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Cyclic multicategories, multivariable adjunctions and mates

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Abstract

A multivariable adjunction is the generalisation of the notion of a 2-variable adjunction, the classical example being the hom/tensor/cotensor trio of functors, to $n+1$ functors of n variables. In the presence of multivariable adjunctions, natural transformations between certain composites built from multivariable functors have "dual" forms. We refer to corresponding natural transformations as multivariable or parametrised mates, generalising the mates correspondence for ordinary adjunctions, which enables one to pass between natural transformations involving left adjoints to those involving right adjoints. A central problem is how to express the naturality (or functoriality) of the parametrised mates, giving a precise characterization of the dualities so-encoded.

We present the notion of "cyclic double multicategory" as a structure in which to organise multivariable adjunctions and mates. While the standard mates correspondence is described using an isomorphism of double categories, the multivariable version requires the framework of "double multicategories". Moreover, we show that the analogous isomorphisms of double multicategories give a cyclic action on the multimaps, yielding the notion of "cyclic double multicategory". The work is motivated by and applied to Riehl's approach to algebraic monoidal model categories.

Contents

Introduction

Frequently in homotopical algebra and algebraic K-theory, one is dealing with model categories with extra structure. In particular, the model structure is often required to be compatible with a closed monoidal structure on the underlying category or an enrichment over another model category. For instance, enriched model categories play an essential role in equivariant homotopy theory [8, 9, 10]. The formal definitions, introduced in [12], generalize Quillen's notion of a simplicial model category and can be expressed in three equivalent ("dual") forms. Although these formulations are well-known, the precise nature of these dualities is not obvious because they involve a two-variable adjunction defined on arrow categories constructed from a two-variable adjunction on the underlying categories and an associated bijective correspondence between certain natural transformations that has never been precisely described.

Indeed, a fully satisfactory account of the dualities for natural transformations involving a "tensor/cotensor/hom" trio of functors demands a generalization from two-variable adjunctions to multivariable adjunctions: the tensor/cotensor/hom combine with the closed monoidal structure on the enriching category to define functors of n-variables with compatibly defined adjoints. Again, in the presence of multivariable adjunctions, natural transformations between certain composites built from multivariable functors (e.g. encoding coherence conditions) have "dual" forms. This sort of structure occurs in work of the third author in the context of monoidal and enriched *algebraic* model category structures [23, 24]. In an algebraic model category structure, the weak factorisation systems involved have certain extra linked algebraic/coalgebraic structure; all cofibrantly generated model categories can be made algebraic in this sense. That research was one direct motivation for the work and results in this paper, which are essential to define the central notion studied in [24].

Our main theorem gives a complete characterisation of the natural transformations involving multivariable adjoints that admit dual forms. It also solves the associated "coherence problem": Certain (but not all) commuting diagrams involving these sorts of natural transformations will have "dual" forms. Our result makes this precise. Before giving a more detailed description of the problem and an outline of our solution, let us introduce a key idea via an analogy.

The pervasive success of homology theories stems from an abstract framework that simultaneously enables computation and generalisation. For example, homology originated from the study of invariants of topological spaces and was extended to associative algebras, Lie algebras, and the extraordinary homology theories appearing in stable homotopy theory, such as K-theory and cobordism.

In these settings, we often start with some basic objects, and then consider additional algebraic structure. Operads are a powerful tool for encoding such structure. This is witnessed by the great progress made in the theory of iterated loop spaces [22] and topological field theories [5], for example.

While operads can be used to generalise notions of algebraic structure, there is still a further useful generalisation: operads themselves come in different flavours, allowing us to embrace yet further notions of algebra. One major example of this is Getzler and Kapranov's notion of cyclic operad. This was introduced in order to generalise cyclic homology to these further types of algebras.

The "cycles" at play here are cycles of inputs and outputs. That is, where operads encode algebraic operations, cycles enable us to exchange inputs and outputs of these operations. One natural way in which such structure arises is in the presence of duality. For example, for finite-dimensional vector spaces, a linear map

$$F\colon V \longrightarrow W$$

corresponds precisely to a map between the duals in the opposite direction, that is

$$F^*\colon W^* \longrightarrow V^*.$$

Of course, vector spaces form not merely a category, but a monoidal category, and the tensor product interacts with the duality as follows. A linear map

$$V_1 \otimes V_2 \otimes \cdots \otimes V_n \longrightarrow V_0$$

corresponds precisely to one as shown below

$$V_2 \otimes \cdots \otimes V_n \otimes V_0^* \longrightarrow V_1^*$$

where the "output" vector space has been exchanged with one of the "inputs", and those two spaces are dualised. We can repeat this process and "cycle" the inputs and outputs round as many times as we like. In this sense the basic version above is the 1-ary version of this cyclic process, which we can do for any $n \geq 1$.

A categorical version of duality is given by adjunctions. An adjunction

$$\begin{tikzcd} A \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & B \arrow[l, shift left=1.5, "G"] \end{tikzcd}$$

is given by the same data as an adjunction

$$\begin{tikzcd} B^{\mathrm{op}} \arrow[r, shift left, "G^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & A^{\mathrm{op}} \arrow[l, shift left, "F^{\mathrm{op}}"] \end{tikzcd}.$$

As for vector spaces, categories form a monoidal category via the cartesian product, and we seek the correct n-ary version of adjunctions.

A ubiquitous example of a 2-variable adjunction is the tensor/hom/cotensor trio of functors mentioned earlier. The tensor/hom adjunction

$$\_ \otimes b \dashv [b, \_]$$

is particularly familiar, and in many enriched cases there is further adjoint, the cotensor

$$a \otimes \_ \dashv a \pitchfork \_$$

These three functors $\otimes, [\_, \_], \pitchfork$ are related by adjunctions in a way that looks somewhat convoluted at first sight. It has the following features:

it involves 3 functors of 2 variables,

each pair of functors is related by a 1-variable adjunction if we fix a variable, and

some care is required over dualities of source and target categories.

In fact, when treated cyclically, the structure becomes transparent; we discuss it in detail in Section 2.2. While the functors in this example have only two variables, composing them results in new functors of higher arity. In fact, any tensored and cotensored category enriched in a closed symmetric monoidal category admits an n-variable adjunction for each natural number n, encoding the interaction between these structures.

Two-variable adjunctions appear in the statement of the pushout-product axiom, which is the crucial component of the definition of a simplicial, or more generally enriched, model category. It is well known that there are three equivalent formulations of this axiom that are somehow dual. The key to this duality is that in the presence of pushouts and pullbacks, the arrow categories admit adjunctions similar to the tensor/hom/cotensor trio. The three cyclic adjoints yield the three forms of the pushout-product axiom. Multivariable adjunctions of this sort are also used in higher category theory, for instance, to define the lifting properties characterising an n-fold quasi-category, a presheaf model for an (∞, n)-category [11].

The correct framework for handling multivariable functors is multicategories. These are just like categories except that morphisms can have many inputs (or none); they still have a single output. Note that non-symmetric operads are multicategories with only one object, and so multicategories are often referred to as "coloured operads".

In fact we desire a richer structure than just multicategories because the "duality" involved in adjunctions extends to 2-cells as well—natural transformations involving left adjoints become natural transformations involving right adjoints via the "mates correspondence". This 2-dimensional duality is at the heart of the three equivalent formulations of an algebraic formulation of the simplicial model category axioms mentioned above.

The mates correspondence is elegantly described using the framework of double categories. Recall that a double category is a form of 2-dimensional category with two types of morphism—horizontal and vertical—and 2-cells that fit inside squares. The double category used to describe the mates correspondence is given as follows:

> 0-cells are categories,
>
> horizontal 1-cells are functors,
>
> vertical 1-cells are adjunctions (pointing in a fixed chosen direction e.g. the direction of the left adjoint), and
>
> 2-cells are certain natural transformations.

Even after we have fixed the direction of the 1-cells, there is a choice for the 2-cells—we could still take the natural transformations to live in the squares involving either the left or right adjoints. This produces *a priori* two different double categories for each choice of 1-cell direction, but the mates correspondence says precisely that there is an isomorphism of double categories between them.

For multivariable adjunctions we thus need to combine the notions of multicategory, double category, and cyclic action. Our vertical 1-cells will now be n-variable adjunctions, so they are the maps of a multicategory with a cyclic action. For example a 2-variable adjunction involves functors

$$\begin{aligned} A \times B &\xrightarrow{F} C^{\mathrm{op}} \\ B \times C &\xrightarrow{G} A^{\mathrm{op}} \\ C \times A &\xrightarrow{H} B^{\mathrm{op}}. \end{aligned}$$

Note the duality that arises as a category "cycles" between the source and target. The essential fact is that each time a category moves between the source and target, it is dualised; this is exactly what happens for vector spaces, and in the tensor/hom/cotensor situation. This is the notion of a "cyclic multicategory"—a multicategory equipped with additional structure in the form of

> an involution (such as $(\ )^{\mathrm{op}}$), and
>
> a cyclic action on homsets, invoking the involution appropriately.

This formulation allows for cyclic structures that do not arise from duals in the sense of dual vector spaces, such n-variable adjunction. (Note that opposite categories are not "duals" in the sense that dual vector spaces are duals.)

We must also implement the cyclic structure on 2-cells, that is, the n-variable version of the mates correspondence. We are interested in a correspondence of natural transformations such as below (for the 2-variable example):

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d, "F"'] \arrow[dr, phantom, "\Nearrow"] & A' \times B' \arrow[d, "F'"] \\ C^{\mathrm{op}} \arrow[r] & C'^{\mathrm{op}} \end{tikzcd} \quad \begin{tikzcd} B \times C \arrow[r] \arrow[d, "G"'] \arrow[dr, phantom, "\Nearrow"] & B' \times C' \arrow[d, "G'"] \\ A^{\mathrm{op}} \arrow[r] & A'^{\mathrm{op}} \end{tikzcd} \quad \begin{tikzcd} C \times A \arrow[r] \arrow[d, "H"'] \arrow[dr, phantom, "\Nearrow"] & C' \times A' \arrow[d, "H'"] \\ B^{\mathrm{op}} \arrow[r] & B'^{\mathrm{op}} \end{tikzcd}$$

and this indicates the required form of 2-cells and their cyclic structure, in our "cyclic double multicategory". Recall that a double category can be defined succinctly as a category object in **Cat**; similarly a cyclic double multicategory is a category object in the category of cyclic multicategories.

The motivation for this work is the third author's work on algebraic monoidal model categories. In the theory of algebraic model categories [23] the double category framework for 1-variable adjunctions and mates plays a crucial role. For the monoidal version [24], multivariable adjunctions and mates are needed, not simply to describe the equivalent forms the definition of a monoidal algebraic model category might take but to state the correct definition at all. Examples that could now be made algebraic using the results of the present paper include the model structures arising from 2-category theory [18], in particular the monoidal model structure on 2-categories with the Gray tensor product [16, 17]. Similar ideas applied in the context of n-fold quasi-categories would give an "algebraic" model for (∞, n)-categories.

This paper is organised as follows. In Section 1 we recall the standard theory of mates. In Section 2 we define multivariable adjunctions and the multivariable mates correspondence. In Section 3 we give the definition of cyclic double multicategory, building up gradually through multicategories, cyclic multicategories and double multicategories. We show that multivariable adjunctions form a cyclic double multicategory. In Section 4 we describe the application to algebraic monoidal model categories.

Our notion of cyclic multicategory is non-symmetric and thus generalises the notion of (non-symmetric) cyclic operad given in [1]; symmetric cyclic operads are defined in [4] and a multicategory version is mentioned in [13]. Our definition could also be given in a symmetric form but we felt that the new ideas introduced here were highlighted most clearly when the obvious symmetries of the cartesian product on **CAT** were ignored. Cyclic operads support a wide variety of applications, as described in the papers [1] and [4], and so we expect the categorical formalism encoded by our "coloured" version presented here will also be useful in other contexts.

Our notion of double multicategory is not the same as the notion of fc-multicategory (introduced by Leinster in [19] and renamed "virtual double category" by Cruttwell and Shulman in [2]); fc-multicategories do not involve vertical 1-cells of higher arities.

Notation

Throughout this paper we will write $A^\bullet$ for A^{op}. Also, for n-variable adjunctions and cyclic multicategories, we will need to use subscripts cyclically. Thus we will index objects by $0, \ldots, n$ with lists taken cyclically, mod $n+1$. For example we will frequently use the string $a_{i+1}, \ldots, a_{i-1}$ which means

$$a_{i+1}, a_{i+2}, \ldots, a_n, a_0, a_1, \ldots, a_{i-1}.$$

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1 Mates

In this section we describe the situation we will be generalising. Suppose we have the following categories, functors and adjunctions

$$\begin{tikzcd} A \arrow[d, shift right, "F"'] \arrow[d, phantom, "\dashv"] \\ B \arrow[u, shift right, "G"'] \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[d, shift right, "F'"'] \arrow[d, phantom, "\dashv"] \\ B' \arrow[u, shift right, "G'"'] \end{tikzcd}$$

with unit and counit (η, ε) and (η', ε') respectively. Then given functors S and T and a natural transformation α as shown

$$\begin{tikzcd} A \arrow[r, "S"] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha\swarrow"] & A' \arrow[d, "F'"] \\ B \arrow[r, "T"'] & B' \end{tikzcd}$$

its **mate** $\bar{\alpha}$ is the natural transformation

$$\begin{tikzcd} A \arrow[r, "S"] \arrow[dr, phantom, "\bar{\alpha}\searrow"] & A' \\ B \arrow[u, "G"] \arrow[r, "T"'] & B' \arrow[u, "G'"'] \end{tikzcd}$$

obtained as the following composite

$$
\begin{tikzcd}
B \arrow[r, "G"] \arrow[dr, "1"'] & A \arrow[r, "S"] \arrow[d, "F"] \arrow[l, phantom, shift right=3, "\Swarrow\varepsilon"] \arrow[dr, phantom, "\Swarrow\alpha"] & A' \arrow[d, "F'"] \arrow[dr, "1"] & \\
& B \arrow[r, "T"'] & B' \arrow[r, "G'"'] \arrow[ur, phantom, shift left=1, "\Swarrow\eta'"] & A'.
\end{tikzcd}
$$

Conversely we can start with

$$
\begin{tikzcd}
A \arrow[r, "S"] & A' \\
B \arrow[u, "G"] \arrow[r, "T"'] \arrow[ur, phantom, "\beta\Searrow"] & B' \arrow[u, "G'"']
\end{tikzcd}
$$

and obtain the mate

$$
\begin{tikzcd}
A \arrow[r, "S"] \arrow[d, "F"'] \arrow[dr, phantom, "\Swarrow\bar{\beta}"] & A' \arrow[d, "F'"] \\
B \arrow[r, "T"'] & B'
\end{tikzcd}
$$

as the composite

$$
\begin{tikzcd}
& A \arrow[r, "S"] & A' \arrow[r, "F'"] & B' \\
A \arrow[ur, "1"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=3, "\varepsilon\Searrow"] & B \arrow[u, "G"] \arrow[r, "T"'] \arrow[ur, phantom, "\beta\Searrow"] & B'. \arrow[u, "G'"'] \arrow[ur, "1"'] \arrow[ur, phantom, shift left=3, "\Searrow\eta'"] &
\end{tikzcd}
$$

