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A Dual Algorithm for Stochastic Control Problems: Applications to Uncertain Volatility Models and CVA*

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Abstract. We derive an algorithm in the spirit of Rogers [*SIAM J. Control Optim.*, 46 (2007), pp. 1116–1132] and Davis and Burstein [*Stochastics Stochastics Rep.*, 40 (1992), pp. 203–256] that leads to upper bounds for stochastic control problems. Our bounds complement lower biased estimates recently obtained in Guyon and Henry-Labordère [*J. Comput. Finance*, 14 (2011), pp. 37–71]. We evaluate our estimates in numerical examples motivated by mathematical finance.

Key words. optimal stochastic control, duality theory, numerical methods

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1. Introduction.

Solving stochastic control problems, for example, by approximating the Hamilton–Jacobi–Bellman (HJB) equation, is an important problem in applied mathematics. Classical PDE methods are effective tools for solving such equations in low-dimensional settings, but quickly become computationally intractable as the dimension of the problem increases: a phenomenon commonly referred to as "the curse of dimensionality." Probabilistic methods on the other hand such as Monte Carlo simulation are less sensitive to the dimension of the problem. It was demonstrated in Pardoux and Peng [16] and Cheridito et al. [3] that first and second backward stochastic differential equations (in short BSDE) can provide stochastic representations that may be regarded as a nonlinear generalization of the classical Feynman–Kac formula for semilinear and fully nonlinear second order parabolic PDEs.

The numerical implementation of such a BSDE-based scheme associated with a stochastic control problem was first proposed in Bouchard and Touzi [2], also independently in Zhang [19]. Further generalization was provided in Fahim, Touzi, and Warin [8] and in Guyon and Henry-Labordère [10]. The algorithm in [10] requires evaluating high-dimensional conditional expectations, which are typically computed using parametric regression techniques. Solving the BSDE yields a suboptimal estimation of the stochastic control. Performing an additional, independent (forward) Monte Carlo simulation using this suboptimal control, one obtains a biased estimation: a lower bound for the value of the underlying stochastic control problem. Choosing the right basis for the regression step is in practice a difficult task, particularly in high-dimensional settings. In fact, a similar situation arises for the familiar Longstaff–Schwarz

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algorithm, which also requires the computation of conditional expectations with parametric regressions and produces a low-biased estimate.

As the algorithm in [10] provides a biased estimate, i.e., a lower bound, it is of limited use in practice, unless it can be combined with a dual method that leads to a corresponding upper bound. Such a dual expression was obtained by Rogers [17], building on earlier work by Davis and Burstein [4]. While the work of Rogers is in the discrete time setting, it applies to a general class of Markov processes. Previous work by Davis and Burstein [4] linking deterministic and stochastic control using flow decomposition techniques (see also Diehl, Friz, and Gassiat [5] for a rough path approach to this problem) is restricted to the control of a diffusion in its drift term. In the present paper we are also concerned with the control of diffusion processes, but allow the control to act on both the drift and the volatility term in the diffusion equation. The basic idea underlying the dual algorithm in all these works is to replace the stochastic control by a pathwise deterministic family of control problems that are not necessarily adapted. The resulting "gain" of information is compensated for by introducing a penalization analogous to a Lagrange multiplier. In contrast to [4] and [5], we do not consider continuous pathwise, i.e., deterministic, optimal control problems. Instead, we rely on a discretization result for the HJB equation due to Krylov [12] and recover the solution of the stochastic control problem as the limit of deterministic control problems over a finite set of discretized controls.

Our paper is structured as follows. In section 2 we introduce the stochastic control problem and derive the dual bounds in the Markovian setting for European-type payoffs. In section 3.1 we generalize our estimates to a non-Markovian setting, i.e., where the payoff has a path dependence. Finally, in section 3.2 we consider a setting suitable for pricing American style options in a Markov setting. We evaluate the quality of the upper bounds obtained in two numerical examples. First, we consider the pricing of a variety of options in the uncertain volatility model (UVM). Based on our earlier estimates we transform the stochastic optimization problem into a family of suitably discretized deterministic optimizations, which we can in turn approximate, for example, using local optimization algorithms. Second, we consider a problem arising in credit valuation adjustment. In this example, the deterministic optimization can particularly efficiently be solved by deriving a recursive ODE solution to the corresponding Hamilton–Jacobi equations. Our algorithm complements the lower bounds derived in [10] by effectively reusing some of the quantities already computed when obtaining the lower bounds (cf. Remark 2.8).

2. Duality result for European options.

2.1. Notations. We begin by introducing some basic notations. For any $k \in \mathbb{N}$ let

$$\Omega^k := \{\omega : \omega \in C([0,T], \mathbb{R}^k), \omega_0 = 0\}.$$

Let $d, m \in \mathbb{N}$ and $T > 0$. Define $\Omega := \Omega^d$, $\Theta := [0,T] \times \Omega$, and let B denote the canonical process on Ω^m with $\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$ the filtration generated by B. Finally, denote by $\mathbb{P}_0$ the Wiener measure.

For $h > 0$, consider a finite partition $\{t_i^h\}_i$ of $[0,T]$ with mesh less than h, i.e., such that $t_{i+1}^h - t_i^h \le h$ for all i. For some $M > 0$, let A be a compact subset of

$$O_M := \{x \in \mathbb{R}^k : |x| \le M\} \quad \text{for some} \quad k \in \mathbb{N},$$

and N^h be a finite h-net of A, i.e., for all $a, b \in N^h \subset A$, we have $|a-b| \le h$. We define sets

- $\mathcal{A} := \{\varphi : \Theta \to \mathbb{R}^k : \ \varphi \text{ is } \mathbb{F}\text{-adapted, and takes values in } A\}$;
- $\mathcal{A}_h := \{\varphi \in \mathcal{A} : \ \varphi \text{ is constant on } [t_i^h, t_{i+1}^h) \text{ for } i, \text{ and takes values in } N^h\}$;
- $\mathcal{U} := \{\varphi : \Theta \to \mathbb{R}^d : \ \varphi \text{ is bounded and } \mathbb{F}\text{-adapted}\}$;
- $\mathcal{D}_h := \{f : [0,T] \to \mathbb{R}^k : \ f \text{ is constant on } [t_i^h, t_{i+1}^h) \text{ for } i, \text{ and takes values in } N^h\}$.

For the following it is important to note that $\mathcal{D}_h$ is a finite set of piecewise constant functions.

We would like to emphasize that, throughout this paper, C denotes a generic constant, which may change from line to line. For example the reader may find $2C \le C$, without any contradiction as the left-hand side C is different from the right-hand side C.

2.2. The Markovian case. We consider stochastic control problems of the form

$$u_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_0}\left[\int_0^T R_t^\alpha f(t, \alpha_t, X_t^\alpha)dt + R_T^\alpha g(X_T^\alpha)\right], \tag{2.1}$$

where $R_t^\alpha := e^{-\int_0^t r(s, \alpha_s, X_s^\alpha)ds}$, X^α is a d-dimensional controlled diffusion defined by

$$X^\alpha := \int_0^\cdot \mu(t, \alpha_t, X_t^\alpha)dt + \int_0^\cdot \sigma(t, \alpha_t, X_t^\alpha)dB_t,$$

and the functions μ, σ, f, r satisfy the following assumption.

Assumption 2.1. The functions μ, σ, f, r defined on $\mathbb{R}^+ \times A \times \mathbb{R}^d$ take values in $\mathbb{R}^d$, $\mathbb{R}^{d\times m}$, $\mathbb{R}$, $\mathbb{R}$, respectively. Assume that

- μ, σ, f, r are uniformly bounded, and continuous in α;
- μ, σ, f, r are uniformly δ_0-Hölder continuous in t for some fixed constant $\delta_0 \in (0,1]$;
- μ, σ are uniformly Lipschitz in x, and f, r are uniformly δ_0-Hölder continuous in x;
- $g : \mathbb{R}^d \to \mathbb{R}$ is continuous.

Remark 2.2. Our assumptions match the assumptions on the continuity of the coefficients in Krylov [12, 13], and allow us to apply his results.

Our main result is a duality in the spirit of [4] that allows us to replace the stochastic control problem by a family of suitably discretized deterministic control problems. We first discretize the control problem through the following lemma which is a direct consequence of Theorem 2.3 in Krylov [12].

Define the function

$$u_0^h := \sup_{\alpha \in \mathcal{A}_h} \mathbb{E}^{\mathbb{P}_0}\left[\int_0^T R_t^\alpha f(t, \alpha_t, X_t^\alpha)dt + R_T^\alpha g(X_T^\alpha)\right].$$

Lemma 2.3. *Suppose Assumption 2.1 holds and g is bounded. We have for any family of partition of $[0,T]$ with mesh tending to zero that*

$$u_0 = \lim_{h \to 0} u_0^h. \tag{2.2}$$

Remark 2.4. Theorem 2.3 in [12] also gives a rate of convergence for the discretization in Lemma 2.3, i.e., there exists a constant $C > 0$ such that

$$\left|u_0 - u_0^h\right| \le C h^{\frac{1}{3}} \quad \text{for all} \quad 0 < h \le 1.$$

For the following statement, we introduce

$$\begin{aligned} v^h &:= \inf_{\varphi\in\mathcal{U}} \mathbb{E}^{\mathbb{P}_0}\left[\max_{a\in\mathcal{D}_h} \Phi^{a,\varphi}\right] \quad \text{with} \\ \Phi^{a,\varphi} &:= R_T^a g(X_T^a) + \int_0^T R_t^a f(t, a_t, X_t^a) dt - \int_0^T R_t^a \varphi_t(X^a)^\intercal \sigma(t, a_t, X_t^a) dB_t. \end{aligned} \tag{2.3}$$

Remark 2.5. It is noteworthy that stochastic integrals are defined in L^2-space, so it is in general meaningless to take the pathwise supremum of a family of stochastic integrals. However, as we mentioned before, the set $\mathcal{D}_h$ is of finite elements. So there is a unique random variable in L^2 equal to the maximum value of the finite number of stochastic integrals, $\mathbb{P}_0$-a.s.

The next theorem allows us to recover the stochastic optimal control problem as a limit of discretized deterministic control problems.

Theorem 2.6. *Suppose Assumption 2.1 holds and g is bounded. Then we have*

$$u_0 = \lim_{h\to 0} v^h.$$

Proof. We first prove that $u_0 \le \underline{\lim}_{h\to 0} v^h$. Recall u_0^h defined in (2.2). Since R^α, σ are bounded, for all $\varphi \in \mathcal{U}$ the process $\int_0^\cdot R_t^\alpha \varphi_t(X^\alpha)^\intercal \sigma(t, \alpha_t, X_t^\alpha) dB_t$ is a martingale. So we have

$$u_0^h = \sup_{\alpha\in\mathcal{A}_h} \mathbb{E}^{\mathbb{P}_0}\left[\Phi^{\alpha,\varphi}\right].$$

Since $\Phi^{\alpha,\varphi} \le \max_{a\in\mathcal{D}_h} \Phi^{a,\varphi}$ for all $\alpha \in \mathcal{A}_h$, we have

$$u_0^h \le \mathbb{E}^{\mathbb{P}_0}\left[\max_{a\in\mathcal{D}_h} \Phi^{a,\varphi}\right].$$

The required result follows.

