathrm{Nil}(M/A) = B/A$ for an $S[T;f]$-submodule B of M. Again we have $B \in \mathcal{A}(M)$ and $B \subseteq N$, so $B = N$. Pick any $(A/N \cap \mathrm{Nil}(M))^* \subseteq (N/N \cap \mathrm{Nil}(M))^*$ and assume $(A/N \cap \mathrm{Nil}(M))^* \neq 0$; again we have $A^* = B^*$ and $(N/N \cap \mathrm{Nil}(M))^* = (A/N \cap \mathrm{Nil}(M))^* \subseteq A/N \cap \mathrm{Nil}(M) \subseteq (N/N \cap \mathrm{Nil}(M))^*$ so $A/(N \cap \mathrm{Nil}(M)) = (N/N \cap \mathrm{Nil}(M))^*$. □

We now produce quasimaximal filtrations of E_S when it is T-torsion free. These are described in terms of prime E_S-ideals. Recall that in this case the set of E_S ideals coincides with the set of E_S-special ideals (cf. §6 in [K]) and that this set is finite (cf. Theorem 3.10 in [S1]).

Corollary 4.4. *Assume that E_S is T-torsion free and let $P_1, \dots, P_n$ be all its prime E_S-ideals ordered so that $P_i \not\subseteq P_j$ for all $1 \leq i < j \leq n$. The chain*

$$0 \subset \mathrm{ann}_{E_S} P_1 \subset \cdots \subset \mathrm{ann}_{E_S} \bigcap_{j=1}^{i} P_j \subset \cdots \subset \mathrm{ann}_{E_S} \bigcap_{j=1}^{n} P_j \subset E_S.$$

is a quasimaximal filtration of E_S. Therefore, the set of annihilators of the factors of any quasimaximal filtration of E_S is $\{P_1, \dots, P_n\}$.

Proof. Notice that the ordering above of $\mathcal{I} = \{P_1, \dots, P_n\}$ can always be achieved: start with $P_{i_1}, \dots, P_{i_{n_1}}$ maximal with respect to inclusion in $\mathcal{I}$, then list $P_{i_{n_1}}, \dots, P_{i_{n_2}}$ maximal with respect to inclusion in $\mathcal{I} \setminus \{P_{i_1}, \dots, P_{i_{n_1}}\}$, etc.

Write $A_i = \mathrm{ann}_{E_S} \bigcap_{j=1}^{i} P_j$ for all $0 \leq i \leq n$; notice that our ordering guarantees that these form a strictly ascending chain. For all $1 \leq i \leq n$, A_i is a graded-annihilator submodule of E_S; since E_S is T-torsion free, so is E_S/A_i and $A_i \in \mathcal{A}(E_S)$.

Now any $S[T;f]$-submodule between A_{i-1} and A_i would be a graded-annihilator submodule of the form $B = \mathrm{ann}_{E_S} J$ where $J = P_{j_1} \cap \cdots \cap P_{j_m}$ with $i \leq j_1, \dots, j_m \leq n$ is a proper E_S-special and

$$P_1 \cap \cdots \cap P_{i-1} \cap P_i \subseteq J \subseteq P_1 \cap \cdots \cap P_{i-1}.$$

The first inclusion above shows that for all $1 \leq k \leq m$, P_{i_k} contains one of $P_1, \dots, P_i$ and our ordering then shows that $i_k \leq i$. The second inclusion above now shows that either

$J = P_1 \cap \cdots \cap P_{i-1} \cap P_i$ or $P_1 \cap \cdots \cap P_{i-1}$, i.e., $B = A_{i-1}$ or $B = A_i$. We deduce that the factors A_i/A_{i-1} are simple for all $1 \leq i \leq n$ and so our chain of of modules in $\mathcal{A}(E_S)$ is saturated. The result now follows from Theorem 4.3. □

The rest of this section will describe quasimaximal filtrations of E_S in the presence of T-torsion.

