Article:

Brzezniak, Zdzislaw orcid.org/0000-0001-8731-6523, Ondreját, Martin and Seidler, J. (2016) Invariant measures for stochastic nonlinear beam and wave equations. Journal of Differential Equations. 4157–4179. ISSN 0022-0396

https://doi.org/10.1016/j.jde.2015.11.007

INVARIANT MEASURES FOR STOCHASTIC NONLINEAR BEAM AND WAVE EQUATIONS

ZDZISŁAW BRZEŹNIAK, MARTIN ONDREJÁT, JAN SEIDLER

Abstract. Existence of an invariant measure for a stochastic extensible beam equation and for a stochastic damped wave equation with polynomial nonlinearities is proved. Toward this end, it is shown first that the corresponding transition semigroups map the space of all bounded sequentially weakly continuous functions on the state space into itself and then by a Lyapunov functions approach solutions bounded in probability are found.

1. Introduction

In this paper, we aim at showing existence of an invariant (probability) measure for a stochastic extensible beam equation and for damped stochastic wave equations with polynomial nonlinearities (both on $\mathbb{R}^d$ and on bounded domains).

Our approach is based on the classical Krylov-Bogolyubov procedure, let us recall it in a context relevant for us. Let X be a separable Hilbert space and $U = (U_t)$ a transition semigroup on X. There exists an invariant measure for U, provided the semigroup is Feller, that is, U_t maps the space $C_b(X)$ of all bounded continuous functions on X into itself for all $t \geq 0$, and the set of measures

$$\left\{ \frac{1}{T_n} \int_0^{T_n} U_s^* \nu \, ds; \; n \geq 1 \right\} \tag{1.1}$$

is tight on X for some $T_n \nearrow \infty$ and a probability measure ν on X. Transition semigroups associated with stochastic partial differential equations may be quite often easily shown to be Feller but tightness of the set (1.1) is a difficult problem for equations with solutions of low spatial regularity like beam and wave equations. The situation completely changes if the space X is endowed with its weak topology. Then tightness of the set (1.1) follows from existence of a solution that is bounded in probability (in the mean) on $[\tau, \infty)$ for some $\tau > 0$, a property which may be verified by direct calculations with Lyapunov functions in many cases. On the other hand, it is not obvious why $U_t f$ should be weakly continuous for a bounded weakly continuous function f on X. In fact, except for linear equations, only sequential weak continuity can be usually established. Let us denote by *bw* the bounded weak topology on X, i.e. the finest topology that agrees with the weak topology on every closed ball; note that a real function on X is *bw*-continuous if and only if it is sequentially weakly continuous and if and only if its restriction to any ball is weakly

The second and the third named authours were supported by GAČR grant P201/10/0752, part of the research was done at Isaac Newton Institute for Mathematical Sciences in Cambridge, Great Britain and at Centre Interfacultaire Bernoulli at the EPFL, Lausanne, Switzerland.

continuous. Hence to carry out the Krylov-Bogolyubov procedure in X with its weak topology, it is necessary to check that $U_t(C_b(X,bw)) \subseteq C_b(X,bw)$ for every $t \geq 0$. We shall call transition semigroups with this property bw-Feller.

It is straightforward to prove that a bw-Feller semigroup such that the set (1.1) is tight on (X,bw) has an invariant probability measure (see [?, Proposition 3.1]), however, it is not straightforward to identify stochastic PDEs for which the associated transition semigroups are bw-Feller. Up to our knowledge, the first to address this problem was A. Ichikawa [?, Theorem 3.1] who considered equations with coefficients depending only on finite dimensional projections of solutions. G. Leha and G. Ritter ([?, ?]) studied thoroughly (yet in different terms) general results concerning bw-Feller and related semigroups. In the field of stochastic PDEs, however, they considered only a bit particular stochastic reaction-diffusion equation. In [?] (see also [?]) the bw-Feller property was shown for semigroups corresponding to parabolic problems in bounded domains and to equations reducible to bw-Feller ones via the Girsanov transform, neither of these results applies to hyperbolic or beam equations.

There are several other papers containing implicitly considerations related to the bw-Feller property. E.g., J. U. Kim in [?], when studying invariant measures for the von Karman equation with an additive white noise, used an argument which can be recast so that it might fit within the scheme above, if bw is replaced with a suitable mixed topology on X, but he proceeded in a different way. In [?], Kolmogorov operators L corresponding to generalized Burgers equations (in one spatial dimension and with additive noise) are studied in the space $C(X,bw)$, in particular, invariant measures are find by solving the equation $L^*\mu = 0$; cf. also [?].

We shall establish bw-Feller property of transition semigroups corresponding to stochastic nonlinear beam and wave equations by a new method, whose main ingredients are bw-continuity of nonlinear terms on X (if the target space is endowed with a suitable weak-type topology, this follows from the fact the equations of the second order in time are dealt with), uniform boundedness in probability on compact intervals for solutions starting in a given ball, and results on convergence of sequences of local martingales (invoked in a form that was used in [?] to construct weak solutions of stochastic differential equations). Combining results on the bw-Feller property with fairly standard estimates obtained in terms of Lyapunov functions we arrive at theorems on existence of invariant measures; these theorems are stated below in Section 1.1. The Lyapunov functions we employ are direct generalizations to nonlinear problems of the one introduced by A. J. Pritchard and J. Zabczyk (cf. [?, Proposition 3.5]).

It seems that no results on invariant measures for stochastic beam equations have been hitherto available. Invariant measures for stochastic nonlinear wave equations were studied in several papers. In [?], a damped stochastic wave equation in a bounded domain with nonlinear, but globally Lipschitz continuous nonlinear terms in drift and diffusion is dealt with. If the Lipschitz constants are sufficiently small (compared with the damping coefficient), existence, uniqueness and stability of an invariant measure are proved by the "early start" method. A damped wave equation in a bounded domain in $\mathbb{R}^d$, $d \leq 3$, with a drift of polynomial growth and additive noise is studied in [?]. An invariant measure is found, its uniqueness being established under an additional (quite restrictive) hypothesis on the drift. If the noise is in addition finite dimensional, existence of an invariant measure is also a consequence of existence of a random attractor that was proved in [?]. An invariant measure for a similar model, but with a nonlinear damping term of polynomial growth, is constructed (under rather different sets of hypotheses) in [?] and [?], moreover, in [?] a sufficient condition for uniqueness of the invariant measure is provided. Existence of invariant measures for a damped wave equation on $\mathbb{R}^3$ with a polynomial drift and a bounded Lipschitz continuous diffusion term is studied in [?]. Only the papers [?] and [?] have a more substantial intersection with the present paper, we provide a more detailed comparison after Theorem 1.2 and Theorem 1.3, respectively.

1.1. **Main results.** First, let us consider the stochastic extensible beam equation

$$u_{tt} + A^2 u + \beta u_t + m(\|B^{1/2}u\|_H^2)Bu = G(u)\,dW, \tag{1.2}$$

assuming

- (b1) A and B are selfadjoint operators on a separable Hilbert space H; W is a standard cylindrical Wiener proces on a real separable Hilbert space $\mathcal{X}$, defined on a stochastic basis $(O, \mathcal{G}, (\mathcal{G}_t), \mathbb{P})$;
- (b2) $B > 0$, $A \geq \mu I$ for some $\mu > 0$, $\operatorname{Dom} A \subseteq \operatorname{Dom} B$ and $B \in \mathcal{L}(\operatorname{Dom} A, H)$;
- (b3) $m : \mathbb{R}_+ \to \mathbb{R}_+$ is C^1 and $\beta \geq 0$;
- (b4) $G : \operatorname{Dom} A \to \mathcal{L}_2(\mathcal{X}, H)$ such that there exist constants L and (L_n) such that, for every $x, y, z \in \operatorname{Dom} A$,

$$\|G(x)\|_{\mathcal{L}_2(\mathcal{X},H)} \leq L(1 + \|x\|_{\operatorname{Dom} A}), \qquad \|G(y) - G(z)\|_{\mathcal{L}_2(\mathcal{X},H)} \leq L_n \|y - z\|_{\operatorname{Dom} A}$$

holds for every $\|y\|_{\operatorname{Dom} A} \leq n$, $\|z\|_{\operatorname{Dom} A} \leq n$ and every $n \in \mathbb{N}$.

Here $\mathcal{L}_2$ denotes the ideal of Hilbert-Schmidt operators, $\operatorname{Dom} A$ is equipped with the graph norm and (1.2) is interpreted in a standard way as a system of two first-order equations in the state space $X = \operatorname{Dom} A \times H$. (See Section 11 for details and an example showing that (1.2) really covers (a multidimensional version of) the stochastic extensible beam equation with either clamped or hinged boundary conditions.) It was proved in [?] that under the hypotheses (b1)–(b4) there exists a pathwise unique mild solution to (1.2) for any deterministic initial condition in X and (1.2) defines a Feller Markov process on X with a transition semigroup U. To show the bw-Feller property, we need two additional assumptions (which are satisfied almost automatically in applications to the beam equation, see again Example 11.8):

- (b5) $\operatorname{Dom} B$ is compactly embedded into H;
- (b6) $G : (\operatorname{Dom} A, \|\cdot\|_{\operatorname{Dom} B^{1/2}}) \to \mathcal{L}_2(\mathcal{X}, H)$ is continuous;

Finally, to find a solution bounded in probability via Lyapunov functions we employ the hypothesis

- (b7) Either $\beta > 0$ and $\|G\|_{\mathcal{L}_2(\mathcal{X},H)}$ is bounded on $\operatorname{Dom} A$, or $L^2 < \beta$.