To show $u_0 \ge \overline{\lim}_{h\to 0} v^h$ we construct an explicit minimizer φ^*. First note that under Assumption 2.1, it is easy to verify that u_t defined as

$$u(t,x) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_0}\left[\int_t^T \frac{R_s^\alpha}{R_t^\alpha} f(s, \alpha_s, X_s^\alpha) ds + \frac{R_T^\alpha}{R_t^\alpha} g(X_T^\alpha) \Big| X_t^\alpha = x\right]$$

is a viscosity solution to the Dirichlet problem of the HJB equation:

$$\begin{aligned} &-\partial_t u - \sup_{b\in A}\left\{\mathcal{L}^b u + f(t,b,x)\right\} = 0, \quad u_T = g, \\ &\text{where} \quad \mathcal{L}^b u := \mu(t,b,x)\cdot\partial_x u + \frac{1}{2}\mathrm{Tr}\big((\sigma\sigma^\intercal)(t,b,x)\partial_{xx}^2 u\big) - r(t,b,x)u. \end{aligned} \tag{2.4}$$

We next define the mollification $u^{(\varepsilon)} := u * K^{(\varepsilon)}$ of u, where K is a smooth function with compact support in $(-1,0)\times O_1$ (O_1 is the unit ball in $\mathbb{R}^d$), and $K^{(\varepsilon)}(x) := \varepsilon^{-n-2}K(t/\varepsilon^2, x/\varepsilon)$. Clearly, $u^{(\varepsilon)} \in C_b^\infty$ and $u^{(\varepsilon)}$ converges uniformly to u. As mentioned in Remark 2.2, Assumption 2.1 matches the assumptions in [13], where the author proved in his Theorem 2.1 that $u^{(\varepsilon)}$ is a classical supersolution to the HJB equation (2.4). Denote

$$\varphi_t^\varepsilon(\omega) := \partial_x u^{(\varepsilon)}(t,\omega_t). \tag{2.5}$$

Since $u^{(\varepsilon)} \in C_b^\infty$, it follows from Ito's formula that

$$\begin{aligned} R_T^a u^{(\varepsilon)}(T,X_T^a) - u_0^{(\varepsilon)} &= \int_0^T R_t^a\big(\partial_t u^{(\varepsilon)} + \mathcal{L}^{a_t}u^{(\varepsilon)}(t,X_t^a)\big)dt \\ &\quad + \int_0^T R_t^a \varphi_t^\varepsilon(X^a)^\intercal \sigma(t,a_t,X_t^a)dB_t \quad \text{for all } a\in\mathcal{D}_h, \quad \mathbb{P}_0\text{-a.s.} \end{aligned}$$

Then, by the definition of $\Phi^{a,\varphi^\varepsilon}$ in (2.3), we obtain

$$\begin{aligned} \Phi^{a,\varphi^\varepsilon} = R_T^a g(X_T^a) &+ \int_0^T R_t^a\Big(f(t,a_t,X_t^a) + \big(\partial_t u^{(\varepsilon)} + \mathcal{L}^{a_t}u^{(\varepsilon)}\big)(t,X_t^a)\Big)dt \\ &- R_T^a u^{(\varepsilon)}(T,X_T^a) + u_0^{(\varepsilon)} \quad \text{for all } a\in\mathcal{D}_h, \quad \mathbb{P}_0\text{-a.s.} \end{aligned}$$

Since $u^{(\varepsilon)}$ is a supersolution to the HJB equation (2.4), it follows that

$$\Phi^{a,\varphi^\varepsilon} \le R_T^a\big(g(X_T^\alpha) - u^{(\varepsilon)}(T,X_T^\alpha)\big) + u_0^{(\varepsilon)} \quad \text{for all } a\in\mathcal{D}_h, \quad \mathbb{P}_0\text{-a.s.} \tag{2.6}$$

By Assumption 2.1 and the fact that g is bounded,

$$\Phi^{a,\varphi^\varepsilon} \text{ is uniformly bounded from above.} \tag{2.7}$$

Also, it is easy to verify that the function u is continuous and therefore uniformly continuous on $S_L := [0,T]\times\{|x|\le L\}$ for any $L>0$ and that $u^{(\varepsilon)}$ converges uniformly to u on S_L. In particular,

$$\begin{aligned} &u_0^{(\varepsilon)} \to u_0, \\ &\rho_L(\varepsilon) := \max\nolimits_{|x|\le L}\left|g(x) - u^{(\varepsilon)}(T,x)\right| \to 0, \end{aligned} \qquad \text{as } \ \varepsilon\to 0. \tag{2.8}$$

It follows from (2.6), (2.7), and (2.8) that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_0}\Big[\max_{a\in\mathcal{D}_h}\Phi^{a,\varphi^\varepsilon}\Big] &= \mathbb{E}^{\mathbb{P}_0}\Big[\max_{a\in\mathcal{D}_h}\Phi^{a,\varphi^\varepsilon}; \max_{a\in\mathcal{D}_h}|X_T^a|\le L\Big] + \mathbb{E}^{\mathbb{P}_0}\Big[\max_{a\in\mathcal{D}_h}\Phi^{a,\varphi^\varepsilon}; \max_{a\in\mathcal{D}_h}|X_T^a| > L\Big] \\ &\le C\rho_L(\varepsilon) + u_0^{(\varepsilon)} + C\mathbb{P}_0\Big[\max_{a\in\mathcal{D}_h}|X_T^a|>L\Big], \end{aligned}$$

where C is a constant independent of L and ε. Therefore

$$v^h \le \varliminf_{\varepsilon\to0}\mathbb{E}^{\mathbb{P}_0}\Big[\max_{a\in\mathcal{D}_h}\Phi^{a,\varphi^\varepsilon}\Big] \le u_0 + C\mathbb{P}_0\Big[\max_{a\in\mathcal{D}_h}|X_T^a|>L\Big] \quad \text{for any } \ L>0.$$

Further, since

$$\mathbb{P}_0\left[\max_{a\in\mathcal{D}_h}|X_T^a|>L\right] \le \sum_{a\in\mathcal{D}_h}\mathbb{P}_0\big[|X_T^a|>L\big] \to 0, \quad \text{as } \ L\to\infty,$$

we conclude that $v^h \le u_0$. So the required inequality follows. ∎

The boundedness assumption on g may be relaxed by means of a simple cutoff argument.

Corollary 2.7. *Assume that g is of polynomial growth, i.e.,*

$$|g(x)| \leq C\big(1+|x|^p\big) \quad \textit{for some} \quad C, p \geq 0.$$

Let $M > 0$, g^M a continuous compactly supported function that agrees with g on $O_M \subseteq \mathbb{R}^d$ and satisfies $|g^M| \leq |g|$. Let $v^{h,M}$ denote the approximations defined in (2.3) with respect to g^M in place of g. Then we have

$$\lim_{M\to 0} \left| u_0 - \lim_{h\to 0} v^{h,M} \right| = 0.$$

Proof. Define u_0^M as in (2.1) by using the approximation g^M, i.e.,

$$u_0^M := \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_0}\left[\int_0^T R_t^\alpha f(t,\alpha_t,X_t^\alpha)dt + R_T^\alpha g^M(X_T^\alpha)\right].$$

By Theorem 2.6, we know that $u_0^M = \lim_{h\to 0} v^{h,M}$.

Further, we have

$$\begin{aligned}|u_0 - u_0^M| &\leq C \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_0}\Big[g(X_T^\alpha) - g^M(X_T^\alpha)\Big]\\ &\leq C \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_0}\Big[|X_T^\alpha|^p + 1; |X_T^\alpha| \geq M\Big].\end{aligned}$$

Assume $M \geq 1$. Then we obtain

$$|u_0 - u_0^M| \leq C \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_0}\Big[|X_T^\alpha|^p; |X_T^\alpha| \geq M\Big] \leq C \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_0}\left[\frac{|X_T^\alpha|^{p+1}}{M}\right]. \tag{2.9}$$

Since μ, σ are both bounded, we have

$$\mathbb{E}^{\mathbb{P}_0}\Big[|X_T^\alpha|^{p+1}\Big] \leq C\mathbb{E}^{\mathbb{P}_0}\left[\left|\int_0^T \mu(t,\alpha_t,X_t^\alpha)dt\right|^{p+1} + \left|\int_0^T \sigma(t,\alpha_t,X_t^\alpha)dB_t\right|^{p+1}\right] \leq CT. \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\lim_{M\to\infty} |u_0 - u_0^M| = 0.$$

The proof is completed. ∎

We conclude the section with two remarks, both relevant to the numerical simulation of the approximation derived in Theorem 2.6.

Remark 2.8. To approximate v^h in our numerical examples we will as in the proof of Theorem 2.6 use fixed functions φ^* for the minimization. The definition (2.5) makes it clear that the natural choice for these minimizers are (the numerical approximations of) the function $\partial_x u$. Note that these approximations are readily available from the numerical scheme in [10] that is used to compute the complementary lower bounds.

Remark 2.9. In the proof of Theorem 2.6 we showed that $u_0^h \le v^h \le u_0$. It therefore follows from Remark 2.4 that there exists a constant $C > 0$ such that

$$\left|u_0 - v^h\right| \le Ch^{\frac{1}{3}}$$

for all $0 < h \le 1 \wedge T$.

3. Some extensions.

3.1. The non-Markovian case. In our first extension we consider stochastic control problems of the form

$$u_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_0}\Big[g(X^{\alpha}_{T\wedge\cdot})\Big],$$

where X^α is a d-dimensional diffusion defined by $X^\alpha := \int_0^\cdot \mu(t,\alpha_t)dt + \int_0^\cdot \sigma(t,\alpha_t)dB_t$. Note that in this setting μ and σ only depend on α and t, but the payoff function g is path dependent.

Remark 3.1. The arguments in this subsection are based on the "frozen-path" approach developed in Ekren, Touzi, and Zhang [6]. In order to apply their approach, we have restricted the class of diffusions X^α we consider, compared to the Markovian control problem.

Writing $\mathbb{P}_\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1}$, we have

$$u_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha}\Big[g(B_{T\wedge\cdot})\Big].$$

Throughout this subsection we will impose the following regularity assumptions.

Assumption 3.2. The functions $\mu, \sigma : \mathbb{R}^+ \times A \to E$ (E is the respective metric space) and $g : \Omega^d \to \mathbb{R}$ are uniformly bounded such that
- μ, σ are continuous in α;
- μ, σ are δ_0-Hölder continuous in t, for some constant $\delta_0 \in (0,1]$;
- g is uniformly continuous.

