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A Top Dog Tale with Preference Complementarities

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Abstract

The emergence of a winner-take-all (top dog) equilibrium outcome is generally due to political or institutional constraints or to specific technological features which favour the performance of just one individual. In this paper we provide a different explanation for the occurrence of a top-dog equilibrium in exchange economies. We show that once heterogeneous complementarities (i.e. Scarf's preferences) are analysed with general endowment distributions, a variety of equilibria different from the well-known symmetric outcome with full utilisation of resources can emerge. Specifically, we show that stable corner equilibria with a winner-take-all (top dog) individual arise that are Pareto optima although the remaining individuals are no better off than with zero consumption and resources can be unused. Because of heterogenous complementarities, market mechanisms are weak and cannot overcome the top dog's power. Voting mechanisms or taxation policies can reduce the top dog's privileged position.

Keywords: Exchange economy; Complements; Top dog allocation.

JEL classification: D50; D61.

1 Introduction

The emergence of an individual with an exceptional performance (a winner-take-all, a superstar – we address her as a "top dog") has been extensively analysed in different contexts of individual interaction, e.g. voting systems (Lizzeri and Persico 2005; Lizzeri and Persico 2001; Lindbeck and Weibull 1987), innovation industries (for example, De Nicolò and Franzoni 2010; Moldovanu and Sela 2001; Maurer and Scotchmer 2002; Fullerton and Preston McAfee 1999), sport or art markets (Rosen (1981)), hierarchical societies (Piccione and Rubinstein 2007; Feldman and Serrano 2006). The occurrence of a top dog equilibrium is generally the result of political or institutional constraints, or of specific supply and technological features, which magnify the performance of just one individual. It may depend on

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the rules of the electoral system, on policy decisions such as the choice of a patent system vs more permissive protection mechanisms in innovation races, on talent and technology, or even on an exogenous ranking scheme defining the distribution of power between individuals, in which the more powerful individual can seize the endowment of the least powerful (Piccione and Rubinstein (2007), Feldman and Serrano (2006)).

In a competitive market setting with equilibrium outcomes defined in terms of fixed individual endowments and preferences, can a stable top dog equilibrium also emerge? In this paper, we find that a combination of bundled preference complementarities[1] and asymmetry in the endowment distributions can generate such outcomes in which only one individual gets the highest consumption shares or all the endowments of the desired goods and the resources left do not yield any utility to the other individuals. We show that such equilibria are Pareto efficient, although they may imply unused resources and some consumers being no better off than with zero consumption. Asymmetry of the endowment and the bundled complementarity entail that some individuals have nothing of value to trade in equilibrium since some (part of the) endowments have no utility raising power for any individual.

We start from Scarf's economy (1960). This economy was introduced to highlight the possibility of instability of general equilibrium. Scarf showed that for a special initial endowment distribution between individuals, there is a unique market equilibrium with equal prices of the goods that is globally unstable. In the sense above this is not a top dog equilibrium since each individual has an equal share of the aggregate endowment of the goods he values. His work had a strong impact on the development of general equilibrium theory, since it was the first clear example of global instability of the tatonnement process.

Market equilibrium in this exchange economy has been later investigated with various but still special initial endowment restrictions. For example, Hirota (1981) analyses the equilibrium assuming that the sum of the initial endowment across goods is equal for each individual. Anderson *et al* (2004) developed an experimental double auction and allowed prices to adjust under a nontatonnement rule, based on the same endowment restrictions as those imposed by Hirota. Mukherji (2007) also uses Scarf's initial endowments but with the second good as a parameter. In all these contributions only the symmetric interior

[1] Bundled complementarities exist if each individual utility depends only on fixed coefficient combinations of different pairs (or groups) of goods.

equilibrium emerges with equal positive prices of all goods and full utilisation of resources.

In this paper, we provide a complete analysis of the efficiency, equilibrium and stability properties of these economies with Scarf's preferences, allowing for more general endowment distribution. Once a more general environment is considered, a variety of Pareto optima and equilibria different from the symmetric outcome can arise. We identify all these Pareto efficient allocations and corresponding individual endowment distributions and prices that will decentralise them through an exchange economy equilibrium.

Specifically, we show that three classes of Pareto optima emerge. There is the symmetric Pareto optimum with the full utilisation of endowments in which all individuals equally share the aggregate endowment of the goods they want (Class I, i.e. the Scarf/Hirota case). In all the other cases, corner Pareto optima arise in which the whole or part of the aggregate endowment of one good generates no incremental utility for any individual (Class II and III). These Pareto optima involve allocations of goods where a top dog individual gets a higher share of all the goods that he wants than the other individuals do. In extreme cases (Class III), the top dog is a winner-take-all-individual getting the whole aggregate endowment of the goods she values, while the bottom dog individuals can do no better than the utility level corresponding to zero consumption. In the other corner optima there is a similar but less extreme phenomena. We show that it is possible to uniquely identify the set of prices that decentralise the different Pareto optima. Finally we conduct stability analysis of these different market equilibria, showing the stability of the extreme top dog Pareto equilibrium in which the second class citizens are no better off than with zero consumption.