Proposition 4.5. *For any E_S-ideal $J \subseteq R$, and any $e \geq 0$ we have*

$$I_e(u^{\nu_e}J) \supseteq I_{e+1}(u^{\nu_{e+1}}J).$$

Proof. First, $u^{\nu_{e+1}}J = u^{p\nu_e}uJ \subseteq u^{p\nu_e}J^{[p]}$, so

$$I_{e+1}\left(u^{\nu_{e+1}}J\right) \subseteq I_{e+1}\left(u^{p\nu_e}J^{[p]}\right).$$

Now $I_e(u^{\nu_e}J)^{[p^{e+1}]} \supseteq (u^{\nu_e}J)^{[p]} = u^{p\nu_e}J^{[p]}$ and the minimality of $I_{e+1}\left(u^{p\nu_e}J^{[p]}\right)$ implies that $I_{e+1}\left(u^{p\nu_e}J^{[p]}\right) \subseteq I_e(u^{\nu_e}J)$. □

For any E_S-ideal J the sequence $\{I_e(u^{\nu_e}J)\}_{e\geq 0}$ is decreasing and we can introduce the following definition.

Definition 4.6. For any E_S-ideal $J \subseteq R$ let

$$J^{\sharp u} = \bigcap_{e\geq 0} I_e(u^{\nu_e}J) + I.$$

Notice that $R^\sharp = \bigcap_{e\geq 0} I_e(u^{\nu_e}R) + I$ defines the submodule of nilpotent elements, i.e., $\mathrm{Nil}(E_S) = \mathrm{ann}_{E_S} R^\sharp$ (cf. Theorem 4.6 in [K]).

Lemma 4.7. *For any E_S-ideal $J \subseteq R$, $J^{\sharp u}$ is an E_S-ideal.*

Proof. It is enough to show that for all $e \geq 0$, $uI_e(u^{\nu_e}J) \subseteq I_{e+1}(u^{\nu_{e+1}}J)^{[p]}$. Now

$$\left(I_{e+1}(u^{\nu_{e+1}}J)^{[p]}\right)^{[p^e]} = I_{e+1}(u^{\nu_{e+1}}J)^{[p^{e+1}]} \supseteq u^{\nu_{e+1}}J$$

so $I_{e+1}(u^{\nu_{e+1}}J)^{[p]} \supseteq I_e(u^{\nu_{e+1}}J) = I_e(u^{p^e}u^{\nu_e}J)$ so it is enough to show that for any $a \in R$ and any ideal $B \subseteq R$ we have $I_e(a^{p^e}B) = aI_e(B)$.

Now $a^{p^e}B \subseteq I_e(a^{p^e}B)^{[p^e]}$ so

$$B \subseteq \left(I_e(a^{p^e}B)^{[p^e]} :_R a^{p^e}\right) = \left(I_e(a^{p^e}B) :_R a\right)^{[p^e]}$$

and so $I_e(B) \subseteq \left(I_e(a^{p^e}B) :_R a\right)$, and, therefore, $aI_e(B) \subseteq I_e(a^{p^e}B)$. On the other hand, $a^{p^e}B \subseteq (aI_e(B))^{[p^e]}$, so $I_e(a^{p^e}B) \subseteq aI_e(B)$. □

Theorem 4.8. *Let M be an $S[T;f]$-submodule of E_S and write $M = \mathrm{ann}_{E_S} J$ for an E_S-ideal J. Then* $\mathrm{Nil}(E_S/M) = \mathrm{ann}_{E_S} J^{\sharp u}/M$.

Proof. Let N_e be the $S[T;f]$-submodule of E_S/M consisting of all elements killed by T^e.

An application of the functor Δ^e (cf. section 4 in [K]) to the short exact sequence $0 \to M \to E_S \to E_S/M \to 0$ yields the following short exact sequence in $\mathcal{D}^e$

(3)
$$\begin{tikzcd} 0 \arrow[r] & \frac{J}{I} \arrow[r] \arrow[d, "u^{\nu_e}"] & \frac{R}{I} \arrow[r] \arrow[d, "u^{\nu_e}"] & \frac{R}{J} \arrow[r] \arrow[d, "u^{\nu_e}"] & 0\ . \\ 0 \arrow[r] & \frac{J^{[p^e]}}{I^{[p^e]}} \arrow[r] & \frac{R}{I^{[p^e]}} \arrow[r] & \frac{R}{J^{[p^e]}} \arrow[r] & 0 \end{tikzcd}$$

Write $J_e = I_e(u^{\nu_e}J) + I$ and consider the following exact sequence in $\mathcal{D}^e$