Example 3.3. Arguing as in Corollary 2.7 we may also consider unbounded payoffs. Hence, possible path-dependent payoffs that fit our framework include, e.g., the maximum $\max_{s\in[0,T]} \omega_s$ and Asian options $\frac{1}{T}\int_0^T \omega_s ds$.

Let

$$\Lambda_\varepsilon := \big\{t_0 = 0, t_1, t_2, \dots, t_n = T\big\}$$

be a partition of $[0,T]$ with mesh bounded above by ε. For $k \le n$ and $\pi_k = (x_1 = 0, x_2, \dots, x_k) \in \mathbb{R}^{d\times k}$, denote by $\Gamma_\varepsilon^{\Lambda_\varepsilon,k}(\pi_k)$ the path generated by the linear interpolation of the points $\{(t_i, x_i)\}_{0\le i\le k}$. Where no confusion arises with regards to the underlying partition we will in the following drop the superscript Λ_ε and write $\Gamma_\varepsilon^k(\pi_k)$ in place of $\Gamma_\varepsilon^{\Lambda_\varepsilon,k}(\pi_k)$, but it must be emphasized that the entire analysis in this subsection is carried out with a fixed but arbitrary partition Λ_ε in mind. Define the interpolation approximation of g by

$$g^\varepsilon(\pi_n) := g\Big(\Gamma_\varepsilon^n(\pi_n)\Big)$$

and define an approximation of the value function by letting

$$\theta_0^\varepsilon := \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha}\Big[g^\varepsilon\big((B_{t_i})_{0\le i\le n}\big)\Big].$$

The following lemma justifies the use of linear interpolation for approximating dependent payoff.

Lemma 3.4. *Under Assumption 3.2, we have*

$$\lim_{\varepsilon\to 0}\theta_0^\varepsilon = u_0.$$

Proof. Recall that g is uniformly continuous. Let ρ be a modulus of continuity of g. If necessary, we may choose ρ to be concave (by taking the concave envelope). Further, we define

$$\mathbf{w}_B(\varepsilon,T) := \sup_{s,t\le T;|s-t|\le\varepsilon}|B_s - B_t|.$$

Clearly, we have

$$\begin{aligned}|\theta_0^\varepsilon - u_0| &= \left|\sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mathbb{P}_\alpha}\big[g^\varepsilon\big((B_{t_i})_{0\le i\le n}\big)\big] - \sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mathbb{P}_\alpha}\big[g(B_{T\wedge\cdot})\big]\right|\\ &\le \sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mathbb{P}_\alpha}\big[\rho\big(\mathbf{w}_B(\varepsilon,T)\big)\big] \le \rho\left(\sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mathbb{P}_\alpha}\big[\mathbf{w}_B(\varepsilon,T)\big]\right).\end{aligned}$$

It is proved in Theorem 1 in Fisher and Nappo [9] that

$$\mathbb{E}^{\mathbb{P}_\alpha}\big[\mathbf{w}_B(\varepsilon,T)\big] \le C\left(\varepsilon\ln\frac{2T}{\varepsilon}\right)^{\frac{1}{2}},$$

where C is a constant only dependent on the bound of μ and σ. Thus,

$$\lim_{\varepsilon\to 0}\sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mathbb{P}_\alpha}\big[\mathbf{w}_B(\varepsilon,T)\big] = 0.$$

The proof is completed. ∎

We next define the controlled diffusion with time-shifted coefficients by setting

$$X^{\alpha,t} := \int_0^s \mu(t+r,\alpha_r)dr + \int_0^s \sigma(t+r,\alpha_r)dB_r,\ \ s\in[0,T-t],\ \ \mathbb{P}_0\text{-a.s.},$$

and the corresponding law

$$\mathbb{P}_\alpha^t := \mathbb{P}_0\circ(X^{\alpha,t})^{-1}.$$

Further, for $1\le k\le n-2$ let

$$\eta_k := t_{k+1} - t_k,$$

and define recursively a family of stochastic control problems:

$$\theta^\varepsilon(\pi_{n-1};t,x) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_{n-1}+t}}\Big[g^\varepsilon\big((\pi_{n-1}, x_{n-1}+x+B_{\eta_{n-1}-t})\big)\Big], \quad t\in[0,\eta_{n-1}),\ x\in\mathbb{R}^d,$$

$$\theta^\varepsilon(\pi_k;t,x) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_k+t}}\Big[\theta^\varepsilon\big((\pi_k, x_k+x+B_{\eta_k-t}),0,0\big)\Big], \quad t\in[0,\eta_k),\ x\in\mathbb{R}^d. \tag{3.1}$$

Clearly, $\theta^\varepsilon(0;0,0)=\theta_0^\varepsilon$.

Remark 3.5. By freezing the path π_k, we get the value function $\theta^\varepsilon(\pi_k;\cdot,\cdot)$ of a *Markovian* stochastic control problem on the small interval $[0,\eta_k)$. This will allow us to apply the PDE tools which played a key role in proving the dual form in the previous section.

Lemma 3.6. *Fix $\varepsilon>0$. The function $\theta^\varepsilon(\pi;t,x)$ is Borel measurable in all the arguments and uniformly continuous in (t,x) uniformly in π.*

Proof. It follows from the uniform continuity of g and the fact that interpolation with respect to a partition Λ_ε is a Lipschitz function (in this case from $\mathbb{R}^{n\times d}$ into the continuous functions), that g^ε is also uniformly continuous. Denote by ρ^ε a modulus of continuity of g^ε, chosen to be increasing and concave if necessary. For any $\pi_{n-1},\pi'_{n-1}\in\mathbb{R}^{(n-1)\times d}$, given $t\in[0,\eta_{n-1}]$, $x,x'\in\mathbb{R}^d$, we have

$$\begin{aligned}
&|\theta^\varepsilon(\pi_{n-1};t,x)-\theta^\varepsilon(\pi'_{n-1};t,x')|\\
&\quad\le \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_{n-1}+t}}\Big[\Big|g^\varepsilon\big((\pi_{n-1},x_{n-1}+x+B_{\eta_{n-1}-t})\big)-g^\varepsilon\big((\pi'_{n-1},x_{n-1}+x'+B_{\eta_{n-1}-t})\big)\Big|\Big]\\
&\quad\le \rho^\varepsilon(|(\pi_{n-1},x)-(\pi'_{n-1},x')|).
\end{aligned}$$

Similarly, for any $k<n-1$ and $\pi_k,\pi'_k\in\mathbb{R}^{k\times d}$, given $t\in[0,\eta_k]$, $x,x'\in\mathbb{R}^d$, we have

$$\begin{aligned}
&|\theta^\varepsilon(\pi_k;t,x)-\theta^\varepsilon(\pi'_k;t,x')|\\
&\quad\le \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_k+t}}\Big[\Big|\theta^\varepsilon\big((\pi_k,x_k+x+B_{\eta_k-t}),0,0\big)-\theta^\varepsilon\big((\pi'_k,x_k+x'+B_{\eta_k-t}),0,0\big)\Big|\Big]\\
&\quad\le \rho^\varepsilon(|(\pi_k,x)-(\pi'_k,x')|).
\end{aligned} \tag{3.2}$$

For $0\le t^0<t^1\le\eta_k$, it follows from the dynamic programming principle (for a general theory on the dynamic programming principle for sublinear expectations, we refer to Nutz and Van Handel [15]) that

$$\theta^\varepsilon(\pi_k;t^0,x)=\sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mathbb{P}_\alpha^{t_k+t^0}}\Big[\theta^\varepsilon(\pi_k;t^1,x+B_{t^1-t^0}))\Big] \tag{3.3}$$

and from (3.3) and (3.2) we deduce that

$$\begin{aligned}
|\theta^\varepsilon(\pi_k;t^0,x)-\theta^\varepsilon(\pi_k;t^1,x)| &\le \sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mathbb{P}_\alpha^{t_k+t^0}}\Big[\Big|\theta^\varepsilon(\pi_k;t^1,x+B_{t^1-t^0})-\theta^\varepsilon(\pi_k;t^1,x)\Big|\Big]\\
&\le \sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mathbb{P}_\alpha^{t_k+t^0}}\big[\rho^\varepsilon(|B_{t^1-t^0}|)\big]\\
&\le \rho^\varepsilon\left(\sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mathbb{P}_\alpha^{t_k+t^0}}\big[|B_{t^1-t^0}|\big]\right).
\end{aligned} \tag{3.4}$$

Similarly to (2.10), we have the estimate

$$\sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_k+t^0}}\Big[|B_{t^1-t^0}|\Big] = \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_0}\Big[|X^{\alpha,t_k+t^0}_{t^1-t^0}|\Big] \le C\big(t^1-t^0\big), \tag{3.5}$$

where C is a constant only dependent on the bound of μ and σ. It follows from (3.4) and (3.5) that

$$|\theta^\varepsilon(\pi_k;t^0,x)-\theta^\varepsilon(\pi_k;t^1,x)| \ \le\ \rho^\varepsilon\Big(C(t^1-t^0)\Big)\ .$$

Hence, combining (3.2) and (3.5) we conclude that $\theta^\varepsilon(\pi_k;t,x)$ is uniformly continuous in (t,x) uniformly in π_k. ∎

The functions $\theta^\varepsilon(\pi_k;\cdot,\cdot)$ are defined as the value functions of stochastic control problems, and one can easily check that they are viscosity solutions to the corresponding HJB equations. For $k=1,\dots,n-1$, we define a family of PDEs by letting

$$\begin{aligned} &-\mathbf{L}^k\theta = 0 \quad \text{on} \quad [0,\eta_k)\otimes\mathbb{R}^d, \ \text{where}\\ &\mathbf{L}^k\theta := \partial_t\theta + \sup_{b\in A}\left\{\mu\big(t_k+\cdot,b\big)\cdot\partial_x\theta + \frac{1}{2}\mathrm{Tr}\Big((\sigma\sigma^\intercal)(t_k+\cdot,b)\partial^2_{xx}\theta\Big)\right\}. \end{aligned} \tag{3.6}$$

The following proposition links the stochastic control problems with the PDE and applies, analogously to the Markovian case, a mollification argument.