The paper is organised as follows. After the introduction of the base scenario (Scarf's preferences), we find the three classes of Pareto optima of this economy. In Section 2.2, we analyse the feasible types of market equilibria. We next define the set of prices and initial endowment distributions that can decentralise the different Pareto optima (Section 3 and 4). We conduct a stability analysis of the market equilibria (Section 5).

2 The Economy

For the sake of simplicity, we consider the original Scarf economy with perfect complements and cyclical preferences of three individuals and three goods (X,Y,Z). In an appropriate normalisation, individual preferences are given by

$$\begin{aligned} u_1(x_1, y_1, z_1) &= \min\{y_1, z_1\}, \\ u_2(x_2, y_2, z_2) &= \min\{x_2, z_2\}, \\ u_3(x_3, y_3, z_3) &= \min\{x_3, y_3\}. \end{aligned} \tag{1}$$

There is an interlocking set of perfect complementarities in preferences between the three goods.

In the benchmark scenario with three individuals and goods, each good enters the preferences of two individuals and each individual gets utility only from two goods, but no pair of individuals care about exactly the same goods. Each individual wants to consume combinations of goods, "packages" of goods, in which the goods are in fixed proportions. But the set of goods each individual desires overlaps just partially, e.g. any two of the individuals have something in common but not everything. Typical examples can be found in the household environment when individuals are sharing something but not everything. Or in an international trade scenario where countries specialise on a set of goods that only partially overlaps with the set of goods of the other countries.

As well as a market exchange economy between consumers, there are other scenarios in which preference complementarities matter and other ways in which such allocations can be realised, e.g. voting or bargaining. A natural application of this setting is in social contracting, when the players are the representatives of social groups or social classes and the goods are local public goods or privately provided public goods. Since the total amount of resources is fixed, the task is to choose how to share a fixed amount of resources when each social group has a pool of priorities that only partially overlaps with the priorities of the other groups.

For convenience we set the aggregate endowments of the economy at 1 unit of each good. Obviously, changing the scale of the economy does not affect the nature of the results.

2.1 Pareto Optima

The set of feasible allocations is given by

$$F = \{x, y, z | \Sigma x_h \leq 1, \Sigma y_h \leq 1, \Sigma z_h \leq 1, x \geq 0, y \geq 0, z \geq 0\},$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3)$.

The set of efficient allocations is most easily shown in terms of the efficient utility distributions. Define

$$P_1 = \{x, y, z | (x, y, z)\varepsilon F, u_1(x_1, y_1, z_1) = 1 - a, u_2(x_2, y_2, z_2) = a, u_3(x_3, y_3, z_3) = a, 0 \leq a < 1/2\},$$
$$P_2 = \{x, y, z | (x, y, z)\varepsilon F, u_1(x_1, y_1, z_1) = a, u_2(x_2, y_2, z_2) = 1 - a, u_3(x_3, y_3, z_3) = a, 0 \leq a < 1/2\},$$
$$P_3 = \{x, y, z | (x, y, z)\varepsilon F, u_1(x_1, y_1, z_1) = a, u_2(x_2, y_2, z_2) = a, u_3(x_3, y_3, z_3) = 1 - a, 0 \leq a < 1/2\}.$$

We assume cardinal and interpersonally comparable utility[2]. Thus P_1 is a set of feasible efficient allocations which favour individual 1, in the sense that, as a varies, u_1 varies in the interval $(1/2, 1]$, while $u_2 = u_3$ vary in the interval $[0, 1/2)$. In this situation, we refer to the most favoured individual as the top dog**.**

The full set of efficient allocations is given by

$$P = P_1 \cup P_2 \cup P_3 \cup P_s$$

where P_s defines the symmetric efficient outcome:

$$P_s = \{x, y, z | (x, y, z)\varepsilon F, u_1(x_1, y_1, z_1) = u_2(x_2, y_2, z_2) = u_3(x_3, y_3, z_3) = 1/2\}.$$

P_s is the set of efficient allocations in which each individual consumes half of the aggregate endowment of each desired good.

The set of efficient allocations is characterised by three types of Pareto optima. Only the first type involves useful consumption of all the resources[3]. In the other cases only some quantities of the initial holdings are usefully consumed by the individuals. The individual is indifferent between keeping or freely disposing the part of an allocation that do not yield any Pareto improvement. Unconsumed resources can occur only in such allocations. Such

[2]If we define the top dog as the individual with the highest welfare and we compare the set of efficient allocations (and later the equilibria) in terms of utility distribution, we should assume cardinal and interpersonally comparable utility. Alternatively, the top dog in our context can be identified also as the individual getting the highest shares of all the goods that she desires and the analysis can be developed in terms of the individual consumption allocations (and later equilibria). This requires only ordinal preferences. However, for easy of exposition, we assume cardinal and interpersonally comparable utility. Of course our main results on the full set of efficient and equilibrium allocations and stability of the equilibria only depend on ordinal preferences.

[3]We use "usefully" consumed or utilised resources or "useful" consumption or resources to indicate economic activities or resources that yield utility. When (some quantities of) the goods are "not useful"or are "useless" the individual is indifferent between consuming or freely disposing them.

Pareto optima are also attainable in an economy with a lower aggregate endowment of one good.

