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Extension of Co-Prime Arrays Based on the Fourth-Order Difference Co-Array Concept

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***Abstract*—An effective sparse array extension method for maximizing the number of consecutive lags in the fourth-order difference co-array is proposed, leading to a novel enhanced sparse array structure based on co-prime arrays with significantly increased number of degrees of freedom (DOFs). One method to exploit the increased DOFs based on non-stationary signals is also proposed, with simulation results provided to demonstrate the effectiveness of the proposed structure.**

***Index Terms*—Sparse array, fourth-order difference co-array, direction of arrival, compressive sensing, co-prime array.**

I. Introduction

In the past, sparse arrays have been proposed for more effective array processing [1]–[4], and co-array equivalence plays an important role in designing various sparse structures for underdetermined direction-of-arrival (DOA) estimation. One class of arrays employing this concept is the co-prime array (CPA) [5] and its recent generalizations [6], where both the spatial smoothing [7], [8] based subspace methods [9], [10] and compressive sensing (CS) based methods [6], [11]–[13] can be used for DOA estimation. Co-prime frequencies are utilized to generate equivalent CPAs based on a ULA in [14], [15], and multi-frequency techniques have been presented for DOA estimation using CPAs [16], [17].

On the other hand, fourth-order cumulant-based DOA estimation has been proposed to resolve more sources than the number of physical sensors [3], [18], and the virtual array concept for the fourth-order cumulant-based method is presented in [19]. Based on the $2q$-th order cumulants [20]–[22], the $2q$-th order difference co-array concept is proposed in [23] for nested arrays. However, such cumulant-based methods can not be applied to Gaussian sources and how to optimize the high-order co-arrays effectively is still an unsolved problem.

If we check the DOFs provided by a CPA at the fourth-order level, it is much smaller than a two-level nested array (TL-NA) or a four-level nested array (FL-NA). This is not surprising since the CPA is not designed for the fourth-order case and so far the study of CPAs has always been limited to the second-order. Therefore, we here focus on the problem of how to extend the CPA structure further from the viewpoint of fourth-order difference co-array, and propose a novel sparse array construction method, leading to an extended structure called sparse array with fourth-order difference co-array enhancement based on CPA (SAFE-CPA). We first revisit the link between the second-order and the fourth-order difference co-arrays, and offer some insights in constructing array structures for the fourth-order difference co-array by considering it as a result of applying the second-order difference co-array operation twice. Then, the extension method is developed by maximizing the number of consecutive lags at the fourth-order stage. The resultant SAFE-CPA has a significantly increased number of DOFs, exceeding that of a nested array.

In our second contribution, to exploit the increased DOFs of the new structure, instead of using the existing cumulant-based method (for non-Gaussian stationary signals), we here assume the signals are non-stationary (but not necessarily non-Gaussian) and extend the second-order statistics based method in [24] to the fourth order, developing a new CS-based DOA estimation method for handling both Gaussian and non-Gaussian sources.

This letter is organized as follows. A review of DOA estimation for co-prime arrays is presented in Sec. II. The sparse array extension method is proposed in Sec. III, while the application with non-stationary signals is introduced in Sec. IV. Simulation results are provided in Sec. V, and conclusions drawn in Sec. VI.

II. DOA Estimation for Co-Prime Arrays

For an N-sensor linear array with a unit spacing d, the set of sensor positions S is expressed as

$$S = \left\{\eta_0 \cdot d, \eta_1 \cdot d, \ldots, \eta_{N-1} \cdot d\right\}, \tag{1}$$

where $\eta_n \cdot d$, $n = 0, \ldots, N-1$, is the position of the n-th sensor, with η_n being an integer in our following study.