(4)
$$\begin{tikzcd} \frac{J}{I} \arrow[r] \arrow[d, "u^{\nu_e}"] & \frac{J}{J_e} \arrow[r] \arrow[d, "u^{\nu_e}"] & 0\ . \\ \frac{J^{[p^e]}}{I^{[p^e]}} \arrow[r] & \frac{J^{[p^e]}}{J_e^{[p^e]}} \arrow[r] & 0 \end{tikzcd}$$

Write $N'_e = \Psi^e\left(\frac{J}{J_e} \xrightarrow{u^{\nu_e}} \frac{J^{[p^e]}}{J_e^{[p^e]}}\right)$ and note that it is an $S[T;f]$-submodule of E_S/M. The definition of J_e implies that the rightmost map in (4) is zero, hence $T^e N'_e = 0$ so $N'_e \subseteq N_e$. On the other hand, an application of Δ^e to the exact sequence $0 \to N_e \to E_S/M$ yields an exact sequence in $\mathcal{D}^e$

(5)
$$\begin{tikzcd} \frac{J}{I} \arrow[r] \arrow[d, "u^{\nu_e}"] & \frac{J}{L_e} \arrow[r] \arrow[d, "u^{\nu_e}"] & 0\ . \\ \frac{J^{[p^e]}}{I^{[p^e]}} \arrow[r] & \frac{J^{[p^e]}}{L_e^{[p^e]}} \arrow[r] & 0 \end{tikzcd}$$

for some E_S-ideal L_e such that $I \subseteq L_e \subseteq J$ and for which $u^{\nu_e}J \subseteq L_e^{[p^e]}$. Now the minimality of $J_e = I_e(u^{\nu_e}J) + I$ implies that $I_e(u^{\nu_e}J) \subseteq L_e$ and hence

$$N_e = \left(\frac{J}{L_e}\right)^{\vee} = \frac{\operatorname{ann}_E L_e}{M} \subseteq \frac{\operatorname{ann}_E J_e}{M} = N'_e$$

and we deduce that $N_e = N'_e$.

We now conclude the proof by observing that

$$\operatorname{Nil}(E_S/M) = \bigcup_{e\geq 0} N_e = \bigcup_{e\geq 0} N'_e = \bigcup_{e\geq 0} \frac{\operatorname{ann}_{E_S} J_e}{M} = \frac{\operatorname{ann}_{E_S} \bigcap_{e\geq 0} J_e}{M} = \frac{\operatorname{ann}_{E_S} J^{\sharp u}}{M}.$$

$\square$

Corollary 4.9. *For any E_S-ideal $J \subseteq R$, $E_S/\operatorname{ann}_{E_S} J^{\sharp u}$ is T-torsion free and $\left(J^{\sharp u}\right)^{\sharp u} = J^{\sharp u}$.*

Definition 4.10. We define

$$\mathcal{I}^\sharp = \left\{ J^{\sharp u} \,|\, J \subseteq R \text{ is an } E_S \text{ ideal} \right\}$$

and call a chain $J_0^{\sharp u} \subset J_1^{\sharp u} \subset \cdots \subset J_\ell^{\sharp u}$ of ideals in $\mathcal{I}^\sharp$ $\sharp$-*saturated* if it cannot be refined by adding an ideal in $\mathcal{I}^\sharp$.

Theorem 4.11. *Let $I = 0^{\sharp u} = J_0^{\sharp u} \subset J_1^{\sharp u} \subset \cdots \subset J_\ell^{\sharp u} = R^{\sharp u}$ be a $\sharp$-saturated chain. Then*

$$0 \subset J_{\ell-1}^{\sharp u} \subset \cdots \subset \operatorname{ann}_{E_S} J_1^{\sharp u} \subset \operatorname{ann}_{E_S} J_0^{\sharp u} = E_S$$

is a quasi-maximal filtration of E_S.

Proof. Notice that $\mathcal{A}(E_S) = \{\operatorname{ann}_{E_S} J \,|\, J \in \mathcal{I}^\sharp\}$. Any finite strictly ascending chain in $\mathcal{A}(E_S)$ can be refined to saturated chain and all these have the same length, namely the quasilength of E_S (cf. Theorem 4.6 in [L]). So finite saturated chains as in the statement of the theorem do exist and now the theorem follows from Theorem 4.3. □

The ideals $J^{\sharp u}$ will play a central role in calculating tight closure in E_S as described in the following section.