By triangle identities these processes of "conjugation" are inverse to one another. Furthermore, the correspondence respects both horizontal and vertical composition in the following sense. Given adjunctions

$$
\begin{tikzcd}
A_1 \arrow[d, shift right=2, "F_1"'] \arrow[d, phantom, "\dashv"] \\
B_1 \arrow[u, shift right=2, "G_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[d, shift right=2, "F_2"'] \arrow[d, phantom, "\dashv"] \\
B_2 \arrow[u, shift right=2, "G_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A_3 \arrow[d, shift right=2, "F_3"'] \arrow[d, phantom, "\dashv"] \\
B_3 \arrow[u, shift right=2, "G_3"']
\end{tikzcd}
$$

and natural transformations

$$
\begin{tikzcd}
A_1 \arrow[r, "S_1"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\Swarrow\alpha_1"] & A_2 \arrow[r, "S_2"] \arrow[d, "F_2"] \arrow[dr, phantom, "\Swarrow\alpha_2"] & A_3 \arrow[d, "F_3"] \\
B_1 \arrow[r, "T_1"'] & B_2 \arrow[r, "T_2"'] & B_3
\end{tikzcd}
$$

we have

$$\overline{\alpha_2 * \alpha_1} = \overline{\alpha_2} * \overline{\alpha_1}$$

where $*$ is to be interpreted with the appropriate whiskering, so in fact the honest equality is

$$\overline{T_2\alpha_1 \circ \alpha_2 S_1} = \overline{\alpha_2}T_1 \circ S_2\overline{\alpha_1}.$$

For "vertical" composition, given adjunctions

$$\begin{tikzcd} A_1 \arrow[d, shift right, "F_1"'] \arrow[d, phantom, "\dashv"] & & A_2 \arrow[d, shift right, "F_2"'] \arrow[d, phantom, "\dashv"] \\ B_1 \arrow[u, shift right, "G_1"'] \arrow[d, shift right, "H_1"'] \arrow[d, phantom, "\dashv"] & & B_2 \arrow[u, shift right, "G_2"'] \arrow[d, shift right, "H_2"'] \arrow[d, phantom, "\dashv"] \\ C_1 \arrow[u, shift right, "K_1"'] & & C_2 \arrow[u, shift right, "K_2"'] \end{tikzcd}$$

and natural transformations

$$\begin{tikzcd} A_1 \arrow[r, "S"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\swArrow\alpha_1"] & A_2 \arrow[d, "F_2"] \\ B_1 \arrow[r, "T"'] \arrow[d, "H_1"'] \arrow[dr, phantom, "\swArrow\alpha_2"] & B_2 \arrow[d, "H_2"] \\ C_1 \arrow[r, "U"'] & C_2 \end{tikzcd}$$

we have

$$\overline{\alpha_2 \circ \alpha_1} = \overline{\alpha_2} \circ \overline{\alpha_1}$$

which actually means

$$\overline{\alpha_2 F_1 \circ H_2\alpha_1} = G_2\overline{\alpha_2} \circ \overline{\alpha_1}K_1.$$

Both of these facts are easily checked using 2-pasting diagrams and triangle identities.

This situation is conveniently formalised using double categories. In the following definition we have chosen the direction of the vertical 1-cells to correspond to the direction of the left adjoints.

Definition 1.1. We define two double categories $\mathbb{L}\mathbf{Adj}$ and $\mathbb{R}\mathbf{Adj}$ with the same 0- and 1-cells, but different 2-cells. In both cases the 0-cells are categories, the horizontal 1-cells are functors, and a vertical 1-cell $A \longrightarrow B$ is an adjunction

$$\begin{tikzcd} A \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & B \arrow[l, shift left, "G"] \end{tikzcd}.$$

A 2-cell
$$\begin{tikzcd}
A_1 \arrow[rr, "S"] \arrow[d, shift right=2, "F_1"'] \arrow[d, phantom, "\dashv"] & & A_2 \arrow[d, shift right=2, "F_2"'] \arrow[d, phantom, "\dashv"] \\
B_1 \arrow[rr, "T"'] \arrow[u, shift right=2, "G_1"'] \arrow[urr, phantom, "\Downarrow"] & & B_2 \arrow[u, shift right=2, "G_2"']
\end{tikzcd}$$
is given in each case as follows.

In $\mathbb{L}\mathbf{Adj}$ such a 2-cell is a natural transformation
$$\begin{tikzcd}
A_1 \arrow[r, "S"] \arrow[d, "F_1"'] & A_2 \arrow[d, "F_2"] \arrow[dl, phantom, "\alpha \swarrow"] \\
B_1 \arrow[r, "T"'] & B_2.
\end{tikzcd}$$
In $\mathbb{R}\mathbf{Adj}$ such a 2-cell is a natural transformation
$$\begin{tikzcd}
A_1 \arrow[r, "S"] \arrow[dr, phantom, "\alpha \searrow"] & A_2 \\
B_1 \arrow[r, "T"'] \arrow[u, "G_1"] & B_2. \arrow[u, "G_2"']
\end{tikzcd}$$

Theorem 1.2. [15, Proposition 2.2]
There is an isomorphism of double categories
$$\mathbb{L}\mathbf{Adj} \cong \mathbb{R}\mathbf{Adj}$$
which is the identity on 0- and 1-cells (horizontal and vertical); on 2-cells it is given by taking mates.

We now look at this from a slightly different point of view that seems a little contrived here, but leads to a natural framework for the n-variable generalisation. The idea is to notice that an adjunction
$$\begin{tikzcd}
A \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & B \arrow[l, shift left=1.5, "G"]
\end{tikzcd}$$
is equivalently an adjunction
$$\begin{tikzcd}
B^\bullet \arrow[r, shift left=1.5, "G^\bullet"] \arrow[r, phantom, "\perp"] & A^\bullet. \arrow[l, shift left=1.5, "F^\bullet"]
\end{tikzcd}$$
Now, we could deal with this by introducing yet another pair of double categories $\mathbb{L}\mathbf{Adj}_R$ and $\mathbb{R}\mathbf{Adj}_R$ as above but whose vertical 1-cells point in the direction of the *right* adjoints; the 2-cell directions must also be changed accordingly. We would then get isomorphisms of double categories
$$(\ )^\bullet \colon \mathbb{L}\mathbf{Adj} \longrightarrow \mathbb{R}\mathbf{Adj}_R$$
$$(\ )^\bullet \colon \mathbb{R}\mathbf{Adj} \longrightarrow \mathbb{L}\mathbf{Adj}_R.$$

However, we can actually express all this structure using one single version of the above four isomorphic double categories, as follows.

Given a 2-cell in $\mathbb{L}\mathbf{Adj}$, that is, a natural transformation

$$\begin{tikzcd} A_1 \arrow[r, "S"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\overset{\alpha}{\swArrow}"] & A_2 \arrow[d, "F_2"] \\ B_1 \arrow[r, "T"'] & B_2 \end{tikzcd}$$

its mate

$$\begin{tikzcd} A_1 \arrow[r, "S"] \arrow[dr, phantom, "\underset{\bar{\alpha}}{\seArrow}"] & A_2 \\ B_1 \arrow[r, "T"'] \arrow[u, "G_1"] & B_2 \arrow[u, "G_2"'] \end{tikzcd}$$

is not *a priori* a 2-cell of $\mathbb{L}\mathbf{Adj}$ as its source and target involve *right* adjoints G_1 and G_2. However, it can be dualised to give

$$\begin{tikzcd} A_1^\bullet \arrow[r, "S^\bullet"] \arrow[dr, phantom, "\nwArrow \bar{\alpha}^\bullet"] & A_2^\bullet \\ B_1^\bullet \arrow[r, "T^\bullet"'] \arrow[u, "G_1^\bullet"] & B_2^\bullet \arrow[u, "G_2^\bullet"'] \end{tikzcd}$$

where we must reverse the 2-cell direction as the target category has been dualised. Thus, turning the diagram round so that the left adjoints point downwards, we have

$$\begin{tikzcd} B_1^\bullet \arrow[r, "T^\bullet"] \arrow[d, "G_1^\bullet"'] \arrow[dr, phantom, "\bar{\alpha}^\bullet \swArrow"] & B_2^\bullet \arrow[d, "G_2^\bullet"] \\ A_1^\bullet \arrow[r, "S^\bullet"'] & A_2^\bullet \end{tikzcd}$$

which *is* a 2-cell of $\mathbb{L}\mathbf{Adj}$ as $G_1^\bullet$ and $G_2^\bullet$ are *left* adjoints. Thus the mates correspondence actually gives us some extra structure on $\mathbb{L}\mathbf{Adj}$ in the form of isomorphisms:

$\mathbb{L}\mathbf{Adj}_v(A, B) \cong \mathbb{L}\mathbf{Adj}_v(B^\bullet, A^\bullet)$, and

$\mathbb{L}\mathbf{Adj}_2(S, T) \cong \mathbb{L}\mathbf{Adj}_2(T^\bullet, S^\bullet)$.

Here the subscript v indicates the hom-set of vertical 1-cells, and the subscript 2 indicates the hom-set of 2-cells with respect to their horizontal 1-cell boundaries.

We will see that these isomorphisms are the beginning of a cyclic structure: the 1-ary part. The situation has a slightly different flavour, without technically being different, if we put it in the language of "mutual left adjoints".

Definition 1.3. Consider functors

$$\begin{array}{lcl} A \xrightarrow{F} B^{\bullet} & \text{so} & A^{\bullet} \xrightarrow{F^{\bullet}} B \\ B \xrightarrow{G} A^{\bullet} & & B^{\bullet} \xrightarrow{G^{\bullet}} A. \end{array}$$

A **mutual left adjunction** of F and G is an adjunction

$$F^{\bullet} \dashv G$$

or equivalently

$$G^{\bullet} \dashv F.$$

Note that this is given by isomorphisms

$$B(Fa, b) \cong A(Gb, a)$$

natural in a and b. If we started with

$$\begin{array}{ll} A^{\bullet} \xrightarrow{F} B & \text{and} \\ B^{\bullet} \xrightarrow{G} A & \end{array}$$

then the adjunctions

$$F^{\bullet} \dashv G \quad \text{or} \quad G^{\bullet} \dashv F$$

as above would be given by isomorphisms

$$B(b, Fa) \cong A(a, Gb),$$

which is called a **mutual right adjoint**.

Note that the unit and counit for a mutual left adjoint as above have components

$$\begin{array}{l} \eta_a \colon GFa \longrightarrow a \in A \quad \text{and} \\ \varepsilon_b \colon FGb \longrightarrow b \in B, \end{array}$$

whereas for a mutual right adjoint the components are

$$\eta_a \colon a \longrightarrow GFa \in A \quad \text{and}$$

$$\varepsilon_b \colon b \longrightarrow FGb \in B.$$

Remark 1.4. The unit and counit given above are for an adjunction

$$F^{\bullet} \dashv G$$

whereas for the (equivalent) adjunction

$$G^{\bullet} \dashv F$$

the unit and counit are the other way round, that is,

$$\varepsilon_a \colon GFa \longrightarrow a \in A \quad \text{and}$$

$$\eta_b \colon FGb \longrightarrow b \in B.$$

In the spirit of symmetry, we will refer to all natural transformations involved in a mutual adjunction as ε; it will be clear from the source and target for which adjunction this is actually a counit.

We can now express the mates correspondence for mutual left adjunctions. Given a mutual left adjunction between

$$A \xrightarrow{F} B^\bullet \quad \text{and}$$
$$B \xrightarrow{G} A^\bullet$$

the mates correspondence together with duality as above gives us a correspondence between natural transformations

$$\begin{tikzcd} A \arrow[r, "S"] \arrow[d, "F"'] \arrow[dr, phantom, "\nearrow"] & A' \arrow[d, "F'"] \\ B^\bullet \arrow[r, "T^\bullet"'] & B^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r, "T"] \arrow[d, "G"'] \arrow[dr, phantom, "\nearrow"] & B'^\bullet \arrow[d, "G'"] \\ A^\bullet \arrow[r, "S^\bullet"'] & A'^\bullet. \end{tikzcd}$$

This is obtained from the ordinary mates correspondence by taking some appropriate duals. This is the $n = 1$ part of the n-variable case, in which we look at natural transformations

$$\begin{tikzcd} A_1 \times \cdots \times A_n \arrow[r, "S_1 \times \cdots \times S_n"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & A'_1 \times \cdots \times A'_n \arrow[d] \\ A_0^\bullet \arrow[r, "S_0^\bullet"'] & A_0'^\bullet \end{tikzcd}$$

and

$$\begin{tikzcd} A_2 \times \cdots \times A_0 \arrow[r, "S_2 \times \cdots \times S_0"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & A'_2 \times \cdots \times A'_0 \arrow[d] \\ A_1^\bullet \arrow[r, "S_1^\bullet"'] & A_1'^\bullet \end{tikzcd}$$

and every cyclic variant.

2 Multivariable adjunctions

In this section we define multivariable adjunctions. The basic idea is that for an "n-variable adjunction" we have $n+1$ categories $A_0, \cdots, A_n$ and $n+1$ multifunctors, each of which has one of the $A_i^\bullet$ as its target, and the product of the other n categories as its source. These multifunctors can all be restricted to functors with a single category as their source, by fixing an object in each of the other categories. For every pair $i \neq j$ there is a pair of contravariant functors obtained in this way involving A_i and A_j. These should be in a specified adjunction; moreover, of course, all these adjunctions should be coherent in an appropriate way.

We first give the definition of this structure, and then immediately prove Theorem 2.2 giving a more "economical" characterisation, in which *a priori* we specify only one multifunctor, and a family of 1-variable adjoints for it. Using standard results about parametrised representability, these 1-variable adjoints then extend uniquely to n-variable multifunctors with the required structure. It is the characterisation in Theorem 2.2 that we will use in the rest of the work.

2.1 Definition of multivariable adjunctions

Definition 2.1. Let $n \in \mathbb{N}$. An n**-variable (mutual) left adjunction** is given by the following data and axioms.

Categories $A_0, \ldots, A_n$.

Functors

$$\begin{array}{lcl} A_1 \times A_2 \times \cdots \times A_{n-1} \times A_n & \xrightarrow{F_0} & A_0^\bullet \\ A_2 \times A_3 \times \cdots \times A_n \times A_0 & \xrightarrow{F_1} & A_1^\bullet \\ & \vdots & \\ A_{i+1} \times \cdots \times A_{i-1} & \xrightarrow{F_i} & A_i^\bullet \\ & \vdots & \\ A_0 \times \cdots \times A_{n-1} & \xrightarrow{F_n} & A_n^\bullet. \end{array}$$

Here the subscripts are all to be taken mod $n+1$. Where possible, we will adopt the convention that the subscript on a multifunctor matches the subscript of its target category.

