International Mathematics Research Notices Advance Access published June 14, 2016
V. Beresnevich *et al.* (2016) "Diophantine Approximation on Manifolds and the Distribution of Rational Points: Contributions to the Convergence Theory",
International Mathematics Research Notices, Vol. 2016, No. 00, pp. 1–24
doi: 10.1093/imrn/rnv389

Diophantine Approximation on Manifolds and the Distribution of Rational Points: Contributions to the Convergence Theory

Victor Beresnevich[1,*], **Robert C. Vaughan**[2], **Sanju Velani**[1], **and Evgeniy Zorin**[1]

[1]Department of Mathematics, University of York, Heslington, York, YO10 5DD, UK and [2]Department of Mathematics, McAllister Building, Pennsylvania State University, University Park, PA 16802-6401, USA

**Correspondence to be sent to: e-mail: victor.beresnevich@york.ac.uk*

In memory of Klaus Roth (October 29, 1925 – November 10, 2015).

In this article, we develop the convergence theory of simultaneous, inhomogeneous Diophantine approximation on manifolds. A consequence of our main result is that if the manifold $\mathcal{M} \subset \mathbb{R}^n$ is of dimension strictly greater than $(n+1)/2$ and satisfies a natural non-degeneracy condition, then $\mathcal{M}$ is of Khintchine type for convergence. The key lies in obtaining essentially the best possible upper bound regarding the distribution of rational points near manifolds.

1 Introduction and Statement of Results

1.1 The setup

Throughout, we suppose that $m \le d$, $n = m + d$ and that $\mathbf{f} = (f_1, \ldots, f_m)$ is defined on $\mathcal{U} = [0,1]^d$. Suppose further that $\partial\mathbf{f}/\partial\alpha_i$ and $\partial^2\mathbf{f}/\partial\alpha_i\partial\alpha_j$ exist and are continuous on $\mathcal{U}$,

Received June 30, 2015; Revised December 15, 2015; Accepted December 23, 2015
Communicated by Umberto Zannier

and that there is an $\eta > 0$ such that for all $\boldsymbol{\alpha} \in \mathcal{U}$

$$\left| \det \left(\frac{\partial^2 f_j}{\partial \alpha_1 \partial \alpha_i}(\boldsymbol{\alpha}) \right)_{\substack{1\le i\le m\\ 1\le j\le m}} \right| \ge \eta. \tag{1.1}$$

Throughout $\mathbb{R}^+ = [0, +\infty)$ is the set of non-negative real numbers. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\psi(r) \to 0$ as $r \to \infty$ and $\boldsymbol{\theta} = (\boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \mathbb{R}^d \times \mathbb{R}^m$. Now for a fixed $q \in \mathbb{N}$, consider the set

$$\mathcal{R}(q, \psi, \boldsymbol{\theta}) := \left\{ (\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^d \times \mathbb{Z}^m : \begin{array}{l} \frac{\mathbf{a}+\boldsymbol{\lambda}}{q} \in \mathcal{U}, \\ |q\mathbf{f}\left(\frac{\mathbf{a}+\boldsymbol{\lambda}}{q}\right) - \boldsymbol{\gamma} - \mathbf{b}| < \psi(q) \end{array} \right\} \tag{1.2}$$

and let

$$A(q, \psi, \boldsymbol{\theta}) := \#\mathcal{R}(q, \psi, \boldsymbol{\theta}) .$$

The map $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ naturally gives rise to the d-dimensional manifold

$$\mathcal{M}_{\mathbf{f}} := \{(\alpha_1, \ldots, \alpha_d, f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})) \in \mathbb{R}^n : \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathcal{U}\} \tag{1.3}$$

embedded in $\mathbb{R}^n$. Recall that by the Implicit Function Theorem, any smooth manifold $\mathcal{M}$ can be locally defined in this manner; i.e., with a Monge parametrization. The upshot is that, $A(q, \psi, \boldsymbol{\theta})$ counts the number of shifted rational points

$$\left(\tfrac{a_1+\lambda_1}{q}, \ldots, \tfrac{a_d+\lambda_d}{q}, \tfrac{b_1+\gamma_1}{q}, \ldots, \tfrac{b_m+\gamma_m}{q}\right) \in \mathbb{R}^n$$

that lie (up to an absolute constant) within the $\psi(q)/q$-neighbourhood of $\mathcal{M}_{\mathbf{f}}$. Before stating our counting results, it is worthwhile to compare condition (1.1) imposed on the Jacobian of $\mathbf{f}$ with that of non-degeneracy as defined by Kleinbock and Margulis [11] in their pioneering work. In this article, they prove the Baker–Sprindžuk "extremality" conjecture in the theory of Diophantine approximation on manifolds.

The above map $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m : \boldsymbol{\alpha} \mapsto \mathbf{f}(\boldsymbol{\alpha}) = (f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha}))$ is said to be *l-non–degenerate at $\boldsymbol{\alpha} \in \mathcal{U}$* if there exists some integer $l \ge 2$ such that $\mathbf{f}$ is l times continuously differentiable on some sufficiently small ball centred at $\boldsymbol{\alpha}$ and the partial derivatives of $\mathbf{f}$ at $\boldsymbol{\alpha}$ of orders 2 to l span $\mathbb{R}^m$. The map $\mathbf{f}$ is called *non–degenerate* if it is l-non–degenerate at almost every (in terms of d–dimensional Lebesgue measure) point in $\mathcal{U}$; in turn the manifold $\mathcal{M}_{\mathbf{f}}$ is also said to be non–degenerate. Non-degenerate manifolds are smooth sub-manifolds of $\mathbb{R}^n$ which are sufficiently curved so as to deviate from any hyperplane at a polynomial rate see [1, Lemma 1(c)]. As is well known [11, p. 341], any real connected analytic manifold not contained in any hyperplane of $\mathbb{R}^n$ is non–degenerate.

It follows from the definition of l-non-degeneracy that condition (1.1) imposed on $\mathbf{f}$ implies that $\mathbf{f}$ is 2-non-degenerate at every point. Although (1.1) is fairly generic, the converse is not always true even if we allow rotations of the coordinate system. The submanifold $(x, y, z_1, \ldots, z_k, x^2, xy, y^2)$ of $\mathbb{R}^{k+5}$ provides a counterexample.

1.2 Results on counting rational points

Throughout, the Vinogradov symbols $\ll$ and $\gg$ will be used to indicate an inequality with an unspecified positive multiplicative constant. If $a \ll b$ and $a \gg b$, we write $a \asymp b$ and say that the two quantities a and b are comparable. Throughout the article, the constants will only depend on the dimensions n and d and the map $\mathbf{f}$.

Observe that for q sufficiently large so that $\psi(q) \le 1/2$, we have that

$$A(q, \psi, \boldsymbol{\theta}) = \# \left\{ \mathbf{a} \in \mathbb{Z}^d : \begin{array}{l} \frac{\mathbf{a}+\boldsymbol{\lambda}}{q} \in \mathcal{U}, \\ \|q\mathbf{f}\big(\frac{\mathbf{a}+\boldsymbol{\lambda}}{q}\big) - \boldsymbol{\gamma}\| < \psi(q) \end{array} \right\} \tag{1.4}$$

where as usual $\|\mathbf{x}\| := \max_{1 \le i \le m} \|x_i\|$ for any $\mathbf{x} \in \mathbb{R}^m$. In particular, when $0 < \psi(q) \le 1/2$, the obvious heuristic argument leads us to the following estimate:

$$A(q, \psi, \boldsymbol{\theta}) \asymp q^n \left(\frac{\psi(q)}{q}\right)^m = \psi(q)^m q^d . \tag{1.5}$$

We establish the following upper bound result.

Theorem 1. Suppose that $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ satisfies (1.1) and $\boldsymbol{\theta} \in \mathbb{R}^n$. Suppose that $0 < \psi(q) \le 1/2$. Then

$$A(q, \psi, \boldsymbol{\theta}) \ll \psi(q)^m q^d + (q\,\psi(q))^{-1/2} q^d \max\{1, \log(q\,\psi(q))\} , \tag{1.6}$$

where the implied constant is independent of q, $\boldsymbol{\theta}$, and ψ but may depend on $\mathbf{f}$. □

The following is a straightforward consequence of the theorem. It states that the upper bound (1.6) coincides with the heuristic estimate if $\psi(q)$ is not too small.