5. Tight closure in E_S

In this section we give an explicit description of the tight closure of certain submodules of E_S, including $0^*_{E_S}$, which holds whenever, the S-algebra $\mathcal{F}(E_S)$ is generated by one element. The class of complete local rings S with this property includes those which are quasi-Gorenstein, but it is strictly larger than this as is illustrated by the example at the end of this section.

We shall henceforth use the natural isomorphism $\mathcal{F}^e(M) \cong \operatorname{Hom}_S(ST^e \otimes_S M, M)$ which maps a $\phi \in \mathcal{F}^e(M)$ to the S-linear map $\widetilde{\phi} : ST^e \otimes M \to M$ determined by $\widetilde{\phi}(s \otimes m) = s\phi(m)$ (cf. section 3 in [LS]). Conversely, the element $\widetilde{\phi} : ST^e \otimes M \to M$ corresponds under this isomorphism to the map $\phi \in \mathcal{F}^e(M)$ given by $\phi(m) = \widetilde{\phi}(1 \otimes m)$. We shall henceforth identify these two S-modules using this notation.

We can think of the tight closure of ideals $L \subseteq S$ as the set of all elements $s \in S$ such that for some $c \in S$ not in any minimal prime we have $c\phi(s) \in S\phi(L)$ for all $e \gg 0$ and all $\phi \in \mathcal{F}^e(S)$. This is because for each $e \geq 0$, $\mathcal{F}^e(S)$ is generated by the e^{th} iterated Frobenius map on S. Our first aim is to show that this also yields the tight closure of submodules of E_S, and to do so we shall need weak test elements for testing tight closure in this setup.

Definition 5.1. Let M be an S-module and let $N \subseteq M$ be an S-submodule. We call $c \in S$ not in any minimal prime a p^η-weak test element for the pair (N, M) if $a \in N^*_M$ if an only if $c \otimes a \in ST^e \otimes M$ is in the image of $ST^e \otimes N$ in $ST^e \otimes M$ for all $e \geq \eta$. Henceforth $N_M^{[p^e]}$ (or just $N^{[p^e]}$ when it will not lead to confusion) will denote the image of $ST^e \otimes N$ in $ST^e \otimes M$.

These test elements mentioned in the definition above are known to exist when S is F-pure (cf. section 3 in [S2]), and I believe they exist in much wider generality.

Proposition 5.2. *Let N be any S-submodule of E_S, let $c \in S$ and fix an $a \in M$. For all $e \geq 0$, $c \otimes a \in ST^e \otimes E_S$ is in $N^{[p^e]}$ if and only if for all $\phi \in \mathcal{F}^e(E_S)$ we have $c\phi(a) \in S\phi(N)$.*

Consequently, if c is a p^η-weak test element for the pair (N, E_S) then $a \in N^*_{E_S}$ if and only if for all $e \geq \eta$ and all $\phi \in \mathcal{F}^e(E_S)$ we have $c\phi(a) \in S\phi(N)$.

Proof. Note that for all $\phi \in \mathcal{F}^e(E_S)$ we have $\widetilde{\phi}(N^{[p^e]}) = S\phi(N)$.

Assume first that $c \otimes a \in N^{[p^e]}$. Now for all $\phi \in \mathcal{F}^e(E_S)$ we have

$$c\phi(a) = \widetilde{\phi}(c \otimes a) \subseteq \widetilde{\phi}\left(N^{[p^e]}\right) = S\phi(N).$$

Assume now that $c\phi(a) \in S\phi(N)$ for all $\phi \in \mathcal{F}^e(E_S)$. Let $M \subseteq ST^e \otimes_S E_S$ be the S-submodule generated by $N^{[p^e]}$ and $c \otimes a$. The inclusion above yields a surjection $\operatorname{Hom}_S(ST^e \otimes_S E_S, E_S) \twoheadrightarrow \operatorname{Hom}_S(M, E_S)$; we now recall that $\mathcal{F}^e(E_S) = \operatorname{Hom}_S(ST^e \otimes_S E_S, E_S)$ and deduce that for all $\widetilde{\phi} \in \operatorname{Hom}_S(ST^e \otimes_S E_S, E_S)$ we have $\widetilde{\phi}(M) = \widetilde{\phi}(N^{[p^e]} + S(c \otimes a)) = S\phi(N) = \widetilde{\phi}(N^{[p^e]})$.