Proposition 3.7. *There exists a function $u^{(\varepsilon)}:(\pi,t,x)\mapsto\mathbb{R}$ such that $u^{(\varepsilon)}(0,0,0)=\theta^\varepsilon_0+\varepsilon$ and for all π_k, $u^{(\varepsilon)}(\pi_k;\cdot,\cdot)$ is a classical supersolution to the PDE (3.6) and the boundary condition*

$$\begin{aligned} u^{(\varepsilon)}(\pi_k;\eta_k,x) &= u^{(\varepsilon)}\big((\pi_k,x);0,0\big) && \textit{if } k<n-1,\\ u^{(\varepsilon)}(\pi_k;\eta_k,x) &\ge g^\varepsilon\big((\pi_k,x)\big) && \textit{if } k=n-1. \end{aligned}$$

Proof. Define $\theta^{\varepsilon,\delta}(\pi_k;\cdot,\cdot):=\theta^\varepsilon(\pi_k;\cdot,\cdot)*K^\delta$ for all $\pi_k\in\mathbb{R}^{k\times d}$, $k\le n$, where K is a smooth function with compact support in $(-1,0)\times O_1$ (O_1 is the unit ball in $\mathbb{R}^d$), and $K^\delta(t,x):=\delta^{-d-2}K(t/\delta^2,x/\delta)$. By Lemma 3.6, $\theta^{\varepsilon,\delta}(\pi_k;\cdot,\cdot)$ converges uniformly to $\theta^\varepsilon(\pi_k;\cdot,\cdot)$ uniformly in π_k, as $\delta\to0$. Take δ small enough so that $\|\theta^{\varepsilon,\delta}-\theta^\varepsilon\|\le\frac{\varepsilon}{2n}$. Further, Assumption 3.2 implies that all the shifted coefficients $\mu(t_k+\cdot,\cdot),\sigma(t_k+\cdot,\cdot)$ satisfy the assumptions on the continuity of the coefficients in [13], where the author proved that

$$\theta^{\varepsilon,\delta}(\pi_k;\cdot,\cdot)\ \text{is a classical supersolution for}\ (3.6)\,.$$

Note that $\theta^{\varepsilon,\delta}(\pi_k;\cdot,\cdot)+C$ is still a supersolution for any constant C. So we may define a smooth function $v^\varepsilon(0;\cdot,\cdot):=\theta^{\varepsilon,\delta}(0;\cdot,\cdot)+C_0$ on $[0,t_1]\times\mathbb{R}^d$ with some constant C_0 such that

$$v^\varepsilon(0;0,0)=\theta^\varepsilon(0;0,0)+\frac{\varepsilon}{n},\quad v^\varepsilon(0;\cdot,\cdot)\ge\theta^\varepsilon(0;\cdot,\cdot).$$

Similarly, we define smooth functions $v^\varepsilon(\pi_k;\cdot,\cdot):=\theta^{\varepsilon,\delta}(\pi_k;\cdot,\cdot)+C_{\pi_k}$ on $[0,\eta_k]\times\mathbb{R}^d$ for $1\le k\le n-1$ with some constants C_{π_k} such that

$$v^\varepsilon(\pi_k;0,0)=v^\varepsilon(\pi_{k-1};\eta_{k-1},x_k-x_{k-1})+\frac{\varepsilon}{n},\quad v^\varepsilon(\pi_k;\cdot,\cdot)\ge\theta^\varepsilon(\pi_k;\cdot,\cdot)\ .$$

Finally, we define for $\pi_k \in \mathbb{R}^{k\times d}$ and $(t,x) \in [0,\eta_k) \times \mathbb{R}^d$

$$u^{(\varepsilon)}(\pi_k; t, x) := v^{\varepsilon}(\pi_k; t, x) + \frac{n-k+1}{n}\varepsilon.$$

It is now straightfoward to check that $u^{(\varepsilon)}$ satisfies the requirements. ■

The discrete framework we just developed may be linked to path space by means of linear interpolation along the partition Λ_ε. Recall that Θ was defined to be $[0,T]\times\Omega$.

Corollary 3.8. *Define* $\bar{u}^{(\varepsilon)} : \Theta \to \mathbb{R}$ *by*

$$\bar{u}^{(\varepsilon)}(t,\omega) := u^{(\varepsilon)}\big((\omega_{t_i})_{0\le i\le k}; t-t_k, \omega_t - \omega_{t_k}\big) \quad \textit{for} \;\; t\in[t_k,t_{k+1}).$$

There exist adapted processes $\lambda_t(\omega), \varphi_t(\omega), \eta_t(\omega)$ *such that for all* $\alpha\in\mathcal{A}$

$$\begin{aligned}\bar{u}^{(\varepsilon)}(T,X^\alpha) = \bar{u}^{(\varepsilon)}_0 &+ \int_0^T \left(\lambda_t + \mu(t,\alpha_t)\varphi_t + \frac{1}{2}\mathrm{Tr}\big((\sigma\sigma^\intercal)(t,\alpha_t)\eta_t\big)\right)(X^\alpha)dt \\ &+ \int_0^T \varphi_t(X^\alpha)^\intercal \sigma(t,\alpha_t) dB_t,\end{aligned}$$

$\mathbb{P}_0$*-a.s., and*

$$\left(\lambda_t + \mu(t,\alpha_t)\varphi_t + \frac{1}{2}\mathrm{Tr}\big((\sigma\sigma^\intercal)(t,\alpha_t)\eta_t\right)(\omega) \le 0 \quad \textit{for all} \;\; \alpha\in\mathcal{A}, (t,\omega)\in\Theta.$$

Proof. By Itô's formula, we have

$$\begin{aligned}\bar{u}^{(\varepsilon)}(t,X^\alpha) = \bar{u}^{(\varepsilon)}(t_k,X^\alpha) &+ \int_{t_k}^t \left(\lambda_s + \mu(s,\alpha_s)\varphi_s + \frac{1}{2}\mathrm{Tr}\big((\sigma\sigma^\intercal)(s,\alpha_s)\eta_s\big)\right)(X^\alpha)ds \\ &+ \int_{t_k}^t \varphi_s(X^\alpha)^\intercal\sigma(s,\alpha_s)dB_s \quad \text{for } t\in[t_k,t_{k+1}), \quad \mathbb{P}_0\text{-a.s.,}\end{aligned}$$

with

$$\left.\begin{aligned}\lambda_t(\omega) &:= \partial_t u^{(\varepsilon)}\big((\omega_{t_i})_{0\le i\le k}; t-t_k, \omega_t-\omega_{t_k}\big), \\ \varphi_t(\omega) &:= \partial_x u^{(\varepsilon)}\big((\omega_{t_i})_{0\le i\le k}; t-t_k, \omega_t-\omega_{t_k}\big), \\ \eta_s(\omega) &:= \partial^2_{xx} u^{(\varepsilon)}\big((\omega_{t_i})_{0\le i\le k}; t-t_k, \omega_t-\omega_{t_k}\big)\end{aligned}\right. \quad \text{for } t\in[t_k,t_{k+1}).$$

By the supersolution property of $u^{(\varepsilon)}$ proved in Proposition 3.7, we have

$$\begin{aligned}&\left(\lambda_t + \mu(t,\alpha_t)\varphi_t + \frac{1}{2}\mathrm{Tr}\big((\sigma\sigma^\intercal)(t,\alpha_t)\eta_t\big)\right)(\omega) \\ &\qquad\qquad \le \mathbf{L}^k u^{(\varepsilon)}\big((\omega_{t_i})_{0\le i\le k};\cdot,\cdot\big)(t-t_k,\omega_t-\omega_{t_k}) \;\le\; 0.\end{aligned}$$

The proof is completed. ■

Finally, we prove an approximation analogous to Theorem 2.6 in our non-Markovian setting.

Theorem 3.9. *Suppose Assumption 3.2 holds. Then we have*

$$u_0 = \lim_{h\to 0} v^h, \quad \text{where } v^h := \inf_{\varphi\in\mathcal{U}} \mathbb{E}^{\mathbb{P}_0}\left[\sup_{a\in\mathcal{D}_h}\left\{g(X^a_{T\wedge\cdot}) - \int_0^T \varphi_t(X^a)^\intercal \sigma(t,a_t)dB_t\right\}\right].$$

Proof. Arguing as in the proof of Theorem 2.6, one can easily deduce using the Ito formula that $u_0 \le \lim_{h\to 0} v^h$.

Consider the function $\bar{u}^{(\varepsilon)}$ and let φ be the process defined in Corollary 3.8. We have

$$\begin{aligned} v^h &\le \mathbb{E}^{\mathbb{P}_0}\left[\sup_{a\in\mathcal{D}_h}\left\{g(X^a_{T\wedge\cdot}) - \int_0^T \varphi_t(X^a)^\intercal\sigma(t,a_t)dB_t\right\}\right] \\ &\le \mathbb{E}^{\mathbb{P}_0}\left[\sup_{a\in\mathcal{D}_h}\left\{g(X^a_{T\wedge\cdot}) - \bar{u}^{(\varepsilon)}_T(X^a) + \bar{u}^{(\varepsilon)}_0\right\}\right] \\ &\le \mathbb{E}^{\mathbb{P}_0}\left[\sup_{a\in\mathcal{D}_h}\left\{g(X^a_{T\wedge\cdot}) - g^\varepsilon\left((X^a_{t_i})_{0\le i\le n}\right)\right\}\right] + \theta^\varepsilon_0 + \varepsilon. \end{aligned}$$

For the last inequality, we use the fact that $\bar{u}^{(\varepsilon)}_0 = u^{(\varepsilon)}(0;0,0) = \theta^\varepsilon_0 + \varepsilon$. Note that there are only finite elements in the set $\mathcal{D}_h$. Therefore, by Lemma 3.4

$$\begin{aligned} &\overline{\lim_{\varepsilon\to 0}}\left(\mathbb{E}^{\mathbb{P}_0}\left[\sup_{a\in\mathcal{D}_h}\left\{g(X^a_{T\wedge\cdot}) - g^\varepsilon\left((X^a_{t_i})_{0\le i\le n}\right)\right\}\right] + \theta^\varepsilon_0 + \varepsilon\right) \\ &\le \overline{\lim_{\varepsilon\to 0}}\left(\sum_{a\in\mathcal{D}_h}\mathbb{E}^{\mathbb{P}_0}\left[\left|g(X^a_{T\wedge\cdot}) - g^\varepsilon\left((X^a_{t_i})_{0\le i\le n}\right)\right|\right] + \theta^\varepsilon_0 + \varepsilon\right) \\ &= u_0. \end{aligned}$$

We conclude that $v^h \le u_0$ for all $h \in (0, 1\wedge T]$. ■

3.2. Example of a duality result for an American option. In this subsection we give an indication how our approach may be extended to American options. To this end we consider a toy model, in which the d-dimensional controlled diffusion X^α takes the particular form $X^\alpha := \int_0^\cdot \alpha^0_t dt + \int_0^\cdot \alpha^1_t dB_t$ and carry out the analysis in this elementary setting. The stochastic control problem is now

$$u_0 = \sup_{\alpha\in\mathcal{A},\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{P}_0}\left[g(X^\alpha_\tau)\right],$$

where $\mathcal{T}_T$ is the set of all stopping times smaller than T. Throughout this subsection we will make the following assumption.

Assumption 3.10. Suppose $g : \mathbb{R}^d \to \mathbb{R}$ to be bounded and uniformly continuous.

