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# FREE MONOIDS ARE COHERENT

VICTORIA GOULD, MIKLÓS HARTMANN, AND NIK RUŠKUC

ABSTRACT. A monoid  $S$  is said to be *right coherent* if every finitely generated subact of every finitely presented right  $S$ -act is finitely presented. *Left coherency* is defined dually and  $S$  is *coherent* if it is both right and left coherent. These notions are analogous to those for a ring  $R$  (where, of course,  $S$ -acts are replaced by  $R$ -modules). Choo, Lam and Luft have shown that free rings are coherent. In this note we prove that, correspondingly, any free monoid is coherent, thus answering a question posed by the first author in 1992.

## 1. INTRODUCTION AND PRELIMINARIES

The notion of right coherency for a monoid  $S$  is defined in terms of finitary properties of right  $S$ -acts, corresponding to the way in which right coherency is defined for a ring  $R$  via properties of right  $R$ -modules. Namely,  $S$  is said to be *right (left) coherent* if every finitely generated subact of every finitely presented right (left)  $S$ -act is finitely presented. If  $S$  is both right and left coherent then we say that  $S$  is *coherent*. Chase [1] gave equivalent internal conditions for right coherency of a ring  $R$ . The analogous result for monoids states that a monoid  $S$  is right coherent if and only if for any finitely generated right congruence  $\rho$  on  $S$ , and for any  $a, b \in S$ , the right annihilator congruence

$$r(a\rho) = \{(u, v) \in S \times S : au \rho av\}$$

is finitely generated, and the subact  $(a\rho)S \cap (b\rho)S$  of the right  $S$ -act  $S/\rho$  is finitely generated (if non-empty) [4]. *Left coherency* is defined for monoids and rings in a dual manner; a monoid or ring is *coherent* if it is both right and left coherent. Coherency is a rather weak finitary condition on rings and monoids and as demonstrated by Wheeler [7], it is intimately related to the model theory of  $R$ -modules and  $S$ -acts.

A natural question arises as to which of the important classes of infinite monoids are (right) coherent? This study was initiated in [4], where it is shown that the free commutative monoid on any set  $\Omega$  is coherent. For a (right) noetherian ring  $R$ , the free monoid ring  $R[\Omega^*]$  over  $R$  is (right) coherent [2, Corollary 2.2]. Since the free ring on  $\Omega$  is the monoid ring  $\mathbb{Z}[\Omega^*]$  [6], it follows immediately that free rings are coherent. The question of whether the free monoid  $\Omega^*$  itself is coherent was left open in [4]. The purpose of this note is to provide a positive answer to that question:

**Theorem 1.** *For any set  $\Omega$  the free monoid  $\Omega^*$  is coherent.*

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Our proof of Theorem 1, given in Section 2, provides a blueprint for the proof in [5] that free left ample monoids are right coherent. Further comments are provided in Section 3.

A few words on notation and technicalities follow. If  $H$  is a set of pairs of elements of a monoid  $S$ , then we denote by  $\langle H \rangle$  the right congruence on  $S$  generated by  $H$ . It is easy to see that if  $a, b \in S$ , then  $a \langle H \rangle b$  if and only if  $a = b$  or there is an  $n \geq 1$  and a sequence

$$(c_1, d_1, t_1; c_2, d_2, t_2; \dots; c_n, d_n, t_n)$$

of elements of  $S$ , with  $(c_i, d_i) \in H$  or  $(d_i, c_i) \in H$ , such that the following equalities hold:

$$a = c_1 t_1, d_1 t_1 = c_2 t_2, \dots, d_n t_n = b.$$

Such a sequence will be referred to as an  $H$ -sequence (of length  $n$ ) connecting  $a$  and  $b$ . It is convenient to allow  $n = 0$  in the above sequence; the empty sequence is interpreted as asserting equality  $a = b$ . Where convenient we will use the fact that  $\Omega^*$  is a submonoid of the free group  $\text{FG}(\Omega)$  on  $\Omega$ , in order to give the natural meaning to expressions such as  $yx^{-1}$ , where  $x, y \in \Omega^*$  and  $x$  is a suffix of  $y$ .