Corollary 1. Suppose that $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ satisfies (1.1) and $\boldsymbol{\theta} \in \mathbb{R}^n$. Suppose that

$$q^{-1/(2m+1)} (\log q)^{2/(2m+1)} \le \psi(q) \le 1/2 .$$

Then for integers $q \geq 2$ we have that

$$A(q, \psi, \boldsymbol{\theta}) \ll \psi(q)^m q^d . \tag{1.7}$$

□

1.3 Results on metric Diophantine approximation

Given a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and a point $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$, let $\mathcal{S}_n(\psi, \boldsymbol{\theta})$ denote the set of $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ for which there exists infinitely many $q \in \mathbb{N}$ such that

$$\|q\mathbf{y} - \boldsymbol{\theta}\| = \max_{1 \leq i \leq n} \|qy_i - \theta_i\| < \psi(q) .$$

In the case that the inhomogeneous part $\boldsymbol{\theta}$ is the origin, the corresponding set $\mathcal{S}_n(\psi) := \mathcal{S}_n(\psi, \mathbf{0})$ is the usual homogeneous set of simultaneously ψ–approximable points in $\mathbb{R}^n$. In the case ψ is $\psi_\tau : r \to r^{-\tau}$ with $\tau > 0$, let us write $\mathcal{S}_n(\tau, \boldsymbol{\theta})$ for $\mathcal{S}_n(\psi, \boldsymbol{\theta})$ and $\mathcal{S}_n(\tau)$ for $\mathcal{S}_n(\tau, \mathbf{0})$. Note that in view of Dirichlet's theorem (n-dimensional simultaneous version), $\mathcal{S}_n(\tau) = \mathbb{R}^n$ for any $\tau \leq 1/n$.

In the general discussion above, we have not made any assumption on ψ regarding monotonicity. Thus, the integer support of ψ need not be $\mathbb{N}$. Throughout, $\mathcal{N} \subset \mathbb{N}$ will denote the integer support of ψ. That is the set of $q \in \mathbb{N}$ such that $\psi(q) > 0$. Regarding the set $\mathcal{S}_n(\psi, \boldsymbol{\theta})$, measure theoretically, this is equivalent to saying that we are only interested in integers q lying in some given set $\mathcal{N}$ such as the set of primes or squares or powers of two. The theory of restricted Diophantine approximation in $\mathbb{R}^n$ is both topical and well developed for certain sets $\mathcal{N}$ of number theoretic interest—we refer the reader to [10, Chp 6] and [3, §12.5] for further details. However, the theory of restricted Diophantine approximation on manifolds is not so well developed.

Armed with Corollary 1, we are able to establish the following convergent statement for the s-dimensional Hausdorff measure $\mathcal{H}^s$ of $\mathcal{M}_\mathbf{f} \cap \mathcal{S}_n(\psi, \boldsymbol{\theta})$. Note that if $s > d = \dim \mathcal{M}_\mathbf{f}$, then $\mathcal{H}^s(\mathcal{M}_\mathbf{f} \cap \mathcal{S}_n(\psi, \boldsymbol{\theta})) = 0$ irrespective of ψ. This follows immediately from the definition of Hausdorff dimension and that fact that

$$\dim(\mathcal{M}_\mathbf{f} \cap \mathcal{S}_n(\psi, \boldsymbol{\theta})) \leq \dim \mathcal{M}_\mathbf{f} .$$

Theorem 2. Let $\boldsymbol{\theta} \in \mathbb{R}^n$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\psi(r) \to 0$ as $r \to \infty$ and

$$\psi(q) \geq q^{-1/(2m+1)} (\log q)^{2/(2m+1)} \qquad \text{for all } q \in \mathcal{N}, \tag{1.8}$$

where as $\mathcal{N} = \{q \in \mathbb{N} : \psi(q) > 0\}$. Let $0 < s \leq d$ and $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ satisfy the following condition

$$\mathcal{H}^s\big(\{\boldsymbol{\alpha} \in \mathcal{U} : \text{the l.h.s. of } (1.1) = 0\}\big) = 0. \tag{1.9}$$

Then

$$\mathcal{H}^s\big(\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\psi, \boldsymbol{\theta})\big) = 0 \qquad \text{whenever} \qquad \sum_{q=1}^{\infty} \left(\tfrac{\psi(q)}{q}\right)^{s+m} q^n \; < \; \infty .$$

□

Remark 1. Recall, that in view of the discussion in §1.1 the condition imposed on **f** in the above theorem and its corollaries below are equivalent to saying that the manifold is 2-non-degenerate everywhere except on a set of Hausdorff s-measure zero. □

Now we consider two special cases of Theorem 2. First suppose the integer support of ψ is along a lacunary sequence. In particular, consider the concrete situation that $\mathcal{N} = \{2^t : t \in \mathbb{N}\}$. The following statement is valid for any $n = d + m$ and to the best of our knowledge is first result of its type even within the setup of planar curves ($d = m = 1$).

Corollary 2. Let $\boldsymbol{\theta} \in \mathbb{R}^n$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\psi(r) \to 0$ as $r \to \infty$ and $\mathcal{N} = \{2^t : t \in \mathbb{N}\}$. Let

$$d - \tfrac{n}{2(m+1)} < s \leq d$$

and assume that $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ satisfies (1.9). Then

$$\mathcal{H}^s\left(\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\psi, \boldsymbol{\theta})\right) = 0 \qquad \textit{if} \qquad \sum_{t=1}^{\infty} \left(2^{-t}\, \psi(2^t)\right)^{s+m} 2^{tn} \; < \; \infty .$$

□

Proof. Consider the auxiliary function

$$\tilde{\psi}(q) = \max\{\psi(q), Cq^{-1/(2m+1)}(\log q)^{2/(2m+1)}\},$$

where $C > 0$ is a sufficiently large constant. Then as is easily verified using the convergence sum condition of Corollary 2

$$\sum_{t=1}^{\infty} \left(2^{-t}\, \tilde{\psi}(2^t)\right)^{s+m} 2^{tn} < \infty$$

and therefore, by Theorem 2, we have that $\mathcal{H}^s\left(\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\tilde{\psi},\boldsymbol{\theta})\right) = 0$. Trivially, we have that $\mathcal{S}_n(\psi,\boldsymbol{\theta}) \subset \mathcal{S}_n(\tilde{\psi},\boldsymbol{\theta})$ and then the required statement follows on using the monotonicity of $\mathcal{H}^s$. ■

Note that (1.9) is always satisfied if $\dim(\{\boldsymbol{\alpha} \in \mathcal{U} : \text{ the l.h.s. of (1.1)} = 0\}) \leq d - \frac{n}{2(m+1)}$.

Let us now consider Theorem 2 under the assumption that ψ is monotonic. Then, without loss of generality, we can assume that $\mathcal{N} = \mathbb{N}$ since otherwise $\psi(q) = 0$ for all sufficiently large q and so $\mathcal{S}_n(\psi,\boldsymbol{\theta})$ is the empty set and there is nothing to prove. Furthermore, we can assume that $\psi(q) \ll q^{-1/n}$ for all $q \in \mathbb{N}$ since otherwise the s-volume sum appearing in the theorem is divergent for $s \leq d$. This is in line with the fact that if $\psi(q) \geq q^{-1/n}$ for all sufficiently large q, then by Dirichlet's theorem we have that $\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\psi) = \mathcal{M}_{\mathbf{f}}$ and so $\mathcal{H}^s(\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\psi)) > 0$ for $s \leq d$. The upshot is that within the context of Theorem 2, for monotonic ψ we can assume that

$$q^{-1/(2m+1)}(\log q)^{2/(2m+1)} \ll \psi(q) < q^{-1/n}.$$

This forces $d > (n+1)/2$.

Corollary 3. Let $\boldsymbol{\theta} \in \mathbb{R}^n$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonic function such that $\psi(r) \to 0$ as $r \to \infty$. Let

$$d > \tfrac{n+1}{2} \quad \text{and} \quad s_0 := \tfrac{dm}{m+1} + \tfrac{n+1}{2(m+1)} < s \leq d$$

and assume that $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ satisfies (1.9). Then

$$\mathcal{H}^s(\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\psi,\boldsymbol{\theta})) = 0 \qquad \text{whenever} \qquad \sum_{q=1}^{\infty} \left(\frac{\psi(q)}{q}\right)^{s+m} q^n < \infty.$$

□

The proof is similar to that of Corollary 2. Note that (1.9) is always satisfied if

$$\dim(\{\boldsymbol{\alpha} \in \mathcal{U} : \text{ l.h.s. of (1.1)} = 0\}) \leq s_0.$$

Also note that the condition $d > (n+1)/2$ guarantees that $s_0 < d$. However, it does mean that the corollary is not applicable when $n = 3$ or $n = 2$. The fact that is not applicable when $n = 2$ is not a concern—see Remark 2 below.