For $\alpha \in \mathcal{A}$ define probability measures $\mathbb{P}_\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1}$, let $\mathcal{P} := \{\mathbb{P}_\alpha : \alpha \in \mathcal{A}\}$, and define the nonlinear expectation $\mathcal{E}[\cdot] := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\cdot]$. It will be convenient to use the shorthand $\alpha^1 \cdot B$ for the stochastic integral $\int_0^\cdot \alpha^1_s dB_s$. We have

$$u_0 = \sup_{\tau \in \mathcal{T}_T} \mathcal{E}\big[g(B_\tau)\big].$$

Further, we define the dynamic version of the control problem

$$u(t,x) := \sup_{\tau \in \mathcal{T}_{T-t}} \mathcal{E}\big[g(x+B_\tau)\big] \quad \text{for} \quad (t,x) \in [-1,T] \times \mathbb{R}^d.$$

The following lemma shows that the function u satisfies a dynamic programming principle (see, for example, Lemma 4.1 of [7] for a proof).

Lemma 3.11. *The value function u is continuous in both arguments, and we have*

$$u(t_1,x) = \sup_{\tau \in \mathcal{T}_{T-t_1}} \mathcal{E}\big[g(x+B_\tau)1_{\{\tau<t_2\}} + u(t_2, x+B_{t_2})1_{\{\tau \geq t_2\}}\big].$$

In particular, $\{u(t,B_t)\}_{t\in[0,T]}$ is a $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathcal{P}$.

Next we apply the familiar mollification technique already employed in section 2.2. Define $u^{(\varepsilon)} := u * K^{(\varepsilon)}$.

Lemma 3.12. *$\{u^{(\varepsilon)}(t,B_t)\}_t$ is a $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathcal{P}$, and $u^{(\varepsilon)} \geq g^{(\varepsilon)} := g * K^{(\varepsilon)}$.*

Proof. For any $s \leq t \leq T$ and $x \in \mathbb{R}$, we have by Lemma 3.11

$$\begin{aligned}
\mathcal{E}\big[u^{(\varepsilon)}(t,x+B_{t-s})\big] &= \mathcal{E}\left[\int u(t-r, x-y+B_{t-s})K^{(\varepsilon)}(r,y)dydr\right] \\
&\leq \int \mathcal{E}\big[u(t-r,x-y+B_{t-s})\big]K^{(\varepsilon)}(r,y)dydr \\
&\leq \int u(t-r-(t-s), x-y)K^{(\varepsilon)}(r,y)dydr \\
&= \int u(s-r,x-y)K^{(\varepsilon)}(r,y)dydr \;=\; u^{(\varepsilon)}(s,x),
\end{aligned}$$

where for the second inequality, we used the $\mathbb{P}$-supermartingale property of $\{u(t,B_t)\}_{t\in[0,T]}$ for all $\mathbb{P} \in \mathcal{P}$. This implies that for all $\mathbb{P} \in \mathcal{P}$ we have

$$\mathbb{E}^{\mathbb{P}}\big[u^{(\varepsilon)}(t,x+B_{t-s})\big] \leq u^{(\varepsilon)}(s,x).$$

Therefore, $\{u^{(\varepsilon)}(t,B_t)\}_t$ is a $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathcal{P}$. On the other hand, it is clear from the definition of u that $u \geq g$ and, hence, $u^{(\varepsilon)} \geq g^{(\varepsilon)}$. ■

Again, the stochastic control problem can be discretized. For technical reasons, we assume here that the partitions of time satisfy the order

$$\{t_i^h\}_{i\leq n_h} \subset \{t_i^{h'}\}_{i \leq n_{h'}} \quad \text{for} \quad h > h', \tag{3.7}$$

where n_h is the number of time grids of the partition.

Lemma 3.13. *Under Assumption 3.10, it holds*

$$u_0 = \lim_{h\to 0} u_0^h, \quad \textit{where } u_0^h := \sup_{\alpha\in\mathcal{A}_h, \tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{P}_0}\Big[g(X_\tau^\alpha)\Big]. \tag{3.8}$$

Proof. We only prove the case $\alpha^0 = 0$ and $\alpha = \alpha^1 \in A^1$, a compact set in $\mathbb{R}$, in particular, $X^\alpha = (\alpha \cdot B)$. The general case follows by a straightfoward generalization of the same arguments. Note that it is sufficient to show that $u_0 \le \underline{\lim}_{h\to 0} u_0^h$. Fix $\epsilon > 0$. There exists $\alpha^\varepsilon \in \mathcal{A}$ such that

$$u_0 < \sup_{\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{P}_0}\big[g\big((\alpha^\varepsilon \cdot B)_\tau\big)\big] + \varepsilon. \tag{3.9}$$

For any h sufficiently small define a process $\tilde{\alpha}^h$ by letting

$$\tilde{\alpha}_t^h := \sum_i \frac{1}{t_{i+1}^h - t_i^h} \int_{t_i^h}^{t_{i+1}^h} \mathbb{E}^{\mathbb{P}_0}\left[\alpha_s^\varepsilon \middle| \mathcal{F}_{t_i^h}\right] ds \mathbf{1}_{[t_i^h, t_{i+1}^h)}(t).$$

Clearly, $\tilde{\alpha}^h$ is piecewise constant on each interval $[t_i^h, t_{i+1}^h)$. We introduce the filtration $\hat{\mathbb{F}} := \{\hat{\mathcal{F}}_h\}_h$, with

$$\hat{\mathcal{F}}_h := \sigma\Big(\Big\{[t_i^h, t_{i+1}^h) \times A : i \le n_h - 1, A \in \mathcal{F}_{t_i^h}\Big\}\Big).$$

In particular, it follows from (3.7) that $\hat{\mathcal{F}}_h \subset \hat{\mathcal{F}}_{h'}$ for $h > h'$. Also, denote the probability $\hat{\mathbb{P}}$ on the product space Θ:

$$\hat{\mathbb{P}}(dt, d\omega) := \frac{1}{T} dt \times \mathbb{P}_0(d\omega).$$

Note that for all $i \le n_h - 1$, $A \in \hat{\mathcal{F}}_{t_i^h}$, and $h' < h$ we have

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{P}}_0}\left[\tilde{\alpha}^{h'} 1_{\{[t_i^h, t_{i+1}^h)\times A\}}\right] &= \mathbb{E}^{\mathbb{P}_0}\left[\frac{1}{T} \sum_{j: t_i^h \le t_j^{h'}, t_{j+1}^{h'} \le t_{i+1}^h} \int_{t_j^{h'}}^{t_{j+1}^{h'}} \mathbb{E}^{\mathbb{P}_0}\left[\alpha_s^\varepsilon \middle| \mathcal{F}_{t_j^{h'}}\right] ds\ 1_A\right] \\
&= \mathbb{E}^{\mathbb{P}_0}\left[\frac{1}{T} \int_{t_i^h}^{t_{i+1}^h} \mathbb{E}^{\mathbb{P}_0}\left[\alpha_s^\varepsilon \middle| \mathcal{F}_{t_i^h}\right] ds\ 1_A\right] \\
&= \mathbb{E}^{\hat{\mathbb{P}}_0}\left[\tilde{\alpha}^h 1_{\{[t_i^h, t_{i+1}^h)\times A\}}\right].
\end{aligned}$$

So $\{\tilde{\alpha}^h\}_h$ is a martingale in the filtrated probability space $(\Theta, \hat{\mathbb{P}}, \hat{\mathbb{F}})$. Note that α^ε and $\tilde{\alpha}^h$ are bounded, so it follows from the martingale convergence theorem that

$$\lim_{h\to 0} \mathbb{E}^{\mathbb{P}_0} \int_0^T (\alpha_s^\varepsilon - \tilde{\alpha}_s^h)^2 ds = 0. \tag{3.10}$$

Further, define $\hat\alpha^h := h\lfloor \frac{\tilde\alpha^h}{h} \rfloor$ and note that we have $\hat\alpha^h \in \mathcal{A}_h$. It follows from (3.10) that

$$\lim_{h\to 0} \mathbb{E}^{\mathbb{P}_0} \int_0^T (\alpha_s^\varepsilon - \hat\alpha_s^h)^2 ds = 0.$$

With ρ an increasing and concave modulus of continuity of g we have

$$\begin{aligned}
&\sup_{\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{P}_0}\big[g\big((\alpha^\varepsilon\cdot B)_\tau\big)\big] - \sup_{\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{P}_0}\big[g\big((\hat\alpha^h\cdot B)_\tau\big)\big] \\
&\quad\le \sup_{\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{P}_0}\Big[\rho\big(|(\alpha^\varepsilon\cdot B)_\tau - (\hat\alpha^h\cdot B)_\tau|\big)\Big] \\
&\quad\le \mathbb{E}^{\mathbb{P}_0}\Big[\rho\big(\|(\alpha^\varepsilon\cdot B) - (\hat\alpha^h\cdot B)\|_\infty\Big] \\
&\quad= \rho\left(\mathbb{E}^{\mathbb{P}_0}\left[\int_0^T (\alpha_s^\varepsilon - \hat\alpha_s^h)^2 ds\right]^{\frac12}\right). \qquad (3.11)
\end{aligned}$$

Combining (3.9), (3.11) we have

$$\begin{aligned}
u_0 &< \sup_{\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{P}_0}\left[g\left((\hat\alpha^h\cdot B)_\tau\right)\right] + \rho\left(\mathbb{E}^{\mathbb{P}_0}\left[\int_0^T (\alpha_s^\varepsilon - \hat\alpha_s^h)^2 ds\right]^{\frac12}\right) + \varepsilon \\
&\le u_0^h + \rho\left(\mathbb{E}^{\mathbb{P}_0}\left[\int_0^T (\alpha_s^\varepsilon - \hat\alpha_s^h)^2 ds\right]^{\frac12}\right) + \varepsilon.
\end{aligned}$$

Letting $h \to 0$ we deduce

$$u_0 \le \underline{\lim}_{h\to 0} u_0^h + \varepsilon$$

for all $\varepsilon > 0$. ∎

We conclude the section by proving the analogous approximation result for American options.