For all $0 \leq i \leq n$, and for all $a_{i+1} \in A_{i+1}, \ldots, a_{i-2} \in A_{i-2}$ a mutual left adjunction between

$$A_i \xrightarrow{F_{i-1}(\_, a_{i+1}, a_{i+2}, \ldots, a_{i-2})} A_{i-1}^\bullet$$
$$A_{i-1} \xrightarrow{F_i(a_{i+1}, a_{i+2}, \ldots, a_{i-2}, \_)} A_i^\bullet$$

thus isomorphisms

$$A_{i-1}\big(F_{i-1}(a_i, \ldots, a_{i-2}), a_{i-1}\big) \cong A_i\big(F_i(a_{i+1}, \ldots, a_{i-1}), a_i\big)$$

natural in a_{i-1} and a_i. If we use the shorthand $\hat{a}_i$ for the sequence $a_{i+1}, \ldots, a_{i-1}$, this isomorphism takes the appealing form

$$A_{i-1}\big(F_{i-1}(\hat{a}_{i-1}), a_{i-1}\big) \cong A_i\big(F_i(\hat{a}_i), a_i\big).$$

The following axioms must be satisfied:

the above isomorphisms must additionally be natural in all variables, and the “cycle” of isomorphisms commutes:

$$
\begin{tikzcd}
 & A_0\big(F_0(\hat{a}_0), a_0\big) \arrow[r, "\sim"] & A_1\big(F_1(\hat{a}_1), a_1\big) \arrow[dr, "\sim"] & \\
A_n\big(F_n(\hat{a}_n), a_n\big) \arrow[ur, "\sim"] & & & A_2\big(F_2(\hat{a}_2), a_2\big) \arrow[dl, "\sim"] \\
 & \ddots \arrow[ul, "\sim"] & A_3\big(F_3(\hat{a}_3), a_3\big) & \\
 & & \iddots &
\end{tikzcd}
$$

We say that the functor F_0 is **equipped with n-variable left adjoints** $F_1, \cdots, F_n$. This terminology makes more sense in the light of the following theorem.

Theorem 2.2. *The following description precisely corresponds to an n-variable left adjunction.*

categories $A_0, \ldots, A_n$

a functor $A_1 \times \cdots \times A_n \xrightarrow{F_0} A_0^\bullet$

for all $0, j, k$ distinct, and for all $a_j \in A_j$, a mutual left adjoint for the functor

$$F_0(a_1, \ldots, a_{k-1}, \_, a_{k+1}, \ldots, a_n) \colon A_k \longrightarrow A_0^\bullet$$

Remark 2.3. Note we say that F is equipped with n-variable **left** adjoints if each of its 1-variable restrictions has a **left** adjoint. F is equipped with n-variable **right** adjoints if each of its 1-variable restrictions has a **right** adjoint.

To prove this we use the following result of Mac Lane [20, IV.7, Theorem 3].

Theorem 2.4. *Given categories A, B, C, a functor $F \colon A \times B \longrightarrow C^\bullet$, and for all $b \in B$ a mutual left adjoint $G(b, \_) : C \longrightarrow A^\bullet$ for the functor*

$$F(\_, b) \colon A \longrightarrow C^\bullet$$

i.e. isomorphisms

$$C(F(a, b), c) \cong A(G(b, c), a) \tag{1}$$

natural in a and c, there is a unique way to extend the functors

$$G(b, \_) \colon C \longrightarrow A^\bullet$$

to a single functor

$$G \colon B \times C \longrightarrow A^\bullet$$

such that the isomorphism (1) is also natural in b.

This is a standard result about parametrised representability; we give a 2-categorical expression of Mac Lane's proof, as this will be useful later.

Proof. We write $1 \xrightarrow{b} B$ for the functor picking out the object $b \in B$. The hypothesis of the theorem then says that for each such b we have a right adjoint for the composite

$$A^\bullet \xrightarrow{1 \times b^\bullet} A^\bullet \times B^\bullet \xrightarrow{F^\bullet} C$$

which we call

$$C \xrightarrow{\quad G(b,\_) \quad} A^\bullet$$

with unit and counit

$$\begin{tikzcd}
C \arrow[r, "G(b{,}_)"] \arrow[ddr, "1"'] & A^\bullet \arrow[d, "1 \times b^\bullet"] \arrow[ddr, "1"] \arrow[dl, phantom, "\varepsilon_b \Swarrow"] & \\
& A^\bullet \times B^\bullet \arrow[d, "F^\bullet"] \arrow[r, phantom, "\Swarrow \eta_b"] & {} \\
& C \arrow[r, "G(b{,}_)"'] & A^\bullet.
\end{tikzcd}$$

Now, extending the individual functors

$$G(b,\_)\colon C \longrightarrow A^\bullet$$

to a functor

$$G\colon B \times C \longrightarrow A^\bullet$$

consists of giving, for each morphism $b_1 \xrightarrow{f} b_2$ in B, a natural transformation

$$\begin{tikzcd}
C \arrow[r, bend left, "G(b_1{,}_)"] \arrow[r, bend right, "G(b_2{,}_)"'] \arrow[r, phantom, "\Downarrow"] & A^\bullet
\end{tikzcd}$$

and checking functoriality. The natural transformation is given as the mate of

$$\begin{tikzcd}
A^\bullet \arrow[r, bend left, "1\times b_2^\bullet"] \arrow[r, bend right, "1\times b_1^\bullet"'] \arrow[r, phantom, "\Downarrow 1\times f^\bullet"] & A^\bullet \times B^\bullet \arrow[r, "F"] & C
\end{tikzcd}$$

that is

$$\begin{tikzcd}
C \arrow[r, "G(b_1,_)"] \arrow[ddr, "1"'] \arrow[dr, phantom, "\varepsilon_{b_1} \Swarrow"] & A^\bullet \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\Leftarrow 1\times f^\bullet"] \arrow[ddr, "1"] & \\
& A^\bullet \times B^\bullet \arrow[d, "F^\bullet"] \arrow[dr, phantom, "\Swarrow \eta_{b_2}"] & \\
& C \arrow[r, "G(b_2,_)"'] & A^\bullet.
\end{tikzcd}$$

Functoriality then follows from the functoriality of the mates correspondence. Now we further need that the isomorphism

$$C(F(a,b),c) \xrightarrow{\sim} A(G(b,c),a)$$

is natural in b. By the Yoneda lemma this is equivalent to the following diagram commuting for all $f\colon b_1 \longrightarrow b_2$ in B:

$$\begin{tikzcd}
G(b_2, F(a,b_1)) \arrow[r, "G(f,1)"] \arrow[d, "G(1,F(1,f))"'] & G(b_1, F(a,b_1)) \arrow[d, "\eta_{b_1,a}"] \\
G(b_2, F(a,b_2)) \arrow[r, "\eta_{b_2,a}"'] & a
\end{tikzcd}$$

or dually an analogous diagram involving ε's:

$$\begin{tikzcd}
F(b_2, G(a,b_1)) \arrow[r, "F(f,1)"] \arrow[d, "F(1,G(1,f))"'] & F(b_1, G(a,b_1) \arrow[d, "\varepsilon_{b_1,a}"] \\
F(b_2, G(a,b_2)) \arrow[r, "\varepsilon_{b_2,a}"'] & a.
\end{tikzcd}$$

2-categorically this is

$$
\begin{tikzcd}
C \arrow[r, bend left, "b_1\times 1"] \arrow[r, bend right, "b_2\times 1"'] \arrow[r, phantom, "\Downarrow f\times 1"] \arrow[ddrr, bend right, "1"'] & B\times C \arrow[r, "G"] & A^\bullet \arrow[d, "1\times b_2^\bullet"] \\
& \arrow[r, phantom, "\swarrow \varepsilon_{b_2}"] & A^\bullet\times B^\bullet \arrow[d, "F^\bullet"] \\
& & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[r, "b_1\times 1"] \arrow[ddrr, bend right, "1"'] & B\times C \arrow[r, "G"] & A^\bullet \arrow[d, bend right, "1\times b_1^\bullet"'] \arrow[d, bend left, "1\times b_2^\bullet"] \arrow[d, phantom, "\overset{\Leftarrow}{1\times f^\bullet}"] \\
& \arrow[r, phantom, "\swarrow \varepsilon_{b_1}"] & A^\bullet\times B^\bullet \arrow[d, "F^\bullet"] \\
& & C.
\end{tikzcd}
\tag{2}
$$

Now by our definition we have

$$
\begin{tikzcd}
C \arrow[r, "{G(b_1,\_)}"] \arrow[ddr, "1"'] \arrow[dr, phantom, "\varepsilon_{b_1}\swarrow"] & A^\bullet \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{\Leftarrow}{1\times f^\bullet}"] \arrow[ddr, "1"] & \\
& A^\bullet\times B^\bullet \arrow[d, "F^\bullet"] \arrow[dr, phantom, "\swarrow\eta_{b_2}"] & \\
& C \arrow[r, "{G(b_2,\_)}"'] & A^\bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[r, bend left, "b_1\times 1"] \arrow[r, bend right, "b_2\times 1"'] \arrow[r, phantom, "\Downarrow f\times 1"] & B\times C \arrow[r, "G"] & A^\bullet
\end{tikzcd}
\tag{3}
$$

since the right-hand side is the definition of G on morphisms of B. Then equation (2) follows from a triangle identity for η_{b_2} and ε_{b_2}; dually the equation for η holds by a triangle identity for η_{b_1} and ε_{b_1}.

For uniqueness we suppose we have a functor G satisfying the naturality condition as shown in diagram (2) above. Then as above, equation (3) must hold, showing that our construction of G is unique. $\square$

Proof of Theorem 2.2. First we show that the structure in the theorem gives rise to an n-variable left adjunction. First we need to define for all $i \neq 0$ a functor

$$F_i\colon A_{i+1}\times\cdots\times A_{i-1} \longrightarrow A_i^\bullet.$$

Now, we have for for all $a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n$ a left adjoint for the functor

$$F_0(a_1,\ldots,a_{i-1},\_,a_{i+1},\ldots,a_n)\colon A_i \longrightarrow A_0^\bullet,$$

equivalently a right adjoint for its opposite

$$F_0^\bullet(a_1,\ldots,a_{i-1},\_,a_{i+1},\ldots,a_n)\colon A_i^\bullet \longrightarrow A_0$$

called, say

$$F_i(a_{i+1},\ldots,a_n,\_,a_1,\ldots,a_{i-1})\colon A_0 \longrightarrow A_i^\bullet.$$

By Theorem 2.4 it extends uniquely to a functor

$$F_i\colon A_{i+1} \times \cdots \times A_{i-1} \longrightarrow A_i^\bullet$$

making the isomorphism

$$A_0\big(F_0(\hat{a}_0), a_0\big) \cong A_i\big(F_i(\hat{a}_i), a_i\big)$$

natural in every variable (where *a priori* it was only natural in a_0 and a_i). This is by putting

$$\begin{array}{rcl} A & = & A_i \\ B & = & A_{i+1} \times \cdots \times A_{i-1} \\ C & = & A_0 \end{array}$$

in the theorem. It remains to show that we have the correct adjunctions. Now by the above hom-set isomorphism we construct the composite isomorphism

$$A_{i-1}\big(F_{i-1}(\hat{a}_{i-1}), a_{i-1}\big) \xrightarrow{\sim} A_0\big(F_0(\hat{a}_0), a_0\big) \xrightarrow{\sim} A_i\big(F_i(\hat{a}_i), a_i\big)$$

which we already know to be natural in every variable, and by construction the cycle of isomorphisms commutes as required.

Conversely given an n-variable adjunction we use the cycle of isomorphisms to specify an isomorphism

$$A_0\big(F_0(\hat{a}_0), a_0\big) \xrightarrow{\sim} A_1\big(F_1(\hat{a}_1), a_1\big) \xrightarrow{\sim} \cdots \xrightarrow{\sim} A_i\big(F_i(\hat{a}_i), a_i\big).$$

Then, fixing all variables except a_i and a_0 we get the required adjunction. $\square$

It is instructive to work through this definition for some small values of n.

Example 2.5. $n = 1$

A 1-variable adjunction is just an ordinary adjunction, but in the notation of the definition it is given by

categories A_0, A_1,

functors $A_1 \xrightarrow{F_0} A_0^\bullet$, and
$A_0 \xrightarrow{F_1} A_1^\bullet$

an adjunction $F_0^\bullet \dashv F_1$.

Example 2.6. $n = 2$

A 2-variable adjunction is given by categories, functors and adjunctions as follows:

$$\begin{array}{lcl} A \times B \xrightarrow{F} C^\bullet & \qquad & F(\_, b)^\bullet \dashv G(b, \_) \\ B \times C \xrightarrow{G} A^\bullet & & G(\_, c)^\bullet \dashv H(c, \_) \\ C \times A \xrightarrow{H} B^\bullet & & H(\_, a)^\bullet \dashv F(a, \_) \end{array}$$

given by a "cycle of isomorphisms"

$$\begin{tikzcd}[column sep=small] C(F(a,b),c) \arrow[rr, phantom, "\cong"] \arrow[dr, phantom, "\cong" sloped] & & A(G(b,c),a) \arrow[dl, phantom, "\cong" sloped] \\ & B(H(c,a),b) & \end{tikzcd}$$

natural in a, b and c.

Theorem 2.2 says that to specify this it is equivalent to specify the functor F along with, for each $a \in A$ and $b \in B$ left adjoints for the functors $F(\_, b)$ and $F(a, \_)$, that is functors

$$G(b, \_) \colon C \longrightarrow A^{\bullet}$$
$$H(\_, a) \colon C \longrightarrow B^{\bullet}$$

and isomorphisms

$$\begin{array}{ll} A(G(b,c),a) \cong C(F(a,b),c) & \quad \text{natural in } a \text{ and } c \\ A(G(b,c),a) \cong B(H(c,a),b) & \quad \text{natural in } b \text{ and } c. \end{array}$$

Note that the original definition has $n+1$ adjunctions specified cyclically, each involving a pair of "numerically adjacent" categories and naturality in all $n+1$ variables; Theorem 2.2 specifies n adjunctions, each involving A_0 and one other category, and natural only in 2 variables.

Remark 2.7. For $n = 0$ it is useful to say that a "0-variable adjunction" is a functor $1 \longrightarrow A$ as these will be the 0-ary maps in our eventual multicategory structure. The fact that these compose is the following lemma.

Lemma 2.8. *Consider an n-variable adjunction as above. Fix $0 \le k \le n$ and $a_k \in A_k$. Then fixing a_k in each functor $F_i, i \ne k$ yields an $(n-1)$-variable adjunction in the evident way.*

Obviously we can repeat this process to fix any number of variables to restrict an multivariable adjunction to one in a smaller number of variables. Note that apart from being a crucial component of the eventual multicategory structure, this fact is also used in the proof of the multivariable mates correspondence (Theorem 2.16).

Proposition 2.9. *An n-variable left adjunction of functors $F_0, \ldots, F_n$ is equivalently an n-variable right adjunction of $F_0^{\bullet}, \ldots, F_n^{\bullet}$.*

2.2 A motivating example

We begin by presenting the standard example of a 2-variable adjunction that we have generalised, the "tensor/hom/cotensor" adjunction. The only slightly tricky thing is taking care of the dualities.

Let $\mathcal{V}$ be a monoidal category, so we have a functor

$$\_ \otimes \_ \colon \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}.$$

Then $\mathcal{V}$ is biclosed if

$\forall b \in \mathcal{V}$ the functor $\_ \otimes b$ has a right adjoint $[\, b \,, \_\,]$ ("hom"), and

$\forall a \in \mathcal{V}$ the functor $a \otimes \_$ has a right adjoint $a \pitchfork \_$ ("cotensor").