(Recall that the constant L is introduced in (b4).) Now we may state our result.

Theorem 1.1. *Let the hypotheses (b1)–(b6) be satisfied. Then the Markov transition semigroup U defined by* (1.2) *is bw-Feller. If in addition (b7) holds then there exists an invariant probability measure for U.*

Further, let us turn to stochastic wave equations with polynomial nonlinearities. Analysis of the linear case and of stochastic oscillators in finite dimensions indicates that one has to consider damped equations in order to get finite invariant measures. Equations in bounded domains and on the whole $\mathbb{R}^d$ may be studied simultaneously, see Section 12, for simplicity, however, we state here results concerning the two cases separately. Let us start with the Cauchy problem

$$u_{tt} = \Delta u - m^2 u - au|u|^{p-1} - \beta u_t + F + \eta g(u)\,\dot{W} \quad \text{on} \quad \mathbb{R}^d \tag{1.3}$$

where $m, \beta \geq 0$, $a > 0$, $F \in L^2(\mathbb{R}^d)$, $\eta \in L^\infty(\mathbb{R}^d)$ and W is a standard cylindrical Wiener process on a separable Hilbert space $\mathcal{X}$, defined on a stochastic basis $(O, \mathcal{G}, (\mathcal{G}_t), \mathbb{P})$. We suppose

(w0) $\mathcal{X}$ is embedded continuously into $L^\infty(\mathbb{R}^d)$ and there exists a constant $\mathbf{c}$ such that
$$\|\xi \mapsto h\xi\|_{\mathcal{L}_2(\mathcal{X}, L^2(\mathbb{R}^d))} \leq \mathbf{c}\|h\|_{L^2(\mathbb{R}^d)}.$$

It is shown in Sections 12 that a spatially homogeneous Wiener process W with finite spectral measure μ satisfies (w0) with $\mathbf{c} = (2\pi)^{-d}\mu(\mathbb{R}^d)$. Note that if $\eta \equiv 1$ then (1.3) is well-posed in $L^2(\mathbb{R}^d)$ only if $g(0) = 0$. So looking for nontrivial invariant measures we have either to resort to nontrivial weight function η or to work in local Sobolev spaces (see [**?**]). The latter choice is much more interesting, but proofs become rather technical and so results for local Sobolev spaces are deferred to a companion paper.

We shall need the following hypotheses.

(w1) $p \in [1, \infty)$ if $d \in \{1, 2\}$ or $p \in [1, \frac{d}{d-2}]$ if $d \geq 3$,
(w2) $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz,
(w3) if $d = 2$ then $|g'|$ grows polynomially, if $d \geq 3$ then $|g'(x)| \leq c(1 + |x|^{\frac{2}{d-2}})$ a.e.,
(w4) $|g(x)|^2 \leq c_0 + c_1|x|^2 + c_2|x|^{p+1}$ for some $c_0, c_1, c_2 \in [0, \infty)$ and all $x \in \mathbb{R}$,
(w5) $c_0\eta \in L^2(D)$,
(w6) $\mathbf{c}^2 c_1 \|\eta\|^2_{L^\infty(D)} < m^2\beta$ and $\mathbf{c}^2 c_2 \|\eta\|^2_{L^\infty(D)} < a\beta$.

Note that (w6) may be satisfied only if $m, \beta > 0$. It was shown in [**?**] that under (w0)–(w5) there exists a unique mild solution to (1.3) in $X = W^{1,2}(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ for any $\mathcal{G}_0$-measurable X-valued initial condition, hence (1.3) defines a transition semigroup U on X.

Theorem 1.2. *Let the assumptions (w0)–(w5) be satisfied. Then the transition semigroup U defined by* (1.3) *is bw-Feller on X. If (w6) is satisfied as well then there exists an invariant probability measure for U.*

The problem (1.3) for $d = 3$ was considered previously by J. U. Kim in [**?**]. He worked with a standard cylindrical Wiener process W in $L^2(\mathbb{R}^3)$ but assumed that the diffusion coefficient is a bounded globally Lipschitz continuous $\mathcal{L}_2$-valued function, $p \in [1, 3)$ and $\eta \in L^\infty(\mathbb{R}^3) \cap W^{1,2}(\mathbb{R}^3)$. Theorem 1.2 covers polynomially growing diffusion terms and the border case $p = 3$, moreover we believe that our approach is simpler.

Further let us turn to a wave equation in a bounded domain $D \subseteq \mathbb{R}^d$ with a C^2-smooth boundary ∂D. We consider an equation

$$u_{tt} = \Delta u - m^2 u - au|u|^{p-1} - \beta u_t + F + g(u)\,\dot{W} \quad \text{in } D \tag{1.4}$$

with a Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial D \tag{1.5}$$

where now $F \in L^2(D)$ and we set $X = W_0^{1,2} \times L^2(D)$. We have to introduce modified hypotheses:

(d0) $\mathcal{X}$ is embedded continuously into $L^\infty(D)$ and there exists a constant $\mathbf{c}$ such that

$$\|\xi \mapsto h\xi\|_{\mathcal{L}_2(\mathcal{X}, L^2(D))} \le \mathbf{c}\|h\|_{L^2(D)}.$$

(d6) $\mathbf{c}^2 c_1 < (m^2 - \lambda)\beta$ and $\mathbf{c}^2 c_2 < a\beta$, where λ is the first eigenvalue of the Dirichlet Laplacian Δ in D.

The assumption (d6) still requires $\beta > 0$ but is compatible with $m = 0$.

Theorem 1.3. *Suppose that (d0) and (w1)–(w4) hold then the transition semigroup U defined by* (1.4), (1.5) *is bw-Feller on X. If (d6) is also satisfied then there exists an invariant probability measure for U.*

Equation (1.4) with a Neumann boundary condition may be studied analogously, see again Section 12.

If g is constant then existence of an invariant measure for (1.4), (1.5) is shown in [**?**], under a hypothesis upon $\mathcal{X}$ less stringent than (d0). Slightly more general assumptions on the drifts are considered, which however coincide with (w1) for polynomial drifts.

2. Notation and Conventions

- $T \in (0, \infty)$.
- If Y is a topological space equipped with a σ-algebra $\mathscr{Y}$ and with a probability measure μ on $\mathscr{Y}$ then we say that μ is Radon provided that
$$\mu(A) = \sup\{\mu(K) : K \text{ compact}, \ K \subseteq A, \ K \in \mathscr{Y}\}, \qquad \forall A \in \mathscr{Y}.$$
- If Y is a topological space then $\mathscr{B}(Y)$ denotes the Borel σ-algebra over Y and $\mathscr{B}_*(Y) = \{V \subseteq Y : V \cap K \in \mathscr{B}(Y), \ \forall K \subseteq Y \text{ compact}\}$.
- If Y is a topological space then $C_b(Y)$ denotes the space of real bounded continuous functions on Y.
- Denote by $\mathscr{P}_*(Y)$ the space of Radon probability measures on $\mathscr{B}_*(Y)$ and equip it with the topology generated by the maps $\mu \mapsto \int_Y f\,d\mu$ for $f \in C_b(Y)$.
- If Y is a Hilbert space then we denote by $Y_w = (Y, \text{weak})$.
- If $\xi : \Omega \to Y$ is a mapping to a topological space Y then we denote by $\sigma(\xi)$ the σ-algebra $\{\xi^{-1}[B] : B \in \mathscr{B}(Y)\}$, i.e. the σ-algebra generated by ξ.
- If X and Y are linear topological spaces then we denote by $\mathcal{L}(X, Y)$ the space of linear continuous operators from X to Y.

- If X and Y are Hilbert spaces then we denote by $\mathcal{L}_2(X,Y)$ the space of Hilbert-Schmidt operators from X to Y.
- $C_0([0,T])$ denotes the space $\{h\in C([0,T]) : h(0)=0\}$.
- If Y is a topological space and $t\in[0,T]$ then we denote by $\mathscr{B}_t(C([0,T];Y))$ the smallest σ-algebra on $C([0,T];Y)$ for which the mappings $C(\mathbb{R}_+;Y)\to Y : h\mapsto h(s)$, $s\in[0,t]$ are $\mathscr{B}(Y)$-measurable.
- All measures in this paper are σ-additive.

3. Quasi-Polish spaces

Let P be a topological space such that there exist a sequence of real continuous functions (ξ_n) on P that separates points of P (such spaces will be called *quasi-Polish* in the sequel for their similarity with Polish spaces, as far as many properties of which some are listed below). Then P has many properties of Polish spaces. Considering the embedding $\xi=(\xi_n) : P\to\mathbb{R}^{\mathbb{N}}$, we can easily prove that

(1) every compact in P is metrizable,
(2) a set in P is compact iff it is sequentially compact,
(3) the σ-algebra $\sigma(\xi)$ contains σ-compact subsets of P, i.e. $\sigma(\text{compacts in } P)\subseteq\sigma(\xi)$,
(4) if $B\in\mathscr{B}_*(P)$ and S is σ-compact then $B\cap S\in\sigma(\xi)$ and $\xi[B\cap S]\in\mathscr{B}(\mathbb{R}^{\mathbb{N}})$,
(5) every probability measure on $\sigma(\xi)$ sitting on a σ-compact has a unique extension to a probability measure on $\mathscr{B}_*(P)$,
(6) every probability measure on $\mathscr{B}_*(P)$ sitting on a σ-compact is Radon,
(7) if (μ_n) is a tight sequence of probability measures on $\mathscr{B}_*(P)$ then there exists a subsequence n_k and $\mu\in\mathscr{P}_*(P)$ such that $\mu_{n_k}\to\mu$ in $\mathscr{P}_*(P)$.