A typical CPA consists of two uniform linear sub-arrays. The first sub-array has N_2 sensors with an inter-element spacing of $N_1 d$, and the second one has $2N_1$ sensors with a spacing of $N_2 d$ (another layout uses N_1 sensors and our proposed method is applicable to both configurations). With a shared sensor at the zeroth positions, there are $2N_1 + N_2 - 1$ sensors in total. We use S_1 and S_2 to represent the two sets of sensor positions, i.e. $S = S_1 \bigcup S_2$.

$$\begin{aligned} S_1 &= \left\{N_1 n_2 d,\ 0 \le n_2 \le N_2 - 1, n_2 \in \mathbf{Z}\right\}, \\ S_2 &= \left\{N_2 n_1 d,\ 1 \le n_1 \le 2N_1 - 1, n_1 \in \mathbf{Z}\right\}. \end{aligned} \tag{2}$$

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Assume there are K mutually uncorrelated narrowband signals $s_k(t)$ impinging from the directions $\theta_k,\ k=1,\ldots,K$. Then we obtain the following array signal model

$$\mathbf{x}[i] = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}[i] + \overline{\mathbf{n}}[i] \ , \quad (3)$$

where $\mathbf{x}[i]$ is the observed signal vector, $\mathbf{s}[i]$ is the source signal vector, and $\overline{\mathbf{n}}[i]$ the noise vector. $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)]$ is the steering matrix, with each column vector $\mathbf{a}(\theta_k)$ representing the corresponding steering vector

$$\mathbf{a}(\theta_k) = \left[e^{-j\frac{2\pi\eta_0 d}{\lambda}\sin(\theta_k)}, \ldots, e^{-j\frac{2\pi\eta_{N-1} d}{\lambda}\sin(\theta_k)}\right]^T . \quad (4)$$

The correlation matrix of the received signals is given by

$$\mathbf{R_{xx}} = \mathrm{E}\left\{\mathbf{x}[i]\mathbf{x}^H[i]\right\} = \sum_{k=1}^{K} \sigma_k^2 \mathbf{a}(\theta_k)\mathbf{a}^H(\theta_k) + \sigma_{\overline{n}}^2 \mathbf{I}_N \ , \quad (5)$$

where $\mathrm{E}\{\cdot\}$ is the expectation operator, σ_k^2 is the power of the k-th signal, $\sigma_{\overline{n}}^2$ represents the noise power, and $\mathbf{I}_N$ is the $N \times N$ identity matrix. Vectorizing $\mathbf{R_{xx}}$ yields

$$\mathbf{z} = \mathrm{vec}\left\{\mathbf{R_{xx}}\right\} = \widetilde{\mathbf{A}}\widetilde{\mathbf{s}} + \sigma_{\overline{n}}^2 \widetilde{\mathbf{I}}_{N^2} \ , \quad (6)$$

where $\widetilde{\mathbf{A}} = [\widetilde{\mathbf{a}}(\theta_1), \ldots, \widetilde{\mathbf{a}}(\theta_K)]$ with each column vector $\widetilde{\mathbf{a}}(\theta_k) = \mathbf{a}^*(\theta_k) \otimes \mathbf{a}(\theta_k)$ ($\otimes$ is the Kronecker product), and $\widetilde{\mathbf{s}} = \left[\sigma_1^2, \ldots, \sigma_K^2\right]^T$. The $N^2 \times 1$ vector $\widetilde{\mathbf{I}}_{N^2}$ is obtained by vectorizing $\mathbf{I}_N$. For the virtual array in (6), CS-based methods can be applied for DOA estimation [6], [11]–[13].

III. Sparse Array Extension Based on the Fourth-Order Difference Co-array Concept

A. The fourth-order difference co-array perspective

Definition 1: For the array with sensor positions S in (1), the *second-order difference co-array* (known as *difference co-array*) set is defined as $\mathbb{C}_{\mathbb{A}} = \Phi_A \cdot d$, where the set of difference co-array lags $\Phi_A = \{\eta_{n_1} - \eta_{n_2}, 0 \le n_1, n_2 \le N-1\}$.

Definition 2: The *fourth-order difference co-array* set is defined as $\mathbb{C}_{\mathbb{B}} = \Phi_B \cdot d$, with the set of fourth-order difference co-array lags $\Phi_B = \{\eta_{n_1} + \eta_{n_2} - \eta_{n_3} - \eta_{n_4}\}$, for $0 \le n_1, n_2, n_3, n_4 \le N-1$.