The first adjunction gives us isomorphisms

$$\mathcal{V}(a \otimes b, c) \cong \mathcal{V}(a, [b, c])$$

natural in a and c; by parametrised representability the functor

$$[b, \_] \colon \mathcal{V} \longrightarrow \mathcal{V}$$

extends to a functor

$$[\_, \_] \colon \mathcal{V}^{\bullet} \times \mathcal{V} \longrightarrow \mathcal{V}$$

uniquely making the isomorphisms natural in b as well.

Similarly for the second adjunction we get a functor

$$\_ \pitchfork \_ \colon \mathcal{V}^{\bullet} \times \mathcal{V} \longrightarrow \mathcal{V}$$

making the isomorphism

$$\mathcal{V}(a \otimes b, c) \cong \mathcal{V}(b, a \pitchfork c)$$

natural in all three variables.

Note that usually in the non-enriched setting "hom" is called "right hom" and "cotensor" is called "left hom".

More generally for categories A, B, C a tensor/hom/cotensor adjunction consists of functors and adjunctions

$$\begin{array}{lclclcl} A \times B & \xrightarrow{\_ \otimes \_} & C & \qquad \forall a \in A & a \otimes \_ & \dashv & a \pitchfork \_ \\ B^{\bullet} \times C & \xrightarrow{[\_, \_]} & A & \qquad \forall b \in B & \_ \otimes b & \dashv & [\, b \,, \_\,] \\ A^{\bullet} \times C & \xrightarrow{\_ \pitchfork \_} & B & \qquad \forall c \in C & [\,\_ \,, c\,]^{\bullet} & \dashv & \_ \pitchfork c \end{array}$$

and by parametrised representability it follows that the following isomorphisms are natural in all three variables:

$$A(a, [b, c]) \cong B(b, a \pitchfork c) \cong C(a \otimes b, c).$$

For our standard framework with functors

$$\begin{array}{l} A_1 \times A_2 \xrightarrow{F_0} A_0^{\bullet} \\ A_2 \times A_0 \xrightarrow{F_1} A_1^{\bullet} \\ A_0 \times A_1 \xrightarrow{F_2} A_2^{\bullet} \end{array}$$

we can put

$$\begin{aligned} A_1 &= A^\bullet & F_0 &= (\_ \otimes \_)^\bullet \\ A_2 &= B^\bullet & F_1 &= [\_,\_] \\ A_0 &= C & F_2 &= \_ \pitchfork \_ \end{aligned}$$

and F_0, F_1, F_2 then form a 2-variable left adjunction (although $\_ \pitchfork \_$ now has domain $C \times A^\bullet$ instead of $A^\bullet \times C$).

This 2-variable adjunction is is the starting point of the discussion in Section 4.

2.3 Composition

Just as ordinary adjunctions can be composed (with care over directions) so can n-variable adjunctions, with care over directions, dualities and arities. The only difficulty in the following theorem is the notation. The idea is to compose n-variable adjunctions in the manner of multimaps in a multicategory; indeed this is what they will be in Section 3. In this section all multivariable adjunctions are left adjunctions; of course the right adjunctions follow dually.

Theorem 2.10. *Suppose we have the following multivariable left adjunctions.*

$$\begin{array}{lll} A_{11} \times \cdots \times A_{1n_1} \xrightarrow{F_{10}} A_{10}^\bullet = B_1 & \textit{with } n_1\textit{-variable adjoints} & F_{11}, \ldots, F_{1n_1} \\ A_{21} \times \cdots \times A_{2n_2} \xrightarrow{F_{20}} A_{20}^\bullet = B_2 & \textit{with } n_2\textit{-variable adjoints} & F_{21}, \ldots, F_{2n_2} \\ \vdots & & \\ A_{k1} \times \cdots \times A_{kn_k} \xrightarrow{F_{k0}} A_{k0}^\bullet = B_k & \textit{with } n_k\textit{-variable adjoints} & F_{k1}, \ldots, F_{kn_k} \end{array}$$

and

$$\begin{array}{lll} B_1 \times \cdots \times B_k \xrightarrow{G_0} B_0^\bullet = A_{00}^\bullet & \textit{with } k\textit{-variable adjoints} & G_1, \ldots, G_k \end{array}$$

Then the composite functor

$$G_0(F_{10}, F_{20}, \ldots, F_{k0})\colon A_{11} \times \cdots \times A_{1n_1} \times \cdots \times A_{k1} \times \cdots \times A_{kn_k} \longrightarrow B_0^\bullet$$

is canonically equipped with $(n_1 + \cdots + n_k)$-variable adjoints. This composition makes categories and multivarible adjunctions into a multicategory.

Proof. We write $n_1 + \cdots + n_k = m$ and call the above composite H_{00}. We must construct m-variable adjoints for H_{00}, so first we need m functors which we call

$$\begin{array}{c} H_{11}, \ldots, H_{1n_1} \\ H_{21}, \ldots, H_{2n_2} \\ \vdots \\ H_{k1}, \ldots, H_{kn_k} \end{array}$$

where H_{ij} has target category $A_{ij}^{\bullet}$ and its source is then determined cyclically.

As the notation is rather complex we will give one example with all variables written down and then convert to a shorthand for convenience. We define H_{11} by

$$\begin{aligned} &H_{11}(a_{12}, \ldots, a_{1n_1}, \ldots, a_{kn_1}, b_0) = \\ &\qquad F_{11}\big(a_{12}, \ldots, a_{1n_1}, G_1(F_{20}(a_{21}, \ldots, a_{2n_2}), \ldots, F_{k0}(a_{k1}, \ldots, a_{kn_k}), b_0)\big) \end{aligned}$$

where each $a_{ij} \in A_{ij}$ and $b_0 \in B_0$. We think this is clearer if we do not write the variables explicitly, giving

$$H_{11} = F_{11}\big(\;\underline{\hspace{2cm}}\;, G_1(F_{20}, \ldots, F_{k0}, \bullet)\big).$$

Here the long line indicates a string of variables and a dot indicates a single variable. From the sources and targets of all the relevant functors it is unambiguous what the variables need to be, though somewhat tedious to write them out. The remaining functors H_{ij} can then be written like this:

$$\begin{array}{lcl} H_{12} & = & F_{12}\big(\;\underline{\hspace{2cm}}\;, G_1(F_{20}, \ldots, F_{k0}, \bullet), \bullet\big) \\ H_{13} & = & F_{13}\big(\;\underline{\hspace{1.7cm}}\;, G_1(F_{20}, \ldots, F_{k0}, \bullet), \bullet, \bullet\big) \\ H_{14} & = & F_{14}\big(\;\underline{\hspace{1.4cm}}\;, G_1(F_{20}, \ldots, F_{k0}, \bullet), \bullet, \bullet, \bullet\big) \\ \quad\vdots & & \\ H_{1n_1} & = & F_{1n_1}\big(G_1(F_{20}, \ldots, F_{k0}, \bullet), \;\underline{\hspace{2cm}}\;\big) \\ & & \\ H_{21} & = & F_{21}\big(\;\underline{\hspace{2cm}}\;, G_2(F_{30}, \ldots, F_{k0}, \bullet, F_{10})\big) \\ H_{22} & = & F_{22}\big(\;\underline{\hspace{1.7cm}}\;, G_2(F_{30}, \ldots, F_{k0}, \bullet, F_{10}), \bullet\big) \\ H_{23} & = & F_{23}\big(\;\underline{\hspace{1.4cm}}\;, G_2(F_{30}, \ldots, F_{k0}, \bullet, F_{10}), \bullet, \bullet\big) \\ \quad\vdots & & \\ H_{2n_2} & = & F_{2n_2}\big(G_2(F_{30}, \ldots, F_{k0}, \bullet, F_{10}), \;\underline{\hspace{2cm}}\;\big) \\ \quad\vdots & & \\ H_{k1} & = & F_{k1}\big(\;\underline{\hspace{2cm}}\;, G_k(\bullet, F_{10}, \ldots, F_{k-1,0})\big) \\ \quad\vdots & & \\ H_{kn_k} & = & F_{kn_k}\big(G_k(\bullet, F_{10}, \ldots, F_{k-1,0}), \;\underline{\hspace{2cm}}\;\big) \end{array}$$

It remains to exhibit the adjunctions required, which will take the form of the following isomorphisms.

$$\begin{array}{rcl} A_{00}\big(H_{00}(\hat{a}_{00}), a_{00}\big) & \cong & A_{11}\big(H_{11}(\hat{a}_{11}), a_{11}\big) \\ & \cong & A_{12}\big(H_{12}(\hat{a}_{12}), a_{12}\big) \\ & & \quad\vdots \\ & \cong & A_{kn_k}\big(H_{kn_k}(\hat{a}_{kn_k}), a_{kn_k}\big) \end{array}$$

The schematic diagram in Table 1 indicates which adjunctions of F_{ij}'s and G_l's are involved with each of the adjunctions for the H_{ij}'s. The vertical arrows indicate individual adjunctions.

Table 1: Individual adjunctions forming composite multivariable adjunctions

$$
\begin{array}{r|cccccc}
H_{00} & G_0 & F_{10} & F_{20} & F_{30} & \cdots & F_{k0} \\
 & \updownarrow & \updownarrow & & & & \\
H_{11} & G_1 & F_{11} & F_{20} & F_{30} & \cdots & F_{k0} \\
 & & \updownarrow & & & & \\
H_{13} & G_1 & F_{13} & F_{20} & F_{30} & \cdots & F_{k0} \\
 & & \updownarrow & & & & \\
H_{14} & G_1 & F_{14} & F_{20} & F_{30} & \cdots & F_{k0} \\
\vdots & & & & & & \\
 & & \updownarrow & & & & \\
H_{1n_1} & G_1 & F_{1n_1} & F_{20} & F_{30} & \cdots & F_{k0} \\
 & \updownarrow & \updownarrow & \updownarrow & & & \\
H_{21} & G_2 & F_{11} & F_{21} & F_{30} & \cdots & F_{k0} \\
 & & & \updownarrow & & & \\
H_{22} & G_2 & F_{11} & F_{22} & F_{30} & \cdots & F_{k0} \\
 & & & \updownarrow & & & \\
H_{23} & G_2 & F_{11} & F_{23} & F_{30} & \cdots & F_{k0} \\
\vdots & & & & & & \\
 & & & \updownarrow & & & \\
H_{2n_2} & G_2 & F_{11} & F_{2n_2} & F_{30} & \cdots & F_{k0} \\
 & \updownarrow & & \updownarrow & \updownarrow & & \\
H_{31} & G_3 & F_{11} & F_{21} & F_{31} & \cdots & F_{k0} \\
 & & & & \updownarrow & & \\
H_{32} & G_3 & F_{11} & F_{21} & F_{32} & \cdots & F_{k0} \\
\vdots & & & & & & \\
 & \updownarrow & & & \updownarrow & & \\
H_{k1} & G_k & F_{11} & F_{21} & F_{31} & \cdots & F_{k1} \\
 & & & & & & \updownarrow \\
H_{k2} & G_k & F_{11} & F_{21} & F_{31} & \cdots & F_{k2} \\
\vdots & & & & & & \\
 & & & & & & \updownarrow \\
H_{kn_k} & G_k & F_{11} & F_{21} & F_{31} & \cdots & F_{kn_k} \\
 & \updownarrow & & & & & \updownarrow \\
H_{00} & G_0 & F_{10} & F_{20} & F_{30} & \cdots & F_{k0}
\end{array}
$$

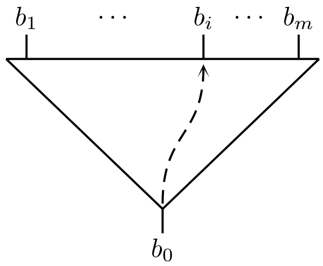
This is much easier to construct formally using Theorem 2.2: we just need to exhibit mutual left *1-variable* adjoints for each of the m functors obtained from H_{00} by fixing all but one of the variables. Now fixing every variable except a_{ij} in the functor

$$H_{11} = G_0(F_{10}, \ldots, F_{k0})$$

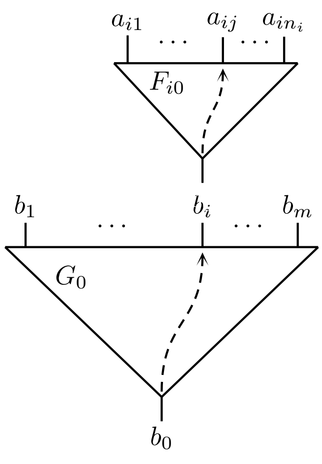
we construct a mutual left adjoint using

> the mutual left adjoint for F_{i0} with all but the jth variable fixed, and
>
> the mutual left adjoint for G_0 with all but the ith variable fixed.

These compose to give the adjoint required. We can depict this schematically as follows. We depict the latter as



so then composing this with the former looks like the diagram below, where F_{i0} and G_0 are the multifunctors pointing downwards, and the 1-variable left adjoint is indicated as the dotted arrow pointing upwards.



That is, starting from the functor

$$F_{i0} \colon A_{i1} \times \cdots \times A_{in_i} \longrightarrow A_{i0}^{\bullet}$$

we fix all but the jth variable and have a mutual left adjoint, that is right adjoint for

$$F^{\bullet}_{i0}(a_{i1}, \ldots, a_{i,j-1}, \_, a_{i,j+1}, \ldots, a_{in_i})\colon A^{\bullet}_{ij} \longrightarrow A_{i0}$$

which is

$$F_{ij}(a_{i,j+1}, \ldots, a_{in_i}, \_, a_{i1}, \ldots, a_{i,j-1})\colon A_{i0} \longrightarrow A^{\bullet}_{ij}.$$

Also consider

$$G^{\bullet}_{0}(b_1, \ldots, b_{i-1}, \_, b_{i+1}, \ldots, b_k)\colon B^{\bullet}_{i} \longrightarrow B_0$$

with its right adjoint

$$G_i(b_{i+1}, \ldots, b_k, \_, b_1, \ldots, b_{i-1})\colon B_0 \longrightarrow B^{\bullet}_{i} = A_{i0}.$$

Now we simply compose the functors

$$B_0 \longrightarrow A_{i0} \longrightarrow A^{\bullet}_{ij}$$

setting each

$$b_q = F_{q0}(a_{q1}, \ldots, a_{qn_q}).$$

Using the previous shorthand this is the composite

$$F_{ij}\big(\ \underline{\qquad\quad}\ , G_i(F_{i+1,0}, \ldots, F_{k0}, \bullet, F_{10}, \ldots, F_{i0}),\ \underline{\qquad\quad}\ \big).$$

This completes the construction of composition. Identities are given by identity adjunctions, which obviously satisfy unit conditions. Associativity follows from associativity of composition of n-variable functors (with one another) and of 1-variable adjunctions (with one another). $\Box$

Special cases

1. If any $n_i = 0$ this amounts to fixing the ith variable of G. If all but 1 of the n_i is 0 then we have fixed every variable except one, and if we do this for each n_i in turn we have effectively characterised the composite multi-variable adjunction by producing the necessary 1-variable adjunctions as in Theorem 2.2.

2. If we compose with the identity adjunction (as a 1-ary adjunction) for all but one of the i's, we have effectively composed in just one position.

3. If we take every $n_i = 1$ or $k = 1$ this says we can compose an n-variable adjunction with 1-variable adjunctions (pre- or post-) to get a new n-variable adjunction; this example is mentioned for composing 2-ary with 1-ary adjunctions in [24].