Remark 2. It is conjectured that the conclusion of Corollary 3 is valid for any non-degenerate manifold (i.e., $d \geq 1$) and $\frac{dm}{(m+1)} < s \leq d$ – see for example [2, §8]. For planar curves ($d = m = 1$), this is known to be true [5, 14]. To the best of our knowledge, beyond planar curves, the corollary represents the first significant contribution in favour of the conjecture. □

Remark 3. Corollary 3 together with the definition of Hausdorff dimension implies that if $d > (n+1)/2$, then for $1/n \leq \tau \leq 1/(2m+1)$

$$\dim\left(\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\tau, \boldsymbol{\theta})\right) \leq \tfrac{n+1}{\tau+1} - m\,.$$

□

Remark 4. Corollary 3 with $s = d$ implies that if $d > (n+1)/2$ then

$$|\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\psi, \boldsymbol{\theta})|_{\mathcal{M}_{\mathbf{f}}} = 0 \qquad \text{whenever} \qquad \sum_{q=1}^{\infty} \psi(q)^n < \infty\,, \tag{1.10}$$

where $|\,.\,|_{\mathcal{M}_{\mathbf{f}}}$ is the induced d-dimensional Lebesgue measure on $\mathcal{M}_{\mathbf{f}}$. In other words, it proves that the 2-non-degenerate submanifold $\mathcal{M}_{\mathbf{f}}$ of $\mathbb{R}^n$ with dimension strictly greater than $(n+1)/2$ is of Khintchine-type for convergence [4]. Apart from the planar curve results referred to in Remark 2, the current state of the convergent Khintchine theory is somewhat ad hoc. Either a specific manifold or a special class of manifolds satisfying various constraints is studied. For example, it has been shown that (1) manifolds which are a topological product of at least four non–degenerate planar curves are Khintchine type for convergence [7] as are (2) the so called 2–convex manifolds of dimension $d \geq 2$ [9], and (3) straight lines through the origin satisfying a natural Diophantine condition [12]. □

Remark 5. In view of the conjecture mentioned above in Remark 2, we expect (1.10) to remain valid for any non-degenerate manifold without any restriction on its dimension. Note that it is relatively straightforward to establish that this is indeed the case for almost all $\boldsymbol{\theta}$. Moreover, we do not need to assume that ψ is monotonic or even that $\mathcal{M}_{\mathbf{f}}$ is non-degenerate. In other words, for any C^1 submanifold (By a C^1 submanifold, we mean an immersed manifold into $\mathbb{R}^n$ by a C^1 map, that is, the image of a C^1 map $\mathbf{f} : \mathcal{U} \to \mathbb{R}^n$.) $\mathcal{M}_{\mathbf{f}}$ of $\mathbb{R}^n$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, we have that (1.10) is valid for almost all $\boldsymbol{\theta} \in \mathbb{R}^n$. This is an immediate consequence of the following even more general "doubly metric" result. □

Proposition 1. Let $\mathbf{f} : \mathcal{U} \to \mathbb{R}^n$ be any continuous map. Given $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, let

$$\mathcal{D}(\mathbf{f}, \psi) := \{(\mathbf{x}, \boldsymbol{\theta}) \in \mathcal{U} \times \mathbb{R}^n : \|q\mathbf{f}(\mathbf{x}) - \boldsymbol{\theta}\| < \psi(q) \text{ for i.m. } q \in \mathbb{N}\}$$

and let $| \,.\, |_{d+n}$ denote $(d+n)$-dimensional Lebesgue measure. Then

$$|\mathcal{D}(\mathbf{f},\psi)|_{d+n} = 0 \qquad \text{whenever} \qquad \sum_{q=1}^{\infty} \psi(q)^n \ < \ \infty\,. \tag{1.11}$$

□

Proof. The proposition is pretty much a direct consequence of Fubini's theorem. Without loss of generality, we can assume that $\boldsymbol{\theta}$ is restricted to the unit cube $[0,1]^n$. For $q \in \mathbb{N}$, let

$$\delta_q(\mathbf{x}) := \begin{cases} 1 & \text{if} \quad \|\mathbf{x}\| < \psi(q) \\ 0 & \text{otherwise} \end{cases}$$

and

$$D_q(\mathbf{f},\psi) := \{(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{U} \times [0,1]^n : \delta_q(q\mathbf{f}(\mathbf{x}) - \boldsymbol{\theta}) = 1\}\,.$$

Notice that

$$\mathcal{D}(\mathbf{f},\psi) = \limsup_{q\to\infty} D_q(\mathbf{f},\psi)\,,$$

and that by Fubini's theorem

$$\begin{aligned}|D_q(\mathbf{f},\psi)|_{d+n} &= \int_{\mathcal{U}} \Big(\int_{[0,1]^n} \delta_q(q\mathbf{f}(\mathbf{x}) - \boldsymbol{\theta}) d\boldsymbol{\theta} \Big)\, d\mathbf{x} \\ &= |\mathcal{U}|_d\ (2\,\psi(q))^n = (2\,\psi(q))^n\,.\end{aligned}$$

Hence

$$\sum_{q=1}^{\infty} |D_q(\mathbf{f},\psi)|_{d+n} \ \asymp \ \sum_{q=1}^{\infty} \psi(q)^n \ < \ \infty\,,$$

and the Borel–Cantelli lemma implies the desired measure zero statement. ■

1.4 Restricting to hypersurfaces

As already mentioned, the condition $d > (n+1)/2$ means that Corollary 3 is not applicable when $n = 3$. We now attempt to rectify this. In the case $m = 1$, so that the manifold $\mathcal{M}_{\mathbf{f}}$ associated with $\mathbf{f}$ is a hypersurface, we can do better than Theorem 1 if we assume that $\mathcal{M}_{\mathbf{f}}$ is genuinely curved. More precisely, in place of (1.1) we suppose that there is an $\eta > 0$ such that for all $\boldsymbol{\alpha} \in \mathcal{U}$

$$\left| \det \left(\frac{\partial^2 f}{\partial \alpha_i \partial \alpha_j}(\boldsymbol{\alpha}) \right)_{\substack{1 \le i \le d \\ 1 \le j \le d}} \right| \ge \eta \tag{1.12}$$

where for brevity we have written f for f_1. It is not too difficult to see that this condition imposed on the determinant (Hessian) is valid for spheres but not for cylinders with a flat base. We will refer to the hypersurface $\mathcal{M}_{\mathbf{f}}$ with $\mathbf{f}$ satisfying (1.12) as *genuinely curved*. Throughout the rest of this section we will assume that $m = 1$ and so $d = n - 1$.

Theorem 3. Suppose that $\mathbf{f} : \mathcal{U} \to \mathbb{R}$ satisfies (1.12) and $\boldsymbol{\theta} \in \mathbb{R}^n$. Suppose that $0 < \psi(q) \le 1/2$. Then

$$A(q, \psi, \boldsymbol{\theta}) \ll \psi(q)\, q^d + (q\,\psi(q))^{-d/2} q^d \max\{1, (\log(q\,\psi(q)))^d\} \tag{1.13}$$

where the implied constant is independent of q, $\boldsymbol{\theta}$ and ψ but may depend on $\mathbf{f}$. □

A simple consequence of this theorem is the following analogue of Corollary 1.

Corollary 4. Suppose that $\mathbf{f} : \mathcal{U} \to \mathbb{R}$ satisfies (1.12) and $\boldsymbol{\theta} \in \mathbb{R}^n$. Suppose that

$$q^{-d/(2+d)} (\log q)^{2d/(2+d)} \le \psi(q) \le 1/2\,.$$

Then for integers $q \ge 2$ we have that

$$A(q, \psi, \boldsymbol{\theta}) \ll \psi(q)\, q^d\,. \tag{1.14}$$

□

It is easily seen that Theorem 1 with $m = 1$ and Theorem 3 coincide when $n = 2$ but for $n \ge 3$ the second term on the R.H.S. in (1.13) is smaller than the corresponding term in (1.6). In particular,

$$q^{-d/(2+d)} (\log q)^{2d/(2+d)} < q^{-1/3} (\log q)^{2/3}$$

and so Corollary 4 is stronger than Corollary 1 for f satisfying (1.12). Corollary 4 enables us to obtain the analogue of Theorem 2 for genuinely curved hypersurfaces in which the condition that $\psi(q) \gg q^{-1/(2m+1)} (\log q)^{2/(2m+1)}$ for $q \in \mathcal{N}$ is replaced by

$\psi(q) \gg q^{-d/(2+d)}(\log q)^{2d/(2+d)}$ for $q \in \mathcal{N}$. In turn for monotonic functions, we have the following statement. It represents a strengthening of Corollary 3 in the case of genuinely curved hypersurfaces and is valid when $n = 3$.

Corollary 5. Suppose that $\mathbf{f} : \mathcal{U} \to \mathbb{R}$ and $\boldsymbol{\theta} \in \mathbb{R}^n$. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonic function such that $\psi(r) \to 0$ as $r \to \infty$. Let

$$n \geq 3 \qquad \textit{and} \qquad \tfrac{n-1}{2} + \tfrac{n+1}{2n} < s \leq n-1$$

and assume that

$$\mathcal{H}^s\big(\big\{\boldsymbol{\alpha} \in \mathcal{U} : \text{the l.h.s. of (1.12)} = 0\big\}\big) = 0.$$

Then

$$\mathcal{H}^s\left(\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\psi, \boldsymbol{\theta})\right) = 0 \qquad \text{whenever} \qquad \sum_{q=1}^{\infty} \left(\frac{\psi(q)}{q}\right)^{s+1} q^n \; < \; \infty \, .$$

□

The conjectured lower bound for s above is $(n-1)/2$—see Remark 2 preceding the statement of Corollary 3. The proof of the above corollary is similar to that of Corollary 2.