By permutation invariance, Φ_B can be rewritten as

$$\Phi_B = \{(\eta_{n_1} - \eta_{n_3}) - (\eta_{n_4} - \eta_{n_2})\} = \{\mu_1 - \mu_2\} \ , \quad (7)$$

where $\mu_1, \mu_2 \in \Phi_A$.

As a result, the fourth-order difference co-array can be obtained by applying the second-order difference operation again to the virtual array at the difference co-array stage with virtual sensors distributed in $\mathbb{C}_{\mathbb{A}}$. The maximum number of consecutive lags indicates the maximum number of virtual uniform linear array (ULA) sensors generated, and with an appropriate unit spacing between adjacent virtual sensors to avoid spatial aliasing, DOFs provided by this ULA part can be easily exploited through various DOA estimation methods. Therefore, in the following, we consider how to maximize the achievable number of consecutive virtual sensors for quantitative evaluation, comparison, and optimal design.

The difference co-array lags in Φ_A of the CPA can reach consecutive integers from $-N_1N_2 - N_1 + 1$ to $N_1N_2 + N_1 - 1$ [6]. Note there are several non-consecutive lags in Φ_A, and the maximum and minimum difference co-array lags in Φ_A are $(2N_1-1)N_2$ and $-(2N_1-1)N_2$, respectively. With these non-uniform features, we can derive that the fourth-order difference co-array lags in Φ_B can reach every integer from $-3N_1N_2 - N_1 + N_2 + 1$ to $3N_1N_2 + N_1 - N_2 - 1$, and a higher number of DOFs is then achieved.

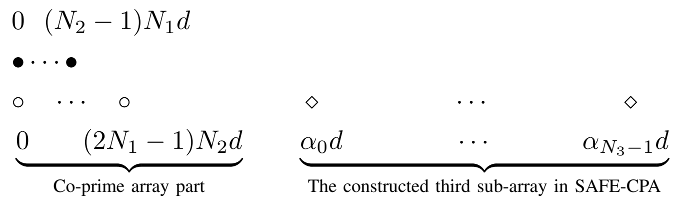


Fig. 1. A general structure of the proposed SAFE-CPA, consisting of three uniform linear sub-arrays, with their sensors expressed as $\bullet$, $\circ$, and $\diamond$, respectively. $\alpha_0 = 6N_1N_2 + 2N_1 - 2N_2 + 1$ and $\alpha_{N_3-1} = 4N_1N_2N_3 + 2N_1N_2 + 2N_1N_3 - N_2N_3 - N_2 + N_3$.

B. Sparse array with fourth-order difference co-array enhancement based on CPA (SAFE-CPA)

The non-uniform features at the difference co-array stage provided by the CPA are limited since the CPA structure is not optimized for the fourth-order co-array. To exploit the advantages provided by the fourth-order difference co-array concept, a further developed sparse array structure called SAFE-CPA is proposed by adding to it a third linear sub-array. By maximizing the number of consecutive integer lags in the resultant Φ_B, we show in the following that the third sub-array should start from the position $\left[6N_1N_2 + 2N_1 - 2N_2 + 1\right]d$ with an inter-element spacing $\left[4N_1N_2 + 2N_1 - N_2 + 1\right]d$, and for the range of consecutive integers in Φ_B, we have:

Proposition 1: For the proposed SAFE-CPA in Fig. 1, the range of consecutive integers in Φ_B is from $-M_0$ to M_0 with

$$\begin{aligned} M_0 = & 4N_1N_2N_3 + 3N_1N_2 + 2N_1N_3 \\ & - N_2N_3 + N_1 - N_2 + N_3 - 1 \ . \end{aligned} \quad (8)$$

Proof: Consider constructing the third sub-array with sensor positions $\alpha_{n_3} d, 0 \le n_3 \le N_3 - 1$ by examining the consecutive lags at each stage associated with each newly added physical sensor, where N_3 is the sensor number of the third sub-array. Since the difference co-array lags and the fourth-order difference co-array lags are both symmetric about 0, we only consider the positive part. In Φ_A, except for the self-difference co-array of the third sub-array under construction, the minimum and maximum positive cross-difference co-array lags associated with the n_3-th sensor are $\alpha_{n_3} - (2N_1-1)N_2$ and α_{n_3}, respectively. Then, the covered range of consecutive integers at the fourth-order difference co-array stage associated with the n_3-th sensor is given by

$$\begin{aligned} \phi_{\alpha_{n_3}} &= \{\mu, \nu_{n_3} \le \mu \le \zeta_{n_3}\} \ , \ \text{ with} \\ \nu_{n_3} &= \alpha_{n_3} - 3N_1N_2 - N_1 + N_2 - 1 \ , \\ \zeta_{n_3} &= \alpha_{n_3} + N_1N_2 + N_1 - 1 \ . \end{aligned} \quad (9)$$