If $c \otimes a \notin N^{[p^e]}$ we can find a non-zero $\overline{\psi} \in \operatorname{Hom}_S(M/N^{[p^e]}, E_S)$. The short exact sequence

$$0 \to \operatorname{Hom}_S(\frac{M}{N^{[p^e]}}, E_S) \to \operatorname{Hom}_S(M, E_S) \to \operatorname{Hom}_S(N^{[p^e]}, E_S) \to 0$$

enables us to identify $\overline{\psi}$ with a non-zero $\psi \in \operatorname{Hom}_S(M, E_S)$ for which $\psi(N^{[p^e]}) = 0$. Since E_S is injective, we can extend ψ to an element $\widetilde{\psi} \in \operatorname{Hom}_S(ST^e \otimes_S E_S, E_S)$. Now $\widetilde{\psi}(c \otimes m) \neq 0$, otherwise $\overline{\psi} = 0$, and hence $\widetilde{\psi}(M) \neq 0$ and so is not equal to $\widetilde{\psi}(N^{[p^e]}) = 0$, contradicting the conclusion of the previous paragraph.

The final conclusion follows directly from the definition of weak test elements. □

The proposition above gives a method for translating the calculation of the tight closure of an S-submodule $N \subseteq E_S$ to a calculation involving ideals of R as follows. Assume $c \in S$ be a p^η-weak test element for the pair (N, E_S). Fix an $e \geq \eta$, $\phi \in \mathcal{F}^e(E_S)$ and the corresponding $S[\Theta; f^e]$-module structure on E_S corresponding to $v \in (I^{[p^e]} : I)$. Define $N_\phi = \{m \in E_S \,|\, c\Theta m \in S\Theta N\}$ and write $N_\phi = \operatorname{ann}_{E_S} L_\phi$ for some ideal $L_\phi \subseteq R$. Notice that N_ϕ is the largest submodule of E_S with the property $cS\Theta N_\phi \subseteq S\Theta N$, i.e., $c \operatorname{ann}_{E_S}(0 :_R S\Theta N_\phi) \subseteq \operatorname{ann}_{E_S}(0 :_R S\Theta N)$ which, using the $S[\Theta; f^e]$-module analogue of Theorem 2.4(b), translates to $c \operatorname{ann}_{E_S}(L_\phi^{[p^e]} : v) \subseteq \operatorname{ann}_{E_S}(J^{[p^e]} : v)$, or, equivalently, $(L_\phi^{[p^e]} : v) \supseteq c(J^{[p^e]} : v)$, i.e., $L_\phi^{[p^e]} \supseteq cv(J^{[p^e]} : v)$. We deduce that L_ϕ is the minimal ideal $L \subseteq R$ containing I for which $L^{[p^e]} \supseteq cv(J^{[p^e]} : v)$, i.e., $L_\phi = I_e\left(cv(J^{[p^e]} : v)\right) + I$. We can now express $N^*_{E_S}$ as the annihilator in E_S of the sum of all these ideals L_ϕ.

In some simple cases this gives directly a fairly explicit expression for the tight closure on N. For example, if I is generated by a regular sequence $g_1, \ldots, g_m$ and $N = 0$, then for all $e \geq 0$ we have $(I^{[p^e]} : I) = g^{p^e - 1} + I^{[p^e]}$ where $g = g_1 \cdot \ldots \cdot g_m$ and, if c is a test element for $(0, E_S)$, then

$$0^*_{E_S} = \operatorname{ann}_{E_S} \sum_{e \geq 0} I_e(cg^{p^e - 1}) + I.$$

The rest of this section applies Proposition 5.2 under additional hypothesis to produce explicit expressions for $N^*_{E_S}$: we shall first restrict our attention to $\mathcal{F}(E_S)$ submodules $N \subseteq E_S$ (which includes the interesting case where $N=0$) and later we shall impose the additional condition that the S-algebra $\mathcal{F}(E_S)$ is generated by one element.