Theorem 3.14. *Suppose Assumption 3.10 holds. Then we have*

$$u_0 = \lim_{h\to 0} v^h, \quad \textit{where } v^h := \inf_{\varphi\in\mathcal{U}} \mathbb{E}^{\mathbb{P}_0}\left[\sup_{\alpha\in\mathcal{D}_h, t\in[0,T]} \left\{ g(X_t^\alpha) - \int_0^t \varphi_s(X^\alpha)^\intercal \alpha_s dB_s \right\}\right].$$

Proof. We first prove that the left-hand side is smaller. Recall u_0^h defined in (3.8). For all $\varphi \in \mathcal{U}$, the process $\int_0^\cdot \varphi_t(X^\alpha)^\intercal \alpha_t^1 dB_t$ is a martingale, and we have

$$u_0^h \le \sup_{\alpha\in\mathcal{A}_h, \tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{P}_0}\left[g(X_\tau^\alpha) - \int_0^\tau \varphi_t(X^\alpha)^\intercal \alpha_t^1 dB_t\right] \quad \text{for all } \varphi\in\mathcal{U}.$$

Since for any $\alpha \in \mathcal{A}_h$ and $\tau \in \mathcal{T}_T$ we have

$$g(X_\tau^\alpha) - \int_0^\tau \varphi_t(X^\alpha)^\intercal \alpha_t^1 dB_t \le \sup_{a\in\mathcal{D}_h, t\in[0,T]} \{g(X_t^a) - \int_0^t \varphi_s(X^a)^\intercal a_s^1 dB_s\},$$

we obtain

$$u_0^h \le \mathbb{E}^{\mathbb{P}_0}\left[\sup_{a\in\mathcal{D}_h, t\in[0,T]}\left\{g(X_t^a) - \int_0^t \varphi_s(X^a)^\intercal a_s^1 dB_s\right\}\right] \quad \text{for all } \varphi\in\mathcal{U}.$$

The required result follows by Lemma 3.13.

For the converse, recall that $u^{(\varepsilon)}(t,B_t)$ is a $\mathbb{P}$-supermartingale for all $\mathbb{P}\in\mathcal{P}$ (Lemma 3.12). Further, since $u^{(\varepsilon)}\in C^{1,2}$, we have

$$\partial_t u^{(\varepsilon)} + \sup_{(b^0,b^1)\in A}\left\{b^0\partial_x u^{(\varepsilon)} + \frac{1}{2}\mathrm{Tr}\big(b^1(b^1)^\intercal\partial_{xx}^2 u^{(\varepsilon)}\big)\right\} \le 0.$$

Hence, for all $h>0$

$$\begin{aligned}
v_h &\le \mathbb{E}^{\mathbb{P}_0}\left[\sup_{a\in\mathcal{D}_h, t\in[0,T]}\left\{g(X_t^a) - \int_0^t \partial_x u_s^{(\varepsilon)}(X^a)^\intercal a_s^1 dB_s\right\}\right]\\
&\le \mathbb{E}^{\mathbb{P}_0}\Bigg[\sup_{a\in\mathcal{D}_h, t\in[0,T]}\Bigg\{g(X_t^a) - u_t^{(\varepsilon)}(X_t^a) + u_0^{(\varepsilon)}\\
&\qquad + \int_0^t \left(\partial_t u_s^{(\varepsilon)}(X_s^a) + a_s^0\cdot\partial_x u_s^{(\varepsilon)}(X_s^a) + \frac{1}{2}\mathrm{Tr}\big(a_s^1(a_s^1)^\intercal\partial_{xx}^2 u_s^{(\varepsilon)}(X_s^a)\big)\right) ds\Bigg\}\Bigg]\\
&\le \mathbb{E}^{\mathbb{P}_0}\left[\sup_{a\in\mathcal{D}_h, t\in[0,T]}\left\{g(X_t^a) - g^{(\varepsilon)}(X_t^a)\right\}\right] + u_0^{(\varepsilon)},
\end{aligned}$$

where we have used Ito's formula and the inequality $u^{(\varepsilon)}\ge g^{(\varepsilon)}$ proved in Lemma 3.12. It is straightforward to check that

$$\lim_{\varepsilon\to 0}\left(\mathbb{E}^{\mathbb{P}_0}\left[\sup_{a\in\mathcal{D}_h, t\in[0,T]}\left\{g(X_t^a) - g^{(\varepsilon)}(X_t^a)\right\}\right] + u_0^{(\varepsilon)}\right) = u_0. \qquad \blacksquare$$

4. Examples.

4.1. UVM. As a first example, we consider a UVM, first considered in [1] and [14]. Let $A\subseteq\mathbb{R}^d\times\mathbb{R}^{d\times d}$ be a compact domain such that for all $\left(\sigma^i,\rho^{ij}\right)_{1\le i,j\le d}\in A$, the matrix

$$\left(\rho^{ij}\sigma^i\sigma^j\right)_{1\le i,j\le d}$$

is positive semidefinite, $\rho^{ij}=\rho^{ji}\in[-1,1]$, and $\rho^{ii}=1$. If $d=2$ an example of such a domain is obtained by setting

$$A = \left(\prod_{i=1}^2[\underline{\sigma}^i,\overline{\sigma}^i]\right)\times\left\{\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix} : \rho\in[\underline{\rho},\overline{\rho}]\right\},$$

where $0 \le \underline{\sigma}^i \le \overline{\sigma}^i$ and $-1 \le \underline{\rho} \le \overline{\rho} \le 1$. Recall the definition of $\mathcal{A}$, i.e., an adapted process $(\sigma, \rho) = (\sigma_t, \rho_t)_{0 \le t \le T} \in \mathcal{A}$ if it takes values in A. In the UVM the stock prices follow the dynamics

$$d(X_t^{\sigma,\rho})^i = \sigma_t^i (X_t^{\sigma,\rho})^i dW_t^i, \quad d\langle W^i, W^j \rangle_t = \rho^{ij} dt, \quad 1 \le i < j \le d,$$

where W^i is a one-dimensional Brownian motion for all $i \le d$, and $(\sigma, \rho) \in \mathcal{A}$ is the unknown volatility process and correlation. The value of the option at time t in the UVM, interpreted as a superreplication price under uncertain volatilities, is given by

$$u_t = \sup_{(\sigma,\rho) \in \mathcal{A}} \mathbb{E}\big[\xi_T(X^{\sigma,\rho}) | \mathcal{F}_t\big]. \tag{4.1}$$

For European payoffs, $\xi_T(\omega) = g(\omega_T)$, the value $u(t, x)$ is then the unique viscosity solution (under suitable conditions on g) of the nonlinear PDE:

$$\partial_t u(t, x) + H(x, D_x^2 u(t, x)) = 0, \quad u(T, x) = g(x), \tag{4.2}$$

with the Hamiltonian

$$H(x, \gamma) = \frac{1}{2} \max_{(\sigma^i, \rho^{ij})_{1 \le i,j \le d} \in A} \sum_{i,j=1}^{d} \rho^{ij} \sigma^i \sigma^j x^i x^j \gamma^{ij} \quad \text{for all } \ x \in \mathbb{R}^d, \gamma \in \mathbb{R}^{d \times d}.$$

Second order BSDE (2BSDE). Fix constants $\hat{\sigma} = (\hat{\sigma}^i)_{1 \le i \le d}$ and $\hat{\rho} = (\hat{\rho}^{i,j})_{1 \le i,j \le d}$. Denote a new diffusion process $\widehat{X}$,

$$d\widehat{X}_t^i = \hat{\sigma}^i \widehat{X}_t^i d\widehat{W}_t^i, \quad d\langle \widehat{W}^i, \widehat{W}^j \rangle_t = \hat{\rho}^{ij} dt, \ 1 \le i \le j \le d,$$

where $\widehat{W}^i$ is one-dimensional Brownian motion for all $1 \le i \le d$. Consider the dynamics

$$\begin{aligned} dZ_t &= \Xi_t dt + \Gamma_t d\widehat{X}_t, \\ dY_t &= -H\big(\widehat{X}_t, \Gamma_t\big) dt + Z_t d\widehat{X}_t + \frac{1}{2}\big(\hat{\sigma}\widehat{X}_t\big)^\intercal \Gamma_t \big(\hat{\sigma}\widehat{X}_t\big) dt, \end{aligned} \tag{4.3}$$

where (Y, Z, Γ, Ξ) is a quadruple taking values in $\mathbb{R}$, $\mathbb{R}^d$, S^d (the space of symmetric $d \times d$ matrices) and $\mathbb{R}^d$, respectively. In particular, if the HJB equation (4.2) has a smooth solution, it follows from Itô's formula that

$$Y_t := u(t, \widehat{X}_t), \quad Z_t := \partial_x u(t, \widehat{X}_t), \quad \Gamma_t := \partial_{xx}^2 u(t, \widehat{X}_t) \tag{4.4}$$

satisfy the dynamics (4.3) with a certain process Ξ. In Cheridito et al. [3], the authors studied the existence and uniqueness of the quadruple (Y, Z, Γ, Ξ) satisfying the dynamics (4.3) with the terminal condition $Y_T = g(\widehat{X}_T)$, without assuming the existence of a smooth solution to the HJB equation (4.2), and they gave the name 2BSDE to this problem. For the readers interested in the theory of 2BSDE, we refer to [3] and Soner, Touzi, and Zhang [18] for more details.

Numerical scheme for 2BSDE. We are interested in solving the 2BSDE numerically. In the existing literature, one may find several different numerical schemes for this problem (see, for example, [3, 8, 10]). Here we recall the one proposed in Guyon and Henry-Labordère [10]. Introduce the partition $\{t_i\}_{i\le n}$ on the interval $[0,T]$, and denote $\Delta t_i = t_i - t_{i-1}$, $\Delta W_{t_i} = W_{t_i} - W_{t_{i-1}}$. First, the diffusion $\widehat{X}$ can be written explicitly:

$$\widehat{X}^j_{t_i} = \widehat{X}^j_0 e^{-(\hat\sigma^j)^2 \frac{t_i}{2} + \hat\sigma^j W^j_{t_i}} \quad \text{with} \quad \Delta W^j_{t_i} \Delta W^k_{t_i} = \hat\rho_{jk} \Delta t_i.$$

Denote by $\widehat{Y}, \widehat{\Gamma}$ the numerical approximations of Y, Γ. In the backward scheme in [10], we set $\widehat{Y}_{t_n} = g\big(\widehat{X}_{t_n}\big)$, and then compute

$$\hat\sigma^j \hat\sigma^k \widehat{X}^j_0 \widehat{X}^k_0 \, \widehat{\Gamma}^{jk}_{t_{i-1}} = \mathbb{E}_{i-1}\Big[\widehat{Y}_{t_i}\big(U^j_{t_i} U^j_{t_i} - (\Delta t_i)^{-1} \hat\rho^{-1}_{jk} - \hat\sigma^j U^j_{t_i} \delta_{jk}\big)\Big]$$

$$\text{with} \quad U^j_{t_i} \ := \ \sum_{k=1}^{d} \hat\rho^{-1}_{jk} \Delta W^k_{t_i} / \Delta t_i, \quad \text{and}$$

$$\widehat{Y}_{t_{i-1}} = \mathbb{E}_{i-1}\big[\widehat{Y}_{t_i}\big] + \left(H(\widehat{X}_{t_{i-1}}, \ \widehat{\Gamma}_{t_{i-1}}) - \frac{1}{2} \sum_{j,k=1}^{n} \widehat{X}^j_{t_{i-1}} \widehat{X}^k_{t_{i-1}} \Gamma^{jk}_{t_{i-1}} \hat\rho_{pk} \hat\sigma^j \hat\sigma^k \right) \Delta t_i,$$

where $\mathbb{E}_i$ denotes the conditional expectation with respect to the filtration $\mathcal{F}_{t_i}$. Below, we denote $u_0^{\mathrm{BSDE}} := \widehat{Y}_0$.