For the starting position $\alpha_0 d$, the lower bound ν_0 in the covered range should be the maximum integer in the consecutive lags in Φ_B plus 1 to ensure the covered range by the

TABLE I
COMPARISON OF THE FOURTH-ORDER DIFFERENCE CO-ARRAY LAGS

Array Structures	Number of Sensors	Number of Consecutive Lags
TL-NA [25]	N_1+N_2	$4N_2(N_1+1)-3^{\dagger}$
FL-NA [23]	$\sum_{\mu=1}^{4} N_\mu+1$	$2\prod_{\mu=1}^{4}(N_\mu+1)-1$
CPA	$2N_1+N_2-1$	$6N_1N_2+2N_1-2N_2-1$
SAFE-CPA	$2N_1+N_2-1+N_3$	$2M_0+1^{\ddagger}$

Examples of different structures for comparison

Array Structures	$(N_1,\ldots,N_m)$ $2\le m\le 4$	Number of Sensors	Number of Consecutive Lags
TL-NA	(4,5)	9	97
FL-NA	(2,2,2,2)	9	161
CPA	(3,4)	9	69
SAFE-CPA	(2,3,3)	9	189
CPA	(5,6)	15	177
SAFE-CPA	(3,5,5)	15	705

† Derived similarly as the fourth-order co-array lags of the CPA.
‡ $M_0=4N_1N_2N_3+3N_1N_2+2N_1N_3-N_2N_3+N_1-N_2+N_3-1$.

starting position is adjacent to the consecutive range of the fourth order difference co-array of the CPA, i.e.

$$\nu_0=\alpha_0-3N_1N_2-N_1+N_2-1=3N_1N_2+N_1-N_2.$$

Then we obtain

$$\alpha_0=6N_1N_2+2N_1-2N_2+1\ . \tag{10}$$

For the remaining sensors in the third sub-array, to ensure that the covered ranges $\phi_{\alpha_{n_3}}$, $n_3=0,1,\ldots,N_3-1$, are adjacent to each other with the purpose of maximizing the consecutive lags, we should have $\nu_{n_3}=\zeta_{n_3-1}+1$, where $1\le n_3\le N_3-1$. Finally we obtain

$$\alpha_{n_3}-\alpha_{n_3-1}=4N_1N_2+2N_1-N_2+1\ . \tag{11}$$

So the third sub-array is also a ULA starting from $\left[6N_1N_2+2N_1-2N_2+1\right]d$ with an inter-element spacing $\left[4N_1N_2+2N_1-N_2+1\right]d$. With $M_0=\zeta_{N_3-1}=4N_1N_2N_3+3N_1N_2+2N_1N_3-N_2N_3+N_1-N_2+N_3-1$, the resultant consecutive integers lie in the range from $-M_0$ to M_0, with a total number of $2M_0+1$ integers.

The number of consecutive integers in Φ_B is listed in Table I for the CPA, the proposed SAFE-CPA, the TL-NA and the FL-NA. We can see that with a fixed number of physical sensors, a significant increase in the number of consecutive lags has been achieved by the SAFE-CPA compared with the CPA. As also shown, give the same nine sensors, the proposed structure has provided the largest number of virtual ULA sensors at the fourth-order difference co-array stage.

IV. Application to DOA Estimation for Non-Stationary Signals

We now consider an application to exploit the fourth-order difference co-array by calculating the second-order difference co-array twice. Such a scenario arises when the impinging signals are non-stationary with two associated key assumptions as shown below. This is different from the existing cumulant-based method, which is applicable only to stationary non-Gaussian signals.