1.5 Further remarks and other developments

The upper bound results of §1.2 for the counting function $A(q, \psi, \boldsymbol{\theta})$ are at the heart of establishing the convergence results of §1.3. We emphasize that $A(q, \psi, \boldsymbol{\theta})$ is defined for a fixed q and that Theorem 1 provides an upper bound for this function for any q sufficiently large. It is this fact that enables us to obtain convergent results such as Theorem 2 without assuming that ψ is monotonic. While statements without monotonicity are desirable, considering counting functions for a fixed q does prevent us from taking advantage of any potential averaging over q. More precisely, for $Q > 1$ consider the counting function

$$\begin{aligned} N(Q, \psi, \boldsymbol{\theta}) \;&:=\; \#\left\{(q, \mathbf{a}, \mathbf{b}) \in \mathbb{N} \times \mathbb{Z}^d \times \mathbb{Z}^m : \begin{array}{l} Q < q \leq 2Q, \; \frac{\mathbf{a}+\boldsymbol{\lambda}}{q} \in \mathcal{U}\,, \\ |q\mathbf{f}\left(\frac{\mathbf{a}+\boldsymbol{\lambda}}{q}\right) - \boldsymbol{\gamma} - \mathbf{b}| < \psi(q) \end{array} \right\} \\ &= \sum_{Q < q \leq 2Q} A(q, \psi, \boldsymbol{\theta}) \, . \end{aligned} \tag{1.15}$$

If ψ is monotonic, then $\psi(q) \leq \psi(Q)$ for $Q < q \leq 2Q$ and the obvious heuristic "volume" argument leads us to the following estimate:

$$N(Q, \psi, \boldsymbol{\theta}) \ \ll \ \psi(Q)^m \, Q^{d+1} \,. \tag{1.16}$$

Clearly, the upper bound (1.7) for $A(q, \psi, \boldsymbol{\theta})$ as obtained in Corollary 1 implies (1.16). The converse is unlikely to be true. However, for monotonic ψ establishing (1.16) suffices to prove convergence results such as Corollary 3. Indeed, the fact that we have a complete convergence theory for planar curves (see Remark 2 in Section 1.3) relies on the fact that we are able to establish (1.16) with $m = 1 = d$. Note that the counting result obtained in this article for $A(q, \psi, \boldsymbol{\theta})$ is not strong enough to imply any sort of convergent Khintchine type result for planar curves with ψ monotonic. Furthermore, it is worth pointing out that averaging over q when considering $N(Q, \psi, \boldsymbol{\theta})$ also has the potential to weaken the lower bound condition (1.8) on ψ appearing in Theorem 2. This in turn would increase the range of s within Corollaries 3 and 5.

Regarding lower bounds for the counting function $N(Q, \psi, \boldsymbol{\theta})$, if ψ is monotonic, then $\psi(q) \geq \psi(Q)$ for $\frac{1}{2}Q < q \leq Q$ and the heuristic "volume" argument leads us to the following estimate:

$$N(\tfrac{1}{2}Q, \psi, \boldsymbol{\theta}) \ \gg \ \psi(Q)^m \, Q^{d+1} \,. \tag{1.17}$$

In the homogeneous case (i.e., when $\boldsymbol{\theta} = \mathbf{0}$), the lower bound given by (1.17) is established in [2] for any analytic non-degenerate manifold $\mathcal{M}$ embedded in $\mathbb{R}^n$ and ψ satisfying $\lim_{q\to\infty} q\psi(q)^m = \infty$. When $\mathcal{M}$ is a curve, the condition on ψ can be weakened to $\lim_{q\to\infty} q\psi(q)^{(2n-1)/3} = \infty$. Moreover, it is shown in [2] that the rational points $\mathbf{a}/q$ associated with $N(\frac{1}{2}Q, \psi, \mathbf{0})$ are "ubiquitously" distributed for analytic non-degenerate manifolds. This together with the lower bound estimate is very much at the heart of the divergent Khintchine type results obtained in [2] for analytic non-degenerate manifolds. In a forthcoming paper [6], we establish the lower bound estimate (1.17) and show that shifted rational points $\frac{\mathrm{a}+\lambda}{q}$ associated with $N(\frac{1}{2}Q, \psi, \boldsymbol{\theta})$ are "ubiquitously" distributed for any C^{n+1} non-degenerate curve in $\mathbb{R}^n$ and arbitrary $\boldsymbol{\theta}$. As a consequence, we obtain a divergent Khintchine type theorem for Hausdorff measures. More specifically, let $\mathbf{f} = (f_1, \ldots, f_{n-1}) : [0, 1] \to \mathbb{R}^{n-1}$ be a C^{n+1} function such that for almost all $\alpha \in [0, 1]$

$$\det\left(f_j^{(i+1)}(\alpha)\right)_{1\leq i,j\leq n-1} \ \neq \ 0\,. \tag{1.18}$$

Let $\frac{1}{2} < s \le 1$, $\boldsymbol{\theta} \in \mathbb{R}^n$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonic function such that $\psi(r) \to 0$ as $r \to \infty$. It is established in [6] that

$$\mathcal{H}^s\big(\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\psi,\boldsymbol{\theta})\big) = \mathcal{H}^s(\mathcal{M}_{\mathbf{f}}) \qquad \text{whenever} \qquad \sum_{q=1}^{\infty} \left(\frac{\psi(q)}{q}\right)^{s+n-1} q^n \;=\; \infty\,.$$

In view of the conditions imposed on $\mathbf{f}$ above, the associated manifold $\mathcal{M}_{\mathbf{f}}$ is by definition a C^{n+1} non-degenerate curve in $\mathbb{R}^n$. When s is strictly less than one, non-degeneracy can be replaced by the condition that (1.18) is satisfied for at least one point $\alpha \in [0,1]$. In other words, all that is required is that there exists at least one point on the curve that is non-degenerate. Using fibering techniques, it is also shown in [6] that the above statement for non-degenerate curve in $\mathbb{R}^n$ can be readily extended to accommodate a large class of non-degenerate manifolds beyond the analytic ones considered in [2].

2 Preliminaries to the Proofs of Theorems 1 and 3

To establish Theorems 1 and 3, we adapt an argument of Sprindžuk [13, Chp2 §6]. In our view, the adaptation is non-trivial.

Without loss of generality suppose $0 < \psi(q) \le 1/4$ and recall that $\boldsymbol{\theta} = (\boldsymbol{\lambda},\boldsymbol{\gamma}) \in \mathbb{R}^d \times \mathbb{R}^m$. Recall also that $A(q,\psi,\boldsymbol{\theta})$ is given by (1.4). Given $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_d) \in \mathbb{R}^d$, let $\tilde{\boldsymbol{\lambda}} := (\{\lambda_1\},\dots,\{\lambda_d\}) \in [0,1)^d$ denote the fractional part of $\boldsymbol{\lambda}$. Then, it follows that

$$A(q,\psi,\boldsymbol{\theta}) = \#\mathcal{A}(q,\psi,\boldsymbol{\theta}) \tag{2.19}$$

where

$$\mathcal{A}(q,\psi,\boldsymbol{\theta}) := \{\mathbf{a} \in \mathbb{Z}(q) \;:\; \|q\,\mathbf{f}\big(\tfrac{\mathbf{a}+\tilde{\boldsymbol{\lambda}}}{q}\big) - \boldsymbol{\gamma}\| < \psi(q)\,\}$$

and

$$\mathbb{Z}(q) := \prod_{i=1}^{d} \big([0,q_i] \cap \mathbb{Z}\big) \quad \text{and} \quad q_i = \begin{cases} q & \text{if } \tilde{\lambda}_i = 0 \\ q-1 & \text{otherwise.} \end{cases}$$

Let δ be a sufficiently small positive constant that will be determined later and depends on $\mathbf{f}$. Without loss of generality, we can assume that

$$\delta q \psi(q) > 1\,.$$

Otherwise, the error term associated with (1.6) is, up to a multiplicative constant, larger than the trivial bound

$$A(q,\psi,\boldsymbol{\theta}) \leq (q+1)^d$$

and there is nothing to prove. Now define

$$r := \lfloor (\delta q \psi(q))^{1/2} \rfloor \tag{2.20}$$

and for each $\mathbf{a} \in \mathbb{Z}(q)$ write

$$\mathbf{a} = r\mathbf{u}(\mathbf{a}) + \mathbf{v}(\mathbf{a})$$

where $\mathbf{u}(\mathbf{a})$, $\mathbf{v}(\mathbf{a})$ satisfy $u_i(\mathbf{a}) = \lfloor a_i/r \rfloor$ and $0 \leq v_i(\mathbf{a}) < r$ $(1 \leq i \leq d)$. In particular

$$0 \leq u_i(\mathbf{a}) \leq s$$

where

$$s := \lfloor q/r \rfloor.$$

For $\mathbf{u} \in \mathbb{Z}^d$, define

$$\mathcal{A}(q,\psi,\boldsymbol{\theta},\mathbf{u}) := \{\mathbf{a} \in \mathcal{A}(q,\psi,\boldsymbol{\theta}) : \mathbf{u}(\mathbf{a}) = \mathbf{u}\}$$

and

$$A(q,\psi,\boldsymbol{\theta},\mathbf{u}) := \#\mathcal{A}(q,\psi,\boldsymbol{\theta},\mathbf{u}).$$