## 2. PROOF OF THEOREM 1

Let  $\Omega$  be a set; it is clearly enough to show that  $\Omega^*$  is right coherent. To this end let  $\rho$  be the right congruence on  $\Omega^*$  generated by a finite subset  $H$  of  $\Omega^* \times \Omega^*$ , which without loss of generality we assume to be symmetric.

**Definition 2.** A quadruple  $(a, u; b, v)$  of elements of  $S$  is said to be *irreducible* if  $(au, bv) \in \rho$  and for any common non-empty suffix  $x$  of  $u$  and  $v$  we have that  $(aux^{-1}, bvx^{-1}) \notin \rho$ .

**Definition 3.** An  $H$ -sequence  $(c_1, d_1, t_1; \dots; c_n, d_n, t_n)$  with

$$au = c_1 t_1, d_1 t_1 = c_2 t_2, \dots, d_n t_n = bv$$

is *irreducible* with respect to  $(a, u; b, v)$  if  $u, t_1, \dots, t_n, v \in \Omega^*$  do not have a common non-empty suffix. Clearly, this is equivalent to one of  $u, t_1, \dots, t_n, v$  being  $\epsilon$ .

Throughout this note for an  $H$ -sequence as above we define  $a = d_0, u = t_0, c_{n+1} = b$  and  $t_{n+1} = v$ . It is clear that if the quadruple  $(a, u; b, v)$  is irreducible then any  $H$ -sequence connecting  $au$  and  $bv$  must be irreducible with respect to  $(a, u; b, v)$ .

We define

$$K = \max\{|p| : (p, q) \in H\}.$$

**Lemma 4.** Let the  $H$ -sequence  $(c_1, d_1, t_1; \dots; c_n, d_n, t_n)$  with

$$au = c_1 t_1, d_1 t_1 = c_2 t_2, \dots, d_n t_n = bv$$

be irreducible with respect to  $(a, u; b, v)$ . Then either the empty  $H$ -sequence is irreducible with respect to  $(a, u; c_1, t_1)$  (in which case  $|u| \leq \max(|b|, K)$  and  $u = \epsilon$  or  $t_1 = \epsilon$ ) or there exist an index  $1 \leq i \leq n$  such that  $t_{i+1} = \epsilon$  (so that  $au \rho c_{i+1}$ ) and  $x \in \Omega^+$  such that  $|x| \leq \max(|b|, K)$ , the sequence

$$(c_1, d_1, t_1 x^{-1}; \dots; c_{i-1}, d_{i-1}, t_{i-1} x^{-1})$$

satisfies

$$aux^{-1} = c_1t_1x^{-1}, d_1t_1x^{-1} = c_2t_2x^{-1}, \dots, d_{i-1}t_{i-1}x^{-1} = c_it_ix^{-1},$$

and is an irreducible  $H$ -sequence with respect to  $(a, ux^{-1}; c_i, t_ix^{-1})$ .

*Proof.* If the empty sequence is irreducible with respect to  $(a, u; c_1, t_1)$  then either  $u = \epsilon$  or  $t_1 = \epsilon$ . In both cases we have that  $|u| \leq \max(|b|, K)$ . Suppose therefore that the empty sequence is not irreducible with respect to  $(a, u; c_1, t_1)$ . Let  $i \in \{1, \dots, n\}$  be the smallest index such that  $t_{i+1} = \epsilon$  (such an index exists, because our original sequence is irreducible), and let  $x$  be the longest common non-empty suffix of  $u = t_0, t_1, \dots, t_i$ . Then the sequence

$$(c_1, d_1, t_1x^{-1}; \dots; c_{i-1}, d_{i-1}, t_{i-1}x^{-1})$$

clearly satisfies

$$aux^{-1} = c_1t_1x^{-1}, d_1t_1x^{-1} = c_2t_2x^{-1}, \dots, d_{i-1}t_{i-1}x^{-1} = c_it_ix^{-1}$$

and is irreducible with respect to  $(a, ux^{-1}; c_i, t_ix^{-1})$ . Furthermore, since  $t_{i+1} = \epsilon$ , we have that  $d_it_i = c_{i+1}$ , so  $x$  is a suffix of  $c_{i+1}$ . If  $i < n$  then  $(c_{i+1}, d_{i+1}) \in H$ , while if  $i = n$  we have  $c_{i+1} = b$ . In either case  $|x| \leq |c_{i+1}| \leq \max(|b|, K)$ .  $\square$

We deduce immediately that one condition for coherency of  $\Omega^*$  is fulfilled.