Remark 3.1. The claim (7) is an application of the Prokhorov theorem on $\mathbb{R}^{\mathbb{N}}$. In fact, by a straightforward generalization of the Mapping Theorem [**?**, Theorem 2.7] to sequences of functions, we obtain the following result.

Proposition 3.2. *Let P be a quasi-Polish space, let (μ_n) be a tight sequence of probability measures converging to some μ in $\mathscr{P}_*(P)$. Then the following holds: If $h_n, h : P\to\mathbb{R}$ are uniformly bounded, $\mathscr{B}_*(P)$-measurable and $\mu^*(D)=0$ where*

$$D=\{x\in P : \exists\, x_n\in P,\ x_n\to x,\ \limsup_{n\to\infty}|h_n(x_n)-h(x)|>0\}$$

and μ^* is the outer measure associated to μ then

$$\int_P h_n\,d\mu_n\to\int_P h\,d\mu.$$

4. The path space $C([0,T];X_w)$

These considerations lead us to formulate the following general conventions:

- X is a separable Hilbert space and we denote by $X_w=(X,\text{weak})$,
- we equip $C([0,T];X_w)$ with a locally convex topology generated by the system of pseudonorms $\|h\|_\varphi=\sup_{t\in[0,T]}|\langle\varphi,h(t)\rangle_X|$ where $\varphi\in X$,

- we recall that $\mathscr{B}_T(C([0,T];X_w))$ is the σ-algebra on $C([0,T];X_w)$ generated by the mappings $C([0,T];X_w) \to X : h \mapsto h(s)$ for $s \in [0,T]$.

Remark 4.1. Observe that if $\varphi_n \in X$ are such that

$$\|x\|_X = \sup\{|\langle \varphi_n, x\rangle| : n \in \mathbb{N}\}$$

then $\xi_{n,q} : C([0,T];X_w) \to \mathbb{R} : f \mapsto \varphi_n(f(q))$ for $q \in \mathbb{Q}_+ \cap [0,T]$ and $n \in \mathbb{N}$ constitutes a countable family of continuous functions separating points of $C([0,T];X_w)$ for which $\sigma(\xi_{n,q}) = \mathscr{B}_T(C([0,T];X_w))$. In particular, $C([0,T];X_w)$ is a quasi-Polish space and all conclusions of Section 3 are valid and applicable to the σ-algebra $\mathscr{B}_T(C([0,T];X_w))$.

Remark 4.2. Fix $a \geq 0$. The rational span of $\{\varphi_n\}$ is dense in X and $B_a = \{x \in X : \|x\|_X \leq a\}$ equipped with the weak topology is a compact space metrizable by a metric induced by the pseudonorms $x \mapsto |\langle \varphi_n, x\rangle|$. The topology on $C([0,T];X_w)$ generated by the metric ζ induced by the pseudonorms $\|\cdot\|_{\varphi_n}$ is weaker than that of $C([0,T];X_w)$ defined above, $\mathscr{B}(C([0,T];X_w),\zeta) = \mathscr{B}_T(C([0,T];X_w))$, the traces of the topologies of $C([0,T];X_w)$ and $(C([0,T];X_w),\zeta)$ coincide on the ζ-closed set

$$K_a = \{u \in C([0,T];X_w) : \sup_{t\leq T}\|u(t)\|_X \leq a\}.$$

Since ζ is complete on K_a, it is a Polish space.

Corollary 4.3. *Every probability measure on $\mathscr{B}_T(C([0,T];X_w))$ is Radon.*

In the following, define the modulus of continuity

$$\delta(f,\varepsilon) = \sup\{|f(a) - f(b)| : \, a,b \in [0,T], \, |a-b| \leq \varepsilon\}.$$

Proposition 4.4. *If $\alpha \geq 0$, $\beta = (\beta^k_n)_{k,n\in\mathbb{N}}$, $\lim_{n\to\infty}\beta^k_n = 0$ for every $k \in \mathbb{N}$, $\{\phi_k\}_{k\in\mathbb{N}} \subseteq X^*$ separate points of X and $K^{\alpha,\beta,\phi}$ is the set of all $h \in C([0,T];X_w)$ such that*

- $\sup\{\|h(t)\|_X : t \leq T\} \leq \alpha$
- $\delta(\phi_k \circ h, 1/n) \leq \beta^k_n$ *for every* $k,n \in \mathbb{N}$

then $K^{\alpha,\beta,\phi}$ is compact in $C([0,T];X_w)$. If $K \subseteq C([0,T];X_w)$ is compact and $\{\phi_k\}_{k\in\mathbb{N}}$ separate points of X then there exist α, β such that $K \subseteq K^{\alpha,\beta,\phi}$.

Proof. The demonstration follows the proof of the Arzela-Ascoli theorem relying on the fact that bounded sets in X are sequentially weakly compact. □

Remark 4.5. The sets $K^{\alpha,\beta,\phi}$ will be called *maximal compacts.*

5. The stochastic equation

5.1. **Solution.** Let $I = [0,T]$ or $I = \mathbb{R}_+$. We consider a separable Hilbert space $\mathcal{X}$, an infinitesimal generator $\mathbf{A}$ of a C_0-semigroup $(e^{\mathbf{A}t})_{t\geq 0}$ on X, Borel mappings $\mathbf{F} : X \to X$, $\mathbf{G} : X \to \mathcal{L}_2(\mathcal{X}, X)$, a stochastic basis $(\mathcal{O}, \mathcal{G}, (\mathcal{G}_t), \nu)$, a standard cylindrical $(\mathcal{G}_t)$-Wiener process W on $\mathcal{X}$ and an equation

$$du = (\mathbf{A}u + \mathbf{F}(u))\,dt + \mathbf{G}(u)\,dW. \tag{5.1}$$

We impose a boundedness condition that will be assumed throughout the paper:

$$\mathbf{F} : X \to X \quad \text{and} \quad \mathbf{G} : X \to \mathcal{L}_2(\mathcal{X}, X) \quad \text{are bounded on bounded sets of } X. \tag{5.2}$$

A $(\mathcal{G}_t)$-adapted X-valued process u with weakly continuous trajectories is called *a mild solution* of (5.1) provided that

$$u(t) = e^{\mathbf{A}t}u(0) + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{F}(u(s))\,ds + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{G}(u(s))\,dW \tag{5.3}$$

holds ν-a.e. for every $t \in I$.

A $(\mathcal{G}_t)$-adapted X-valued process u with weakly continuous trajectories is called *a weak solution* of (5.1) provided that

$$\langle \varphi, u(t)\rangle = \langle \varphi, u(0)\rangle + \int_0^t \langle \mathbf{A}^*\varphi, u(s)\rangle\,ds + \int_0^t \langle \varphi, \mathbf{F}(u(s))\rangle\,ds + \int_0^t \langle \varphi, \mathbf{G}(u(s))\,dW\rangle \tag{5.4}$$

holds $\mathbb{P}$-a.e. for every $t \in I$ and every $\varphi \in \mathrm{Dom}(\mathbf{A}^*)$.

Proposition 5.1. *Let* (5.2) *hold. A* $(\mathcal{G}_t)$*-adapted* X*-valued process* u *with weakly continuous trajectories is a weak solution of* (5.1) *iff* u *is a mild solution of* (5.1).

Proof. See [**?**, Theorem 13]. □

Remark 5.2. Since X is assumed to be a separable Hilbert space, there always exists a countable set $H \subseteq \mathrm{Dom}(\mathbf{A}^*)$ which is dense in the graph norm of $\mathrm{Dom}(\mathbf{A}^*)$. If H is any such set then a $(\mathcal{G}_t)$-adapted X-valued process u with weakly continuous trajectories is a weak solution of (5.1) iff (5.4) holds $\mathbb{P}$-a.e. for every $t \in I$ and every $\varphi \in H$. In particular, the infinite dimensional equation (5.1) is reduced to a countable number of real stochastic equations (5.4).

In view of the above remark, we fix a suitable countable set $\{\varphi_\gamma\}_{\gamma\in\mathbb{N}} \subseteq \mathrm{Dom}(\mathbf{A}^*)$ which is dense in the graph norm of $\mathrm{Dom}(\mathbf{A}^*)$ and plays the role of regular "test functions", we also fix an orthonormal basis (ξ_i) in $\mathcal{X}$ and we define nonlinearities $f_\gamma : X \to \mathbb{R}$ and $g_\gamma = (g_{\gamma,i}) : X \to \ell_2$ by

$$f_\gamma(x) = \langle \mathbf{A}^*\varphi_\gamma, x\rangle + \langle \varphi_\gamma, \mathbf{F}(x)\rangle, \qquad g_{\gamma,i}(x) = \langle \varphi_\gamma, \mathbf{G}(x)\xi_i\rangle, \qquad x \in X, \quad \gamma, i \in \mathbb{N}$$

and we consider a system of real valued equations

$$d\varphi_\gamma(u(t)) = f_\gamma(u(t))\,dt + \sum_{i=1}^{\infty} g_{\gamma,i}(u(t))\,dW_i, \qquad t \in I, \quad \gamma \in \mathbb{N} \tag{5.5}$$

where $W = \{W_1, W_2, W_3 \dots\}$ is a family of independent standard $(\mathcal{G}_t)$-Wiener processes defined by $W_i = W(\xi_i)$.