Lower and upper bound for the value function. Once $\widehat{\Gamma}$ is computed, one gets a (suboptimal) estimation of the controls $(\hat\sigma^*, \hat\rho^*)$:

$$\big(\hat\sigma^*_{t_i}, \hat\rho^*_{t_i}\big) \ := \ \operatorname{argmax}_{(\sigma^j, \rho^{jk})_{1\le j,k\le d} \in A} \sum_{j,k=1}^{d} \rho^{jk} \sigma^j \sigma^k \widehat{X}^j_{t_i} \widehat{X}^k_{t_i} \widehat{\Gamma}^{jk}_{t_i} \quad \text{for} \ \ 0 \le i \le n.$$

Performing a second independent (forward) Monte Carlo simulation using this suboptimal control, we obtain a lower bound for the value function (4.1):

$$u_0^{\mathrm{LS}} := \mathbb{E}\big[g(X_T^{\hat\sigma^*, \hat\rho^*})\big] \le u_0.$$

We next calculate the dual bound derived in the current paper. As mentioned in Remark 2.8, we will use the numerical approximation of $\partial_x u$ to serve as the minimizer φ^* in the dual form. Also, we observe from (4.4) that the process Z in the 2BSDE plays the corresponding role of $\partial_x u$, and we can compute the numerical approximation $\widehat{Z}$ of Z:

$$\hat\sigma^j \widehat{X}^j_{t_i} \widehat{Z}^j_{t_i} = \mathbb{E}_{i-1}\big[\widehat{Y}_{t_i} U^j_{t_i}\big].$$

Then we define

$$\varphi^*_t = \sum_{i=1}^{n} \widehat{Z}_{t_{i-1}} 1_{[t_{i-1}, t_i)}(t).$$

Using our candidate φ^* in the minimization, we get an upper bound

$$u_0^{\mathrm{LS}} \le u_0 \le u_0^{\mathrm{dual}} := \lim_{h\to 0} \mathbb{E}\left[\max_{(\sigma,\rho)\in\mathcal{D}_h}\left\{g(X_{t_n}^{\sigma,\rho}) - \sum_{i=1}^n \varphi_{t_{i-1}}^*(X^{\sigma,\rho})(X_{t_i}^{\sigma,\rho} - X_{t_{i-1}}^{\sigma,\rho})\right\}\right].$$

The algorithm. Our whole algorithm can be summarized by the following four steps:

1. Simulate N_1 replications of $\widehat{X}$ with a lognormal diffusion (we choose $\hat{\sigma} = (\overline{\sigma} + \underline{\sigma})/2$).
2. Apply the backward algorithm using a regression approximation. Compute $Y_0 = u_0^{\mathrm{BSDE}}$.
3. Simulate N_2 independent replication of $X^{\hat{\sigma}^*,\hat{\rho}^*}$ using the suboptimal control $(\hat{\sigma}^*, \hat{\rho}^*)$. Give a low-biased estimate u_0^{LS}.
4. Simulate independent increment ΔW_{t_i} and maximize

$$g(X_{t_n}^{\sigma,\rho}) - \sum_{i=1}^n \varphi_{t_{i-1}}^*(X^{\sigma,\rho})(X_{t_i}^{\sigma,\rho} - X_{t_{i-1}}^{\sigma,\rho})$$

over $(\sigma,\rho)\in\mathcal{D}_h$. In our numerical experiments, as the payoff may be non-smooth, we have used a direct search polytope algorithm. Then compute the average.

Numerical experiments. In our experiments, we take $T = 1$ year and for the ith asset, $X_0^i = 100$, $\underline{\sigma}^i = 0.1$, $\overline{\sigma}^i = 0.2$, and we use the constant midvolatility $\hat{\sigma}^i = 0.15$ to generate the first $N_1 = 2^{15}$ replications of $\widehat{X}$. For the second independent Monte Carlo using our suboptimal control, we take $N_2 = 2^{15}$ replications of X and a time step $\Delta_{\mathrm{LS}} = 1/400$. In the backward and dual algorithms, we choose the time step Δ among $\{1/2, 1/4, 1/8, 1/12\}$, which gives the biggest u_0^{LS} and the smallest u_0^{dual}. The conditional expectations at t_i are computed using parametric regressions. The regression basis consists in some polynomial basis. The exact price is obtained by solving the (one- or two-dimensional) HJB equation with a finite-difference scheme.

1. 90–110 call spread $(X_T - 90)^+ - (X_T - 110)^+$, basis = 5-order polynomial:

$$u_0^{\mathrm{LS}} = 11.07 < u_0^{\mathrm{PDE}} = 11.20 < u_0^{\mathrm{dual}} = 11.70,\ u_0^{\mathrm{BSDE}} = 10.30.$$

2. Digital option $1_{X_T \ge 100}$, basis = 5-order polynomial:

$$u_0^{\mathrm{LS}} = 62.75 < u_0^{\mathrm{PDE}} = 63.33 < u_0^{\mathrm{dual}} = 66.54,\ u_0^{\mathrm{BSDE}} = 52.03.$$

3. Outperformer option $(X_T^2 - X_T^1)^+$ with 2 uncorrelated assets:

$$u_0^{\mathrm{LS}} = 11.15 < u_0^{\mathrm{PDE}} = 11.25 < u_0^{\mathrm{dual}} = 11.84,\ u_0^{\mathrm{BSDE}} = 11.48.$$

4. Outperformer option with 2 correlated assets $\rho = -0.5$:

$$u_0^{\mathrm{LS}} = 13.66 < u_0^{\mathrm{PDE}} = 13.75 < u_0^{\mathrm{dual}} = 14.05,\ u_0^{\mathrm{BSDE}} = 14.14.$$

5. Outperformer spread option $(X_T^2 - 0.9X_T^1)^+ - (X_T^2 - 1.1X_T^1)^+$ with 2 correlated assets $\rho = -0.5$:

$$u_0^{\mathrm{LS}} = 11.11 < u_0^{\mathrm{PDE}} = 11.41 < u_0^{\mathrm{dual}} = 12.35,\ u_0^{\mathrm{BSDE}} = 9.94.$$

In examples 3–5 the regression basis we used consisted of

$$\{1, X^1, X^2, (X^1)^2, (X^2)^2, X^1X^2\}.$$

Remark 4.1. The dual bounds we have derived complement the lower bounds derived in [10]. They allow us to access the quality of the regressors used in computing the conditional expectations.

4.2. Credit value adjustment. Our second example arises in credit valuation adjustment. We will show that for this particular example, we can solve the deterministic optimization problems arising in the dual algorithm efficiently by recursively solving ODEs.

CVA interpretation. Let us recall the problem of the unilateral counterparty value adjustment (see [11] for more details). We have one counterparty, denoted by C, that may default and another, B, that cannot. We assume that B is allowed to trade dynamically in the underlying X—that is described by a local martingale

$$dX_t = \sigma(t, X_t)dW_t \quad \text{with } W \text{ a Brownian motion,}$$

under a risk-neutral measure. The default time of C is modeled by an exponential variable τ with an intensity c, independent of W. We denote by u_0 the value at time 0 of B's long position in a single derivative contracted by C, given that C has not defaulted so far. For simplicity, we assume zero rate. Assume that $g(X_T)$ is the payoff of the derivative at maturity T, and that $\tilde{u}$ is the derivative value just after the counterparty has defaulted. Then, we have

$$\begin{aligned} u_0 &= \mathbb{E}\Big[g(X_T)1_{\{\tau>T\}} + \tilde{u}(\tau, X_\tau)1_{\{\tau\leq T\}}\Big] \\ &= \mathbb{E}\left[e^{-cT}g(X_T) + \int_0^T \tilde{u}(t, X_t)ce^{-ct}dt\right]. \end{aligned}$$

Write down the dynamic version:

$$u(t,x) = \mathbb{E}\left[e^{-c(T-t)}g(X_T) + \int_t^T e^{-c(s-t)}c\tilde{u}(s, X_s)ds\Big|X_t = x\right].$$

The function u can be characterized by the equation

$$\partial_t u + \frac{1}{2}\sigma^2(t,x)\partial^2_{xx}u + c\,(\tilde{u} - u) = 0, \quad u(T,x) = g(x).$$

At the default event, in the case of zero recovery, we assume that $\tilde{u}$ is given by

$$\tilde{u} = u^-,$$

where $x^- := \max(0, -x)$. Indeed, if the value of u is positive, meaning that u should be paid by the counterparty, nothing will be received by B after the default. If the value of u is negative, meaning that u should be received by the counterparty, B will pay u in the case of default of C.

Remark 4.2. The funding value adjustment corresponds to a similar nonlinear equation. By the following change of variable

$$u(t,x)^{\mathrm{HJB}} = e^{c(T-t)}u(t,x),$$

the function u^{HJB} satisfies the HJB equation

$$\partial_t u^{\mathrm{HJB}} + \frac{1}{2}\sigma^2(t,x)\partial^2_{xx}u^{\mathrm{HJB}} + c(u^{\mathrm{HJB}})^- = 0, \quad u^{\mathrm{HJB}}(T,x) = g(x). \tag{4.5}$$

The stochastic representation is

$$u^{\mathrm{HJB}}(t,x) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}\Big[e^{-\int_t^T \alpha_s ds} g(X_T)\big|X_t = x\Big] \quad \text{with} \quad A := [0,c].$$

Dual bound. We are interested in deriving an efficient upper bound for $u^{\mathrm{HJB}}(0,X_0)$. Denoting $R_t^a = e^{\int_0^t a_s ds}$, our dual expression is

$$\begin{aligned} u^{\mathrm{HJB}}(0,X_0) &= \lim_{h\to 0}\inf_{\varphi\in\mathcal{U}} \mathbb{E}\left[\sup_{a\in\mathcal{D}_h}\{R_T^a g(X_T) - \int_0^T R_t^a \varphi(t,X_t)dX_t\}\right] \\ &\le \lim_{h\to 0} \mathbb{E}\left[\sup_{a\in\mathcal{D}_h}\{R_T^a g(X_T) - \int_0^T R_t^a \varphi^*(t,X_t)dX_t\}\right], \end{aligned}$$

where φ^* is a fixed strategy. Rewriting the integral in Stratonovich form, we have

$$\begin{aligned} &\int_0^T R_t^a\varphi^*(t,X_t)dX_t \\ &\quad = \int_0^T R_t^a \varphi^*(t,X_t)\circ dX_t - \frac{1}{2}\int_0^T R_t^a \partial_x\varphi^*(t,X_t)\sigma^2(t,X_t)dt. \end{aligned}$$