By the mean value theorem for second derivatives, when $\mathbf{a} \in \mathcal{A}(q,\psi,\boldsymbol{\theta},\mathbf{u})$,

$$f_j\left(\tfrac{\mathbf{a}+\tilde{\boldsymbol{\lambda}}}{q}\right) = f_j\left(\tfrac{r\mathbf{u}+\tilde{\boldsymbol{\lambda}}}{q}\right) + \sum_{i=1}^{d} \frac{v_i}{q}\frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\boldsymbol{\lambda}}}{q}\right) + O\left(\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{v_i v_j}{q^2}\right)$$

for $\mathbf{v} = \mathbf{v}(\mathbf{a}) \in \mathcal{R}^d$ where $\mathcal{R} := [0,r) \cap \mathbb{Z}$. Here the error term is

$$< \; C_1 r^2 q^{-2} \; \leq \; C_1 \delta \psi(q) q^{-1}$$

where C_1 depends at most on d and the size of the second derivatives. Now choose

$$\delta = 1/C_1\,.$$

Thus, for $\mathbf{a} = r\mathbf{u} + \mathbf{v}$ with $\mathbf{a} \in \mathcal{A}(q, \psi, \boldsymbol{\theta}, \mathbf{u})$ we have

$$\left\| q f_j\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right) + \sum_{i=1}^{d} v_i \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right) - \gamma_j \right\| < 2\psi(q) \quad (1 \le j \le m). \tag{2.21}$$

Therefore

$$A(q, \psi, \boldsymbol{\theta}, \mathbf{u}) \le B(q, \psi, \mathbf{u})$$

where $B(q, \psi, \mathbf{u}) := \#\mathcal{B}(q, \psi, \mathbf{u})$ and

$$\mathcal{B}(q, \psi, \mathbf{u}) := \{\mathbf{v} \in \mathcal{R}^d : (2.21) \text{ holds}\}.$$

Let

$$H := \left\lfloor \frac{1}{4\psi(q)} \right\rfloor \tag{2.22}$$

so that $H \ge 1$ and $\mathcal{H} := [-H, H] \cap \mathbb{Z}$. Then

$$\sum_{h \in \mathcal{H}} \frac{H - |h|}{H^2} \, \mathrm{e}(hx) = H^{-2} \left| \sum_{h=1}^{H} e(hx) \right|^2 = \left(\frac{\sin \pi H x}{H \sin \pi x} \right)^2 \ge \frac{4}{\pi^2}$$

whenever $\|x\| \le (2H)^{-1}$. Thus

$$B(q, \psi, \mathbf{u}) \ll B^*(q, \psi, \mathbf{u})$$

where

$$B^*(q, \psi, \mathbf{u}) := \sum_{\mathbf{h} \in \mathcal{H}^m} \frac{H - |h_1|}{H^2} \cdots \frac{H - |h_m|}{H^2} \sum_{\mathbf{v} \in \mathcal{R}^d} \mathrm{e}(\mathbf{h}.(\mathbf{F}(\mathbf{u}, \mathbf{v}) - \boldsymbol{\gamma})) \tag{2.23}$$

and

$$\mathbf{h} := (h_1, \ldots, h_m)\,,$$
$$\mathbf{F} := (F_1, \ldots, F_m)\,,$$
$$F_j(\mathbf{u}, \mathbf{v}) := q f_j\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right) + \sum_{i=1}^{d} v_i \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right).$$

By the definition of $\mathcal{H}$, we have that

$$0 \le \frac{H - |h_1|}{H^2} \cdots \frac{H - |h_m|}{H^2} \le H^{-m}$$

for any $\mathbf{h} = (h_1, \ldots, h_m) \in \mathcal{H}^m$. Therefore, by (2.23), we get that

$$B^*(q, \psi, \mathbf{u}) \leq H^{-m} \sum_{\mathbf{h} \in \mathcal{H}^m} \left| \sum_{\mathbf{v} \in \mathcal{R}^d} \mathrm{e}(\mathbf{h}.(\mathbf{F}(\mathbf{u}, \mathbf{v}) - \boldsymbol{\gamma})) \right| . \tag{2.24}$$

On using the fact that $\mathrm{e}(x_1 + \cdots + x_\ell) = \mathrm{e}(x_1) \cdots \mathrm{e}(x_\ell)$ and $|\mathrm{e}(x)| = 1$ for any real numbers $x, x_1, \ldots, x_\ell$, we find that

$$\begin{aligned}
\left| \sum_{\mathbf{v} \in \mathcal{R}^d} \mathrm{e}(\mathbf{h}.(\mathbf{F}(\mathbf{u}, \mathbf{v}) - \boldsymbol{\gamma})) \right| &= \left| \sum_{\mathbf{v} \in \mathcal{R}^d} \mathrm{e}(\mathbf{h}.\mathbf{F}(\mathbf{u}, \mathbf{v})) \cdot \mathrm{e}(-\mathbf{h}.\boldsymbol{\gamma}) \right| = \left| \sum_{\mathbf{v} \in \mathcal{R}^d} \mathrm{e}(\mathbf{h}.\mathbf{F}(\mathbf{u}, \mathbf{v})) \right| \\
&= \left| \sum_{\mathbf{v} \in \mathcal{R}^d} \mathrm{e}\left(\sum_{j=1}^m h_j \Big(q f_j \big(\tfrac{r\mathbf{u} + \tilde{\boldsymbol{\lambda}}}{q}\big) + \sum_{i=1}^d v_i \frac{\partial f_j}{\partial \alpha_i} \big(\tfrac{r\mathbf{u} + \tilde{\boldsymbol{\lambda}}}{q}\big) \Big) \right) \right| \\
&= \left| \sum_{\mathbf{v} \in \mathcal{R}^d} \mathrm{e}\left(\sum_{j=1}^m \sum_{i=1}^d h_j v_i \frac{\partial f_j}{\partial \alpha_i} \big(\tfrac{r\mathbf{u} + \tilde{\boldsymbol{\lambda}}}{q}\big) \right) \right| \\
&= \left| \sum_{\mathbf{v} \in \mathcal{R}^d} \prod_{i=1}^d \mathrm{e}\left(v_i \sum_{j=1}^m h_j \frac{\partial f_j}{\partial \alpha_i} \big(\tfrac{r\mathbf{u} + \tilde{\boldsymbol{\lambda}}}{q}\big) \right) \right| \\
&= \left| \prod_{i=1}^d \sum_{v_i \in \mathcal{R}} \mathrm{e}\left(v_i \sum_{j=1}^m h_j \frac{\partial f_j}{\partial \alpha_i} \big(\tfrac{r\mathbf{u} + \tilde{\boldsymbol{\lambda}}}{q}\big) \right) \right| .
\end{aligned}$$

Hence

$$\left| \sum_{\mathbf{v} \in \mathcal{R}^d} \mathrm{e}(\mathbf{h}.(\mathbf{F}(\mathbf{u}, \mathbf{v}) - \boldsymbol{\gamma})) \right| = \prod_{i=1}^d \left| \sum_{v \in \mathcal{R}} \mathrm{e}\left(v \sum_{j=1}^m h_j \frac{\partial f_j}{\partial \alpha_i} \big(\tfrac{r\mathbf{u} + \tilde{\boldsymbol{\lambda}}}{q}\big) \right) \right| .$$

Therefore, by (2.24), it follows that

$$B^*(q, \psi, \mathbf{u}) \ \leq \ \frac{1}{H^m} \sum_{\mathbf{h} \in \mathcal{H}^m} \prod_{i=1}^d \left| \sum_{v \in \mathcal{R}} \mathrm{e}\left(v \sum_{j=1}^m h_j \frac{\partial f_j}{\partial \alpha_i} \big(\tfrac{r\mathbf{u} + \tilde{\boldsymbol{\lambda}}}{q}\big) \right) \right| . \tag{2.25}$$

Since $\mathcal{R} = [0, r) \cap \mathbb{Z}$, for any given $\rho \in \mathbb{R}$ we have that $\left|\sum_{v \in \mathcal{R}} \mathrm{e}(v\rho)\right| \leq r$ and also that

$$\left| \sum_{v \in \mathcal{R}} \mathrm{e}(v\rho) \right| = \left| \frac{\mathrm{e}(r\rho) - 1}{\mathrm{e}(\rho) - 1} \right| \leq \frac{2}{|\mathrm{e}(\rho) - 1|} \ll \|\rho\|^{-1} ,$$

where the implied constant is absolute. Hence, on taking $\rho = \sum_{j=1}^m h_j \frac{\partial f_j}{\partial \alpha_i}\left(\frac{r\mathbf{u}+\tilde{\lambda}}{q}\right)$ we have that

$$\left|\sum_{v\in\mathcal{R}} \mathrm{e}\left(v\sum_{j=1}^m h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right)\right)\right| \ll \min\left(r, \left\|\sum_{j=1}^m h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right)\right\|^{-1}\right).$$