Remark 5.3. A $(\mathcal{G}_t)$-adapted X-valued process u with weakly continuous trajectories satisfies (5.5) iff u is a mild or weak solution of (5.1). We will therefore speak about *a solution* of (5.1) from now on.

Remark 5.4. Let $(\mathcal{G}_t^0)$ denote the augmentation of $(\mathcal{G}_t)$ by ν-zero sets in $\mathcal{G}$. Then a family (W_i) of independent standard $(\mathcal{G}_t)$-Wiener processes defines a unique cylindrical $(\mathcal{G}_t^0)$-Wiener process W on $\mathcal{X}$ such that $W(\xi_i) = W_i$ for every $i \in \mathbb{N}$.

Remark 5.5. The law of u under ν is always defined on the σ-algebra $\mathscr{B}_T(C([0,T];X_w))$.

Theorem 5.6. *Let* (5.2) *hold, let* $(O, \mathcal{G}, (\mathcal{G}_t), \nu)$ *be a stochastic basis,* u *an* X*-valued* $(\mathcal{G}_t)$*-adapted process on* $[0,T]$ *with weakly continuous paths, assume that there exist continuous local* $(\mathcal{G}_t)$*-martingales* (w_i) *such that* $w_i(0) = 0$*, the process*

$$M_\gamma(t) = \varphi_\gamma(u(t)) - \varphi_\gamma(u(0)) - \int_0^t f_\gamma(u(s))\, ds$$

is a local $(\mathcal{G}_t)$*-martingale and* $\langle w_i, w_j \rangle_t = t\delta_{ij}$*,*

$$\langle M_\gamma \rangle_t = \int_0^t \|g_\gamma(u(s))\|_{\ell_2}^2\, ds, \qquad \langle M_\gamma, w_i \rangle_t = \int_0^t g_{\gamma,i}(u(s))\, ds$$

hold for every $\gamma, i, j \in \mathbb{N}$ *and* $t \in [0,T]$*. Then* $w_1, w_2, w_3, \dots$ *are independent standard* $(\mathcal{G}_t)$*-Wiener processes and* $(O, \mathcal{G}, (\mathcal{G}_t), \nu, (w_i), u)$ *is a solution of* (5.1) *on* $[0,T]$*.*

Proof. The processes (w_i) are independent standard $(\mathcal{G}_t)$-Wiener processes by the Lévy characterization theorem and

$$\langle M_\gamma - \int_0^\cdot g_\gamma(u(s))\, dw \rangle_t = \langle M_\gamma \rangle_t - 2\sum_i \int_0^t g_{\gamma,i}(u(s))\, d\langle M_\gamma, w_i \rangle_t + \int_0^t \|g_\gamma(u(s))\|_{\ell_2}^2\, ds.$$

Since the right hand side equals 0 a.s., (5.5) holds. □

6. The working set-up

- Define a quasi-Polish space $\Omega = C([0,T];X_w) \times C_0([0,T];\mathbb{R}^\mathbb{N})$ with the filtration $(\mathscr{F}_t)$ where $\mathscr{F}_t = \mathscr{B}_t(C([0,T];X_w)) \otimes \mathscr{B}_t(C_0([0,T];\mathbb{R}^\mathbb{N}))$ and
- a processes $z : [0,T] \times \Omega \to X$ and $B_j : [0,T] \times \Omega \to \mathbb{R}$

$$z(t,a,b) = a(t), \qquad B_j(t,a,b) = (b(t))_j. \tag{6.1}$$

All the statements in Section 3 hold for the quasi-Polish space Ω, especially the σ-algebra $\mathscr{F}_T$ contains all σ-compact subsets of Ω.

Definition 6.1. *Introduce systems of stopping times* $\tau, \tau^+ : (0,\infty) \times C_0([0,T]) \to (0,\infty]$

$$\tau_r(h) = \inf\{t \in [0,T] : |h(t)| = r\}, \qquad \tau_r^+(h) = \lim_{\varepsilon \to 0+} \tau_{r+\varepsilon}(h),$$

a set $J = \{(r,h) : \tau_r(h) < \tau_r^+(h)\}$ *and a mapping*

$$L_t^r : C_0([0,T]) \to [-r,r] : h \mapsto h(t \wedge \tau_r(h)), \qquad t \geq 0, \quad r > 0.$$

Remark 6.2. See [**?**] for the following observations:

- τ is lower-semicontinuous,
- $r \mapsto \tau_r(h)$ is nondecreasing and left-continuous for every $h \in C_0([0,T])$,
- if $\tau_r(h) = \tau_r^+(h)$ then τ_r and L^r are continuous at h,

- $J \in \mathscr{B}(0,\infty) \otimes \mathscr{B}(C_0([0,T]))$,
- the cut-set J^h is at most countable for every $h \in C_0([0,T])$,
- L_t^r is $\mathscr{B}_t(C_0([0,T]))$-measurable for every $t \geq 0$ and $r > 0$.

Proposition 6.3. *Let (μ_n) be a tight sequence of probability measures on $\mathscr{B}_*(\Omega)$ converging to some μ in $\mathscr{P}_*(\Omega)$. Let M be a continuous $(\mathscr{F}_t)$-adapted process on $[0,T]$ with $M(0) = 0$ and, for every $n \in \mathbb{N}$, M is a continuous local $(\mathscr{F}_t)$-martingale for μ_n. Let $M(\omega_n) \to M(\omega)$ in $C_0([0,T])$ whenever $\omega_n \to \omega$ in Ω. Then M is a local $(\mathscr{F}_t)$-martingale for μ.*

Proof. Consider the sets

$$D_{r,p} = \{\omega : \exists\, \omega_n,\, \omega_n \to \omega,\, \limsup_{n\to\infty} |L_p^r(M(\omega_n)) - L_p^r(M(\omega))| > 0\}.$$

Then $D_{r,p} \subseteq \{\omega : M(\omega) \in J^r\}$ so, for almost every $r > 0$, $\mu(D_{r,p}) = 0$ holds for every $p \in [0,T]$ by the Fubini theorem and, in particular,

$$\int_\Omega GL_p^r(M)\, d\mu_n \to \int_\Omega GL_p^r(M)\, d\mu, \qquad \forall p \in [0,T]$$

holds for every sequentially continuous $G : \Omega \to [0,1]$ by Proposition 3.2. If $0 \leq s < t \leq T$ and G is also $\mathscr{F}_s$-measurable then

$$\int_\Omega [GL_t^r(M) - GL_s^r(M)]\, d\mu = 0.$$

□

Lemma 6.4. *Let M be a real $(\mathscr{F}_t)$-adapted process on $[0,T]$ with continuous paths, $M(0) = 0$, let $(O, \mathcal{G}, (\mathcal{G}_t), \nu, W, u)$ be a solution of (5.1) on $[0,T]$, denote by μ the law of (u,W) on $\mathscr{F}_T$ and assume that $(M(t,(u,W)))_{t\in[0,T]}$ is a local $(\mathcal{G}_t)$-martingale. Then M is a local $(\mathscr{F}_t)$-martingale for μ.*

Proof. The result follows from the equality

$$\int_O [L_t^r(M(u,W)) - L_s^r(M(u,W))]\mathbf{1}_A(u,W)\, d\nu = \int_A [L_t^r(M) - L_t^r(M)]\, d\mu$$

which holds for every $0 \leq s < t \leq T$, $A \in \mathscr{F}_s$ and $r > 0$. □

Theorem 6.5. *Let (5.2) hold, let $(O, \mathcal{G}, (\mathcal{G}_t), \nu, W, u)$ be a solution of (5.1) on $[0,T]$ and μ is the law of (u,W) on $\mathscr{F}_T$. Then $(\Omega, \mathscr{F}_T, (\mathscr{F}_t), \mu, B, z)$ is a solution of (5.1) on $[0,T]$.*

Proof. We may apply Lemma 6.4 to

$$M_\gamma(t,\omega) = \varphi_\gamma(z(t,\omega)) - \varphi_\gamma(z(0,\omega)) - \int_0^t f_\gamma(z(s,\omega))\, ds \tag{6.2}$$

$$M_\gamma^2(t,\omega) - \int_0^t \|g_\gamma(s,z(s,\omega))\|_{\ell_2}^2\, ds, \qquad B_i, \qquad B_iB_j - \delta_{ij}t \tag{6.3}$$

$$M_\gamma(t,\omega)B_i(t,\omega) - \int_0^t g_{\gamma,i}(s,z(s,\omega))\, ds \tag{6.4}$$

and then Theorem 5.6. □

7. A tightness criterion

Here we prove that under (5.2), tightness of a set of laws of solutions on $\mathscr{B}_T(C([0,T];X_w))$ is implied by mere uniform boundedness in probability of these solutions (which is also a necessary condition for tightness, see Proposition 4.4). It is usually known whether solutions are uniformly bounded in probability or not so, in this way, we can conclude immediately tightness. For, if $\alpha \geq 0$, we define the closed set in $C([0,T];X_w)$

$$K^\alpha = \{h \in C([0,T];X_w) : \sup_{t\leq T} \|h(t)\|_X \leq \alpha\} \in \mathscr{B}_T(C([0,T];X_w)). \tag{7.1}$$

Observe also that $\{\varphi_\gamma\}_{\gamma\in\mathbb{N}}$ separates points of X as $\{\varphi_\gamma\}_{\gamma\in\mathbb{N}}$ is assumed to be dense in $\mathrm{Dom}(\mathbf{A}^*)$ and $\mathrm{Dom}(\mathbf{A}^*)$ is dense in X.