Therefore, using the classical Zakai approximation of the Stratonovich integral, it follows that

$$\begin{aligned} &\mathbb{E}\left[\sup_{a\in\mathcal{D}_h}\left\{R_T^a g(X_T) - \int_0^T R_t^a\varphi^*(t,X_t)dX_t\right\}\right] \\ &= \lim_{n\to\infty}\mathbb{E}\left[\sup_{a\in\mathcal{D}_h}\left\{R_T^a g(X_T^n) - \int_0^T R_t^a\varphi^*(t,X_t^n)\circ dX_t^n + \frac{1}{2}\int_0^T R_t^a\partial_x\varphi^*(t,X_t^n)\sigma^2(t,X_t^n)dt\right\}\right] \\ &= \lim_{n\to\infty}\mathbb{E}\left[\sup_{a\in\mathcal{D}_h}\left\{R_T^a g(X_T^n) - \int_0^T R_t^a\left(\varphi^*(t,X_t^n)\sigma(t,X_t^n)\dot W_t^n - \frac{1}{2}\partial_x\varphi^*(t,X_t^n)\sigma^2(t,X_t^n)\right)dt\right\}\right] \\ &\le \lim_{n\to\infty}\mathbb{E}\left[\sup_{a\in\tilde{\mathcal{D}}}\left\{R_T^a g(X_T^n) - \int_0^T R_t^a\left(\varphi^*(t,X_t^n)\sigma(t,X_t^n)\dot W_t^n - \frac{1}{2}\partial_x\varphi^*(t,X_t^n)\sigma^2(t,X_t^n)\right)dt\right\}\right], \end{aligned}$$

where $\tilde{\mathcal{D}} := \big\{a : [0,T]\to\mathbb{R}\big|$ a is measurable, and $0\le a_t\le c$ for all $t\in[0,T]\big\}$. For almost every ω we may consider for all n the following deterministic optimization problem. Set

$$\begin{aligned} g_{\omega,n} = g(X_T^n(\omega)), \quad \alpha_{\omega,n}(t) &= -\varphi^*(t,X_t^n(\omega))\sigma(t,X_t^n(\omega))\dot W_t^n(\omega), \\ \beta_{\omega,n}(t) &= \frac{1}{2}\partial_x\varphi^*(t,X_t^n(\omega))\sigma^2(t,X_t^n(\omega)), \end{aligned}$$

and consider the function

$$u^{\mathrm{HJ}}_{\omega,n}(t) = \sup_{a\in\tilde{\mathcal{D}}}\left\{\frac{R^a_T}{R^a_t}g_{\omega,n} + \int_t^T \frac{R^a_s}{R^a_t}\big(\alpha_{\omega,n}(s)+\beta_{\omega,n}(s)\big)ds\right\}.$$

Note that u^{HJ} is the solution of the (pathwise) Hamilton–Jacobi equation

$$(u^{\mathrm{HJ}}_{\omega,n})'(t) + c\left(u^{\mathrm{HJ}}_{\omega,n}(t)\right)^- + \alpha_{\omega,n}(t) + \beta_{\omega,n}(t) = 0, \quad u^{\mathrm{HJ}}_{\omega,n}(T) = g_{\omega,n}.$$

The ODE for $u^{\mathrm{HJ}}_{\omega,n}$ can be solved analytically. Fix a $t^0\in[0,T]$, and let

$$t^* = \sup\left\{s < t^0 : u^{\mathrm{HJ}}_{\omega,n}(t^0)u^{\mathrm{HJ}}_{\omega,n}(s) < 0\right\} \vee 0.$$

For all $t\in[t^*,t_0]$ we get the following recurrence equation:

$$u^{\mathrm{HJ}}_{\omega,n}(t) = \begin{cases} -\displaystyle\int_t^{t^0} e^{-c(s-t)}\big(\alpha_{\omega,n}(s)+\beta_{\omega,n}(s)\big)ds + u^{\mathrm{HJ}}_{\omega,n}(t^0)e^{c(t^0-t)}, & u^{\mathrm{HJ}}_{\omega,n}(t^0) < 0,\\ -\displaystyle\int_t^{t^0}\big(\alpha_{\omega,n}(s)+\beta_{\omega,n}(s)\big)ds + u^{\mathrm{HJ}}_{\omega,n}(t^0), & u^{\mathrm{HJ}}_{\omega,n}(t^0) > 0,\end{cases}$$
$$u^{\mathrm{HJ}}_{\omega,n}(T) = g_{\omega,n}.$$

Finally, we observe that

$$u^{\mathrm{HJB}}(0,X_0) \le \lim_{n\to\infty}\mathbb{E}\big[u^{\mathrm{HJ}}_{\omega,n}(0)\big].$$

We illustrate the quality of our bounds by the following numerical example.

Remark 4.3. This example falls into the framework of [4] and [5]. By virtue of their (continuous) pathwise analysis the upper bounds derived above could in the limit be replaced with equalities. Only the error introduced by the choice of φ^* remains.

Numerical example. We take $\sigma(t,x)=1$, $T=1$ year, $X_0=0$, $g(x)=x$. We use two choices: $\varphi^*(t,x)=e^{-c(T-t)}$ (which corresponds to $\partial_x u^{\mathrm{HJB}}$ at the first order near $c=0$) and $\varphi^*(t,x)=0$. We have computed $\mathbb{E}[u^{\mathrm{HJ}}_{\omega,n}(0)]$ as a function of the time discretization (see Tables 1 and 2). The exact value has been computed using a one-dimensional PDE solver (see

Table 1

The numerical results of $\mathbb{E}[u^{\mathrm{HJ}}_{\omega,n}(0)]$ with the different time steps when $\varphi^*(t,x)=e^{-c(T-t)}$. The numbers in the brackets indicate the CPU times (Intel Core 2.60 GHz) in seconds for the case $c=0.7$ with $N=8192$ Monte Carlo paths.

c, $(1-e^{-cT})$	PDE	1/2	1/4	1/8	1/12	1/24	1/50	1/100	1/200
0.01 (1%)	0.26	0.23	0.25	0.26	0.26	0.26	0.26	0.26	**0.26**
0.05 (4.9%)	1.29	1.14	1.22	1.26	1.27	1.28	1.29	1.29	**1.29**
0.1 (9.5%)	2.52	2.24	2.39	2.46	2.48	2.51	2.52	2.52	**2.52**
0.7 (50.3%)	13.60(0)	12.63(1)	13.25(2)	13.53(5)	13.61(7)	13.71(18)	13.75(44)	13.77(112)	**13.77**

Table 2
The numerical results of $\mathbb{E}[u^{\mathrm{HJ}}_{\omega,n}(0)]$ *when* $\varphi^*(t,x)=0$.

$c\,,(1-e^{-cT})$	PDE	$\mathbb{E}[u^{\mathrm{HJ}}_{\omega,n}(0)]$
0.01 (1%)	0.26	0.40
0.05 (4.9%)	1.30	1.95
0.1 (9.5%)	2.53	3.80
0.7 (50.3%)	13.60	20.08

column PDE). We have used different values of c corresponding to a probability of default at T equal to $(1-e^{-cT})$.

The approximation has two separate sources of error. First, there is the suboptimal choice of the minimizer φ^* for the discretized optimization implying an upper bias. The second error arises from the discretization of the deterministic optimization problems, which could underestimate the true value of the optimization. The choice $\varphi^* = e^{-c(T-t)}$ in our example—as expected—is close to being optimal, so the errors arising from the discretization dominate. To the contrary, the choice $\varphi^* = 0$ is far from being optimal, so the numerical results are much bigger than the value function.

REFERENCES

[1] M. Avellaneda, A. Levy, and A. Paras, *Pricing and hedging derivative securities in markets with uncertain volatilities*, Appl. Math. Finance, 2 (1995), pp. 73–88.
[2] B. Bouchard and N. Touzi, *Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations*, Stochastic Process. Appl., 111 (2004), pp. 175–206.
[3] P. Cheridito, M. Soner, N. Touzi, and N. Victoir, *Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs*, Comm. Pure Appl. Math., 60 (2007), pp. 1081–1110.
[4] M. H. A. Davis and G. Burstein, *A deterministic approach to stochastic optimal control with application to anticipative control*, Stochastics Stochastics Rep., 40 (1992), pp. 203–256.
[5] J. Diehl, P. Friz, and P. Gassiat, *Stochastic control with rough paths*, preprint, arXiv:1303.7160, 2013.
[6] I. Ekren, N. Touzi, and J. Zhang, *Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part* II, Ann. Probab., to appear.
[7] I. Ekren, N. Touzi, and J. Zhang, *Optimal stopping under nonlinear expectation*, Stochastic Process. Appl., 124 (2014), pp. 3277–3311.
[8] A. Fahim, N. Touzi, and X. Warin, *A probabilistic numerical method for fully nonlinear parabolic PDEs*, Ann. Appl. Probab., 21 (2011), pp. 1322–1364.
[9] M. Fisher and G. Nappo, *On the moments of the modulus of continuity of Itô processes*, Stoch. Anal. Appl., 28 (2010), pp. 103–122.
[10] J. Guyon and P. Henry-Labordère, *Uncertain volatility model: A Monte Carlo approach*, J. Comput. Finance, 14 (2011), pp. 37–71.
[11] J. Guyon and P. Henry-Labordère, *Nonlinear Option Pricing*, Chapman & Hall/Financ. Math. Ser., CRC Press, Boca Raton, FL, 2014.
[12] N. V. Krylov, *Approximating value functions for controlled degenerate diffusion processes by using piece-wise constant policies*, Electron. J. Probab., 4 (1999), 2.
[13] N. V. Krylov, *On the rate of convergence of finite-difference approximations for Bellmans equations with variable coefficients*, Probab. Theory Related Fields, 117 (2000), pp. 1–16.
[14] T. Lyons, *Uncertain volatility and the risk-free synthesis of derivatives*, Appl. Math. Finance, 2 (1995), pp. 117–133.
[15] M. Nutz and R. Van Handel, *Constructing sublinear expectations on path space*, Stochastic Process. Appl., 123 (2013), pp. 3100–3121.

[16] E. Pardoux and S. Peng, *Adapted solutions of backward stochastic differential equations*, Systems Control Lett., 14 (1990), pp. 55–61.
[17] L. C. G. Rogers, *Pathwise stochastic optimal control*, SIAM J. Control Optim., 46 (2007), pp. 1116–1132.
[18] M. Soner, N. Touzi, and J. Zhang, *Wellposedness of second order backward SDEs*, Probab. Theory Related Fields, 153 (2012), pp. 149–190.
[19] J. Zhang, *A numerical scheme for backward stochastic differential equations*, Ann. Appl. Probab., 14 (2004), pp. 459–488.