This together with (2.25), implies that

$$B^*(q,\psi,\mathbf{u}) \le \frac{1}{H^m}\sum_{\mathbf{h}\in\mathcal{H}^m}\prod_{i=1}^d \min\left(r, \left\|\sum_{j=1}^m h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right)\right\|^{-1}\right). \tag{2.26}$$

For a given $\mathbf{u}\in[0,s]^d$ we consider the intervals $I_i = [u_i - 1/2, u_i + 1/2]$, unless $u_i = 0$ or $u_i = s$ in which case we consider $[u_i, u_i + 1/2]$ or $[u_i - 1/2, u_i]$, respectively. For $\beta_i \in I_i$ we have

$$\frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right) = \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\beta+\tilde{\lambda}}{q}\right) + O(r/q)$$

by the mean value theorem. Hence

$$\sum_{j=1}^m h_j\left(\frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right) - \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\beta+\tilde{\lambda}}{q}\right)\right) \ll Hr/q$$

where the implicit constant depends at most on m and the size of the second derivatives. Moreover

$$\frac{Hr^2}{q} \le \frac{\delta q\psi(q)}{4q\psi(q)} = \frac{\delta}{4} < \delta\,,$$

where the left hand side inequality follows from the definitions of r and H—see (2.20) and (2.22). Hence

$$\left|\left\|\sum_{j=1}^m h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right)\right\| - \left\|\sum_{j=1}^m h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\beta+\tilde{\lambda}}{q}\right)\right\|\right| \ll \frac{\delta}{r} \ll \frac{1}{r}.$$

Thus

$$\min\left(r, \left\|\sum_{j=1}^m h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right)\right\|^{-1}\right) \ll \min\left(r, \left\|\sum_{j=1}^m h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\beta+\tilde{\lambda}}{q}\right)\right\|^{-1}\right)$$

and furthermore, by considering their product over i, we get that

$$\prod_{i=1}^{d} \min\left(r, \left\| \sum_{j=1}^{m} h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right) \right\|^{-1}\right) \ll \prod_{i=1}^{d} \min\left(r, \left\| \sum_{j=1}^{m} h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\boldsymbol{\beta}+\tilde{\lambda}}{q}\right) \right\|^{-1}\right).$$

Since the measure of $I_1 \times \cdots \times I_d$ is $\asymp 1$, integrating the above inequality over $\boldsymbol{\beta} \in I_1 \times \cdots \times I_d$ gives that

$$\prod_{i=1}^{d} \min\left(r, \left\| \sum_{j=1}^{m} h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right) \right\|^{-1}\right) \ll \int_{I_1 \times \cdots \times I_d} \prod_{i=1}^{d} \min\left(r, \left\| \sum_{j=1}^{m} h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\boldsymbol{\beta}+\tilde{\lambda}}{q}\right) \right\|^{-1}\right) d\boldsymbol{\beta}.$$

Now recall that the rectangles $I_1 \times \cdots \times I_d$ depend on the choice of $\mathbf{u}$. Note that their union taken over integer points $\mathbf{u} \in \mathcal{S}^d$, where $\mathcal{S} := [0, s]$, is exactly $\mathcal{S}^d$. Furthermore, different rectangles can only intersect on the boundary. Hence summing the above displayed inequality over all integer points $\mathbf{u} \in \mathcal{S}^d$ gives

$$\sum_{\mathbf{u} \in \mathcal{S}^d} \prod_{i=1}^{d} \min\left(r, \left\| \sum_{j=1}^{m} h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\mathbf{u}+\tilde{\lambda}}{q}\right) \right\|^{-1}\right) \ll \int_{\mathcal{S}^d} \prod_{i=1}^{d} \min\left(r, \left\| \sum_{j=1}^{m} h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\boldsymbol{\beta}+\tilde{\lambda}}{q}\right) \right\|^{-1}\right) d\boldsymbol{\beta}.$$

Now combining this together with (2.26) we obtain that

$$\sum_{\mathbf{u} \in \mathcal{S}^d} B^*(q, \psi, \mathbf{u}) \ll H^{-m} \sum_{\mathbf{h} \in \mathcal{H}^m} \int_{\mathcal{S}^d} \prod_{i=1}^{d} \min\left(r, \left\| \sum_{j=1}^{m} h_j \frac{\partial f_j}{\partial \alpha_i}\left(\tfrac{r\boldsymbol{\beta}+\tilde{\lambda}}{q}\right) \right\|^{-1}\right) d\boldsymbol{\beta}. \tag{2.27}$$

Now finally observe that

$$A(q, \psi, \boldsymbol{\theta}) \le \sum_{\mathbf{u} \in \mathcal{S}^d} A(q, \psi, \boldsymbol{\theta}, \mathbf{u}) \le \sum_{\mathbf{u} \in \mathcal{S}^d} B(q, \psi, \mathbf{u}) \ll \sum_{\mathbf{u} \in \mathcal{S}^d} B^*(q, \psi, \mathbf{u}). \tag{2.28}$$

3 The Proof of Theorem 1

With reference to Section 2, by (2.27)

$$\sum_{\mathbf{u} \in \mathcal{S}^d} B^*(q, \psi, \mathbf{u}) \ll r^{d-1} H^{-m} \sum_{\mathbf{h} \in \mathcal{H}^m} \int_{\mathcal{S}^d} \min\left(r, \left\| \sum_{j=1}^{m} h_j \frac{\partial f_j}{\partial \alpha_1}\left(\tfrac{r\boldsymbol{\beta}+\tilde{\lambda}}{q}\right) \right\|^{-1}\right) d\boldsymbol{\beta}.$$

Since (1.1) holds we may make the change of variables

$$\omega_j = \frac{\partial f_j}{\partial \alpha_1}\left(\tfrac{r\boldsymbol{\beta}+\tilde{\lambda}}{q}\right) \quad (1 \le j \le m), \qquad \omega_j = \beta_j \quad (m < j \le d).$$

Thus

$$\sum_{\mathbf{u}\in\mathcal{S}^d} B^*(q,\psi,\mathbf{u}) \ll \frac{r^{d-1}}{H^m}\sum_{\mathbf{h}\in\mathcal{H}^m}\left(\frac{q}{r}\right)^m\int_{\mathcal{J}_d}\min\left(r,\left\|\sum_{j=1}^m h_j\omega_j\right\|^{-1}\right)d\boldsymbol{\omega} \tag{3.1}$$

where $\mathcal{J}_d := \mathcal{F}_1\times\cdots\times\mathcal{F}_m\times[0,s]^{d-m}$, $\mathcal{F}_j := [f_j^-, f_j^+]$ and

$$f_j^- := \inf\frac{\partial f_j}{\partial\alpha_1}(\boldsymbol{\alpha})$$

and

$$f_j^+ := \sup\frac{\partial f_j}{\partial\alpha_1}(\boldsymbol{\alpha}).$$

The contribution from $\mathbf{h}=\mathbf{0}$ is

$$\ll \frac{r^{d-1}}{H^m}\left(\frac{q}{r}\right)^m\int_{\mathcal{J}_d} r\,d\boldsymbol{\omega} \ll \frac{r^{d-m}}{H^m}q^m s^{d-m} \ll H^{-m}q^d$$

since $rs\asymp q$. Next observe that

$$M := \int_{\mathcal{F}_1\times\cdots\times\mathcal{F}_m}\min\left(r,\left\|\sum_{j=1}^m h_j\omega_j\right\|^{-1}\right)d\omega_1\ldots d\omega_m$$

is constant with respect to $\omega_{m+1},\ldots,\omega_d$. Hence, by Fubini's theorem and the fact that $\mathcal{J}_d := \mathcal{F}_1\times\cdots\times\mathcal{F}_m\times[0,s]^{d-m}$, integrating M over $(\omega_{m+1},\ldots,\omega_d)\in[0,s]^{d-m}$ gives that

$$\int_{\mathcal{J}_d}\min\left(r,\left\|\sum_{j=1}^m h_j\omega_j\right\|^{-1}\right)d\boldsymbol{\omega} = s^{d-m}M\,. \tag{3.2}$$

If $\mathbf{h}\neq\mathbf{0}$, then assuming, for example, that $h_1\neq 0$ and using Fubini's theorem again we get that

$$\begin{aligned}
M &= \int_{\mathcal{F}_1\times\cdots\times\mathcal{F}_m}\min\left(r,\left\|\sum_{j=1}^m h_j\omega_j\right\|^{-1}\right)d\omega_1\ldots d\omega_m \\
&\ll \sup_{\rho\in[0,1]}\int_{\mathcal{F}_1}\min\left(r,\|h_1\omega_1-\rho\|^{-1}\right)d\omega_1 \\
&\ll \sup_{\rho\in[0,1]}\sum_{\substack{p\in\mathbb{Z}\\|p|\ll h_1}}\int_{\mathcal{F}_1}\min\left(r,|h_1\omega_1-\rho-p|^{-1}\right)d\omega_1
\end{aligned}$$

$$\ll \sup_{\rho\in[0,1]} \sum_{\substack{p\in\mathbb{Z}\\|p|\ll h_1}} \left(\frac{1}{h_1 r} + \frac{1}{h_1}\log r\right)$$

$$\ll \max\{1, \log r\}\,.$$

Hence, by the above inequalities and (3.2), the contribution from the $\mathbf{h} \neq \mathbf{0}$ terms within (3.1) is estimated by

$$\frac{r^{d-1}}{H^m} \sum_{\mathbf{h}\in\mathcal{H}^m} \left(\frac{q}{r}\right)^m s^{d-m} \max\{1, \log r\}$$

$$\ll r^{-1}(rs)^{d-m} q^m \max\{1, \log r\}$$

$$\ll r^{-1} q^d \max\{1, \log r\}.$$

In view of (2.28), it follows that

$$A(q, \psi, \boldsymbol{\theta}) \ \ll\ H^{-m} q^d \ +\ r^{-1} q^d \max\{1, \log r\}\ .$$

Given the definitions of H and r, this gives (1.6) and thereby completes the proof of the theorem.