2.4 Multivariable mates

We now give a multivariable version of the calculus of mates. As for the adjunctions, we start with the 2-variable case and proceed inductively.

Proposition 2.11. *Suppose we have two 2-variable left adjunctions of functors*

$$\begin{array}{lll} F\colon A \times B \longrightarrow C^\bullet & \textit{and} & F'\colon A' \times B' \longrightarrow C'^\bullet \\ G\colon B \times C \longrightarrow A^\bullet & & G'\colon B' \times C' \longrightarrow A'^\bullet \\ H\colon C \times A \longrightarrow B^\bullet & & H'\colon C' \times A' \longrightarrow B'^\bullet, \end{array}$$

together with functors

$$\begin{array}{l} S\colon A \longrightarrow A' \\ T\colon B \longrightarrow B' \\ U\colon C \longrightarrow C' \end{array}$$

and a natural transformation

$$\begin{tikzcd} A \times B \arrow[r, "S \times T"] \arrow[d, "F"'] \arrow[dr, phantom, "\nearrow\alpha"] & A' \times B' \arrow[d, "F'"] \\ C^\bullet \arrow[r, "U^\bullet"'] & C'^\bullet \end{tikzcd}$$

with components

$$\alpha_{a,b}\colon F'(Sa, Tb) \longrightarrow UF(a,b).$$

Then for each $b \in B$ we have a natural transformation

$$\begin{tikzcd} A \arrow[r, "S"] \arrow[d, "F(\_,b)"'] \arrow[dr, phantom, "\nearrow\alpha_{\_,b}"] & A' \arrow[d, "F'(\_,Tb)"] \\ C^\bullet \arrow[r, "U^\bullet"'] & C'^\bullet \end{tikzcd}$$

with mate

$$\begin{tikzcd} C \arrow[r, "U"] \arrow[d, "G(b,\_)"'] \arrow[dr, phantom, "\nearrow\overline{\alpha_{\_,b}}"] & C' \arrow[d, "G'(Tb,\_)"] \\ A^\bullet \arrow[r, "S^\bullet"'] & A'^\bullet. \end{tikzcd}$$

Then in fact the components $(\overline{\alpha_{\_,b}})_c$ are the components of a natural transformation

$$\begin{tikzcd} B \times C \arrow[r, "T \times U"] \arrow[d, "G"'] \arrow[dr, phantom, "\nearrow\bar{\alpha}"] & B' \times C' \arrow[d, "G'"] \\ A^\bullet \arrow[r, "S^\bullet"'] & A'^\bullet. \end{tikzcd}$$

Dually if we start with 2-variable right *adjunctions then the result holds with all the natural transformations pointing in the opposite direction as below.*

$$\begin{tikzcd}
A \times B \arrow[r, "S \times T"] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha \Swarrow"] & A' \times B' \arrow[d, "F'"] \\
C^\bullet \arrow[r, "U^\bullet"'] & C'^\bullet
\end{tikzcd}$$

Proof. We just need to check that the components $\bar{\alpha}_{b,c} = (\overline{\alpha_{\_,b}})_c$ are natural in b; *a priori* they are natural in c. We use the fact that α is natural in b. As with Theorem 2.2 the proof is possible by a 1-dimensional diagram chase, but we provide a 2-categorical proof as it is quite aesthetically pleasing.

Now the natural transformation $\overline{\alpha_{\_,b}}$ is given by the following composite

$$\begin{tikzcd}
C \arrow[r, "b\times 1"] \arrow[ddrr, "1"] \arrow[dddr, bend right, "b\times 1"'] & B \times C \arrow[r, "G"] \arrow[d, phantom, "\Swarrow \varepsilon_b"] & A^\bullet \arrow[r, "S^\bullet"] \arrow[d, "1\times b^\bullet"] & A'^\bullet \arrow[d, "1\times Tb^\bullet"] \arrow[ddrr, "1"] & & \\
 & & A^\bullet \times B^\bullet \arrow[r, "S^\bullet\times T^\bullet"] \arrow[d, "F^\bullet"] \arrow[dr, phantom, "\Swarrow \alpha^\bullet"] & A'^\bullet \times B'^\bullet \arrow[d, "F'^\bullet"'] \arrow[dr, phantom, "\Swarrow \eta'_{Tb}"] & & \\
 & & C \arrow[r, "U"'] & C' \arrow[r, "Tb\times 1"'] & B' \times C' \arrow[r, "G'"'] & A'^\bullet \\
 & B \times C \arrow[urrr, bend right, "T \times U"'] & & & &
\end{tikzcd}$$

taking care over the direction as the target is in $A'^\bullet$. Again we use the fact that a morphism $b_1 \xrightarrow{F} b_2$ in B corresponds to a natural transformation

$$\begin{tikzcd}
1 \arrow[r, bend left, "b_1"] \arrow[r, bend right, "b_2"'] \arrow[r, phantom, "\Downarrow f"] & B
\end{tikzcd}$$

thus to check that the components

$$\begin{tikzcd}
 & B\times C \arrow[dr, "S^\bullet G"] \arrow[d, phantom, "\Downarrow \bar{\alpha}_{b,_}"] & \\
C \arrow[ur, "b\times 1"] \arrow[dr, "b\times 1"'] & {} & A'^\bullet \\
 & B\times C \arrow[ur, "G'(T\times U)"'] &
\end{tikzcd}$$

are natural in b we show that for all $b_1 \xrightarrow{f} b_2$

$$
\begin{tikzcd}
 & B\times C \arrow[dr, "S^\bullet G"] \arrow[d, phantom, "\Downarrow \bar{\alpha}_{b,\_}"] & \\
C \arrow[ur, bend left=50, "b_1\times 1"] \arrow[ur, bend right=30, "b_2\times 1"'] \arrow[ur, phantom, "\Downarrow f\times 1"] \arrow[dr, "b_2\times 1"'] & {} & A'^\bullet \\
 & B\times C \arrow[ur, "G'(T\times U)"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & B\times C \arrow[dr, "S^\bullet G"] \arrow[d, phantom, "\Downarrow \bar{\alpha}_{b,\_}"] & \\
C \arrow[ur, "b_1\times 1"] \arrow[dr, bend left=30, "b_1\times 1"] \arrow[dr, bend right=50, "b_2\times 1"'] \arrow[dr, phantom, "\Swarrow f\times 1"] & {} & A'^\bullet. \\
 & B\times C \arrow[ur, "G'(T\times U)"'] &
\end{tikzcd}
$$

We have

$$
\begin{tikzcd}
C \arrow[r, bend left, "b_1\times 1"] \arrow[r, bend right, "b_2\times 1"'] \arrow[r, phantom, "\Downarrow f\times 1"] \arrow[dddr, "b_2\times 1"'] \arrow[ddrr, "1"'] & B\times C \arrow[r, "G"] \arrow[dr, phantom, "\Swarrow \varepsilon_{b_2}"] & A^\bullet \arrow[r, "S^\bullet"] \arrow[d, "1\times b_2^\bullet"] & A'^\bullet \arrow[d, "1\times Tb_2^\bullet"'] \arrow[ddrr, "1"] & & \\
 & & A^\bullet\times B^\bullet \arrow[r, "S^\bullet\times T^\bullet"] \arrow[d, "F^\bullet"] \arrow[dr, phantom, "\Swarrow \alpha^\bullet"] & A'^\bullet\times B'^\bullet \arrow[d, "F'^\bullet"'] \arrow[r, phantom, "\Swarrow \eta'_{Tb_2}"] & {} & \\
 & & C \arrow[r, "U"'] & C' \arrow[r, "Tb_2\times 1"'] & B'\times C' \arrow[r, "G'"'] & A'^\bullet \\
 & B\times C \arrow[urrr, bend right, "T\times U"'] & & & &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
C \arrow[r, "b_1\times 1"] \arrow[dddr, "b_2\times 1"'] \arrow[ddrr, "1"'] & B\times C \arrow[r, "G"] \arrow[dr, phantom, "\Swarrow \varepsilon_{b_1}"] & A^\bullet \arrow[r, "S^\bullet"] \arrow[d, bend right, "1\times b_1^\bullet"'] \arrow[d, bend left, "1\times b_2^\bullet"] \arrow[d, phantom, "\overset{1\times f^\bullet}{\Leftarrow}"] & A'^\bullet \arrow[d, "1\times Tb_2^\bullet"'] \arrow[ddrr, "1"] & & \\
 & & A^\bullet\times B^\bullet \arrow[r, "S^\bullet\times T^\bullet"] \arrow[d, "F^\bullet"] \arrow[dr, phantom, "\Swarrow \alpha^\bullet"] & A'^\bullet\times B'^\bullet \arrow[d, "F'^\bullet"'] \arrow[r, phantom, "\Swarrow \eta'_{Tb_2}"] & {} & \\
 & & C \arrow[r, "U"'] & C' \arrow[r, "Tb_2\times 1"'] & B'\times C' \arrow[r, "G'"'] & A'^\bullet \\
 & B\times C \arrow[urrr, bend right, "T\times U"'] & & & &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
C \arrow[r, "b_1\times 1"] \arrow[dddr, "b_2\times 1"'] \arrow[ddrr, "1"'] & B\times C \arrow[r, "G"] \arrow[dr, phantom, "\Swarrow \varepsilon_{b_1}"] & A^\bullet \arrow[r, "S^\bullet"] \arrow[d, "1\times b_1^\bullet"] & A'^\bullet \arrow[d, bend right, "1\times Tb_1^\bullet"'] \arrow[d, bend left, "1\times Tb_2^\bullet"] \arrow[d, phantom, "\overset{1\times Tf^\bullet}{\Leftarrow}"] \arrow[ddrr, bend left, "1"] & & \\
 & & A^\bullet\times B^\bullet \arrow[r, "S^\bullet\times T^\bullet"] \arrow[d, "F^\bullet"] \arrow[dr, phantom, "\Swarrow \alpha^\bullet"] & A'^\bullet\times B'^\bullet \arrow[d, "F'^\bullet"'] \arrow[r, phantom, "\Swarrow \eta'_{Tb_2}"] & {} & \\
 & & C \arrow[r, "U"'] & C' \arrow[r, "Tb_2\times 1"'] & B'\times C' \arrow[r, "G'"'] & A'^\bullet \\
 & B\times C \arrow[urrr, bend right, "T\times U"'] & & & &
\end{tikzcd}
$$

$$
=\quad
\begin{tikzcd}
C \arrow[r, "b_1\times 1"] \arrow[dddr, bend right, "b_2\times 1"'] \arrow[ddrr, bend right, "1"] \arrow[ddrr, phantom, "\Swarrow\varepsilon_{b_1}"] & B\times C \arrow[r, "G"] & A^\bullet \arrow[r, "S^\bullet"] \arrow[d, "1\times b_1^\bullet"] & A'^\bullet \arrow[d, "1\times Tb_1^\bullet"'] \arrow[ddrr, bend left, "1"] \arrow[ddr, phantom, "\Swarrow\eta'_{Tb_1}"] & & \\
 & & A^\bullet\times B^\bullet \arrow[r, "S^\bullet\times T^\bullet"] \arrow[d, "F^\bullet"] \arrow[dr, phantom, "\Swarrow\alpha^\bullet"] & A'^\bullet\times B'^\bullet \arrow[d, "F'^\bullet"'] & & \\
 & & C \arrow[r, "U"'] & C' \arrow[r, bend left, "Tb_1\times 1"] \arrow[r, bend right, "Tb_2\times 1"'] \arrow[r, phantom, "\Downarrow Tf\times 1"] & B'\times C' \arrow[r, "G'"'] & A'^\bullet \\
 & B\times C \arrow[urrr, bend right, "T\times U"'] & & & &
\end{tikzcd}
$$

$$
=\quad
\begin{tikzcd}
C \arrow[r, "b_1\times 1"] \arrow[dddr, bend right, "b_2\times 1"'] \arrow[dddr, bend left, "b_1\times 1"] \arrow[dddr, phantom, "\Swarrow f\times 1"] \arrow[ddrr, "1"] \arrow[ddrr, phantom, "\Swarrow\varepsilon_{b_1}"] & B\times C \arrow[r, "G"] & A^\bullet \arrow[r, "S^\bullet"] \arrow[d, "1\times b_1^\bullet"] & A'^\bullet \arrow[d, "1\times Tb_1^\bullet"'] \arrow[ddrr, bend left, "1"] \arrow[ddr, phantom, "\Swarrow\eta'_{Tb_1}"] & & \\
 & & A^\bullet\times B^\bullet \arrow[r, "S^\bullet\times T^\bullet"] \arrow[d, "F^\bullet"] \arrow[dr, phantom, "\Swarrow\alpha^\bullet"] & A'^\bullet\times B'^\bullet \arrow[d, "F'^\bullet"'] & & \\
 & & C \arrow[r, "U"'] & C' \arrow[r, "Tb_1\times 1"'] & B'\times C' \arrow[r, "G'"'] & A'^\bullet. \\
 & B\times C \arrow[urrr, bend right, "T\times U"'] & & & &
\end{tikzcd}
$$

□

Remark 2.12. Note that in the definition of the mate of α we could start by fixing the first variable instead of the second variable and then follow the analogous process to produce a natural transformation $\hat{\alpha}$ as below:

$$
\begin{tikzcd}
C\times A \arrow[r, "U\times S"] \arrow[d, "H"'] \arrow[dr, phantom, "\Nearrow\hat{\alpha}"] & C'\times A' \arrow[d, "H'"] \\
B^\bullet \arrow[r, "T^\bullet"'] & B'^\bullet.
\end{tikzcd}
$$

Note that $\bar{\hat{\alpha}} = \alpha = \hat{\bar{\alpha}}$ by the usual mates correspondence; the following result deals with a less trivial combination of these processes. This can be thought of as the 2-variable version of the mates correspondence.

Proposition 2.13. *Given 2-variable adjunctions and a natural transformation α as above, $\bar{\bar{\bar{\alpha}}} = \hat{\alpha}$.*

The proof of this result is analogous to the 1-variable case, which follows from the triangle identities for the adjunctions in question. Therefore we start by making explicit the 2-variable version of the triangle identities, which must now involve three instances of units/counits.

Lemma 2.14 (Generalised triangle identity). *For a 2-variable adjunction, the following triangles commute, along with all cyclic variants.*

$$\begin{tikzcd} H(F(a,b),a) \arrow[rr, "\varepsilon"] \arrow[dr, "H(1,\varepsilon)"'] & & b \\ & H\big(F(a,b), G(b, F(a,b))\big) \arrow[ur, "\varepsilon"'] & \end{tikzcd} \qquad \begin{tikzcd} H(c, G(b,c)) \arrow[rr, "\varepsilon"] \arrow[dr, "H(\varepsilon,1)"'] & & b \\ & H\big(F(G(b,c),b), G(b,c)\big) \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

Proof. This follows from the "cycle of isomorphisms" as in Example 2.6. $\square$

In the following proof we adopt notational shorthand as below, for simplification, clarity and to save space.

1. All objects have been omitted. The source categories can always be determined from the functors shown, and whenever a variable in A is required, it is understood to be a; likewise for $c \in C$. For example:

   TH means $TH(c,a)$, and

   $G(H,1)$ means $G(H(c,a),c))$.

2. As in Remark 1.4 we write all units and counits for all adjunctions as ε; the source and target functors uniquely determine which adjunction is being used, and the object at which the component is being taken.