**Corollary 5.** *Let  $a, b \in S$ . Then  $(a\rho)S \cap (b\rho)S$  is empty or finitely generated.*

*Proof.* Let us suppose that  $(a\rho)S \cap (b\rho)S \neq \emptyset$  and let

$$X = \{a\rho, b\rho, c\rho : (c, d) \in H\} \cap (a\rho)S \cap (b\rho)S.$$

We claim that  $X$  generates  $(a\rho)S \cap (b\rho)S$ . It is enough to show that for every irreducible quadruple  $(a, u; b, v)$  we have that  $(au)\rho \in X$ . For this, let  $(c_1, d_1, t_1; \dots; c_n, d_n, t_n)$  be an  $H$ -sequence with

$$au = c_1t_1, \dots, d_nt_n = bv.$$

Note that this sequence is necessarily irreducible with respect to  $(a, u; b, v)$ . Then by Lemma 4, either  $u = \epsilon$ , or  $t_i = \epsilon$  for some  $i \in \{1, \dots, n\}$ , or  $v = t_{n+1} = \epsilon$ . In each of these cases we see that  $(au)\rho \in X$ .  $\square$

It remains to show that for any  $a \in \Omega^*$ , the right congruence  $r(a\rho)$  is finitely generated. To this end we first present a technical result.

**Lemma 6.** *Let  $(c_1, d_1, t_1; \dots; c_n, d_n, t_n)$  with*

$$au = c_1t_1, \dots, d_nt_n = bv$$

*be an irreducible  $H$ -sequence with respect to  $(a, u; b, v)$ . Then either  $u = \epsilon$ , or there exist a factorisation  $u = x_k \dots x_1$  and indices  $n+1 \geq \ell_1 > \ell_2 > \dots > \ell_k \geq 1$  such that for all  $1 \leq j \leq k$ :*

- (i)  $0 < |x_j| \leq \max(|b|, K)$  and
- (ii)  $aux_1^{-1} \dots x_{j-1}^{-1} \rho c_{\ell_j}$  (note that for  $j = 1$  we have  $au \rho c_{\ell_1}$ ).

*Proof.* We proceed by induction on  $|u|$ : if  $|u| = 0$  the result is clear. Suppose that  $|u| > 0$  and the result is true for all shorter words. If the empty sequence is irreducible with respect to  $(a, u; c_1, t_1)$ , then  $t_1 = \epsilon$  and the factorisation  $u = x_1$  satisfies the required conditions, with  $k = 1$  and  $\ell_1 = 1$ . Otherwise, by Lemma 4, there exist an index  $1 \leq i \leq n$  such that  $t_{i+1} = \epsilon$ , so that  $au \rho c_{i+1}$ , and  $x_1 \in \Omega^+$  such that  $|x_1| \leq \max(|b|, K)$  and the sequence

$$(c_1, d_1, t_1 x_1^{-1}; \dots; c_{i-1}, d_{i-1}, t_{i-1} x_1^{-1})$$

satisfies

$$aux_1^{-1} = c_1 t_1 x_1^{-1}, d_1 t_1 x_1^{-1} = c_2 t_2 x_1^{-1}, \dots, d_{i-1} t_{i-1} x_1^{-1} = c_i t_i x_1^{-1}$$

and is an irreducible  $H$ -sequence with respect to  $(a, ux_1^{-1}; c_i, t_i x_1^{-1})$ . Put  $\ell_1 = i + 1$ . Since  $|ux_1^{-1}| < |u|$ , the result follows by induction.  $\square$

**Lemma 7.** *Let  $a \in \Omega^*$ . Then  $r(a\rho)$  is finitely generated.*

*Proof.* Let  $K' = \max(K, |a|) + 1$ ,  $L = 2|H| + 2$ ,  $N = K'L$  and define

$$X = \{(u, v) : |u| + |v| \leq 3N\} \cap r(a\rho).$$

We claim that  $X$  generates  $r(a\rho)$ . It is clear that  $\langle X \rangle \subseteq r(a\rho)$ .

