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# FREE MONOIDS ARE COHERENT

### VICTORIA GOULD, MIKLÓS HARTMANN, AND NIK RUŠKUC

ABSTRACT. A monoid S is said to be *right coherent* if every finitely generated subact of every finitely presented right S-act is finitely presented. Left coherency is defined dually and S is coherent if it is both right and left coherent. These notions are analogous to those for a ring R (where, of course, S-acts are replaced by R-modules). Choo, Lam and Luft have shown that free rings are coherent. In this note we prove that, correspondingly, any free monoid is coherent, thus answering a question posed by the first author in 1992.

### 1. INTRODUCTION AND PRELIMINARIES

The notion of right coherency for a monoid S is defined in terms of finitary properties of right S-acts, corresponding to the way in which right coherency is defined for a ring R via properties of right R-modules. Namely, S is said to be right (left) coherent if every finitely generated subact of every finitely presented right (left) S-act is finitely presented. If S is both right and left coherent then we say that S is coherent. Chase [1] gave equivalent internal conditions for right coherency of a ring R. The analogous result for monoids states that a monoid S is right coherent if and only if for any finitely generated right congruence  $\rho$  on S, and for any  $a, b \in S$ , the right annihilator congruence

$$r(a\rho) = \{(u, v) \in S \times S : au \,\rho \,av\}$$

is finitely generated, and the subact  $(a\rho)S \cap (b\rho)S$  of the right S-act  $S/\rho$  is finitely generated (if non-empty) [4]. Left coherency is defined for monoids and rings in a dual manner; a monoid or ring is coherent if it is both right and left coherent. Coherency is a rather weak finitary condition on rings and monoids and as demonstrated by Wheeler [7], it is intimately related to the model theory of *R*-modules and *S*-acts.

A natural question arises as to which of the important classes of infinite monoids are (right) coherent? This study was initiated in [4], where it is shown that the free commutative monoid on any set  $\Omega$  is coherent. For a (right) noetherian ring R, the free monoid ring  $R[\Omega^*]$  over R is (right) coherent [2, Corollary 2.2]. Since the free ring on  $\Omega$  is the monoid ring  $\mathbb{Z}[\Omega^*]$  [6], it follows immediately that free rings are coherent. The question of whether the free monoid  $\Omega^*$  itself is coherent was left open in [4]. The purpose of this note is to provide a positive answer to that question:

**Theorem 1.** For any set  $\Omega$  the free monoid  $\Omega^*$  is coherent.

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Our proof of Theorem 1, given in Section 2, provides a blueprint for the proof in [5] that free left ample monoids are right coherent. Further comments are provided in Section 3.

A few words on notation and technicalities follow. If H is a set of pairs of elements of a monoid S, then we denote by  $\langle H \rangle$  the right congruence on S generated by H. It is easy to see that if  $a, b \in S$ , then  $a \langle H \rangle b$  if and only if a = b or there is an  $n \ge 1$  and a sequence

$$(c_1, d_1, t_1; c_2, d_2, t_2; \ldots; c_n, d_n, t_n)$$

of elements of S, with  $(c_i, d_i) \in H$  or  $(d_i, c_i) \in H$ , such that the following equalities hold:

$$a = c_1 t_1, d_1 t_1 = c_2 t_2, \ldots, d_n t_n = b.$$

Such a sequence will be referred to as an *H*-sequence (of length *n*) connecting *a* and *b*. It is convenient to allow n = 0 in the above sequence; the empty sequence is interpreted as asserting equality a = b. Where convenient we will use the fact that  $\Omega^*$  is a submonoid of the free group FG( $\Omega$ ) on  $\Omega$ , in order to give the natural meaning to expressions such as  $yx^{-1}$ , where  $x, y \in \Omega^*$  and x is a suffix of y.