For example, the above two triangles become:

$$\begin{tikzcd} H(F,1) \arrow[rr, "\varepsilon"] \arrow[dr, "H(1,\varepsilon)"'] & & 1 \\ & H(F, G(1,F)) \arrow[ur, "\varepsilon"'] & \end{tikzcd} \qquad \begin{tikzcd} H(1,G) \arrow[rr, "\varepsilon"] \arrow[dr, "H(\varepsilon,1)"'] & & 1 \\ & H(F(G,1),G). \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

Proof of Proposition 2.13. It suffices to show that these two natural transformations have the same component at $(c,a) \in C \times A$. This is shown in the following (large) commutative diagram in which the top edge is the component $\bar{\bar{\alpha}}_{c,a}$ and the bottom edge $\hat{\alpha}_{c,a}$.

Regions (3) and (4) are naturality squares, (5) and (6) are functorality of H', (2) and (7) are generalised triangle identities and (1) commutes by extranaturality of ε as follows. The counit ε in question has components

$$H'(c, G'(b,c)) \longrightarrow b$$

and is natural in b but extranatural in c. Region (1) is obtained by writing out the extranaturality condition for the morphism

$$F'(S,TH) \xrightarrow{\alpha} UF(1,H) \xrightarrow{U\varepsilon} U.$$

$$
\begin{tikzcd}
 & & & & & H'(U,G'(TH,UF(G(H,1),H))) \arrow[ddrr, "{H'(1,G'(1,U\varepsilon))}"] & & & \\
 & & & H'(U,G'(TH,F'(SG(H,1),TH))) \arrow[urr, "{H'(1,G'(1,\alpha))}"] & & & & & \\
 & & & & & & & H'(U,G'(TH,U)) \arrow[dddr, "\varepsilon"] & \\
 & H'(U,SG(H,1)) \arrow[uurr, "{H'(1,\varepsilon)}"] & & & & & & & \\
 & & & & H'(U,G'(TH,UF(1,H))) \arrow[uuuur, "{H'(1,G'(1,UF(\varepsilon,1)))}"] \arrow[uurrr, "{H'(1,G'(1,U\varepsilon))}"'] \arrow[uur, phantom, "\textcircled{2}"] & & & & \\
H'(U,S) \arrow[uur, "{H'(1,S\varepsilon)}"] \arrow[rr, "{H'(1,\varepsilon)}"'] \arrow[dddrrr, "{H'(U\varepsilon,1)}"'] \arrow[uuuurrr, phantom, "\textcircled{4}"] \arrow[drrrr, phantom, "\textcircled{5}"] & & H'(U,G'(TH,F'(S,TH))) \arrow[uuuur, "{H'(1,G'(1,F'(S\varepsilon,1)))}"] \arrow[urr, "{H'(1,G'(1,\alpha))}"'] \arrow[drr, "{H'(U\varepsilon,1)}"] \arrow[uuuuurrr, phantom, "\textcircled{3}"] \arrow[rrrrrr, phantom, "\textcircled{1}"] & & & & & & TH \\
 & & & & H'(UF(1,H),G'(TH,F'(S,TH))) \arrow[drrr, "{H'(\alpha,1)}"] & & & & \\
 & & & & & & & H'(F'(S,TH),G'(TH,F'(S,TH))) \arrow[uur, "\varepsilon"] & \\
 & & & H'(UF(1,H),S) \arrow[uur, "{H'(1,\varepsilon)}"] \arrow[rrr, "{H'(\alpha,1)}"'] \arrow[urrrr, phantom, "\textcircled{6}"] & & & H'(F'(S,TH),S) \arrow[ur, "{H'(1,\varepsilon)}"] \arrow[uuurr, "\varepsilon"'] \arrow[uurr, phantom, "\textcircled{7}"] & &
\end{tikzcd}
$$

□

Now by allowing B to be a finite product of categories, we get a notion of n-variable mates with respect to an n-variable adjunction.

Theorem 2.15. *Suppose we have functors*

$$\begin{aligned} F_0 &\colon A_1 \times \cdots \times A_n \longrightarrow A_0^\bullet, \\ F_0' &\colon A_1' \times \cdots \times A_n' \longrightarrow A_0'^\bullet \end{aligned}$$

equipped with n-variable left adjoints, and for all $0 \le i \le n$ a functor

$$S_i \colon A_i \longrightarrow A_i'.$$

Then a natural transformation

$$\begin{tikzcd} A_1 \times \cdots \times A_n \arrow[r, "S_1 \times \cdots \times S_n"] \arrow[d, "F_0"'] \arrow[dr, phantom, "\Nearrow\alpha"] & A_1' \times \cdots \times A_n' \arrow[d, "F_0'"] \\ A_0^\bullet \arrow[r, "S_0^\bullet"'] & A_0'^\bullet \end{tikzcd}$$

has for all $1 \le i \le n$ a mate

$$\begin{tikzcd} A_{i+1} \times \cdots \times A_{i-1} \arrow[r, "S_{i+1} \times \cdots \times S_{i-1}"] \arrow[d, "F_i"'] \arrow[dr, phantom, "\Nearrow\alpha_i"] & A_{i+1}' \times \cdots \times A_{i-1}' \arrow[d, "F_i'"] \\ A_i^\bullet \arrow[r, "S_i^\bullet"'] & A_i'^\bullet \end{tikzcd}$$

given as in Proposition 2.11 with

$$\begin{aligned} A &= A_i \\ B &= A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n \\ C &= A_0. \end{aligned}$$

We now give the n-variable version of the mates correspondence, which follows from the 2-variable case (Proposition 2.13). First we need to fix our notation carefully.

Notation for n-variable mates.

Suppose we have functors

$$\begin{aligned} F_0 &\colon A_1 \times \cdots \times A_n \longrightarrow A_0^\bullet, \\ F_0' &\colon A_1' \times \cdots \times A_n' \longrightarrow A_0'^\bullet \end{aligned}$$

equipped with n-variable left adjoints, and for all $0 \le i \le n$ a functor

$$S_i \colon A_i \longrightarrow A_i'.$$

Then for any $0 \leq i \leq n$, given a natural transformation α as below

$$\begin{tikzcd}
A_{i+1} \times \cdots \times A_{i-1} \arrow[r, "S_{i+1} \times \cdots \times S_{i-1}"] \arrow[d, "F_i"'] \arrow[dr, phantom, "\Nearrow\alpha"] & A'_{i+1} \times \cdots \times A'_{i-1} \arrow[d, "F'_i"] \\
A_i^\bullet \arrow[r, "S_i^\bullet"'] & A_i'^\bullet
\end{tikzcd}$$

and any $j \neq i$ we denote by α_{ij} the mate

$$\begin{tikzcd}
A_{j+1} \times \cdots \times A_{j-1} \arrow[r, "S_{j+1} \times \cdots \times S_{j-1}"] \arrow[d, "F_j"'] \arrow[dr, phantom, "\Nearrow\alpha_{ij}"] & A'_{j+1} \times \cdots \times A'_{j-1} \arrow[d, "F'_j"] \\
A_j^\bullet \arrow[r, "S_j^\bullet"'] & A_j'^\bullet
\end{tikzcd}$$

produced by Theorem 2.15. Note that in this notation, the mate called α_i in the theorem would be called α_{0i}.

Theorem 2.16 (The n-variable mates correspondence). *Given a pair of n-variable adjunctions, any distinct i, j, k and a natural transformation α as above, we have*

$$(\alpha_{ij})_{jk} = \alpha_{ik}.$$

Proof. Restricting to the functors F_i, F_j, F_k and fixing all variables except those in A_i, A_j, A_k we get a 2-variable adjunction. The result is then simply an instance of Proposition 2.13 since it suffices to check it componentwise. □

Corollary 2.17. *Given a pair of n-variable adjunctions as above and a natural transformation*

$$\begin{tikzcd}
A_1 \times \cdots \times A_n \arrow[r, "S_1 \times \cdots \times S_n"] \arrow[d, "F_0"'] \arrow[dr, phantom, "\Nearrow\alpha"] & A'_1 \times \cdots \times A'_n \arrow[d, "F'_0"] \\
A_0^\bullet \arrow[r, "S_0^\bullet"'] & A_0'^\bullet
\end{tikzcd}$$

we have

$$((\cdots((\alpha_{01})_{12})\cdots)_{n-1,n})_{n0} = \alpha.$$

That is, taking mates $n+1$ times is the identity.

Note that the n-variable mates correspondence respects horizontal and vertical composition. For horizontal composition this follows immediately from the analogous result for 1-variable mates. For vertical composition a little more effort is required, but mainly just to make precise the meaning of "respects vertical composition". However this is only a matter of indices. The idea is not hard: composition of multivariable adjunctions is defined by fixing variables and composing the resulting 1-variable adjunctions, and the mates correspondence follows likewise.

To put this result in a more precise framework we will show that we have the structure of a cyclic double multicategory.

3 Cyclic double multicategories

In this section we give the definition of "cyclic double multicategory", the structure into which multivariable adjunctions and mates organise themselves. The idea is to combine the notions of double category and cyclic multicategory so that in our motivating example the cyclic action expresses the multivariable mates correspondence.

Recall that a double category can be defined as a category object in **Cat**; similarly a double multicategory is a category object in the category **Mcat** of multicategories, and a cyclic double multicategory is a category object in the category **CMcat** of "cyclic multicategories". (Note that this could be called a "double cyclic multicategory" but this might sound as if there are two cyclic actions.)

We build up to the definition step by step, with some examples.

3.1 Plain multicategories

We begin by recalling the definition of plain (non-symmetric) multicategories.

Definition 3.1. Let T be the free monoid monad on **Set**. Write T**-Span** for the bicategory in which

0-cells are sets,

1-cells $A \longrightarrow\!\!\!\!\!\!\circ\;\; B$ are T-spans
$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "t"] & \\ TA & & B \end{tikzcd}$$

2-cells are maps of T-spans.

Composition is by pullback using the multiplication for T: the composite

$$A \xrightarrow[\circ]{X} B \xrightarrow[\circ]{Y} C$$

is given by the span

$$\begin{tikzcd} & & & \cdot \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\ & & TX \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & \\ & T^2A \arrow[dl, "\mu_A"'] & & TB & & C \\ TA & & & & & \end{tikzcd}$$

A **multicategory** A is a monad in T**-Span**, thus

a 0-cell A_0,

a T-span
$$\begin{tikzcd} & A_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ TA_0 & & A_0 \end{tikzcd}$$

equipped with unit and multiplication 2-cells. Explicitly, this gives

a set A_0 of objects,

for all $n \geq 0$ and objects $a_1, \ldots, a_n, a_0 \in A_0$ a set $A(a_1, \ldots, a_n; a_0)$ of n-ary "multimaps" (in the case $n = 0$ the source is empty)

equipped with

composition: for all sets of k ordered strings $a_{i1}, \ldots, a_{im_i}, a_{i0}$ and a_{00} in A_0 a function

$$A(a_{10}, \ldots, a_{k0}; a_{00}) \times A(\underline{a}_1; a_{10}) \times \cdots \times A(\underline{a}_k; a_{k0}) \longrightarrow A(\underline{a}_1, \ldots, \underline{a}_k; a_{00})$$

where we have written $\underline{a}_i$ for the string $a_{i1}, \ldots, a_{in_i}$, and

identities: for all $a \in A_0$ a function

$$1 \xrightarrow{1_a} A(a; a)$$

satisfying the usual associativity and unit axioms.

Note that we can define composition at the ith input by composing with identities at every other input; this will be useful when giving the axioms for a cyclic multicategory and we denote it $\circ_i$.

Examples 3.2.

1. Multifunctors: take objects to be categories and k-ary multimaps to be multifunctors, that is functors of the form

$$A_1 \times \cdots \times A_k \longrightarrow A_0.$$

2. Multicategories from monoidal categories: given any monoidal category C there is a multicategory M_C with the same objects, and with

$$M_C(x_1, \ldots, x_k; x_0) = C(x_1 \otimes \cdots \otimes x_k, x_0).$$

3. Profunctors: we might try to use profunctors instead of functors in the above example, but this would form some sort of "weak multicategory" or "multi-bicategory" as profunctor composition is not strictly associative and unital. However this is a pertinent case to consider. A profunctor

$$A_1 \times \cdots \times A_k \longrightarrow\!\!\!\!\!\!\!\!+\;\;\; A_0$$

is by definition a functor

$$A_1 \times \cdots \times A_k \times A_0^\bullet \longrightarrow \mathbf{Set}.$$

But this also gives rise to a profunctor

$$\hat{A}_i \times A_0^\bullet \longrightarrow\!\!\!\!\!\!\!+\;\; A_i^\bullet$$

for each $i \neq 0$ where here $\hat{A}_i$ denotes the product

$$A_1 \times \cdots A_{i-1} \times A_{i+1} \times \cdots \times A_k.$$

Strictness aside, this is the sort of cyclic action we will be considering. (In fact, the cyclic action in this example is strict although the composition is not.)

3.2 Cyclic multicategories

We now introduce the notion of a "cyclic action" on a multicategory. Symmetric multicategories are multicategories with a symmetric group action that can be thought of as permuting the source elements of a given multimap. Cyclic multicategories have a *cyclic* group action that permutes the inputs *and* outputs cyclically. There is also a "duality" that is invoked each time an object moves between the input and output sides of a map under the cyclic action, as in the example with profunctors sketched above.

Throughout, we work with C_n the cyclic group of order n considered as a subgroup of the symmetric group S_n with canonical generator the cycle $\sigma_n = (123 \cdots n)$. We will often write this as σ with its order being understood from the context.

Definition 3.3. A **cyclic multicategory** X is a multicategory equipped with

an involution on objects

$$\begin{array}{ccc} X_0 & \longrightarrow & X_0 \\ x & \longmapsto & x^* \end{array}$$

for every $n \geq 1$ and ordered string $x_0, x_1, \ldots, x_n$ an isomorphism

$$\sigma = \sigma_{n+1} \colon X(x_1, \cdots, x_n; x_0) \xrightarrow{\sim} X(x_2, \ldots, x_n, x_0^*; x_1^*)$$

such that the following axioms are satisfied.

1. Each isomorphism σ_n is cyclic so that $(\sigma_n)^n = 1$.
2. The identity is preserved by σ_2, that is, the following diagram commutes

$$\begin{tikzcd} 1 \arrow[r, "1_x"] \arrow[dr, "1_{x^*}"'] & X(x;x) \arrow[d, "\sigma_2"] \\ & X(x^*;x^*) \end{tikzcd}$$

3. Interaction between σ and composition.
   Let c_i denote composition at the ith input only, that is

   $$c_i\colon X(y_1,\dots,y_m;y_0)\times X(x_1,\dots,x_n;y_i)\longrightarrow X(y_1,\dots,y_{i-1},x_1,\dots,x_n,y_{i+1},\dots,y_m;y_0)$$

   Then the following diagrams commute.