(a) *Class I: Useful consumption of all the resources,* (i.e. $u_1 = u_2 = u_3 = 1/2$). This is the symmetric Pareto optimum in which each individual has an equal share of the two goods that she values in her utility.

$$\begin{aligned} y_1 &= z_1 = 1/2, \\ x_2 &= z_2 = 1/2, \\ x_3 &= y_3 = 1/2, \end{aligned}$$

and all endowments of all goods yield utility.

(b) *Class II: Partially useful consumption of the resources.* The aggregate endowment of one good can be partially unused or free disposed. There is an infinite number of other efficient allocations which can be reached without consuming the total endowment of one of the goods. For example set $u_1 = u_2 = a, u_3 = 1 - a$. This can be attained by consumptions

	x_h	y_h	z_h	u_h
$h = 1$	0	a	a	a
$h = 2$	a	0	a	a
$h = 3$	$1 - a$	$1 - a$	0	$1 - a$
Total	1	1	$2a$	

.

So long as $0 < a < 1/2$, these allocations are feasible and they cannot be bettered. There is a surplus of good Z available, but it cannot usefully be consumed by either individual 3 (he does not want it) nor by individuals 1 and 2 (since there is no matching remaining amount of their complementary good available). For example, if $a = 1/4$, individual 3 is the top dog and 50% of good Z yields no utility. Similarly, there are two alternative Pareto optima in which only half of one good is not usefully consumed but in which there is a different top dog individual.

(c) *Class III: One good is totally useless.* It can be totally unused or free disposed. This class is characterised by three Pareto optima in which one individual gets the total endowment of two goods and the third good does not provide any utility to any individual. For example

$$u_1 = 1 \text{ with } y_1 = z_1 = 1; u_2 = u_3 = 0.$$

Here 1 uses all of Y, Z which since these are essential goods for $2, 3$, it means that $2, 3$ are restricted to the utility associated with zero consumption of the goods they care about.

2.2 Market Equilibria

Initial endowments for h are given by $\omega_h = (X_h, Y_h, Z_h)$, for consumer $h = 1, 2, 3$. Prices are $p = (p_x, p_y, p_z)$. Note also that homogeneity of degree zero in prices implies that we can impose a price normalisation. The two most common are either to set one price equal to unity (but this assumes that any equilibrium will have a positive price in that particular market, i.e. the numeraire good is not in excess supply in equilibrium) or $\Sigma p_i = 1$. Here we use the latter normalisation.

All goods are owned by some individual so that, as the aggregate endowment of each good is unity,

$$\Sigma_h X_h = \Sigma_h Y_h = \Sigma_h Z_h = 1.$$

Define the individual demand for each good f_{ih}, with $i = X, Y, Z$, $h = 1, 2, 3$. They are given by

$$\begin{aligned} f_{x1} &= 0, f_{y1} = f_{z1} = \frac{p_x X_1 + p_y Y_1 + (1 - p_x - p_y) Z_1}{p_y + (1 - p_x - p_y)}, \\ f_{y2} &= 0, f_{x2} = f_{z2} = \frac{p_x X_2 + p_y Y_2 + (1 - p_x - p_y) Z_2}{p_x + (1 - p_x - p_y)}, \\ f_{z3} &= 0, f_{x3} = f_{y3} = \frac{p_x X_3 + p_y Y_3 + (1 - p_x - p_y) Z_3}{p_x + p_y}. \end{aligned}$$

These are continuous in prices for $p_x, p_y, p_z > 0$, they satisfy the individual budget constraints with equality and they are homogeneous of degree zero in p.

Note that f_{ih} is a correspondence when one price is zero. If two prices are zero, one individual will demand an infinite amount of the two desired goods.

For a fixed initial endowment distribution across individuals, an equilibrium is a price vector p, such that there is no aggregate excess demand E_i (with $i = X, Y, Z$), and for any good i, if there is excess supply at p of good i, then $p_i = 0$. Acutally, goods which in equilibrium are in excess supply are priced at zero. Formally, for a given initial endowment distribution between individuals, an equilibrium is a set of prices p_i and quantities such that

$$E_i \leq 0, p_i \geq 0, p_i E_i = 0 \quad i = X, Y, Z.$$

In the next sections, we find different combinations of endowments and prices which decentralise the different Pareto optima as a market equilibrium.

3 The Decentralisation of Pareto Optimum Class I

Here we have that all goods are consumed. To represent this as a market equilibrium, there must be an initial endowment distribution and prices such that all excess demands are zero (as each good is fully consumed) and prices are all positive.