Proposition 7.1. *Let $\alpha \geq 0$, $\varepsilon > 0$ and let* (5.2) *hold. Then there exist a maximal compact K in $C([0,T];X_w)$ such that $\nu\,[u \notin K, u \in K^\alpha] \leq \varepsilon$ holds for every solution $(\mathcal{O},\mathcal{G},(\mathcal{G}_t),\nu,W,u)$ of* (5.1) *on* $[0,T]$.

Proof. Let λ and p be positive numbers such that $p^{-1} < \lambda$ and $\lambda + p^{-1} < \frac{1}{2}$. Fix $0 < \rho < \lambda - p^{-1}$, $\gamma \in \mathbb{N}$ and define $B = \{x \in X : \|x\|_X \leq \alpha\}$. Define the $(\mathcal{G}_{t+})$-stopping time $\tau = \inf\{t \in [0,T] : u(t) \notin B\}$ and the process $\tilde{u}(t) = u(t \wedge \tau)$. Then, by the Garsia-Rodemich-Rumsey lemma [**?**], there exists C_γ such that the modulus of continuity satisfies

$$\mathbb{E}\,[\delta(\langle\varphi_\gamma,\tilde{u}\rangle,\theta)]^p \leq C_\gamma \theta^{\lambda p}, \qquad \forall \theta \in [0,1].$$

So, since $[u \in K^\alpha] \subseteq [\tau \geq T]$, we get by the Tchebyschev inequality,

$$\nu\left([u \in K^\alpha] \cap \bigcup_{n=1}^{\infty} [\delta(\langle\varphi_\gamma,u\rangle,n^{-1}) > \beta_n(\vartheta)]\right) \leq c_\gamma \vartheta, \qquad \forall \vartheta > 0$$

where $\beta_n(\vartheta) = \vartheta^{-\frac{1}{p}} n^{-\rho}$. Choosing $\vartheta_\gamma > 0$ so that $\sum_{\gamma=1}^\infty c_\gamma \vartheta_\gamma \leq \varepsilon$, we can set $K = K^{\alpha,\{\beta_n(\vartheta_\gamma)\},\{\varphi_\gamma\}}$. ☐

8. bw-continuous dependence

Let us introduce a bw-continuity assumption:

$$f_\gamma,\ g_{\gamma,i},\ \|g_\gamma\|_{\ell_2} \quad \text{are sequentially weakly continuous on } X \text{ for every } \gamma, i \in \mathbb{N} \tag{8.1}$$

We are going to study continuous dependence of solutions on the coefficients.

(a) Let $(\Omega^n,\mathscr{F}^n,(\mathscr{F}^n_t),\mathbb{P}^n,W^n,u^n)$ be solutions of (5.1) on $[0,T]$ such that the laws of u^n on $\mathscr{B}_T(C([0,T];X_w))$ are tight and denote by μ_n the laws of (u^n,W^n) on $\mathscr{F}_T$ and extend them to $\mathscr{B}_*(\Omega)$ (see Section 3).

(b) Let n_k be some subsequence and $\mu \in \mathscr{P}_*(\Omega)$ such that $\mu_{n_k} \to \mu$ in $\mathscr{P}_*(\Omega)$.

Theorem 8.1. *Let (a), (b),* (5.2) *and* (8.1) *hold. Then $(\Omega,\mathscr{F}_T,(\mathscr{F}_t),\mu,(B_i),z)$ is a solution of* (5.1) *on* $[0,T]$.

Proof. By Theorem 6.5 and Proposition 6.3, the assumptions on Theorem 5.6 are satisfied. ☐

9. *bw*-Feller semigroup

(1) Let (5.1) have global solution $(O^x,\mathcal{G}^x,(\mathcal{G}^x_t),\mathbb{P}^x,W^x,u^x)$ on $\mathbb{R}_+$ with $u^x(0)=x$ for every $x\in X$.
(2) Let weak uniqueness hold for (5.1) in the class of solutions with the initial law δ_x, whenever $x\in X$.
(3) Let for $\forall\varepsilon>0\ \forall\tau>0\ \forall r>0\ \exists R>0$ such that $\mathbb{P}^x\left[\sup_{t\in[0,\tau]}\|u^x(t)\|_X\geq R\right]\leq\varepsilon$ holds for every $\|x\|_X\leq r$.

Define the Markov operators for bounded Borel functions $\psi:X\to\mathbb{R}$

$$(U_t\psi)(x)=\int_{O^x}\psi(u^x(t))\,d\mathbb{P}^x,\qquad t\in\mathbb{R}_+ \tag{9.1}$$

and denote by μ^x the law of $(u^x|_{[0,T]},W^x|_{[0,T]})$ on $\mathscr{F}_T$ and extend it to $\mathscr{B}_*(\Omega)$.

Remark 9.1. It is well known that $(U_t)_{t\geq0}$ is a well defined semigroup of operators on bounded Borel functions on X under (1) and (2) above, see e.g. [**?**].

Remark 9.2 (Joint uniqueness in law). If $(\Omega,\mathscr{F}_T,(\mathscr{F}_t),\mu^i,B,z)$ are solutions of (5.1) for $i=1,2$ and $\mu^1[z(0)=x]=\mu^2[z(0)=x]=1$ for some $x\in X$ then the condition (2) above on weak uniqueness implies that $\mu^1=\mu^2$ on $\mathscr{F}_T$ by the Cherny theorem [**?**, Theorem 4].

Remark 9.3. Pathwise uniqueness of (5.1) implies the condition (2) above by the Yamada-Watanabe theorem [**?**].

Theorem 9.4. *Let (1)-(3) above hold, let* (5.2) *and* (8.1) *be satisfied and let $t_n\to t$ in $\mathbb{R}_+$, $x_n\to x$ weakly in X and $\psi:X\to\mathbb{R}$ is a bounded sequentially weakly continuous function. Then $U_{t_n}\psi(x_n)\to U_t\psi(x)$.*

Proof. We may assume that $t_n\leq T$ holds for every $n\in\mathbb{N}$. Then $(\Omega,\mathscr{F}_T,(\mathscr{F}_t),\mu^x,B,z)$ is a solution of (5.1) on $[0,T]$ by Theorem 6.5. The measures μ^{x_n} are tight on $\mathscr{F}_T$ by Proposition 7.1 and $\mu^{x_n}\to\mu^x$ in $\mathscr{P}_*(\Omega)$ by Theorem 8.1 and Remark 9.2. Now apply Proposition 3.2. $\square$

10. Invariant measure

Let us recall a consequence of Proposition 3.1 in [**?**].

Theorem 10.1. *Under the assumptions of Thereom 9.4, let there exist a global solution $(O,\mathcal{G},(\mathcal{G}_t),\nu,u,W)$ of* (5.1) *such that, for every $\varepsilon>0$, there exists $R>0$ and*

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\nu\left[\|u(s)\|_X\geq R\right]ds\leq\varepsilon. \tag{10.1}$$

Then there exists an invariant measure for $(U_t)_{t\geq0}$ defined in (9.1).

11. Stochastic beam equation

Consider the equation

$$u_{tt} + A^2 u + \beta u_t + m(\|B^{1/2}u\|_H^2)Bu = G(u)\,dW, \tag{11.1}$$

with the hypotheses (b1)-(b4) set up in Section 1.1 and define $X = \operatorname{Dom} A \times H$.

Remark 11.1. If C is a closed operator on H then we consider $\operatorname{Dom} C$ as a Hilbert space with $\|x\|^2_{\operatorname{Dom} C} = \|x\|^2_X + \|Cx\|^2_X$ for $x \in \operatorname{Dom} C$.

Remark 11.2. We may also define $\operatorname{Dom} \mathbf{A} = \operatorname{Dom} A^2 \times \operatorname{Dom} A$,

$$\mathbf{A} = \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix}, \quad \mathbf{F}(u,v) = \begin{pmatrix} 0 \\ -m(\|B^{1/2}u\|_H^2)Bu - \beta v \end{pmatrix}, \quad \mathbf{G}(u,v)\xi = \begin{pmatrix} 0 \\ G(u)\xi \end{pmatrix}$$

and rewrite (11.1) as a stochastic evolution equation

$$d\phi = (\mathbf{A}\phi + \mathbf{F}(\phi))\,dt + \mathbf{G}(\phi)\,dW \tag{11.2}$$

in the Hilbert space X. On the other hand, let $(h_\gamma)_{\gamma\in\mathbb{N}}$ be some dense subset in $\operatorname{Dom} A^2$, (ξ_i) an orthonormal basis in $\mathcal{X}$ and define, for $\gamma, i \in \mathbb{N}$ and $(u,v) \in X$,

$$f_\gamma(u,v) = -\langle u, A^2 h_\gamma\rangle_H - \langle m(\|B^{1/2}u\|_H^2)Bu + \beta v, h_\gamma\rangle_H, \qquad f_{-\gamma}(u,v) = \langle v, h_\gamma\rangle_H$$

$$g_{\gamma,i}(u,v) = \langle G(u)\xi_i, h_\gamma\rangle_H, \quad g_{-\gamma}(u,v) = 0, \quad \varphi_{-\gamma}(u,v) = \langle u, h_\gamma\rangle_H, \quad \varphi_\gamma(u,v) = \langle v, h_\gamma\rangle_H.$$

Now, according to Remark 5.3, the equation (11.2) is equivalent to

$$d\varphi_\gamma(u(t)) = f_\gamma(u(t))\,dt + \sum_{i=1}^{\infty} g_{\gamma,i}(u(t))\,dW_i, \qquad t \in I, \quad \gamma \in \mathbb{N} \tag{11.3}$$

where $W_i = W(\xi_i)$.