   For $i=1$, that is, for composition at the first input:

$$\begin{tikzcd}
X(y_1,\dots,y_m;y_0)\times X(x_1,\dots,x_n;y_1) \arrow[r, "c_1"] \arrow[d, "\sigma\times\sigma"'] & X(x_1,\dots,x_n,y_2,\dots,y_m;y_0) \arrow[dd, "\sigma"] \\
X(y_2,\dots,y_m,y_0^*;y_1^*)\times X(x_2,\dots,x_n,y_1^*;x_1^*) \arrow[d, "\cong"'] & \\
X(x_2,\dots,x_n,y_1^*;x_1^*)\times X(y_2,\dots,y_m,y_0^*;y_1^*) \arrow[r, "c_n"'] & X(x_2,\dots,x_n,y_2,\dots,y_m,y_0^*;x_1^*).
\end{tikzcd}$$

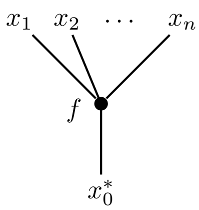
   If $i\neq 1$

$$\begin{tikzcd}
X(y_1,\dots,y_m;y_0)\times X(x_1,\dots,x_n;y_i) \arrow[r, "c_i"] \arrow[d, "\sigma\times 1"'] & X(y_1,\dots,y_{i-1},x_1,\dots,x_n,y_{i+1},\dots,y_m;y_0) \arrow[d, "\sigma"] \\
X(y_2,\dots,y_m,y_0^*;y_1^*)\times X(x_1,\dots,x_n;y_i) \arrow[r, "c_{i-1}"'] & X(y_2,\dots,y_{i-1},x_1,\dots,x_n,y_{i+1},\dots,y_m,y_0^*;y_1^*).
\end{tikzcd}$$

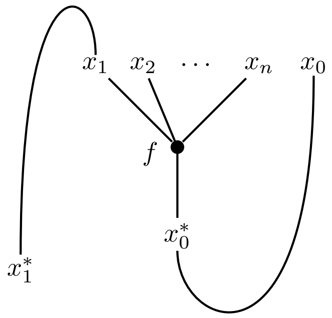
   In algebra

$$\sigma(g\circ_i f)=\begin{cases}(\sigma f)\circ_n(\sigma g) & i=1\\ (\sigma g)\circ_{i-1} f & 2\leq i\leq m.\end{cases}$$

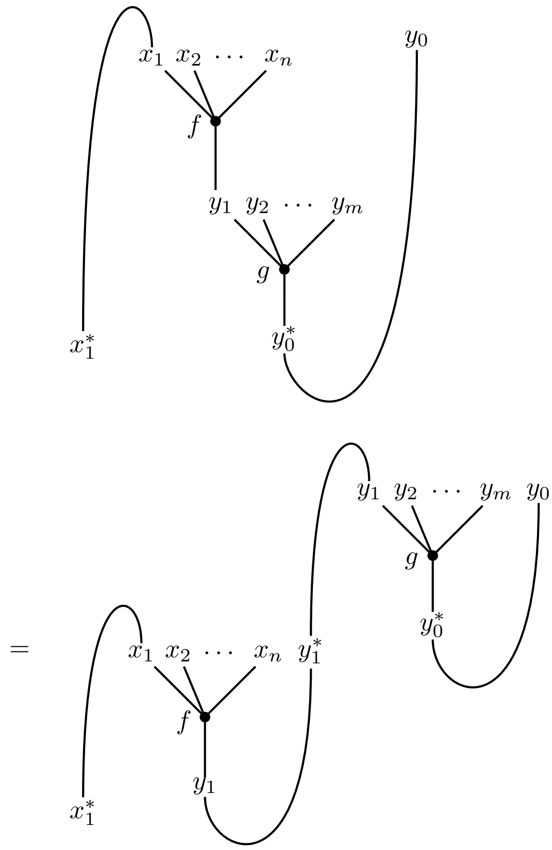
We can depict the axioms (3) pictorially as follows. Depicting $f\in X(x_1,\dots,x_n;x_0^*)$ as



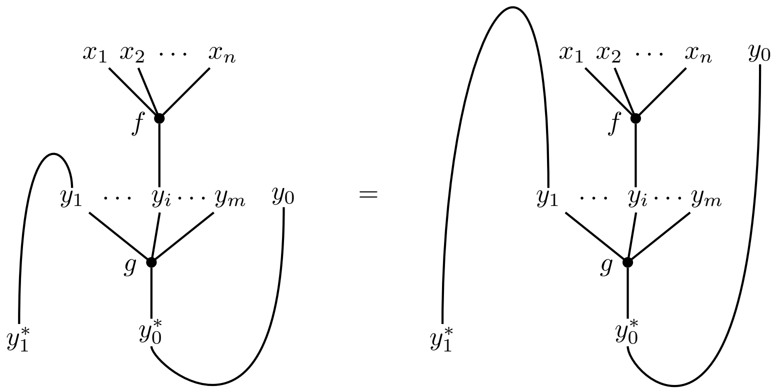
we depict σf as



Then the first axiom is depicted as shown below.



The second is a little is a little less satisfying to depict pictorially, but is shown below.



Note that these two axioms are equivalent to a single axiom involving the cyclic action and composition at every variable.

Examples 3.4. First we give some slightly degenerate examples.

1. If X is a cyclic multicategory with only one object (so the involution must be the identity) then we have a notion of non-symmetric cyclic operad as given by Batanin and Berger in [1]. This is in contrast to the definition of (symmetric) cyclic operad in [14].

2. More generally, the involution can be the identity even for a non-trivial set of objects.

Example 3.5. We define a cyclic multicategory **MAdj** as follows. Take objects to be categories, and a multimap

$$A_1, \ldots, A_n \longrightarrow A_0$$

to be a functor

$$A_1 \times \cdots \times A_n \xrightarrow{F_0} A_0$$

equipped with all n-variable left adjoints, $F_1, \ldots, F_n$. The involution $()^*$ is then given by $()^\bullet$ and the cyclic action is given by

$$\sigma\colon F_i \longmapsto F_{i+1}$$

and the axioms are satisfied by construction. We could also do this with n-variable right adjoints.

Note that **MAdj** can be expressed using profunctors. Recall our profunctor example that was not quite a true example (Example 3.2.3) as profunctor composition is not strictly associative or unital; nevertheless it has a strict cyclic action. In fact n-variable adjunctions can be thought of as n-variable profunctors F_0 such that F_0 and all its cyclic versions in **Prof** are representable, or, more precisely, equipped with representations as follows.

Proposition 3.6. *Let $P : A_1 \times \cdots \times A_n \times A_0 \longrightarrow$ **Set** be a profunctor equipped with a representation for each $P(a_1, \ldots, a_{i-1}, \_, a_{i+1}, \ldots, a_n, a_0)$. That is, given*

$$a_{i+1}, \ldots, a_{i-1}$$

an object $F_i(a_{i+1}, \ldots, a_{i-1}) \in A_i$ and an isomorphism

$$P(a_1, \ldots, a_0) \cong A_i(F_i(a_{i+1}, \ldots, a_{i-1}), a_i) \tag{4}$$

natural in a_i.

Then the F_i canonically extend to functors

$$A_{i+1} \times \cdots \times A_{i-1} \xrightarrow{F_i} A_i^{\bullet}$$

forming an n-variable (left) adjunction.

Proof. By standard results about parametrised representability, each F_i extends to a functor

$$A_{i+1} \times \cdots \times A_{i-1} \xrightarrow{F_i} A_i^{\bullet}$$

unique making the isomorphism (4) above natural in all variables. For the n-variable adjunction we then compose the isomorphisms

$$A_j(F_j(a_{j+1}, \ldots, a_{j-1}), a_j) \xrightarrow{\cong} P(a_1, \ldots, a_0) \xrightarrow{\cong} A_i(F_i(a_{i+1}, \ldots, a_{i-1}), a_i).$$

□

Remark 3.7. Note that composition of these profunctors matches composition of the corresponding multivariable adjunctions up to isomorphism; this is as strict as we can expect as profunctor composition is only defined up to isomorphism (by coends).

Remark 3.8. The idea is that we consider the functor

$$\mathbf{Cat} \longrightarrow \mathbf{Prof}$$

that is the identity on objects and on morphisms sends a functor

$$A \xrightarrow{G} B$$

to the profunctor

$$A \longrightarrow\!\!\!\!\!\!\!+\;\; B$$

given by

$$\begin{array}{ccc} A \times B^{\bullet} & \longrightarrow & \mathbf{Set} \\ (a, b) & \longmapsto & B(b, Ga). \end{array}$$

With the usual composition in **Prof** this is only a pseudo-functor, giving us a "sub-pseudo-multicategory" of **Prof** that is somehow "equivalent" to **MAdj**. In order to get an honest multicategory we must specifiy data as above, giving us a strict multicategory biequivalent to the more natural arising pseudo-multicategory.

3.3 Cyclic double multicategories

We are now ready to introduce the 2-cells we need. Recall that a double category can be defined very succinctly as a category object in the category of (small) categories. We proceed analogously for the multi-versions.

Definition 3.9. A **double multicategory** is a category object in the category **Mcat** of multicategories.

A **cyclic double multicategory** is a category object in the category **CMcat** of cyclic multicategories.

Note that pullbacks in the category **CMcat** are defined in the obvious way, so this definition makes sense. As with double categories, it is desirable to give an elementary description. A cyclic double multicategory X has as underlying data a diagram

$$\begin{tikzcd} \mathrm{B} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathrm{A} \end{tikzcd}$$

in **CMcat**.

Recall that the underlying data for a multicategory A is in turn a diagram in sets of the following form

$$\begin{tikzcd} & A_1 \arrow[dl, "d"'] \arrow[dr, "c"] & \\ TA_0 & & A_0 \end{tikzcd}$$

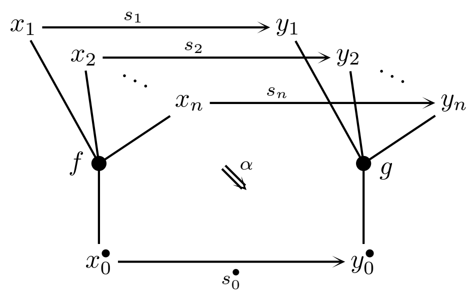
where T is the free monoid monad on **Set**. Thus for a category object in **Mcat** we have a diagram of the following form in **Set**:

$$\begin{tikzcd} & B_1 \arrow[dl, "d"'] \arrow[dr, "c"] \arrow[dd, shift right, "s"'] \arrow[dd, shift left, "t"] & \\ TB_0 \arrow[dd, shift right, "s"'] \arrow[dd, shift left, "t"] & & B_0 \arrow[dd, shift right, "s"'] \arrow[dd, shift left, "t"] \\ & A_1 \arrow[dl, "d"] \arrow[dr, "c"'] & \\ TA_0 & & A_0 \end{tikzcd}$$

where the sets correspond to data as follows:

$$\begin{aligned} A_0 &= \text{0-cells} \\ A_1 &= \text{vertical (multi) 1-cells} \\ B_0 &= \text{horizontal (plain) 1-cells} \\ B_1 &= \text{2-cells.} \end{aligned}$$

Commuting conditions tell us that 2-cells might be depicted as:



Inside the structure of a (cyclic) double multicategory we have two categories given by

> 0-cells, horizontal 1-cells and horizontal composition, and
>
> vertical 1-cells, 2-cells and horizontal composition

and two (cyclic) multicategories with objects and multimaps given by

> 0-cells, vertical 1-cells and vertical multi-composition, and
>
> horizontal 1-cells, 2-cells and vertical multi-composition.

Furthermore these must all be compatible, in the following sense. In addition to the underlying diagram

$$\mathrm{B} \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{A}$$

in **CMcat** we must have an identity map

$$I \colon A \longrightarrow B$$

and a composition map

$$\gamma \colon B \times_A B \longrightarrow B$$

and s, t, I, γ must all be maps of cyclic multicategories, that is, they must respect (co)domains, composition, involution and cyclic actions in passing from B to A. Note that s/t give "horizontal source and target", I gives "horizontal identities" and γ "horizontal composition". Respecting (co)domains and composition is analogous to the axioms for a double category, just with multimaps instead of 1-ary maps where appropriate; notably this gives us interchange between horizontal and vertical composition.

Respecting involution and cyclic actions gives us the following information.

1. Horizontal source and target respect involution: we have an involution on A_0 (0-cells) and an involution on B_0 (horizontal 1-cells), both written $(\ )^*$ under which
$$x \xrightarrow{f} y \qquad \longmapsto \qquad x^* \xrightarrow{f^*} y^*.$$

2. Horizontal identities respect involution: for any 0-cell $x \in A_0$ we have a horizontal 1-cell identity $I_x \colon x \longrightarrow x \in B_0$. This assignation must satisfy the following equality of horizontal 1-cells:

$$I_{x^*} = (I_x)^*.$$

3. Horizontal composition respects involution: given composable horizontal 1-cells in B_0

$$x \xrightarrow{f} y \xrightarrow{g} z$$

we must have the following equality of horizontal 1-cells:

$$(gf)^* = g^* f^*.$$

4. Horizontal source and target respect cyclic action: given a 2-cell $\alpha \in B_1$ we have the following equalities of vertical 1-cells

$$\begin{aligned} s(\sigma\alpha) &= \sigma(s\alpha), \quad \text{and} \\ t(\sigma\alpha) &= \sigma(t\alpha). \end{aligned}$$

5. Horizontal identities respect cyclic action: given a vertical 1-cell $f \in A_1$ we have a horizontal 2-cell identity $I_f \in B_1$. This assignation must satisfy the following equality of 2-cells:

$$\sigma(I_f) = I_{\sigma f}.$$

6. Horizontal composition respects cyclic action: given horizontally composable 2-cells $\alpha, \beta \in B_1$ we have the following equality of 2-cells:

$$\sigma(\beta * \alpha) = \sigma\beta * \sigma\alpha$$

where as usual we write horizontal composition of 2-cells as $\beta * \alpha$.

3.4 Multivariable adjunctions

In this section we show how to organise multivariable adjunctions and mates into a cyclic double multicategory. In fact, just as for the 1-variable case, there are many choices of such a structure on this underlying data. The difference is that now, because of the extra variables, there are also extra choices but many of them are rather unnatural so there is more danger of confusion. The other source of confusion is that the standard notation used in the 1-variable case does not generalise very easily to express all the possible choices in the multivariable case. We begin by giving the most obvious choices of structure.

Theorem 3.10. *There is a cyclic double multicategory $\mathbb{M}\mathbf{Adj}$ extending the cyclic multicategory $\mathbf{MAdj}$ of multivariable (left) adjunctions, given as follows.*

0-cells are categories.

Horizontal 1-cells are functors.

A vertical 1-cell $A_1, \ldots, A_n \xrightarrow{F} A_0^\bullet$ *is a functor* F *equipped with* n*-variable left adjoints.*

2-cells are natural transformations

$$\begin{tikzcd}
A_1 \times \cdots \times A_n \arrow[rr, "S_1 \times \cdots \times S_n"] \arrow[d, "F"'] \arrow[drr, phantom, "\Nearrow\alpha"] & & B_1 \times \cdots \times B_n \arrow[d, "G"] \\
A_0^\bullet \arrow[rr, "S_0^\bullet"'] & & B_0{}^\bullet
\end{tikzcd}$$

(note direction). Here, despite the direction of the natural transformation, the horizontal source of α as a 2-cell of $\mathbb{M}\mathbf{Adj}$ *is* F *and the horizontal target is* G*; the vertical source is* $S_1, \ldots, S_n$ *and the vertical target is* $S_0^\bullet$*.*

The cyclic action on 2-cells is given by the multivariable mates correspondence.