Assumption 1: The uncorrelated source signals $s_k[i]$, $k=1,\ldots,K$ are wide-sense quasi-stationary within the frame length P. Then the local statistical expectation $\sigma_k^2[\tilde{p}]=\mathrm{E}\left\{s_k[i]\cdot s_k^*[i]\right\}$ for $i\in\{\tilde{p}\cdot P,\tilde{p}\cdot P+1,\ldots,(\tilde{p}+1)P-1\}$ can be approximated by

$$\sigma_k^2[\tilde{p}]\approx\frac{1}{P}\sum_{i=\tilde{p}\cdot P}^{(\tilde{p}+1)P-1} s_k[i]\cdot s_k^*[i]\ , \tag{12}$$

where $\tilde{p}=0,\ldots,\widetilde{P}-1$ is the frame index, and $\widetilde{P}$ is the total number of frames.

Assumption 2: $\sigma_k^2[\tilde{p}]$, $k=1,\ldots,K$ are wide-sense stationary and uncorrelated with each other. Then, we obtain

$$\bar{m}_k=\mathrm{E}\left\{\sigma_k^2[\tilde{p}]\right\}\ ,\quad \tilde{\sigma}_k^2=\mathrm{E}\left\{(\sigma_k^2[\tilde{p}]-\bar{m}_k)^2\right\}\ , \tag{13}$$

$$\mathrm{E}\left\{(\sigma_{k_1}^2[\tilde{p}]-\bar{m}_{k_1})\cdot(\sigma_{k_2}^2[\tilde{p}]-\bar{m}_{k_2})\right\}=0,\quad k_1\neq k_2\ . \tag{14}$$

Examples of quasi-stationary signals with uncorrelated stationary powers include many speech and audio signals. In [24], an approach for generating synthetic quasi-stationary signals was given, and will be used in our simulations.

Under *Assumption 1*, we can define the local correlation matrix $\mathbf{R_{xx}}[\tilde{p}]=\mathrm{E}\left\{\mathbf{x}[i]\cdot\mathbf{x}^H[i]\right\}$ within the $\tilde{p}$-th frame

$$\begin{aligned}\mathbf{R_{xx}}[\tilde{p}]&=\sum_{k=1}^{K}\sigma_k^2[\tilde{p}]\mathbf{a}(\theta_k)\mathbf{a}^H(\theta_k)+\sigma_{\bar{n}}^2\mathbf{I}_N\\&\approx\frac{1}{P}\sum_{i=\tilde{p}\cdot P}^{(\tilde{p}+1)P-1}\mathbf{x}[i]\cdot\mathbf{x}^H[i]\ .\end{aligned} \tag{15}$$

Vectorizing $\mathbf{R_{xx}}[\tilde{p}]$ yields

$$\mathbf{z}[\tilde{p}]=\mathrm{vec}\left\{\mathbf{R_{xx}}[\tilde{p}]\right\}=\widetilde{\mathbf{A}}\widetilde{\mathbf{s}}[\tilde{p}]+\sigma_{\bar{n}}^2\widetilde{\mathbf{I}}_{N^2}\ , \tag{16}$$

where $\widetilde{\mathbf{s}}[\tilde{p}]=\left[\sigma_1^2[\tilde{p}],\ldots,\sigma_K^2[\tilde{p}]\right]^T$.

To obtain the fourth-order difference co-array, we apply the difference co-array operation again after transforming the virtual array model in (16) into another virtual model with zero-mean equivalent impinging signals.

With *Assumption 2*, we calculate the following

$$\bar{\mathbf{z}}=\mathrm{E}\left\{\mathbf{z}[\tilde{p}]\right\}=\widetilde{\mathbf{A}}\,\mathrm{E}\left\{\widetilde{\mathbf{s}}[\tilde{p}]\right\}+\sigma_{\bar{n}}^2\widetilde{\mathbf{I}}_{N^2}=\widetilde{\mathbf{A}}\bar{\mathbf{s}}+\sigma_{\bar{n}}^2\widetilde{\mathbf{I}}_{N^2}\ , \tag{17}$$

where $\bar{\mathbf{s}}$ is the expectation of $\widetilde{\mathbf{s}}[\tilde{p}]$.