From Walras law, we can focus on just two excess demands ($E_x = f_{x2} + f_{x3} - 1$ and $E_y = f_{y1} - f_{y3} - 1$). In fact to yield this Pareto optimal allocation, we must have $f_{x2} = f_{y1} = f_{x3} = 1/2$. These equations are not all independent, so we focus on the first two $f_{x2} = f_{y1} = 1/2$. Solving them, we find the price equilibrium levels:

$$\begin{aligned} p_x &= \frac{Y_2 - Z_2 + \frac{1}{4} - Z_2 Z_1 - \frac{1}{2} Y_1 + Y_1 Z_2}{Y_2 X_1 - Y_2 Z_1 + \frac{1}{2} Y_2 - Z_2 X_1 - \frac{1}{2} Z_2 + \frac{1}{2} X_1 - \frac{1}{2} Z_1 + \frac{1}{4} + Y_1 Z_2 - Y_1 X_2 + Z_1 X_2} \quad (2) \\ p_y &= \frac{\frac{1}{2} X_2 - Z_1 X_2 - \frac{1}{2} X_1 + Z_2 X_1 + \frac{1}{2} Z_1 - \frac{1}{4}}{Y_2 X_1 - Y_2 Z_1 + \frac{1}{2} Y_2 - Z_2 X_1 - \frac{1}{2} Z_2 + \frac{1}{2} X_1 - \frac{1}{2} Z_1 + \frac{1}{4} + Y_1 Z_2 - Y_1 X_2 + Z_1 X_2}. \end{aligned}$$

The market equilibrium allocation requires just two equations to be satisfied, whilst there are two normalised prices and six free initial endowment variables that can be selected. So there will be an infinity of ways of decentralising this efficient allocation.

3.1 Supporting Pareto Optimum Class I with Unequal Prices

Here we show that the symmetric Pareto optimum with the full utilisation of resources can be decentralised with positive prices iff the endowment of goods satisfies a set of linear restrictions (see below). These conditions generalises the endowment restrictions used by Hirota and Scarf. Under their restrictions, decentralisation of this Pareto optimum requires equal prices for all goods whereas under our more general restrictions this is not necessary.

Suppose we take an arbitrary initial endowment distribution ω_1, ω_2 with $X_3 = 1 - X_1 - X_2, Y_3 = 1 - Y_1 - Y_2, Z_3 = 1 - Z_1 - Z_2$:

Proposition 1. *The symmetric Pareto optimum with full utilisation of resources is* *supported by unequal prices iff*

$$\begin{aligned}\alpha X_1 + \beta Y_1 + (1-\alpha-\beta) Z_1 &= (1-\alpha)/2, \\ \alpha X_2 + \beta Y_2 + (1-\alpha-\beta) Z_2 &= (1-\beta)/2,\end{aligned} \tag{3}$$

where α, β *are arbitrary constants with* $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha \neq \beta$.

Proof. See appendix.

This endowment distribution leads to a market equilibrium with prices fixed at the value $p_x = \alpha$, $p_y = \beta$.

Lemma 1. *If (3) holds for some numbers* α, β *then*

$$\alpha X_3 + \beta Y_3 + (1-\alpha-\beta) Z_3 = (\alpha+\beta)/2. \tag{4}$$

Proof: *This is simply obtained summing the equations (3)*

$$\alpha(X_1 + X_2) + \beta(Y_1 + Y_2) + (1-\alpha-\beta)(Z_1 + Z_2) = 1 - \alpha/2 - \beta/2.$$

and considering that the aggregate endowment of each good is equal to one, getting (4).

(3) and (4) define the restrictions between individual endowments which lead to the equal equilibrium in which each individual has equal shares of her desired goods. For fixed α, β we have three linear equations which must be satisfied by the individual endowments of the different goods. Alternatively, for each individual, their endowments must satisfy a linear restriction.

Example 1. *The endowment distribution*

$$Z_1 = 0.3; Y_1 = 0.7; X_1 = .04; Z_2 = 0.35; Y_2 = 0.1; X_2 = .59$$

yields $p_x = \alpha = 0.28, p_y = \beta = 0.33$. *But the endowment distribution*

$$Z_1 = 0.3; Y_1 = 0.4; X_1 = .4; Z_2 = 0.35; Y_2 = 0.1; X_2 = .59$$

yields exactly the same equilibrium.

We can use (3) to generate special cases of endowment distributions in which the equilibrium prices have special properties. For example, the Pareto optimum with usefully consumed resources can be supported by p_y costing twice p_x if and only if in (3) $\beta = 2\alpha$. Another case of some interest is that in which in equilibrium goods X and Y are equally expensive. Then two individuals trading these goods between themselves would be in a similar position of relative advantage. The same type of Pareto optimum with full utilisation is supported by $p_x = p_y \neq p_z$ for all goods iff in (3) $\beta = \alpha$.

When $\beta = \alpha$ in (3), the equilibrium prices (2) are

$$\begin{aligned} p_x &= p_y = \frac{Z_3}{(2Z_3 + 1 - X_3 - Y_3)} = \frac{Z_3}{k + 3Z_3}, \\ p_z &= 1 - 2p = \frac{k + Z_3}{k + 3Z_3}, \end{aligned} \tag{5}$$

where $k = 1 - X_3 - Y_3 - Z_3$. This gives a whole family of values of initial endowment distributions supporting a market equilibrium which decentralises this type of Pareto optimum with $p_x = p_y \neq p_z$. Fig. 1 plots these alternative equilibrium prices as a function of $k = 1 - X_3 - Y_3 - Z_3$ and Z_3.