Proposition 5.3. *Fix any $S[\Theta; f^\eta]$-module structure on E_S. Let $c \in R$ and let $Z = \operatorname{ann}_{E_S} J$ be an $S[\Theta; f^\eta]$-submodule, where $I \subseteq J \subseteq R$ is an ideal. Let $Y = \operatorname{ann}_{E_S} L$ be the largest $S[\Theta; f^\eta]$-submodule of E_S contained in $\operatorname{ann}_{E_S} cJ$ where $cJ \subseteq L \subseteq R$ is an ideal. Choose a positive integer j_0 such that $\Theta^{j_0} \operatorname{Nil}(E_S/Y) = 0$. Write*

$$M = \left\{ m \in E_S \,|\, c\Theta^j m \in Z \text{ for all } j \geq j_0 \right\}.$$

Then M is the preimage in E_S of $\operatorname{Nil}(E_S/Y)$

Proof. Clearly, if $m + Y \in \operatorname{Nil}(E_S/Y)$ then for all $j \geq j_0$ we have $L\Theta^j m = 0$ and since $L \supseteq cJ$ we also have $c\Theta^j m \in \operatorname{ann}_{E_S} J = Z$.

Notice that M is an $S[\Theta; f^\eta]$-submodule of E_S. Write $A = (0 :_R S\Theta^{j_0} M)$; Theorem 2.4(d) shows that $\operatorname{ann}_{E_S} A$ is an $S[\Theta; f^\eta]$-submodule of E_S. Furthermore, $cS\Theta^{j_0} M \subseteq \operatorname{ann}_{E_S} J$, i.e., $cJ\Theta^{j_0} M = 0$ and hence $cJ \subseteq A$ implying $L \subseteq A$. Since $S\Theta^{j_0} M \subseteq \operatorname{ann}_{E_S} A$ we have $m + \operatorname{ann}_{E_S} A \in \operatorname{Nil}(E_S/\operatorname{ann}_{E_S} A)$ for all $m \in M$; as $L \subseteq A$ we also have $m + \operatorname{ann}_{E_S} L \in \operatorname{Nil}(E_S/\operatorname{ann}_{E_S} L)$. □

Our next goal is produce an explicit method of calculating tight closure in E_S. The following introduces the main tool.

Definition 5.4. Let $e \geq 0$, fix any $u \in (I^{[p^e]} : I)$ and let $J \subseteq R$ be any ideal containing I.

We write $J^{\star^e u}$ for the smallest ideal L containing J for which $uL \subseteq L^{[p^e]}$ (see section 5 in [K] for a construction of this ideal).

Endow E_S with the structure of an $S[\Theta; f^e]$-module corresponding to u, and let M be an S-submodule of E_S. We define $M^{\star^e}$ to be the largest $S[\Theta; f^e]$-submodule of E_S contained in M.

Note that if $M = \operatorname{ann}_{E_S} J$, $M^{\star^e} = \operatorname{ann}_{E_S} J^{\star^e u}$.

Theorem 5.5. *Suppose that the S-algebra $\mathcal{F}(E_S)$ is generated by one element corresponding to $u \in (I^{[p]} : I)$. Let N be a $S[T; f]$-submodule of E_S and let $Z = \operatorname{ann}_{E_S} J$ be the stable value of the descending chain $\{ST^j N\}_{j \geq 0}$. Assume further that the image of $c \in R$ in S is a weak p^η-test element for the pair (N, E_S) and that η was chosen so large that $Z = ST^\eta N$. We have*

$$N^*_{E_S} = \operatorname{ann}_{E_S} \left((cJ + I)^{\star^\eta u} \right)^{\sharp u}.$$

Proof. Fix the $S[T; f]$-module structure on E_S corresponding to u. In view of Proposition 5.2 and of the fact that for all $e \geq \eta$, $\mathcal{F}^e(E_S) = ST^e$,

$$N^*_{E_S} = \cap_{e \geq \eta} \{ m \in E_S \,|\, cT^e m \in ST^e N \}$$

for all $\eta \geq \eta_0$.