# 2. Proof of Theorem 1

Let  $\Omega$  be a set; it is clearly enough to show that  $\Omega^*$  is right coherent. To this end let  $\rho$  be the right congruence on  $\Omega^*$  generated by a finite subset H of  $\Omega^* \times \Omega^*$ , which without loss of generality we assume to be symmetric.

**Definition 2.** A quadruple (a, u; b, v) of elements of S is said to be *irreducible* if  $(au, bv) \in \rho$  and for any common non-empty suffix x of u and v we have that  $(aux^{-1}, bvx^{-1}) \notin \rho$ .

**Definition 3.** An *H*-sequence  $(c_1, d_1, t_1; \ldots; c_n, d_n, t_n)$  with

$$au = c_1 t_1, d_1 t_1 = c_2 t_2, \dots, d_n t_n = bv$$

is *irreducible* with respect to (a, u; b, v) if  $u, t_1, \ldots, t_n, v \in \Omega^*$  do not have a common nonempty suffix. Clearly, this is equivalent to one of  $u, t_1, \ldots, t_n, v$  being  $\epsilon$ .

Throughout this note for an *H*-sequence as above we define  $a = d_0, u = t_0, c_{n+1} = b$  and  $t_{n+1} = v$ . It is clear that if the quadruple (a, u; b, v) is irreducible then any *H*-sequence connecting au and bv must be irreducible with respect to (a, u; b, v).

We define

 $K = \max\{|p| : (p,q) \in H\}.$ 

**Lemma 4.** Let the *H*-sequence  $(c_1, d_1, t_1; \ldots; c_n, d_n, t_n)$  with

$$au = c_1 t_1, d_1 t_1 = c_2 t_2, \dots, d_n t_n = bv$$

be irreducible with respect to (a, u; b, v). Then either the empty H-sequence is irreducible with respect to  $(a, u; c_1, t_1)$  (in which case  $|u| \leq max (|b|, K)$  and  $u = \epsilon$  or  $t_1 = \epsilon$ ) or there exist an index  $1 \leq i \leq n$  such that  $t_{i+1} = \epsilon$  (so that  $au \rho c_{i+1}$ ) and  $x \in \Omega^+$  such that  $|x| \leq max (|b|, K)$ , the sequence

$$(c_1, d_1, t_1 x^{-1}; \ldots; c_{i-1}, d_{i-1}, t_{i-1} x^{-1})$$

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satisfies

$$aux^{-1} = c_1t_1x^{-1}, d_1t_1x^{-1} = c_2t_2x^{-1}, \dots, d_{i-1}t_{i-1}x^{-1} = c_it_ix^{-1},$$

and is an irreducible H-sequence with respect to  $(a, ux^{-1}; c_i, t_i x^{-1})$ .

*Proof.* If the empty sequence is irreducible with respect to  $(a, u; c_1, t_1)$  then either  $u = \epsilon$  or  $t_1 = \epsilon$ . In both cases we have that  $|u| \leq \max(|b|, K)$ . Suppose therefore that the empty sequence is not irreducible with respect to  $(a, u; c_1, t_1)$ . Let  $i \in \{1, \ldots, n\}$  be the smallest index such that  $t_{i+1} = \epsilon$  (such an index exists, because our original sequence is irreducible), and let x be the longest common non-empty suffix of  $u = t_0, t_1, \ldots, t_i$ . Then the sequence

$$(c_1, d_1, t_1 x^{-1}; \ldots; c_{i-1}, d_{i-1}, t_{i-1} x^{-1})$$

clearly satisfies

$$aux^{-1} = c_1t_1x^{-1}, d_1t_1x^{-1} = c_2t_2x^{-1}, \dots, d_{i-1}t_{i-1}x^{-1} = c_it_ix^{-1}$$

and is irreducible with respect to  $(a, ux^{-1}; c_i, t_ix^{-1})$ . Furthermore, since  $t_{i+1} = \epsilon$ , we have that  $d_i t_i = c_{i+1}$ , so x is a suffix of  $c_{i+1}$ . If i < n then  $(c_{i+1}, d_{i+1}) \in H$ , while if i = n we have  $c_{i+1} = b$ . In either case  $|x| \leq |c_{i+1}| \leq \max(|b|, K)$ .