4 The Proof of Theorem 3

Recall that within Theorem 3, we have that $m = 1$ and $d = n - 1$. Hence, with reference to Section 2, (2.27) becomes

$$\sum_{\mathbf{u}\in\mathcal{S}^d} B^*(q, \psi, \mathbf{u}) \ll H^{-1} \sum_{h\in\mathcal{H}} \int_{\mathcal{S}^d} \prod_{i=1}^{d} \min\left(r, \left\| h \frac{\partial f}{\partial \alpha_i}\left(\tfrac{r\boldsymbol{\beta}+\tilde{\boldsymbol{\lambda}}}{q}\right)\right\|^{-1}\right) d\boldsymbol{\beta}\,,$$

where $\mathbf{f} = f : \mathcal{U} \to \mathbb{R}$. Since (1.12) holds we may make the change of variables

$$\omega_i = \frac{\partial f}{\partial \alpha_i}\left(\tfrac{r\boldsymbol{\beta}+\tilde{\boldsymbol{\lambda}}}{q}\right) \quad (1 \le i \le d).$$

Thus

$$\sum_{\mathbf{u}\in\mathcal{S}^d} B^*(q, \psi, \mathbf{u}) \ll H^{-1} \sum_{h\in\mathcal{H}} \left(\frac{q}{r}\right)^d \int_{\mathcal{J}_d} \prod_{i=1}^{d} \min\left(r, \|h\omega_i\|^{-1}\right) d\boldsymbol{\omega}$$

where $\mathcal{J}_d := \mathcal{F}_1 \times \cdots \times \mathcal{F}_d$, $\mathcal{F}_i := [f_i^-, f_i^+]$ and

$$f_i^- := \inf \frac{\partial f}{\partial \alpha_i}(\boldsymbol{\alpha})$$

and

$$f_i^+ := \sup \frac{\partial f}{\partial \alpha_i}(\boldsymbol{\alpha}).$$

The contribution from $h = 0$ is

$$\ll H^{-1}\left(\frac{q}{r}\right)^d \int_{\mathcal{J}_d} r^d d\boldsymbol{\omega} \ll H^{-1}q^d$$

and the contribution from the remaining terms is

$$\begin{aligned}
&\ll H^{-1} \sum_{h\in\mathcal{H}\setminus\{0\}} \left(\frac{q}{r}\right)^d \int_{\mathcal{J}_d} \prod_{i=1}^{d} \min\left(r, \|h\omega_i\|^{-1}\right) d\boldsymbol{\omega} \\
&= H^{-1} \sum_{h\in\mathcal{H}\setminus\{0\}} \left(\frac{q}{r}\right)^d \prod_{i=1}^{d} \int_{\mathcal{F}_i} \min\left(r, \|h\omega_i\|^{-1}\right) d\omega_i \\
&\ll H^{-1} \sum_{h\in\mathcal{H}\setminus\{0\}} \left(\frac{q}{r}\right)^d \prod_{i=1}^{d} \max\{1, \log r\} \\
&\ll r^{-d} q^d \max\{1, (\log r)^d\}.
\end{aligned}$$

In view of (2.28), it follows that

$$A(q, \psi, \boldsymbol{\theta}) \ \ll\ H^{-1}q^d \ +\ r^{-d}q^d \max\{1, (\log r)^d\}\ .$$

Given the definitions of H and r this gives (1.13) and thereby completes the proof of the theorem.

5 Proof of Theorem 2

Step 1. As mentioned in Section 1, in view of the Implicit Function Theorem, we can assume without loss of generality that the manifold $\mathcal{M}_{\mathbf{f}}$ is of the Monge form (1.3). Note that, since $\mathcal{U}$ is compact and $\mathbf{f}$ is C^1, this implies via the Mean Value Theorem that $\mathbf{f} = (f_1, \ldots, f_m)$ is bi-Lipschitz and so there exists a constant $c_1 \geq 1$ such that

$$\max_{1\leq i\leq m} |f_i(\boldsymbol{\alpha}) - f_i(\boldsymbol{\alpha}')| \ \leq\ c_1\ |\boldsymbol{\alpha} \ -\ \boldsymbol{\alpha}'| \qquad \forall \quad \boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathcal{U} = [0, 1]^d\ . \tag{5.1}$$

Let $\Omega_n^{\mathbf{f}}(\psi,\boldsymbol{\theta})$ denote the projection of $\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\psi,\boldsymbol{\theta})$ onto $\mathcal{U}$; that is,

$$\Omega_n^{\mathbf{f}}(\psi,\boldsymbol{\theta}) := \{\boldsymbol{\alpha} \in \mathcal{U} \,:\, (\boldsymbol{\alpha}, \mathbf{f}(\boldsymbol{\alpha})) \in \mathcal{S}_n(\psi,\boldsymbol{\theta})\}\,.$$

Explicitly, given $\boldsymbol{\theta} = (\boldsymbol{\lambda},\boldsymbol{\gamma}) \in \mathbb{R}^d \times \mathbb{R}^m$, the set $\Omega_n^{\mathbf{f}}(\psi,\boldsymbol{\theta})$ consists of points $\boldsymbol{\alpha} \in \mathcal{U}$ such that the system of inequalities

$$\begin{cases} \left|\alpha_i - \frac{a_i+\lambda_i}{q}\right| < \frac{\psi(q)}{q} & 1 \le i \le d \\ \left|f_j(\boldsymbol{\alpha}) - \frac{b_j+\gamma_j}{q}\right| < \frac{\psi(q)}{q} & 1 \le j \le m \end{cases} \tag{5.2}$$

is satisfied for infinitely many $(q,\mathbf{a},\mathbf{b}) \in \mathbb{N} \times \mathbb{Z}^d \times \mathbb{Z}^m$. Furthermore, there is no loss of generality in assuming that $\frac{\mathbf{a}+\boldsymbol{\lambda}}{q} \in \mathcal{U}$ for solutions of (5.2). In view of (5.1), the sets $\Omega_n^{\mathbf{f}}(\psi,\boldsymbol{\theta})$ and $\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\psi,\boldsymbol{\theta})$ are related by a bi-Lipschitz map and therefore

$$\mathcal{H}^s\left(\mathcal{M}_{\mathbf{f}} \cap \mathcal{S}_n(\psi,\boldsymbol{\theta})\right) = 0 \quad \Longleftrightarrow \quad \mathcal{H}^s(\Omega_n^{\mathbf{f}}(\psi,\boldsymbol{\theta})) = 0\,.$$

Hence, it suffices to show that

$$\mathcal{H}^s(\Omega_n^{\mathbf{f}}(\psi,\boldsymbol{\theta})) = 0\,. \tag{5.3}$$

Step 2. Notice that the set $B = \{\boldsymbol{\alpha} \in \mathcal{U} \,:\, \text{l.h.s. of (1.1)} = 0\}$ is closed and therefore $G = \mathcal{U}\setminus B$ can be written as a countable union of closed rectangles $\mathcal{U}_i$ on which f satisfies (1.1). The constant η associated with (1.1) depends on the particular choice of $\mathcal{U}_i$. For the moment, assume that $\mathcal{H}^s(\Omega_n^{\mathbf{f}}(\psi,\boldsymbol{\theta})\cap\mathcal{U}_i) = 0$ for any $i \in \mathbb{N}$. On using the fact that $\mathcal{H}^s(B) = 0$, we have that

$$\begin{aligned} \mathcal{H}^s(\Omega_n^{\mathbf{f}}(\psi,\boldsymbol{\theta})) &\le \mathcal{H}^s\Big(B \cup \big(\bigcup_{i=1}^{\infty} \Omega_n^{\mathbf{f}}(\psi,\boldsymbol{\theta}) \cap \mathcal{U}_i\big)\Big) \\ &\le \mathcal{H}^s(B) \,+\, \sum_{i=1}^{\infty} \mathcal{H}^s\big(\Omega_n^{\mathbf{f}}(\psi,\boldsymbol{\theta}) \cap \mathcal{U}_i\big) \;=\; 0 \end{aligned}$$

and this establishes (5.3). Thus, without loss of generality, and for the sake of clarity, we assume that f satisfies (1.1) on $\mathcal{U}$.