Remark 11.3. By [**?**], the equation (11.1) has a global solution for every $(\mathcal{G}_0)$-measurable X-valued initial condition, pathwise uniqueness holds and every solution has X-continuous paths almost surely.

11.1. **Weak sequential continuity.** Let us consider the additional hypotheses (b5) and (b6) introduced in Section 1.1.

Lemma 11.4. *Let (b1), (b2), (b5) and (b6) hold. Then f_γ, $g_{\gamma,i}$ and $\|g_\gamma\|^2_{\ell_2}$ are sequentially weakly continuous for every $\gamma \in \mathbb{Z}\setminus\{0\}$ and $i \in \mathbb{N}$.*

Proof. The claim is obvious as

$$\|B^{1/2}x\|_H^2 = \langle Bx, x\rangle_H \le \|Bx\|_H \|x\|_H, \qquad \forall x \in \operatorname{Dom} A$$

and $\operatorname{Dom} B$ is compactly embedded in H. In particular, if $x_n \to x$ weakly in $\operatorname{Dom} A$ then $x_n \to x$ in $\operatorname{Dom} B^{1/2}$. □

11.2. **Weak tightness.** Under (b1)-(b4), define $M(r) = \int_0^r m(s)\,ds$ for $r \geq 0$,

$$V(w) = \frac{1}{2}\|w\|_X^2 + M(\|B^{1/2}w_1\|^2), \qquad w = (w_1, w_2) \in X,$$

$q_k = \inf\{V(w) : \|w\|_X \geq k\}$, let ϕ^x be the unique global mild solution of (11.2) with $\mathbb{P}[\phi^x(0) = x] = 1$ and $\tau_k^x = \inf\{t \geq 0 : \|\phi^x(t)\|_X \geq k\}$ for $x \in X$.

In the course of the proof of Theorem 1.1 in [**?**], it is shown that

$$\mathbb{E}\,V(\phi^x(t\wedge\tau_k^x)) \leq V(x) + 2L^2\int_0^t (1 + \mathbb{E}\,V(\phi^x(s\wedge\tau_k^x)))\,ds, \qquad t \geq 0$$

hence

$$\mathbb{P}[\tau_k^x \leq t] \leq \frac{e^{2L^2t}}{q_k}(1 + V(x)), \qquad t \geq 0. \tag{11.4}$$

by the Gronwall inequality. In particular, if $\varepsilon > 0$ and $R > 0$ are given, there exists $\alpha \geq 0$ such that $\mathbb{P}[\phi^x \notin K^\alpha] \leq \varepsilon$ holds for every $\|x\|_X \leq R$ where K^α was defined in (7.1). Now, as a consequence of Proposition 7.1, we get the following tightness result:

Lemma 11.5. *Let $R > 0$ and $\varepsilon > 0$. Then there exists a maximal compact K in $C([0,T]; X_w)$ (see Remark 4.5) such that $\mathbb{P}[\phi^x \in K] > 1 - \varepsilon$ holds whenever $\|x\|_X \leq R$.*

11.3. **Weak sequential Feller property.** Under (b1)-(b6), ϕ^x denotes the unique mild solution of (11.2) starting from $x \in X$ and we define $U_tF(x) = \mathbb{E}F(\phi^x(t))$ for $(t,x) \in \mathbb{R}_+ \times X$.

Theorem 11.6. *Let (b1)-(b6) hold. Then $(U_t)_{t\geq 0}$ is a semigroup on bounded Borel measurable functions on X and if $t_n \to t$ in $\mathbb{R}_+$ and $x_n \to x$ weakly in X and F is a bounded sequentially weakly continuous function on X then $U_{t_n}F(x_n) \to U_tF(x)$.*

Proof. The assumptions of Theorem 9.4 are satisfied. □

11.4. **Boundedness in probability.** Assuming (b1)-(b4) hold, set

$$\mathbf{P} = \begin{pmatrix} \beta^2 A^{-2} + 2I & \beta A^{-2} \\ \beta I & 2I \end{pmatrix} \in \mathcal{L}(X,X), \qquad M(r) = \int_0^r m(s)\,ds, \quad r \geq 0$$

and

$$V(w) = \frac{1}{2}\langle w, \mathbf{P}w\rangle_X + M(\|B^{1/2}w_1\|^2), \qquad w = (w_1, w_2) \in X.$$

Then $V \geq 0$, $V \in C^2(X)$ and, for $w \in \operatorname{Dom} A^2 \times \operatorname{Dom} A$,

$$\begin{aligned} LV(w) &= \langle \mathbf{A}w + \mathbf{F}(w), V'(w)\rangle_X + \frac{1}{2}\sum_{i=1}^\infty \langle V''(w)\mathbf{G}_i(w_1), \mathbf{G}_i(w_1)\rangle \\ &\leq -\beta\|w\|_X^2 + \|G(w_1)\|^2_{\mathcal{L}_2(\ell_2,H)}. \end{aligned}$$

Therefore

$$\lim_{R\to\infty}[\sup\{LV(w) : w \in \operatorname{Dom} A^2 \times \operatorname{Dom} A,\ \|w\|_X \geq R\}] = -\infty$$

provided the hypothesis (b7) in Section 1.1 holds.

Formal calculations following [?, Theorem 3.7] show that any solution u to (11.1) with $u(0)$ deterministic is bounded in probability in the mean on the interval $[1, \infty)$, that is, (10.1) holds. These calculations may be justified in a straightforward manner by invoking the approximations used in [?, Section 3].

Altogether, we have obtained the following result as a consequence of Theorem 10.1:

Theorem 11.7. *Let (b1)-(b7) hold. Then there exists an invariant measure for* (11.1).

Plainly, Theorem 1.1 follows from Theorems 11.6 and 11.7.

Example 11.8 (Section 4 in [?]). Let $D \subseteq \mathbb{R}^n$ be a bounded domain with a C^∞-boundary and $\Pi : D \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ a Borel function. Let us consider a multidimensional version of the extensible beam equation

$$\frac{\partial^2 u}{\partial t^2} - m\left(\int_D |\nabla u|^2\, dx\right) \Delta u + \gamma \Delta^2 u + \beta \frac{\partial u}{\partial t} = \Pi(x, u, \nabla u) Q^{1/2}\, \dot{W}$$

with either the clamped boundary condition

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial D \tag{11.5}$$

or the hinged boundary condition

$$u = \Delta u = 0 \quad \text{on} \quad \partial D \tag{11.6}$$

where $Q \geq 0$ is a selfadjoint bounded trace class operator on $L^2(D)$ and W is a cylindrical Wiener process on $L^2(D)$. This equation may be rewritten in the form (11.1), a fortiori (5.5). Set $\gamma = 1$ without loss of generality and define $H = \mathcal{X} = L^2(D)$, let $B = -\Delta$ be the Dirichlet Laplacian on D with $\operatorname{Dom} B = W_0^{1,2}(D) \cap W^{2,2}(D)$. Further, set $A = B$ for the boundary condition (11.6) and $A = C^{1/2}$ for the boundary condition (11.5) where $C = \Delta^2$ and $\operatorname{Dom} C = \{\psi \in W^{4,2}(D) : \psi = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D\}$. Then (b1), (b2) and (b5) are satisfied.

Now let us turn to the stochastic term. It was shown in [?] that if Π is bounded, does not depend on the last variable, $1 \leq n \leq 3$ and $|\Pi(x,r) - \Pi(x,\tilde{r})| \leq L|r - \tilde{r}|$ for almost every $x \in D$ and every $r, \tilde{r} \in \mathbb{R}$ and $G(\psi) = \Pi(\cdot, \psi) Q^{1/2}$ then $G : \operatorname{Dom} A \to \mathcal{L}_2(\mathcal{X}, H)$ is a Lipschitz map and (b4) holds. If, moreover, $n = 1$ then

$$\|G(\psi) - G(\tilde{\psi})\|_{\mathcal{L}_2(\mathcal{X},H)} \leq c\|\psi - \tilde{\psi}\|_{W^{1,2}(D)} \tag{11.7}$$

owing to the embedding $W^{1,2}(D) \hookrightarrow L^\infty(D)$. Consequently, (b6) is satisfied since the norm of $\operatorname{Dom} B^{1/2}$ is equivalent to the $W^{1,2}(D)$-norm.

The hypotheses upon Π may be relaxed considerably provided that H has a basis $\{\tilde{e}_i\}_{i \in \mathbb{N}}$ of eigenvectors of Q satisfying $\sup_{i \in \mathbb{N}} \|\tilde{e}_i\|_{L^\infty(D)} < \infty$. Suppose that $|\Pi(x, r, s) - \Pi(x, \tilde{r}, \tilde{s})| \leq c(|r - \tilde{r}| + |s - \tilde{s}|)$ for almost every $x \in D$ and every $r, \tilde{r} \in \mathbb{R}$ and $s, \tilde{s} \in \mathbb{R}^n$. Setting $G(\psi) = \Pi(\cdot, \psi, \nabla\psi) Q^{1/2}$ then, as shown in [?], $G : \operatorname{Dom} A \to \mathcal{L}_2(\mathcal{X}, H)$ satisfies (11.7) as well.