Notice that, if for some $a \in E_S$ and positive integer j the element $c \otimes a \in ST^{j\eta} \otimes E_S$ is in $N^{[p^{j\eta}]}$, then after tensoring on the left with ST^k for $1 \le k \le \eta - 1$ and using the isomorphism $ST^k \otimes ST^{j\eta} \cong ST^{k+j\eta}$, we obtain $c^{p^k} \otimes a \in ST^{j\eta+k} \otimes E_S$ is in $N^{[p^{j\eta+k}]}$ for all $1 \le k \le \eta - 1$ and hence $c^{p^{\eta-1}} \otimes a \in ST^{j\eta+k} \otimes E_S$ is in $N^{[p^{j\eta}+k]}$ for all $1 \le k \le \eta - 1$. For any positive integer j_0, we may replace c with $c^{p^{\eta-1}}$ as a $p^{j_0\eta}$ weak test element and deduce that

$$a \in N^*_{E_S} = \cap_{j \ge j_0} \{m \in E_S \mid cT^{j\eta} m \in ST^{j\eta} N\}.$$

Write $\Theta = T^\eta$ and let $L = (cJ + I)^{\star^\eta u}$. Note that $\operatorname{ann}_{E_S} L$ is the largest $S[\Theta; f^\eta]$-submodule of E_S contained in $\operatorname{ann}_{E_S} cJ$. Pick any positive integer j_0 such that $\Theta^{j_0} \operatorname{Nil}(E_S/\operatorname{ann}_{E_S} L) = 0$.

An application of Proposition 5.3 shows that

$$N^*_{E_S} = \cap_{j \ge j_0} \{m \in E_S \mid c\Theta^j m \in Z\}$$

is the pre-image of $\operatorname{Nil}(E_S/\operatorname{ann}_{E_S} L)$ in E_S, and this is precisely $\operatorname{ann}_{E_S} L^{\sharp u}$. $\square$

One instance when the S-algebra $\mathcal{F}(E_S)$ is generated by one element is when S is Gorenstein, or more generally, when S is quasi-Gorenstein (i.e., $E_S \cong \mathrm{H}^{\dim S}_{\mathfrak{m}S}(S)$) and satisfies Serre's S_2 condition. This is the content of Example 3.6 in [LS]. However, the class of quotients S of R for which the S-algebra $\mathcal{F}(E_S)$ is generated by one element is strictly larger than this. Consider the power series ring $R = \mathbb{K}[\![a, b, c]\!]$, where $\mathbb{K}$ is a field of prime characteristic p, its ideal $I = (ab - bc, bc - b^2, ac - bc)R = (a, b)R \cap (c, b)R \cap (a - c, b - c)R$ and the one dimensional quotient $S = R/I$. We have a minimal resolution

$$0 \to R^2 \xrightarrow{\begin{pmatrix} b & c \\ -a & -c \\ -b & -b \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} ab - bc & bc - b^2 & ac - bc \end{pmatrix}} R \to S \to 0$$

which shows that S is Cohen-Macaulay of type 2, hence non-Gorenstein and not quasi-Gorenstein. For all primes $p \ge 5$,

$$b^{p-1}(b - c)^{p-1}(a - b)^{p-1} \in (I^{[p]} : I)$$

and a calculation with Macaulay2 shows that this element generates the S-module $(I^{[p]} : I)/I^{[p]}$ for all $5 \le p \le 97$.

Corollary 5.6. *Assume that S is equidimensional and quasi-Gorenstein and that it satisfies Serre's S_2 condition. Let $c \in R$ be such that its image in S is a test element for the pair $(0, E_S)$. The test ideal of S is*

$$\operatorname{ann}_S 0^*_{E_S} = ((Rc + I)^{\star u})^{\sharp u} S.$$

Proof. The fact that S is quasi-Gorenstein and equidimensional implies that the finitistic tight closure of 0 in E_S coincides with $0^*_{E_S}$ and hence the test ideal of S is $\operatorname{ann}_S 0^*_{E_S}$ (cf. section 8 in [HH] and Proposition 3.3 in [Sm]). The fact that S satisfies Serre's S_2

condition implies that the S-algebra $\mathcal{F}(E_S)$ is generated by one element. Now the result follows from Theorem 5.5 with $J = R$. $\square$

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Department of Pure Mathematics, University of Sheffield, Hicks Building, Sheffield S3 7RH, United Kingdom

E-mail address: `M.Katzman@sheffield.ac.uk`