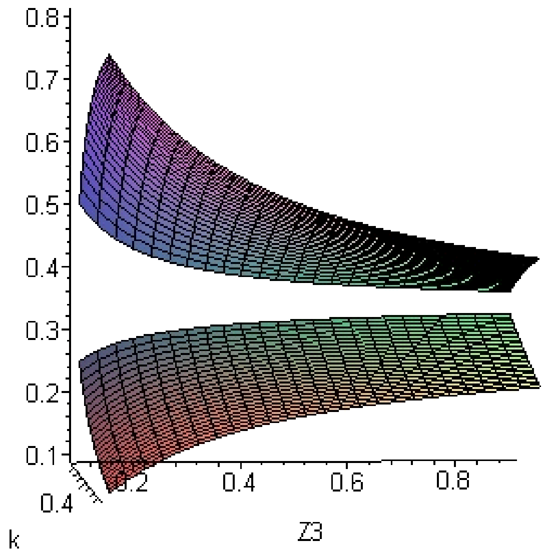

Fig. 1: Equilibrium prices p_x, p_z as a function of k and Z_3.

For example, if $k = 0.2, Z_3 = 0.25$ then $p_x = .263, p_z = .474$. And so on for other combinations.

3.2 Supporting Pareto Optimum Class I with Equal Positive Prices

Scarf and Hirota use The particular distributions of initial endowments by Scarf and Hirota leads to an equilibrium with equal consumption shares and equal prices ($p_x = p_y = p_z = 1/3$). Hirota's class is defined by

$$X_h + Y_h + Z_h = 1 \quad \text{for all } h.$$

In fact Scarf's endowments, i.e. $Y_1 = Z_2 = X_3 = 1$ and with the remaining endowments equal to zero, are a special case of Hirota's class of endowments. Hirota's endowments have the strong interpretation that, when they hold, all individuals have equal wealth if prices are equal for all goods. We can derive this class of endowments from (3) by setting $\alpha = \beta = 1/3$.

We can then ask what is the full set of initial endowment distributions which make $p_x = p_y = p_z = 1/3$ a market equilibrium which decentralises this Pareto optimum.

Proposition 2. *The symmetric Pareto optimum with full utilisation of resources is supported by equal positive prices for all goods iff the Hirota conditions hold.*

Proof. See appendix.

The equilibrium with equal quantities and prices is obtained when the total endowment is equally distributed among individuals. On average, every individual has the same power in trading since every individual has got a third of the total initial endowment. Setting the prices equal allows one unit of any good to exchange for one unit of any other good.

4 Decentralisation of Corner Pareto Optima

By definition, in a corner Pareto optimum one individual has higher consumption of all her desired goods than the other individuals have of their desired goods. This one individual is the top dog. Below we characterise the prices and the exact endowment distribution restriction for each type of corner Pareto optimum. One aspect of the endowment restriction is that the top dog must have a sufficiently large endowment of at least one of the goods which she wishes to consume. Note that the top-dog equilibrium with unused resources which emerges in our context is different from the D-manipulable equilibrium of Postlewaite (1979). In the latter, equilibria without the full utilisation of resources emerge because individuals can gain from destroying part of their ex-ante endowment of resources, manipulating the size of the economy and reducing the trade opportunities for other individuals. In our context, instead, the size of the economy is fixed and the equilibrium with unused resources is an ex-post result of trading with preference complementarities.

4.1 Corner Pareto Optimum Class II

Pareto optima belonging to Class II have the form $u_h = 1-a, u_k = a = u_l$ for $h, k, l = 1, 2, 3$ with $0 < a < 1/2$. If we analyse one case say $u_1 = 1-a, u_2 = a = u_3$ the others will follow.

In this case, $y_1 = z_1 = 1-a; x_2 = z_2 = a; x_3 = y_3 = a$ with other consumptions being zero. Generally, we think of 1 as being the favoured individual so that $a < 1/2$, in which case part of the total endowment of x is not usefully consumed at the Pareto optimum. In market terms, prices must be such that x is in excess supply. To decentralise this class of Pareto Optima as a market equilibrium, it must be that $p_x = 0$. We know that the total endowment of goods y and z is consumed, so in market equilibrium they must exhibit zero excess demand. So we can take $p_y, p_z > 0$ and for example normalise the prices so that $p_x + p_y + p_z = p_y + p_z = 1$.

Proposition 3. *A Pareto optimum with utility distributions* $u_1 = 1-a, u_2 = u_3 = a$ *with* $0 < a < 1/2$ *is supported with prices* $p_x = 0, 0 < p_y = k < 1, 0 < p_z < 1$ *iff*

$$\begin{aligned} kY_1 + (1-k)Z_1 &= 1-a, \\ kY_2 + (1-k)Z_2 &= (1-k)a, \end{aligned} \tag{6}$$

with $k \neq (1-k)$, $p_z \neq k$.

Proof. *(a) In the equilibrium* $u_1 = 1-a, u_2 = u_3 = a < 1/2$. *There is excess supply for good* X. *Thus* $p_x = 0$. *Suppose that* $p_y = k$, *so* $p_y(1-a) + p_z(1-a) = 1-a$ *and* $p_z a = (1-k)a$. *The wealths of the individuals 1 and 2 are*

$$\begin{aligned} kY_1 + (1-k)Z_1 &= 1-a, \\ \frac{(kY_2 + (1-k)Z_2)}{(1-k)} &= a \end{aligned}$$

giving (4) in the text.