12. Stochastic wave equation

In [?], existence of global mild solutions to stochastic wave equations was proved in general domains for coefficients of polynomial growth. [?, Example 10] can be seen as a stochastic generalization of classical by now papers by K. Jörgens [?], F. Browder [?], I. Segal [?] and W. Strauss [?]. Below we will show that a damped version of that equation has an invariant measure. Let us begin by stating the corresponding result from [?].

Let us consider an equation

$$u_{tt} = -A^2 u - au|u|^{p-1} - \beta u_t + F + \eta g(u)\,\dot{W} \qquad \text{on} \quad D \tag{12.1}$$

where either $D = \mathbb{R}^d$ or D is a bounded domain in $\mathbb{R}^d$ and A is a positive selfadjoint operator on $L^2(D)$ with $0 \in \varrho(A)$ and with $\operatorname{Dom} A$ being a closed subspace in $W^{1,2}(D)$, $a > 0$, $p \geq 1$, $\beta \geq 0$, $F \in L^2(D)$, $\eta \in L^\infty(\mathbb{R}^d)$. The equation is considered on some stochastic basis $(O, \mathcal{G}, (\mathcal{G}_t), \mathbb{P})$ with a cylindrical $(\mathcal{G}_t)$-Wiener process on a separable Hilbert space $\mathcal{X}$ embedded continuously in $L^\infty(D)$ such that there exists a constant $\mathbf{c}$ for which

$$\|\xi \mapsto h\xi\|_{\mathcal{L}_2(\mathcal{X}, L^2(D))} \leq \mathbf{c}\|h\|_{L^2(D)} \tag{12.2}$$

holds for every $h \in L^2(D)$. If $D = \mathbb{R}^d$ then we also assume that

$$\{\varphi \in \operatorname{Dom} A^2 : \text{ess-supp}\,\varphi \text{ is compact}\} \text{ is dense in } \operatorname{Dom} A^2. \tag{12.3}$$

Remark 12.1. We can consider the operator $A = \sqrt{-\Delta + m^2 I}$ for some $m > 0$ if $D = \mathbb{R}^d$ or $D \subseteq \mathbb{R}^d$ is bounded with a C^2-smooth boundary and Δ is the Neumann Laplace operator, or with $m \geq 0$ if D is bounded with a C^2-smooth boundary and Δ is the Dirichlet Laplace operator.

Remark 12.2. If $D = \mathbb{R}^d$ then spatially homogeneous $(\mathcal{G}_t)$-Wiener processes W with finite spectral measures μ satisfy (12.2). In other words, W can be an $\mathscr{S}'(\mathbb{R}^d)$-valued process where $\mathscr{S}(\mathbb{R}^d)$ denotes the Schwartz space of smooth rapidly decreasing real functions on $\mathbb{R}^d$ and $\hat{S}$ the Fourier transform of a tempered distribution S, such that

- $W_t\varphi$ is a real $(\mathcal{G}_t)$-Wiener process for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$,
- $W_t(c\varphi_1 + \varphi_2) = cW_t(\varphi_1) + W_t(\varphi_2)$ a.s. for $\forall c \in \mathbb{R}$, $\forall t \in \mathbb{R}_+$ and $\forall \varphi_1, \varphi_2 \in \mathscr{S}(\mathbb{R}^d)$,
- $\mathbb{E}\{W_t\varphi_1 W_t\varphi_2\} = t\langle \widehat{\varphi}_1, \widehat{\varphi}_2\rangle_{L^2(\mu)}$ for all $t \geq 0$ and $\varphi_1, \varphi_2 \in \mathscr{S}(\mathbb{R}^d)$,

see [?, ?] for further details and examples of spatially homogeneous Wiener processes. Let $\mathcal{X} \subseteq \mathscr{S}'(\mathbb{R}^d)$ be the reproducing kernel Hilbert space of the $\mathscr{S}'(\mathbb{R}^d)$-valued random vector $W(1)$, see e.g. [?]. Then W is a cylindrical Wiener process on $\mathcal{X}$, i.e. if we fix an orthonormal basis (ξ_i) in $\mathcal{X}$ then there exist standard real-valued $(\mathcal{G}_t)$-Wiener processes (W_i) such that

$$W_t(\varphi) = \sum_i W_i(t)\langle \varphi, \xi_i\rangle, \qquad t \geq 0, \quad \varphi \in \mathscr{S}(\mathbb{R}^d). \tag{12.4}$$

Moreover, see [?, ?], if we denote by $L^2_s(\mathbb{R}^d, \mu)$ the complex subspace of $L^2_{\mathbb{C}}(\mathbb{R}^d, \mu)$ of all ψ such that $\psi = \psi_s$ where $\psi_s(x) = \overline{\psi(-x)}$, then

$$\mathcal{X} = \{\widehat{\psi\mu} : \psi \in L^2_s(\mathbb{R}^d, \mu)\}, \qquad \langle \widehat{\psi\mu}, \widehat{\varphi\mu}\rangle_{\mathcal{X}} = \langle \psi, \varphi\rangle_{L^2(\mu)}, \qquad \psi, \varphi \in L^2_s(\mathbb{R}^d, \mu).$$

In fact, according to [?], $\mathcal{X}$ is continuously embedded in $C_b(\mathbb{R}^d)$ and if $h \in L^2(\mathbb{R}^d)$ then the multiplication operator $\xi \mapsto h\xi$ is Hilbert-Schmidt from $\mathcal{X}$ to $L^2(\mathbb{R}^d)$ and (12.2) holds with $\mathbf{c}^2 = (2\pi)^{-d}\mu(\mathbb{R}^d)$ even as an equality.

Remark 12.3. If $D \subseteq \mathbb{R}^d$ is bounded then the restriction of spatially homogeneous $(\mathcal{G}_t)$-Wiener process with finite spectral measure on D satisfies (12.2).

Remark 12.4. Stochastic integration with respect to a spatially homogeneous Wiener process is understood in the standard way, see e.g. [?], [?] or [?].

We define the state space $X := \operatorname{Dom} A \times L^2(D)$ which we equip with the Hilbert norm $\|(u,v)\|_X^2 = \|Au\|_2^2 + \|v\|_2^2$. Then the equation (12.1) can be written in an equivalent form as a first order in time evolution equation

$$dz = \mathbf{A}z\,dt + \mathbf{F}(z)\,dt + \mathbf{G}(z)\,dW \tag{12.5}$$

on X where, for $z = (u,v) \in X$, $\xi \in \mathcal{X}$ and with $\operatorname{Dom}\mathbf{A} = \operatorname{Dom} A^2 \times \operatorname{Dom} A$,

$$\mathbf{A} = \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix}, \quad \mathbf{F}(z) = \begin{pmatrix} 0 \\ -au|u|^{p-1} - \beta v + F \end{pmatrix}, \quad \mathbf{G}(z)\xi = \begin{pmatrix} 0 \\ \eta g(u)\xi \end{pmatrix}.$$

Let us consider the following hypotheses:

(w1) $p \in [1,\infty)$ if $d \in \{1,2\}$ or $p \in [1, \frac{d}{d-2}]$ if $d \geq 3$,
(w2) $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz,
(w3) if $d = 2$ then $|g'|$ grows polynomially, if $d \geq 3$ then $|g'(x)| \leq c(1 + |x|^{\frac{2}{d-2}})$ a.e.,
(w4) $|g(x)|^2 \leq c_0 + c_1|x|^2 + c_2|x|^{p+1}$ for some $c_0, c_1, c_2 \in [0,\infty)$ and all $x \in \mathbb{R}$,
(w5) $c_0\eta \in L^2(D)$,
(w6) $\mathbf{c}^2 c_1 \|\eta\|_{L^\infty(D)}^2 \|A^{-1}\|_{\mathcal{L}(L^2(D))}^2 < \beta$ and $\mathbf{c}^2 c_2 \|\eta\|_{L^\infty(D)}^2 < a\beta$.

The following was proved in [?].

Theorem 12.5. *Let (w1)-(w5) hold. Then the equation* (12.5) *has a unique X-valued continuous mild solution for every $\mathcal{G}_0$-measurable X-valued initial condition.*

To be precise only the case $\beta = 0$, $F = 0$, $\eta = 1$ was considered in [?] but the same proof literally applies to the general case as it was later showed in a much more general setting in [?]. In the next two results (with a joint proof) we will show that solutions to the equation (12.1) are locally uniformly bounded in probability and globally bounded in probability under some natural assumptions.

Theorem 12.6. *Let (w1)-(w5) hold and fix $r > 0$ and $t > 0$. Then there exits a number $C_{r,t}$ such that*

$$\mathbb{P}\,[\sup_{s \leq t} \|z(s)\|_X \geq R, \|z(0)\|_X \leq r] \leq R^{-2} C_{r,t}, \qquad \forall R > 0$$

holds for every solution z of (12.1).