Let  $(u, v) \in r(a\rho)$ . We show by induction on  $|u| + |v|$  that  $(u, v) \in \langle X \rangle$ . Clearly, if  $|u| + |v| \leq 3N$ , then  $(u, v) \in X$ . We suppose therefore that  $|u| + |v| > 3N$  and make the inductive assumption that if  $(u', v') \in r(a\rho)$  and  $|u'| + |v'| < |u| + |v|$ , then  $(u', v') \in \langle X \rangle$ . If the quadruple  $(a, u; a, v)$  is not irreducible, it is immediate that  $(u, v) \in \langle X \rangle$ . Without loss of generality we therefore suppose that the quadruple  $(a, u; a, v)$  is irreducible and  $|v| \leq |u|$ , so that  $|u| > N$ . Let  $(c_1, d_1, t_1; \dots; c_n, d_n, t_n)$  with

$$au = c_1 t_1, \dots, d_n t_n = av$$

be an irreducible  $H$ -sequence with respect to  $(a, u; a, v)$ . We apply Lemma 6, noting here that  $a = b$ . Clearly  $u \neq \epsilon$ , so by Lemma 6, there exists a factorisation  $u = x_k \dots x_1$  such that for all  $1 \leq j \leq k$  we have  $0 < |x_j| \leq K'$  and  $aux_1^{-1} \dots x_{j-1}^{-1} \rho c_{\ell_j}$  for some  $1 \leq \ell_j \leq n + 1$ . Since  $|u| > K'L$  we have that  $k > L$ . Note that the number of distinct elements among  $c_1, \dots, c_n$  is less than  $L - 1$ . This in turn implies that there exist two indices  $1 \leq k - L < j < i \leq k$  such that  $c_{\ell_i} = c_{\ell_j}$ , so that

$$aux_1^{-1} \dots x_{i-1}^{-1} \rho c_{\ell_i} = c_{\ell_j} \rho aux_1^{-1} \dots x_{j-1}^{-1}.$$

Since  $i, j > k - L$  we have that  $k - i + 1 \leq L$ , so  $|ux_1^{-1} \dots x_{i-1}^{-1}| = |x_k \dots x_i| \leq K'L$ , and similarly  $|ux_1^{-1} \dots x_{j-1}^{-1}| \leq K'L$ . As a consequence  $(ux_1^{-1} \dots x_{i-1}^{-1}, ux_1^{-1} \dots x_{j-1}^{-1}) \in X$ , and letting  $u' = ux_1^{-1} \dots x_{i-1}^{-1} x_{j-1} \dots x_k$ , we see that

$$(u', u) = (ux_1^{-1} \dots x_{i-1}^{-1}, ux_1^{-1} \dots x_{j-1}^{-1}) x_{j-1} \dots x_1 \in \langle X \rangle.$$

In particular,  $au' \rho au \rho av$ . Note that  $|u'| < |u|$ , because  $j < i$  and  $x_j \neq \epsilon$ . Thus by the induction hypothesis we have that  $(v, u') \in \langle X \rangle$  and so the lemma is proved.  $\square$

In view of the characterisation of coherency given in [4] and cited in the Introduction, Corollary 5 and Lemma 7 complete the proof of Theorem 1.

## 3. COMMENTS

Given that the class of right coherent monoids is closed under retract [5], it follows from the results of that paper that free monoids are coherent. However, as the arguments in [5] for free left ample monoids are burdened with unavoidable technicalities, we prefer to present here the more transparent proof that  $\Omega^*$  is coherent, by way of motivation for the work of [5]. With free objects in mind, we remark that we also show in [5] that the free inverse monoid on  $\Omega$  is not coherent if  $|\Omega| > 1$ .

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