Subtracting $\bar{\mathbf{z}}$ from $\mathbf{z}[\tilde{p}]$ in (16), each equivalent impinging signal in $\widetilde{\mathbf{s}}[\tilde{p}]$ is then transformed into a zero-mean process

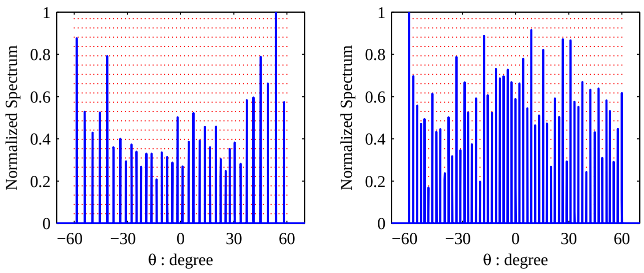
$$\bar{\mathbf{z}}[\tilde{p}]=\mathbf{z}[\tilde{p}]-\bar{\mathbf{z}}=\widetilde{\mathbf{A}}\left\{\widetilde{\mathbf{s}}[\tilde{p}]-\bar{\mathbf{s}}\right\}=\widetilde{\mathbf{A}}\bar{\mathbf{s}}[\tilde{p}], \tag{18}$$

where $\bar{\mathbf{s}}[\tilde{p}]=\widetilde{\mathbf{s}}[\tilde{p}]-\bar{\mathbf{s}}$.

Then we apply the second-order difference co-array concept again, and the correlation matrix is expressed as

$$\mathbf{R_{zz}}=\mathrm{E}\left\{\bar{\mathbf{z}}[\tilde{p}]\cdot\bar{\mathbf{z}}^H[\tilde{p}]\right\}=\sum_{k=1}^{K}\tilde{\sigma}_k^2\widetilde{\mathbf{a}}(\theta_k)\widetilde{\mathbf{a}}^H(\theta_k)\ , \tag{19}$$

where $\tilde{\sigma}_k^2$ is given in (13). Here we are calculating a new correlation matrix based on the virtual signals and the effect of the number snapshots on the estimated new correlation matrix



(a) Results for CPA.

(b) Results for SAFE-CPA.

Fig. 2. DOA estimation results obtained for different array structures.

in (19) is similar to that of the traditional correlation matrix calculation. Vectorizing $\mathbf{R_{zz}}$ yields

$$\mathbf{y} = \text{vec}\left\{\mathbf{R_{zz}}\right\} = \mathbf{Bu} \;, \tag{20}$$

where $\mathbf{B} = [\mathbf{b}(\theta_1), \ldots, \mathbf{b}(\theta_K)]$ with each column vector $\mathbf{b}(\theta_k) = \widetilde{\mathbf{a}}^*(\theta_k) \otimes \widetilde{\mathbf{a}}(\theta_k)$, and $\mathbf{u} = \left[\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_K^2\right]^T$.

(20) represents a further developed virtual array model exploring the fourth-order difference co-array concept, with the equivalent steering matrix $\mathbf{B}$, the equivalent source signal vector $\mathbf{u}$, and the virtual sensors expressed as $\Phi_B \cdot d$.

With a search grid of K_g potential incident angles $\theta_{g,0}, \ldots, \theta_{g,K_g-1}$, a steering matrix is constructed as $\mathbf{B_g} = \left[\mathbf{b}(\theta_{g,0}), \ldots, \mathbf{b}(\theta_{g,K_g-1})\right]$. We also construct a $K_g \times 1$ column vector $\mathbf{u_g}$ with each entry representing a potential source signal at the corresponding incident angle. Then our CS-based DOA estimation employing the fourth-order difference co-array concept is formulated as

$$\min \; \|\mathbf{u_g}\|_1 \;\; \text{subject to} \;\;\; \|\mathbf{y} - \mathbf{B_g u_g}\|_2 \leq \varepsilon \;, \tag{21}$$

where ε is the allowable error bound, $\|\cdot\|_1$ is the l_1 norm and $\|\cdot\|_2$ the l_2 norm. The $K_g \times 1$ vector $\mathbf{u_g}$ represent the DOA estimation results over K_g grid points. The optimization problem can be solved using CVX, a software package for specifying and solving convex problems [26], [27]. Note there are redundant entries in the formulation and those entries can be combined together using the method in [13] to reduce complexity and we will adopt it in our simulations.