(b) Suppose that (6) holds. We have to show that $p_y = k$. The wealth for individual 1 is:

$$p_y Y_1 + (1 - p_y) Z_1 = 1 - a$$

implying

$$p_y = \frac{(1 - a - Z_1)}{(Y_1 - Z_1)}$$

Substituting out $1 - a$ from (6) for individual 1 we get:

$$p_y = k.$$

■

To support the corner Pareto optima, what matters is the endowment/wealth distribution. In the case above, individual 1 is the top dog with most of the endowment. The wealth of individuals 2 and 3 valued at the equilibrium prices is lower than the wealth of individual 1 valued at the equilibrium prices, since $a < 1/2$ and $0 < k < 1$. Note that although the bottom dogs 2 and 3 have equal shares of consumption of their desired goods, in general their wealths valued at equilibrium prices differ. If $k = 1/2$ they have equal wealth, but if $p_y = k < 1/2$ (and so $p_z > 1/2$), individual 3 who wants to consume X and Y has lower wealth than individual 2, who wants to consume X and Z.

4.2 Corner Pareto Optimum Class III

In this class, the Pareto optimum displays extreme inequity since the top dog consumes the aggregate endowment of the two goods she wishes and the remaining two individuals are not better off than can do no better than with zero consumption of each good (for example, $u_1 = 1, u_2 = u_3 = 0$). This can be supported as a market equilibrium only if the top dog, individual 1, has got all the endowment of the two goods that he likes, whatever the distribution of the good that he does not want among the other individuals. The net-trade conditions in this case for individual 2 and 3 are respectively $kY_2 + (1 - k)Z_2 = 0$ and $kY_3 + (1 - k)Z_3 = 0$, which implies that $Y_2 = Z_2 = Y_3 = 0$ (since $0 < k < 1$ and $Y_h \geq 0, Z_h \geq 0$) and so from the aggregate endowment availability: $Y_1 = Z_1 = 1$.

Proposition 4. *The Pareto optimum with utility distribution* $u_1 = 1, u_2 = u_3 = 0$ *is supported with prices* $p_x = 0,$*and* $p_y > 0$, $p_z > 0$ *iff*

$$Y_1 = Z_1 = 1, Y_2 = Z_2 = Y_3 = Z_3 = 0.$$

The tog dog position in such a case is extreme: individuals 2 and 3 have only the endowment of the zero priced good (i.e., X), while individual 1 has the total endowment of the goods with some exchange value.

Non-market mechanisms such as voting can offset the top dog's power. For example consider the allocation process in which starting from the initial endowment distribution, individuals take turns to propose a new feasible allocation as an alternative to the status quo. If the allocation is by simple majority voting, the proposal voted by at least two individuals becomes the new status quo and this process is iterated. The final allocation is one which cannot be defeated in majority vote against any new proposal by any individual. Suppose that the initial endowment distribution is such that there is a top-dog equilibrium. Then the bottom dogs in an unfavorable position can propose an endowment or final allocation change which improve their wellbeing and cannot be defeated by the top dog in majority vote. And there is no alternative feasible allocation which the top dog can propose which will overturn this outcome in a majority vote.

Alternatively, an inequality-averse government may want to offset the top dog power. Faced with an endowment distribution leading to a top dog outcome, the government may wish to use either direct commodity transfers or, failing that, fiscal policy to move to the efficient outcome with equal consumption shares of the desired goods. If the government has the power to redistribute goods, it can also just redistribute directly to the allocation that it wishes to realise. This will then result in a no-trade market equilibrium. Moreover, the government may not have direct redistribution power although it can use commodity taxation. For example, it can tax the top-dog wealth and deliver subsidies or tax credits for the remaining individuals.

5 Stability Of Market Equilibria Under Tatonnement

If the endowment distributions belong to one of the classes stated in Proposition 1-4, the equilibrium is unique. For more general and arbitrary endowment distributions, it is not guaranteed the uniqueness of the equilibrium, since interior and corner solutions may arise. If the equilibrium is not unique, then it can not be globally stable. This is the reason for which in this section we focus on the local stability analysis. We focus on the standard tatonnement process, neglecting some recent developments which consider alternative but less intuitive processes, e.g. price-scaled tatonnement (Yamamoto, Asano, Togawa and Masanori 2008), α-compatible price adjustment (Artik 2003) or integral controller's mechanisms (Ogata 1970; Kumar and Shubik 2004). We confine attention to price adjustment stemming from the sign and level of excess demands similar to Scarf's original setting.

In general, for local stability, the excess demand functions must be downward sloping in their own price and the feedback cross effects between markets should be "small" in comparison with the own price effects. Generally, we can write the Jacobian of the excess demand functions for x and y as

$$J = \begin{bmatrix} \partial E_x/\partial p_x & \partial E_x/\partial p_y \\ \partial E_y/\partial p_x & \partial E_y/\partial p_y \end{bmatrix}, \tag{7}$$

so that

$$det(J) = (\partial E_x/\partial p_x)(\partial E_y/\partial p_y) - (\partial E_x/\partial p_y)(\partial E_y/\partial p_x),$$

and $trace(J) = \partial E_x/\partial p_x + \partial E_y/\partial p_y$. If the excess demand functions are downward sloping in their own price, then the trace is always negative. The condition for the determinant to be positive is that

$$(\partial E_x/\partial p_x)(\partial E_y/\partial p_y) > (\partial E_x/\partial p_y)(\partial E_y/\partial p_x).$$

We can think of this as saying that the aggregate of cross market effects (the LHS) should be small in absolute value relative to the own price effects.