We deduce immediately that one condition for coherency of  $\Omega^*$  is fulfilled.

**Corollary 5.** Let  $a, b \in S$ . Then  $(a\rho)S \cap (b\rho)S$  is empty or finitely generated.

*Proof.* Let us suppose that  $(a\rho)S \cap (b\rho)S \neq \emptyset$  and let

 $X = \{a\rho, b\rho, c\rho : (c, d) \in H\} \cap (a\rho)S \cap (b\rho)S.$ 

We claim that X generates  $(a\rho)S \cap (b\rho)S$ . It is enough to show that for every irreducible quadruple (a, u; b, v) we have that  $(au)\rho \in X$ . For this, let  $(c_1, d_1, t_1; \ldots; c_n, d_n, t_n)$  be an *H*-sequence with

$$au = c_1 t_1, \ldots, d_n t_n = bv.$$

Note that this sequence is necessarily irreducible with respect to (a, u; b, v). Then by Lemma 4, either  $u = \epsilon$ , or  $t_i = \epsilon$  for some  $i \in \{1, \ldots, n\}$ , or  $v = t_{n+1} = \epsilon$ . In each of these cases we see that  $(au)\rho \in X$ .

It remains to show that for any  $a \in \Omega^*$ , the right congruence  $r(a\rho)$  is finitely generated. To this end we first present a technical result.

**Lemma 6.** Let  $(c_1, d_1, t_1; ...; c_n, d_n, t_n)$  with

$$au = c_1 t_1, \ldots, d_n t_n = bv$$

be an irreducible H-sequence with respect to (a, u; b, v). Then either  $u = \epsilon$ , or there exist a factorisation  $u = x_k \dots x_1$  and indices  $n + 1 \ge \ell_1 > \ell_2 > \dots > \ell_k \ge 1$  such that for all  $1 \le j \le k$ :

(i)  $0 < |x_j| \le max (|b|, K)$  and (ii)  $aux_1^{-1} \dots x_{j-1}^{-1} \rho c_{\ell_j}$  (note that for j = 1 we have an  $\rho c_{\ell_1}$ ). *Proof.* We proceed by induction on |u|: if |u| = 0 the result is clear. Suppose that |u| > 0 and the result is true for all shorter words. If the empty sequence is irreducible with respect to  $(a, u; c_1, t_1)$ , then  $t_1 = \epsilon$  and the factorisation  $u = x_1$  satisfies the required conditions, with k = 1 and  $\ell_1 = 1$ . Otherwise, by Lemma 4, there exist an index  $1 \le i \le n$  such that  $t_{i+1} = \epsilon$ , so that  $au \rho c_{i+1}$ , and  $x_1 \in \Omega^+$  such that  $|x_1| \le \max(|b|, K)$  and the sequence

$$(c_1, d_1, t_1 x_1^{-1}; \dots; c_{i-1}, d_{i-1}, t_{i-1} x_1^{-1})$$

satisfies

$$aux_1^{-1} = c_1t_1x_1^{-1}, d_1t_1x_1^{-1} = c_2t_2x_1^{-1}, \dots, d_{i-1}t_{i-1}x_1^{-1} = c_it_ix_1^{-1}$$

and is an irreducible *H*-sequence with respect to  $(a, ux_1^{-1}; c_i, t_ix_1^{-1})$ . Put  $\ell_1 = i + 1$ . Since  $|ux_1^{-1}| < |u|$ , the result follows by induction.