Step 3. For a point $\frac{\mathbf{p}+\boldsymbol{\theta}}{q} \in \mathbb{R}^n$ with $\mathbf{p} = (\mathbf{a},\mathbf{b}) \in \mathbb{Z}^d \times \mathbb{Z}^m$, let $\sigma\big(\frac{\mathbf{p}+\boldsymbol{\theta}}{q}\big)$ denote the set of $\boldsymbol{\alpha} \in \mathcal{U}$ satisfying (5.2). Trivially,

$$\operatorname{diam}\big(\sigma\big(\tfrac{\mathbf{p}+\boldsymbol{\theta}}{q}\big)\big) \ll \psi(q)/q\,, \tag{5.4}$$

where the implied constant depends on n only.

Assume that $\sigma\left(\frac{\mathbf{p}+\boldsymbol{\theta}}{q}\right) \neq \emptyset$. Thus, q lies in the integer support $\mathcal{N}$ of ψ. Let $\boldsymbol{\alpha} \in \sigma\left(\frac{\mathbf{p}+\boldsymbol{\theta}}{q}\right)$. The triangle inequality together with (5.1) and (5.2), implies that

$$\begin{aligned} \left|\mathbf{f}\left(\tfrac{\mathbf{a}+\boldsymbol{\lambda}}{q}\right) - \tfrac{\mathbf{b}+\boldsymbol{\gamma}}{q}\right| &\leq \left|\mathbf{f}\left(\tfrac{\mathbf{a}+\boldsymbol{\lambda}}{q}\right) - \mathbf{f}(\boldsymbol{\alpha})\right| \;+\; \left|\mathbf{f}(\boldsymbol{\alpha}) - \tfrac{\mathbf{b}+\boldsymbol{\gamma}}{q}\right| \\ &< c_1 \left|\boldsymbol{\alpha} - \tfrac{\mathbf{a}+\boldsymbol{\lambda}}{q}\right| \;+\; \psi(q)/q \\ &\leq c_2\, \psi(q)/q\,, \end{aligned}$$

where $c_2 := 1 + c_1$ is a constant. Thus, for q sufficiently large so that $c_2\,\psi(q) < 1/2$ we have that

$$\begin{aligned} &\#\left\{\mathbf{p} \in \mathbb{Z}^n : \sigma\left(\tfrac{\mathbf{p}+\boldsymbol{\theta}}{q}\right) \neq \emptyset\right\} \\ &\quad\leq \#\left\{\mathbf{p} \in \mathbb{Z}^n :\ \tfrac{\mathbf{a}+\boldsymbol{\lambda}}{q} \in \mathcal{U},\ \left|\mathbf{f}\left(\tfrac{\mathbf{a}+\boldsymbol{\lambda}}{q}\right) - \tfrac{\mathbf{b}+\boldsymbol{\gamma}}{q}\right| < c_2\,\psi(q)/q\right\} \\ &\quad= \#\left\{\mathbf{a} \in \mathbb{Z}^d \ :\ \tfrac{\mathbf{a}+\boldsymbol{\lambda}}{q} \in \mathcal{U},\ \left\|q\,\mathbf{f}\left(\tfrac{\mathbf{a}+\boldsymbol{\lambda}}{q}\right) - \boldsymbol{\gamma}\right\| < c_2\psi(q)\right\}\ . \end{aligned}$$

By definition, the right hand side is simply the counting function $A(q, c_2\psi, \boldsymbol{\theta})$. Thus, by Corollary 1, for $q \in \mathcal{N}$ sufficiently large we have that

$$\#\left\{\mathbf{p} \in \mathbb{Z}^n : \sigma\left(\tfrac{\mathbf{p}+\boldsymbol{\theta}}{q}\right) \neq \emptyset\right\} \ll \psi(q)^m\, q^d\,. \tag{5.5}$$

Step 4. For $q > 0$, let

$$\Omega_n^{\mathbf{f}}(\psi, \boldsymbol{\theta}; q) \ := \bigcup_{\mathbf{p}\in\mathbb{Z}^n,\, \sigma\left(\frac{\mathbf{p}+\boldsymbol{\theta}}{q}\right)\neq\emptyset} \sigma\left(\tfrac{\mathbf{p}+\boldsymbol{\theta}}{q}\right)\ .$$

Then $\mathcal{H}^s(\Omega_n^{\mathbf{f}}(\psi, \boldsymbol{\theta})) = \mathcal{H}^s(\limsup_{q\to\infty} \Omega_n^{\mathbf{f}}(\psi, \boldsymbol{\theta}; q))$ and the Hausdorff-Cantelli Lemma [8, p. 68] implies (5.3) if

$$\sum_{q=1}^{\infty} \ \sum_{\mathbf{p}\in\mathbb{Z}^n,\, \sigma\left(\frac{\mathbf{p}+\boldsymbol{\theta}}{q}\right)\neq\emptyset} \left(\operatorname{diam}\left(\sigma\left(\tfrac{\mathbf{p}+\boldsymbol{\theta}}{q}\right)\right)\right)^s < \infty\,. \tag{5.6}$$

In view of (5.4) and (5.5), it follows that

$$\begin{aligned} \text{L.H.S of (5.6)} &\ll \sum_{q\in\mathcal{N}} \ \sum_{\mathbf{p}\in\mathbb{Z}^n,\, \sigma\left(\frac{\mathbf{p}+\boldsymbol{\theta}}{q}\right)\neq\emptyset} (\psi(q)/q)^s \\ &\ll \sum_{q\in\mathcal{N}} (\psi(q)/q)^s \times \psi(q)^m\, q^d \;=\; \sum_{q=1}^{\infty} (\psi(q)/q)^{s+m}\, q^n \;<\; \infty\ . \end{aligned}$$

This completes the proof of Theorem 2.

Funding

This work was supported by EPSRC Programme Grant [EP/J018260/1 to V.B. and S.V.]; NSA [H98230-09-1-0015 and H98230-06-0077 to R.C.V.]; and EPSRC First Grant [EP/M021858/1 to E.Z.]

Acknowledgments

We would like to thank the referees for their thorough reports which enabled us to remove a number of inaccuracies and improve the clarity of the paper. S.V. would like to thank Shoeb Khan for his wonderful friendship and for helping him to keep his curry muscle under control. Also many thanks to the teenagers, Ayesha and Iona, and their mother, Bridget for much happiness, love, and laughter. We are grateful to David Simmons for providing the counterexample mentioned in the end of Section 1.1.

References

[1] Beresnevich, V. "A Groshev type theorem for convergence on manifolds." *Acta Mathematica Hungarica* 94, no. 1–2 (2002): 99–130.

[2] Beresnevich, V. "Rational points near manifolds and metric Diophantine approximation." *Annals of Mathematics (2)* 175, no. 1 (2012): 187–235.

[3] Beresnevich, V., D. Dickinson, and S. Velani. "Measure theoretic laws for lim sup sets." *Memoirs of the American Mathematical Society* 179(846):x+91, 2006.

[4] Beresnevich, V., D. Dickinson, and S. Velani. "Diophantine approximation on planar curves and the distribution of rational points." *Annals of Mathematics (2)* 166, no. 2 (2007): 367–426.

[5] Beresnevich, V., R. C. Vaughan, and S. Velani. "Inhomogeneous Diophantine approximation on planar curves." *Mathematische Annalen* 349, no. 4 (2011): 929–42.

[6] Beresnevich, V., R. C. Vaughan, S. Velani, and E. Zorin. "Diophantine approximation on manifolds and the distribution of rational points: contributions to the divergence theory." In preparation.

[7] Bernik, V. I. "Asymptotic number of solutions for some systems of inequalities in the theory of Diophantine approximation of dependent quantities." *Vescī Akadèmīī Navuk BSSR. Seryja Fīzīka-Matèmatyčnyh Navuk* no. 1 (1973): 10–17 (In Russian).

[8] Bernik, V. I., and M. M. Dodson. "Metric Diophantine Approximation on Manifolds." *Cambridge Tracts in Mathematics*, vol. 137, Cambridge: Cambridge University Press, 1999.

[9] Dodson, M. M., B. P. Rynne, and J. A. G. Vickers. "Khintchine-type theorems on manifolds." *Acta Arithmetica* 57 (1991): 115–30.

[10] Harman, G. *Metric Number Theory, Volume 18 of LMS Monographs New Series*. 1998.

[11] Kleinbock, D. Y., and G. A. Margulis. "Flows on homogeneous spaces and Diophantine approximation on manifolds." *Annals of Mathematics (2)* 148 (1998): 339–60.

[12] Kovalevskaya, E. "On the exact order of simultaneous approximation of almost all points on linear manifold." *Vestsī Natsyyanal'naĭ Akadèmīī Navuk Belarusī. Seryya Fīzīka-Matèmatychnykh Navuk* no. 1. (2000): 23–7 (In Russian).

[13] Sprindžuk, V. *Metric Theory of Diophantine Approximation*. New York: John Wiley & Sons, 1979. (English transl.).

[14] Vaughan, R. C., and S. Velani. "Diophantine approximation on planar curves: the convergence theory." *Inventiones mathematicae* 166, no. 1 (2006): 103–24.