The equilibrium is locally stable if the determinant is positive and the trace is negative. We provide some examples of local stability or instability of the equilibrium.

Example 2. If $Z_1 = 0.3$, $Y_1 = 0.7$, $Z_2 = 0.35$, $Y_2 = 0.1$, $X_1 = .04$, $X_2 = .59$ then $p_x = 0.28$, $p_y = 0.33$. With these values the determinant of the Jacobian is $-.09$ and the trace is $-.94$. In such a case, the equilibrium is locally unstable. On the other hand, if we take the other endowment distribution ($Z_1 = 0.3$, $Y_1 = 0.4$, $Z_2 = 0.35$, $Y_2 = 0.1$, $X_1 = .4$, $X_2 = .59$), then again $p_x = 0.28$, $p_y = 0.33$ but now the determinant has a value of .331 while the trace is equal to -1.448. In this case the equilibrium is locally stable.

Example 3. If we fix $Y_2 = 0.1$, $Z_3 = .25$, $Y_3 = .5$, $X_3 = .3$ we have $p_x = p_y = .357$ and $p_z = 0.286$ whilst the trace t and determinant d are respectively

$$\begin{aligned} t &= -2.769 + 3.111X_2 + 1.444Z_2 \\ d &= 1.219 - 1.084Z_2 - 1.355X_2 \end{aligned}$$

and we can plot these as functions of X_2, Z_2.

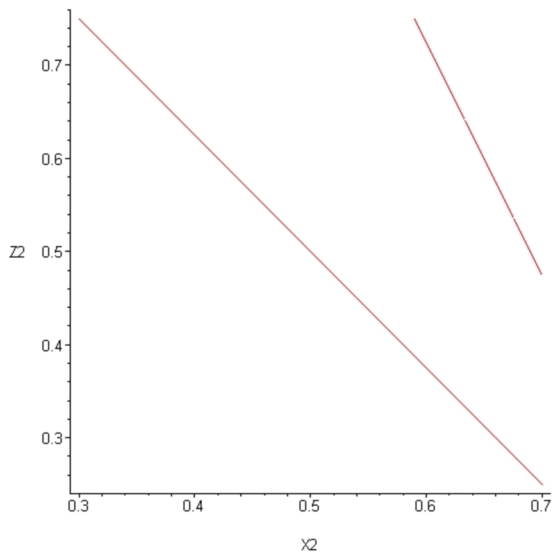

Fig. 2: Trace and determinant as a function of X_2, Z_2.

In Fig. 2, on the line further from the origin the trace is zero whilst on the line closer to the origin the determinant is zero. Below both lines we have the trace negative and the determinant positive so the equilibrium is locally stable; between the lines we have the determinant and trace both negative whilst above both lines the determinant is negative whilst the trace is positive. Thus in either of these second cases we have local instability.

Instead, the equilibrium that emerges with a partial use of resources is locally stable for any initial conditions starting with a zero price for the good which is in excess supply. Indeed, suppose that $p_y = 0$ $p_y = k, p_z = 1 - k$. In such a case, the endowment distribution supporting the equilibrium are $Y_2 = ((1-k)/k)(a - Z_2); Z_3 = (k/(1-k))(a - Y_3); Y_1 = 1/k[1 - a - (1-k)Z_1]$.The excess demand function for y is

$$E_y = \frac{(p_y(1-k)(a - Z_2))/k + (1 - p_y)Z_2)}{(1 - p_y)} + \frac{p_y Y_3 + \frac{(1-p_y)(a-Y_3)k}{1-k}}{p_y} - 1.$$

Computing its derivative and evaluating at $p_y = k$ we obtain

$$\frac{\partial E_y}{\partial p_y} = \frac{-2a + Y_3 + Z_2}{k} < 0. \tag{8}$$

The equilibrium is always stable since $(Y_3 - a) < 0$ and $(Z_2 - a) < 0$.

However starting with arbitrary initial conditions, for the equilibria with some trade which have $p_x = 0$ and individual 1 as the top dog, the sign of the determinant and the trace depends on the initial endowment distribution. For example, when $p_x = k$, $p_y = 1 - k$ and $Y_1 = (-(1-k)Z_1 + (1-a))/k$; $Y_2 = (-(1-k)Z_2 + (1-k)a)/k$, the determinant of (7) is

$$d = \frac{a[2(X_1 + X_2)(1-k) + k - 2(1 + Z_2)] + 2a^2 - 2(1-k)[(Z_2X_1 - Z_1X_2 + X_2)] + (1 - Z_1 + Z_2)])}{k^2(1-k)}$$

The trace is equal to

$$t = \frac{k^2(2a - 1 + 2X_2 - 2Z_2 + X_1) + k(2(1-a) - Z_1 - X_2 - X_1)) - 1 + Z_1 + Z_2}{k^2(1-k)}.$$

Given the endowment distribution of Proposition 4, the extreme corner solution in which one good is completely unused is a no-trade equilibrium displaying globally stability.