Theorem 12.7. *Let (w1)-(w6) hold and $r > 0$ is fixed. Then there exists a constant C_r such that*

$$\mathbb{P}\,[\|z(t)\|_X \geq R, \|z(0)\|_X \leq r] \leq R^{-2} C_r, \qquad \forall R > 0, \quad \forall t \geq 0$$

holds for every solution z of (12.1).

Proof. As in the case of the beam equation, the proof of this result will be based on the Pritchard-Zabczyk trick. Consider the operator $\mathbf{P}$ from Section 11.4. Then $\sqrt{\langle \mathbf{P}z, z\rangle_X}$ is an equivalent norm on X. With the notation $\|\cdot\|_r = \|\cdot\|_{L^r(D)}$, define next the Lyapunov functional for $z = (u, v) \in X$ by

$$\Phi(z) = \frac{\langle \mathbf{P}z, z\rangle_X}{2} + \frac{2a}{p+1}\|u\|_{p+1}^{p+1} = \|Au\|_2^2 + \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|\beta u + v\|_2^2 + \frac{2a}{p+1}\|u\|_{p+1}^{p+1}.$$

The map Φ is of C^2-class on X and, analogously to [**?**] and [**?**] we have,

$$\Phi(z(t)) = \Phi(z(0)) + M(t) - \int_0^t V(z(s))\,ds, \quad M(t) = \int_0^t \langle \beta u(s) + 2v(s), \eta g(u(s))\,dW\rangle_2$$

where, for a splitting $\beta = \beta_1 + \beta_2$ to some $\beta_1, \beta_2 > 0$,

$$\begin{aligned} V(z) &= \beta\|z\|_X^2 + a\beta\|u\|_{p+1}^{p+1} - \langle \beta u + 2v, F\rangle_2 - \|\eta g(u)\|_{\mathcal{L}_2(\mathcal{X}, L^2(D))}^2 \\ &\geq \beta_1\|Au\|_2 + [(\beta_2\|A^{-1}\|_{\mathcal{L}(L^2(D))}^{-2} - \mathbf{c}^2 c_1\|\eta\|_\infty^2)\|u\|_2^2 - \beta\|F\|_2\|u\|_2] \\ &+ (a\beta - \mathbf{c}^2 c_2\|\eta\|_\infty^2)\|u\|_{p+1}^{p+1} + [\beta\|v\|_2^2 - 2\|F\|_2\|v\|_2] - \mathbf{c}^2 c_0\|\eta\|_2^2 \\ &\geq \delta\Phi(z) - \kappa \end{aligned} \tag{12.6}$$

for some $\delta > 0$ and $\kappa > 0$ by (w6). If (w6) is not assumed or $\beta = 0$ then $V(z) \geq \delta\|z\|_X^2 - \kappa$ holds for some $\delta \in \mathbb{R}$ and $\kappa > 0$. Assume that $\|z(0)\|_X \leq r$ holds, without imposing (w6). Since there exists $\gamma > 0$ such that

$$\frac{d\langle M\rangle}{dt} \leq \gamma(1 + \Phi(z(t)))^2, \qquad \forall t \geq 0$$

we conclude by the Itô formula that there exist constants C_k, K_r and $K_{k,r}$

$$\mathbb{E}\,[\Phi(z(t))]^k \leq e^{C_k t}(1 + \mathbb{E}\,[\Phi(z(0))]^k) \leq K_r^k e^{C_k t}, \qquad t \geq 0$$

for every $k \in \mathbb{N}$ and, consequently, by the Doob maximal inequality, for some $\vartheta_k > 0$,

$$\mathbb{E} \sup_{t\in[0,T]} [\Phi(z(t))]^k \leq K_{k,r} e^{t\vartheta_k}, \qquad t > 0, \quad k \in \mathbb{N}.$$

In particular, $\mathbb{E}\,[\Phi(z)]^k$ is continuous on $\mathbb{R}_+$ for every $k \in \mathbb{N}$. Now, assuming (w6), we may conclude from (12.6) that

$$\mathbb{E}\Phi(z(t_2)) \leq \mathbb{E}\Phi(z(t_1)) + \int_{t_1}^{t_2} [\kappa - \delta\mathbb{E}\,\Phi(z(s))]\,ds$$

holds for every $0 \leq t_1 \leq t_2$. Hence, by the comparison theorem for ODEs,

$$\mathbb{E}\,\Phi(z(t)) \leq e^{-\delta t}\mathbb{E}\,\Phi(z(0)) + \frac{\kappa}{\delta}(1 - e^{-\delta t}), \qquad t \geq 0.$$

□

We fix a countable dense subset $\{h_\gamma\}$ of Dom A^2 assuming additionally, only if $D = \mathbb{R}^d$, that each h_γ has a compact support, and an orthonormal basis $\{\xi_i\}$ in $\mathcal{X}$. Now define, for $\gamma, i \in \mathbb{N}$ and $z = (u, v) \in X$,

$$f_\gamma(z) = -\langle u, A^2 h_\gamma\rangle_{L^2(D)} + \langle F - au|u|^{p-1} - \beta v, h_\gamma\rangle_{L^2(D)}, \quad f_{-\gamma}(z) = \langle v, h_\gamma\rangle_{L^2(D)} \tag{12.7}$$

$g_{\gamma,i}(z)=\langle \eta g(u)\xi_i,h_\gamma\rangle_{L^2(D)},\ g_{-\gamma}(z)=0,\ \varphi_{-\gamma}(z)=\langle u,h_\gamma\rangle_{L^2(D)},\ \varphi_\gamma(z)=\langle v,h_\gamma\rangle_{L^2(D)}.$
According to Remark 5.3, the equation (12.5) is equivalent to

$$d\varphi_\gamma(u(t))=f_\gamma(u(t))\,dt+\sum_{i=1}^\infty g_{\gamma,i}(u(t))\,dW_i,\qquad t\in I,\quad \gamma\in\mathbb{N} \tag{12.8}$$

where (W_i) were defined in (12.4).

Remark 12.8. Denote by ϕ^x the unique mild solution of (12.5) for a deterministic inital condition $x\in X$ and $U_tF(x)=\mathbb{E}F(\phi^x)$ for $(t,x)\in\mathbb{R}_+\times X$ and $F:X\to\mathbb{R}$ bounded and measurable.

Corollary 12.9. *The laws of $\phi^x|_{[0,T]}$ for $\|x\|_X\le r$ are tight in $\mathscr{B}_T(C([0,T];X_w))$, for every $r>0$.*

Proof. Apply Theorem 12.6 and the tightness criterion Proposition 7.1 which is applicable as 5.2 holds. □

Lemma 12.10. *Let (w1) - (w3) hold. Then f_γ, $g_{\gamma,i}$ and $\|g_\gamma\|^2_{\ell_2}$ are sequentially weakly continuous for every $\gamma\in\mathbb{Z}\setminus\{0\}$ and $i\in\mathbb{N}$.*

Proof. We use the Sobolev and the Rellich embedding theorems here. Indeed, if $u_n\to u$ weakly in $W^{1,2}(D)$ and $l:\mathbb{R}\to\mathbb{R}$ satisfies (w2) and (w3) then $l(u_n)$ converges to $l(u)$ in $L^r(D)$, resp. $L^r_{loc}(\mathbb{R}^d)$ if $D=\mathbb{R}^d$, for every $r\in[1,2)$, and $l(u_n)$ is bounded in $L^2(D)$, resp. $L^2_{loc}(\mathbb{R}^d)$ if $D=\mathbb{R}^d$. Setting $l(x)=ax|x|^{p-1}$ and $l(x)=g(x)$, this is sufficient to conclude bw-continuity of f_γ and $g_{\gamma,i}$ as (h_γ) belong to $L^{2+\varepsilon}(D)$ by the Sobolev embedding and are compactly supported if $D=\mathbb{R}^d$. Concerning the term $\|g_\gamma\|^2_{\ell_2}$, if $z_n\to z$ weakly in X, we have a majorant $|g_{\gamma,i}(z_n)|\le\kappa\|\eta\xi_ih_\gamma\|_{L^2(D)}$. So $\|g_\gamma\|^2_{\ell_2}$ is bw-continuous by the Lebesgue dominated convergence theorem as (12.2) holds. □

Theorem 12.11. *Let (w1)-(w5) hold. Then $(U_t)_{t\ge0}$ is a semigroup on bounded Borel measurable functions on X and if $t_n\to t$ in $\mathbb{R}_+$ and $x_n\to x$ weakly in X and F is a bounded sequentially weakly continuous function on X then $U_{t_n}F(x_n)\to U_tF(x)$.*

Proof. The assumptions of Theorem 9.4 are satisfied. □

Finally, we have obtained the following result as a corollary of Theorem 10.1:

Theorem 12.12. *Let (w1)-(w6) hold. Then there exists an invariant measure for* (12.5).

Zdzisław Brzeźniak, Department of Mathematics, University of York, York, YO10 5DD, zb500@york.ac.uk, tel: +44 1904 32 4154, fax: +44 1904 32 3071

Martin Ondreját, Institute of Information Theory and Automation of the ASCR, Pod Vodárenskou věží 4, CZ-182 08, Praha 8, Czech Republic, ondrejat@utia.cas.cz, phone: ++ 420 266 052 284, fax: ++ 420 286 890 378

Jan Seidler, Institute of Information Theory and Automation of the ASCR, Pod Vodárenskou věží 4, CZ-182 08, Praha 8, Czech Republic, seidler@utia.cas.cz, phone: ++ 420 266 052 041, fax: ++ 420 286 890 378