**Lemma 7.** Let  $a \in \Omega^*$ . Then  $r(a\rho)$  is finitely generated.

Proof. Let 
$$K' = \max(K, |a|) + 1$$
,  $L = 2|H| + 2$ ,  $N = K'L$  and define  

$$X = \{(u, v) : |u| + |v| \le 3N\} \cap r(a\rho).$$

We claim that X generates  $r(a\rho)$ . It is clear that  $\langle X \rangle \subseteq r(a\rho)$ .

Let  $(u, v) \in r(a\rho)$ . We show by induction on |u| + |v| that  $(u, v) \in \langle X \rangle$ . Clearly, if  $|u| + |v| \leq 3N$ , then  $(u, v) \in X$ . We suppose therefore that |u| + |v| > 3N and make the inductive assumption that if  $(u', v') \in r(a\rho)$  and |u'| + |v'| < |u| + |v|, then  $(u', v') \in \langle X \rangle$ . If the quadruple (a, u; a, v) is not irreducible, it is immediate that  $(u, v) \in \langle X \rangle$ . Without loss of generality we therefore suppose that the quadruple (a, u; a, v) is irreducible and  $|v| \leq |u|$ , so that |u| > N. Let  $(c_1, d_1, t_1; \ldots; c_n, d_n, t_n)$  with

$$au = c_1 t_1, \ldots, d_n t_n = av$$

be an irreducible *H*-sequence with respect to (a, u; a, v). We apply Lemma 6, noting here that a = b. Clearly  $u \neq \epsilon$ , so by Lemma 6, there exists a factorisation  $u = x_k \dots x_1$ such that for all  $1 \leq j \leq k$  we have  $0 < |x_j| \leq K'$  and  $aux_1^{-1} \dots x_{j-1}^{-1} \rho c_{\ell_j}$  for some  $1 \leq \ell_j \leq n+1$ . Since |u| > K'L we have that k > L. Note that the number of distinct elements among  $c_1, \dots, c_n$  is less than L - 1. This in turn implies that there exist two indices  $1 \leq k - L < j < i \leq k$  such that  $c_{\ell_i} = c_{\ell_j}$ , so that

$$aux_1^{-1}\dots x_{i-1}^{-1} \ \rho \ c_{\ell_i} = c_{\ell_j} \ \rho \ aux_1^{-1}\dots x_{j-1}^{-1}$$

Since i, j > k - L we have that  $k - i + 1 \leq L$ , so  $|ux_1^{-1} \dots x_{i-1}^{-1}| = |x_k \dots x_i| \leq K'L$ , and similarly  $|ux_1^{-1} \dots x_{j-1}^{-1}| \leq K'L$ . As a consequence  $(ux_1^{-1} \dots x_{i-1}^{-1}, ux_1^{-1} \dots x_{j-1}^{-1}) \in X$ , and letting  $u' = ux_1^{-1} \dots x_{i-1}^{-1}x_{j-1} \dots x_k$ , we see that

$$(u', u) = (ux_1^{-1} \dots x_{i-1}^{-1}, ux_1^{-1} \dots x_{j-1}^{-1})x_{j-1} \dots x_1 \in \langle X \rangle.$$

In particular,  $au' \rho au \rho av$ . Note that |u'| < |u|, because j < i and  $x_j \neq \epsilon$ . Thus by the induction hypothesis we have that  $(v, u') \in \langle X \rangle$  and so the lemma is proved.

In view of the characterisation of coherency given in [4] and cited in the Introduction, Corollary 5 and Lemma 7 complete the proof of Theorem 1.

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# 3. Comments

Given that the class of right coherent monoids is closed under retract [5], it follows from the results of that paper that free monoids are coherent. However, as the arguments in [5] for free left ample monoids are burdened with unavoidable technicalities, we prefer to present here the more transparent proof that  $\Omega^*$  is coherent, by way of motivation for the work of [5]. With free objects in mind, we remark that we also show in [5] that the free inverse monoid on  $\Omega$  is not coherent if  $|\Omega| > 1$ .

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