6 Conclusion

The type of heterogeneous complementarities in the Scarf model have previously always been analysed with endowment distributions which lead to the symmetric equilibrium with equal prices and equal shares consumed of each good by all consumers. The instability property of the original Scarf' example is a corner stone in general equilibrium. We show that once a more general endowment distribution is considered, stable corner equilibria arise. Such equilibria are Pareto optima although they may imply unused resources and a few individuals being no better off than with zero consumption. In such a case, a winner-take-all (top dog) individual emerges which consumes all the valuable endowments. Because of heterogenous complementarities, market mechanisms are weak and cannot overcome the basic inequality in the endowment distribution.

Non-market allocation mechanisms such a simple majority voting could neutralise the initial endowment power of a top dog. Alternatively, fiscal tools could be used to reduce the initial inequality in the endowment distribution. Taxing the top-dog's wealth and using tax credits or subsidies for the losers should reach this aim.

For the sake of simplicity, we have developed the analysis considering the original Scarf's setting with three individuals and three goods. Of course, our results can be applied to more general environments. First, all the previous results hold in an economy with n individuals and n goods in which each individual wants to consume $n-1$ goods in fixed proportions and the non desired good differs among individuals. Second, similar results arise assuming cyclical complementarity in synthetic aggregates produced by technologies allowing substitution between inputs or assuming cyclical complements in access costs to markets. Finally, the equilibria with partially useful resources can arise without assuming cyclical preference, i.e. if individuals want to consume all the goods in different fixed proportion with regard to the other individuals or in similar proportion but with different aggregate endowment of each goods.

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A Appendix

Proof of Proposition 1 (Case with different prices)

a) Suppose that the price are unequal and such that: $p_x = \alpha$, $p_y = \beta$ and $(1-\alpha-\beta) = p_z$, with $0 < 1-\alpha-\beta < 1$, and $0 < \alpha \neq \beta \neq \gamma < 1$. The equilibrium conditions become:

$$\begin{aligned} f_{y1} &= \frac{\alpha X_1 + \beta Y_1 + (1-\alpha-\beta)Z_1}{(1-\alpha)} = 1/2, \\ f_{x2} &= \frac{\alpha X_2 + \beta Y_2 + (1-\alpha-\beta)Z_2}{(1-\beta)} = 1/2, \end{aligned} \tag{9}$$

which imply:

$$\begin{aligned} \alpha X_1 + \beta Y_1 + (1-\alpha-\beta)Z_1 &= (1-\alpha)/2 \\ \alpha X_2 + \beta Y_2 + (1-\alpha-\beta)Z_2 &= (1-\beta)/2 \end{aligned}$$

(b) Conversely suppose the conditions (3) hold. Then we have to show that this implies that $p_x = \alpha$; $p_y = \beta$. Again multiplying through (9), we get the linear system:

$$\begin{aligned} p_x X_1 + p_y Y_1 + (1 - p_x - p_y) Z_1 &= (1 - p_x)/2, \\ p_x X_2 + p_y Y_2 + (1 - p_x - p_y) Z_2 &= (1 - p_y)/2. \end{aligned}$$

Solving these linear equations we get:

$$\begin{aligned} p_y &= -\frac{(-\frac{1}{2}X_2 + Z_1X_2 + \frac{1}{2}X_1 - Z_2X_1 - \frac{1}{2}Z_1 + \frac{1}{4})}{(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2)}, \qquad (10) \\ p_x &= \frac{(Y_2 - Z_2 + \frac{1}{4} - Z_2Z_1 - \frac{1}{2}Y_1 + Y_1Z_2)}{(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2)}. \end{aligned}$$

This solution requires that the determinant condition

$$(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2) \neq 0$$

should hold.

Recalling the general Hirota conditions (3):

$$X_1 = (\frac{-\beta Y_1 - (1 - \alpha - \beta)Z_1 + (1 - \alpha)/2}{\alpha}),$$

$$X_2 = (\frac{-\beta Y_2 - (1 - \alpha - \beta)Z_2 + (1 - \beta)/2}{\alpha}),$$

and substituting them in (10) gives $p_x = \alpha; p_y = \beta$. ■

The sufficient and necessary conditions to decentralise the other special cases of the interior Pareto optimum with i) p_y costing twice p_x, or ii) $p_x = p_y$ can be shown by simply assuming in the above proof respectively that i) $p_x = \alpha$, $p_y = 2\alpha$ and $\alpha(X_1 + 2Y_1) + (1 - \alpha - \beta)Z_1 = (1 - \alpha)/2$, $\alpha(X_2 + 2Y_2) + (1 - \alpha - \beta)Z_2 = (1 - \beta)/2$, ii) $p_x = p_y = \alpha$, $\alpha(X_1 + Y_1) + (1 - \alpha - \beta)Z_1 = (1 - \alpha)/2$ and $\alpha(X_2 + Y_2) + (1 - \alpha - \beta)$

The sufficient and necessary conditions are derived using the same procedure as the proof of Proposition 1 and imposing $p_x = p_y = \alpha = \beta = 1/3$.

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