V. Simulation Results

Consider a 9-sensor array with $d = \lambda/2$, and $(3, 4)$ for the CPA, and $(2, 3, 3)$ for the SAFE-CPA. The K source signals are uniformly distributed between -60° and 60°. A grid of $K_g = 3601$ angles is formed within the angle range from -90° to 90° with a step size of 0.05°. ε is chosen to give the best result through trial-and-error in every experiment. The signal power expectation $\bar{m}_k$ in (13) is used to calculate the signal-to-noise ratio (SNR).

For the first set of simulations, the input SNR is 0 dB, and the number of sources $K = 55$. The frame length P is 1000, and the number of frames $\widetilde{P} = 1000$. Fig. 2 gives the results for both the CPA and the proposed SAFE-CPA. Clearly, all the sources have been distinguished successfully by the SAFE-CPA, while the CPA has failed. Under the environment of Intel CPU I7-4700HQ with a clock speed of 2.40 GHz and 4 GB RAM, it took about 2.30s for the CPA and 5.40s for the SAFE-CPA to obtain the results.

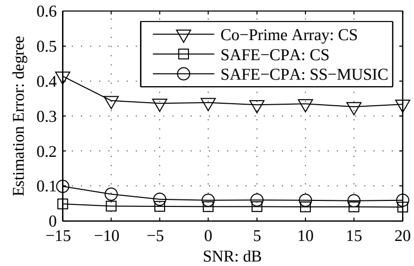
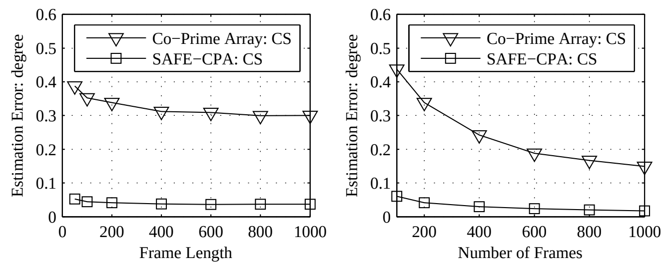


Fig. 3. RMSEs with different array structures versus input SNR.



(a) RMSEs versus P.

(b) RMSEs versus $\widetilde{P}$.

Fig. 4. RMSEs with different frame length P and frame number $\widetilde{P}$.

To compare their estimation accuracy, with $K = 30$ and $P = \widetilde{P} = 200$, the root mean square error (RMSE) results are shown in Fig. 3, where the case for the spatial smoothing based MUSIC (SS-MUSIC) is also provided [5], [9]. Evidently, the higher the input SNR, the higher its estimation accuracy. Furthermore, the physical aperture for the SAFE-CPA is $87d$ while it is $20d$ for the CPA. With a much larger aperture and number of consecutive lags, the SAFE-CPA has consistently outperformed the CPA. For the same SAFE-CPA, the CS-based method has outperformed the SS-MUSIC due to exploration of all unique co-array lags (SS-MUSIC only exploits the consecutive lags).

Finally the values of P and $\widetilde{P}$ are varied with a 0 dB SNR. The RMSE results versus P with $\widetilde{P} = 200$ are shown in Fig. 4 (a), while results versus $\widetilde{P}$ with $P = 200$ are given in Fig. 4 (b). As shown, with the increase of either P or $\widetilde{P}$, the results become more accurate, due to better estimation of the second-order statistics of the involved signals. As expected, the performance of the SAFE-CPA is better than the CPA.

VI. Conclusion

An effective sparse array extension method has been proposed to maximize the number of consecutive lags in the fourth-order co-array. By applying it to the CPA, a new structure with three uniform sub-arrays was derived. To exploit the significantly increased number of DOFs, the DOA estimation problem for non-stationary signals is revisited and a novel two-stage co-array operation is applied. As shown in simulations, the proposed structure consistently outperforms the CPA due to a much larger aperture and number of DOFs.

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