Proof. It only remains to prove that the cyclic composition axioms hold for 2-cells; these are the axioms given in Definition 3.3, applied to the multicategory whose objects are horizontal 1-cells and whose multimaps are 2-cells. We will use the subscript notation for multivariable adjoints of a functor and corresponding mates of a natural transformation, as in Theorem 2.16.

For the first axiom, it suffices to consider the following 2-cells.

$$\begin{tikzcd}
A \times B \arrow[r, "S \times T"] \arrow[d, "F_0"'] \arrow[dr, phantom, "\alpha\Nearrow"] & A' \times B' \arrow[d, "F_0{}'"] \\
C^\bullet \arrow[r, "U^\bullet"'] & C'^\bullet
\end{tikzcd}$$

$$\begin{tikzcd}
C^\bullet \times D \arrow[r, "U^\bullet \times V"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\beta\Nearrow"] & C'^\bullet \times C' \arrow[d, "G_0{}'"] \\
E^\bullet \arrow[r, "W^\bullet"'] & E'^\bullet
\end{tikzcd}$$

This gives the general axiom by considering B and D to be products. Using multicategorical notation, and our previous notation for multivariable mates, we need to show

$$(\beta \circ_1 \alpha)_{01} = \alpha_{01} \circ_2 \beta_{01}.$$

Now the component of $(\beta \circ_1 \alpha)_{01}$ at (b, d, e) is obtained as follows:

1. fix b and d in the composite $\beta \circ_1 \alpha$,

2. take the 1-variable mate,

3. evaluate at e.

Now step (1) is the same as fixing b in α, d in β and then composing the squares vertically. So the axiom is an instance of 1-variable mates respecting vertical composition.

For the second axiom it suffices to consider the following 2-cells.

$$\begin{tikzcd} A \arrow[r, "S"] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha\nearrow"] & A' \arrow[d, "F'"] \\ C^\bullet \arrow[r, "U^\bullet"'] & C'^\bullet \end{tikzcd}$$

$$\begin{tikzcd} B \times C^\bullet \arrow[r, "T \times U^\bullet"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\beta\nearrow"] & B' \times C'^\bullet \arrow[d, "{G_0}'"] \\ D^\bullet \arrow[r, "V^\bullet"'] & D'^\bullet \end{tikzcd}$$

This gives the general axiom by letting A and C be products.

We need to show

$$(\beta \circ_2 \alpha)_{01} = \beta_{01} \circ_1 \alpha.$$

Note that the α on the right hand side is not a mate, as in the axiom given in Definition 3.3.

The component of $(\beta \circ_2 \alpha)_{01}$ at (a, d) is obtained as follows:

1. fix a in the composite $\beta \circ_2 \alpha$,

2. take the 1-variable mate, and

3. evaluate at d.

Step (1) is the same as taking the following horizontal composite:

$$\begin{tikzcd} B \arrow[r, "T"] \arrow[d, "{G_0(\_, Fa)}"'] \arrow[dr, phantom, "\beta_{\_, Fa}\nearrow"] & B' \arrow[r, "1"] \arrow[d, "{G'_0(\_, UFa)}"] \arrow[dr, phantom, "\nearrow G'_0(\_, \alpha_a)"] & B' \arrow[d, "{G'_0(\_, F'Sa)}"] \\ D^\bullet \arrow[r, "V^\bullet"'] & D'^\bullet \arrow[r, "1"'] & D'^\bullet \end{tikzcd}$$

and the axiom then follows from the fact that 1-variable mates respect horizontal composition, together with the fact that the mate of $G'_0(\_, \alpha_a)$ is $G'_1(\alpha_a, \_)$. To show this last fact, we show that, more generally, for any morphism $f \colon c_2 \longrightarrow c_1$ the mate of $G_0(\_, f)$ is $G_1(f, \_)$ (omitting the primes as they are not relevant to this general result). This is seen from the following diagram, where the top edge is the mate of $G_0(\_, f)$ and the bottom is $G_1(f, \_)$.

$$
\begin{tikzcd}
& G_1\big(c_2, G_0(G_1(c_1,b),c_1\big) \arrow[rr, "{G_1(1,G_0(1,f))}"] \arrow[dr, "{G_1(f,1)}"] & & G_2\big(c_2, G_0(G_1(c_1,b),c_2)\big) \arrow[dr, "\varepsilon"] \arrow[d, phantom, "\textcircled{2}"] & \\
G_1(c_1,b) \arrow[ur, "{G_1(1,\varepsilon)}"] \arrow[dr, "{G_1(f,1)}"'] \arrow[rr, phantom, "\textcircled{1}"] & & G_1\big(c_1, B_0(G_1(c_1,b),c_1)\big) \arrow[rr, "\varepsilon"] & \arrow[d, phantom, "\textcircled{3}"] & G_1(c_1,b) \\
& G_1(c_1,b) \arrow[ur, "{G_1(1,\varepsilon)}"'] \arrow[urrr, bend right] & & &
\end{tikzcd}
$$

Region (1) is functoriality of G_1, region (2) is extranaturality of ε, and region (3) is a triangle identity. □

Remark 3.11. The direction of the natural tranformation for 2-cells is crucial so that the multivariable mates correspondence can be applied. There is a cyclic double multicategory involving multivariable *right* adjunctions in which the 2-cells must be given by natural transformations pointing down, as in

$$
\begin{tikzcd}
A_1 \times \cdots \times A_n \arrow[rr, "S_1 \times \cdots \times S_n"] \arrow[d, "F"'] & \arrow[d, phantom, "\overset{\alpha}{\swarrow}"] & B_1 \times \cdots \times B_n \arrow[d, "G"] \\
A_0^\bullet \arrow[rr, "S_0^\bullet"'] & & B_0{}^\bullet
\end{tikzcd}
$$

To be precise we write $\mathbb{M}\mathbf{Adj}_L$ for the multivariable left adjunctions and $\mathbb{M}\mathbf{Adj}_R$ for the multivariable right adjunctions. We will need the latter construction in the next section.

Theorem 3.12. *There is an isomorphism of double multicategories*

$$(\ )^\bullet \colon \mathbb{M}\mathbf{Adj}_L \longrightarrow \mathbb{M}\mathbf{Adj}_R.$$

This isomorphism is analogous to the isomorphism of double categories

$$\mathbb{L}\mathbf{Adj} \cong \mathbb{L}\mathbf{Adj}_R.$$

We now discuss isomorphisms analogous to the isomorphism of double categories

$$\mathbb{L}\mathbf{Adj} \cong \mathbb{R}\mathbf{Adj}.$$

Recall that these double categories have the same 0- and 1-cells, but the 2-cells are natural transformations living in squares involving the left adjoints, for

$\mathbb{L}\mathbf{Adj}$, and the right adjoints, for $\mathbb{R}\mathbf{Adj}$. For the n-variable version we have instead of left and right adjoints, a cycle of $n+1$ possible mutual adjoints. This gives us many possible variants of the cyclic double multicategory $\mathbb{M}\mathbf{Adj}$.

For the multivariable case the situation is further complicated by the fact that we have a choice of 2-cell convention for *each arity* n, and these can all be chosen independently. These choices are the w_n in the following theorem. This theorem might seem unnatural and/or contrived to state; we include it emphasise the fact that the $\mathbb{L}\mathbf{Adj} \cong \mathbb{R}\mathbf{Adj}$ isomorphism is not the natural one to generalise to multivariables.

Theorem 3.13. *Suppose we have fixed for each $n \in \mathbb{N}$ an integer w_n with $0 \leq w_n \leq n$. Write this infinite sequence of natural numbers as $\mathbf{w}$. Then we have a cyclic double multicategory $\mathbb{M}\mathbf{Adj}_{\mathbf{w}}$ with the same 0- and 1-cells as $\mathbb{M}\mathbf{Adj}$ (with multivariable left adjunctions, say) but where for each n an n-ary 2-cell is as shown below*

$$\begin{tikzcd}
A_{w_n+1} \times \cdots \times A_{w_n-1} \arrow[r, "S_{w_n+1} \times \cdots \times S_{w_n-1}"] \arrow[d, "F_{w_n}"'] \arrow[dr, phantom, "\Nearrow\alpha"] & B_{w_n+1} \times \cdots \times B_{w_n-1} \arrow[d, "G_{w_n}"] \\
A^{\bullet}_{w_n} \arrow[r, "S^{\bullet}_{w_n}"'] & B^{\bullet}_{w_n}
\end{tikzcd}$$

(note direction). We emphasise that the horizontal source is still F_0 and the horizontal target is G_0; the vertical source is $S_1, \ldots, S_n$ and the vertical target is $S^{\bullet}_0$. If each $w_n = 0$ we get the original version of $\mathbb{M}\mathbf{Adj}$.

Composition proceeds via the mates correspondence.

Then for all $\mathbf{w}$ there is an isomorphism of cyclic double multicategories

$$\mathbb{M}\mathbf{Adj} \cong \mathbb{M}\mathbf{Adj}_{\mathbf{w}}$$

which is the multivariable generalisation of the double category isomorphism

$$\mathbb{L}\mathbf{Adj} \cong \mathbb{R}\mathbf{Adj}.$$

4 Application to algebraic monoidal model categories

One aim of this work is to study an algebraic version of Hovey's notion of monoidal model category [12]. In such a model category we have hom and tensor structures that must interact well with the given model structure. One such interaction requirement is that the 2-variable adjunction for hom and tensor should be a morphism of the underlying algebraic weak factorisation systems of the model category. An important consequence of the defining axioms is that the total derived functors of the 2-variable adjunction given by the tensor and hom define a closed monoidal structure on the homotopy category of the model category.

A model category has, among other things, two weak factoristion systems. In an *algebraic* model category [23] these are *algebraic* weak factorisation systems [7]. In this case, elements in the left and right classes of the weak factorisation systems specifying the model structure become coalgebras and algebras for the comonads and monads of the algebraic weak factorisation systems. An algebraic model category with a closed monoidal structure is a *monoidal algebraic model category* [24] just when the tensor/hom/cotensor 2-variable adjunction is a "2-variable adjunction of algebraic weak factorisation systems". This notion makes use of the definition of parametrised mates and motivates much of the present work.

As in [23], we abbreviate "algebraic weak factorisation system" to "awfs". First we recall the definition of awfs and of a standard (1-variable) adjunction of awfs. Throughout this section, given a category A we write A for the category whose objects are morphisms of A, and whose morphisms are commuting squares. That is, A is the category $\mathbf{Cat}(\ , A)$ where denotes the category containing a single non-trivial arrow. We have domain and codomain projections $\mathbf{dom}, \mathbf{cod} : A \longrightarrow A$.

A functorial factorisation on a category A is given by a pair of functors $L, R\colon A \longrightarrow A$ with $\mathbf{dom}\, L = \mathbf{dom}$, $\mathbf{cod}\, R = \mathbf{cod}$, and $\mathbf{cod}\, L = \mathbf{dom}\, R$. We call this last functor E, so we can write the factorisation of a morphism f as below.

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "Lf"'] & & b \\ & Ef \arrow[ur, "Rf"'] & \end{tikzcd}$$

An awfs on a category A is given by a functorial factorisation together with extra structure making

L a comonad on A , and

R a monad on A , such that

the canonical map $LR \longrightarrow RL$ given by multiplication and comultiplication is a distributive law.

The idea is that the L-coalgebras are the left maps (equipped with structure specifying their liftings) and the R-algebras are the right maps.

Definition 4.1. A adjunction of awfs

$$\begin{array}{ccc} (L_1, R_1) & \longrightarrow & (L_2, R_2) \\ \text{on } \ A_1 & & \text{on } \ A_2 \end{array}$$

consists of the following.

An adjunction

$$\begin{tikzcd} A_1 \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & A_2 \arrow[l, shift left, "G"] \end{tikzcd}$$

Natural transformations λ and ρ making

1. $(F\ \ ,\lambda)$ into a colax comonad map $L_1 \longrightarrow L_2$, and
2. $(G\ \ ,\rho)$ into a lax monad map $R_2 \longrightarrow R_1$

where

$$\begin{aligned} \lambda &= (1, \alpha), \quad \text{and} \\ \rho &= (\overline{\alpha}, 1). \end{aligned}$$

Here $\overline{\alpha}$ denotes the mate of α, about which some further comments are called for. *A priori* the natural transformations λ and ρ are as shown below

$$\begin{tikzcd} A_1 \arrow[r, "L_1"] \arrow[d, "F"'] \arrow[dr, phantom, "\swarrow\lambda"] & A_1 \arrow[d, "F"] \\ A_2 \arrow[r, "L_2"'] & A_2 \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r, "R_1"] \arrow[dr, phantom, "\rho\searrow"] & A_1 \\ A_2 \arrow[u, "G"] \arrow[r, "R_2"'] & A_2 \arrow[u, "G"'] \end{tikzcd}$$

but it turns out that such λ and ρ are completely determined by respective natural transformations as below

$$\begin{tikzcd} A_1 \arrow[r, "E_1"] \arrow[d, "F"'] \arrow[dr, phantom, "\swarrow"] & A_1 \arrow[d, "F"] \\ A_2 \arrow[r, "E_2"'] & A_2 \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r, "E_1"] \arrow[dr, phantom, "\searrow"] & A_1 \\ A_2 \arrow[u, "G"] \arrow[r, "E_2"'] & A_2. \arrow[u, "G"'] \end{tikzcd}$$

It is these that are required to be mates α and $\overline{\alpha}$ respectively, under the adjunctions $F\ \ \dashv G\ \$ and $F \dashv G$. (Note that $(\ )\ \ $ is actually the 2-functor **Cat**$(\ \ ,\_)$ so preserves adjunctions.)

It turns out that the appropriate generalisation for the n-variable case involves generalising the functor $(\ )\ \ $ as well, as follows.

Definition 4.2. Let $F\colon A_1 \times \cdots \times A_n \longrightarrow A_0$ be an n-variable functor, and assume that each category A_i has appropriate colimits. We define a functor

$$\hat{F}\colon A_1 \times \cdots \times A_n \longrightarrow A_0$$

as follows. Consider morphisms

$$a_{i0} \xrightarrow{f_i} a_{i1} \in A_i$$

for each $1 \leq i \leq n$. We need to define a morphism $\hat{F}(f_1, \ldots, f_n)$ in A_0. Consider the commuting hypercube in $A_1 \times \cdots \times A_n$ built from f_i's as follows.

Vertices are given by $(a_{1k_1}, \ldots, a_{nk_n})$ where each $k_i = 0$ or 1 (thus, the ith term is either the source or target of f_i).

Edges are given by $(1, \ldots, 1, f_i, 1, \ldots, 1)$ for some $1 \leq i \leq n$.

Each face of this hypercube clearly commutes.

We apply F to this diagram and take the "obstruction" map induced by the colimit over the diagram obtained by removing the terminal vertex (and all morphisms involving it). We call this map $\hat{F}(f_1, \ldots, f_n)$ in A_0; its domain is the above colimit and its codomain is $(a_{11}, \ldots, a_{n1})$.

The action on morphisms is then induced in the obvious way. In fact $\hat{(\ )}$ is a pseudo-functor so preserves adjunctions. Furthermore, a straightforward but notationally involved proof shows that $\hat{(\ )}$ preserves n-variable adjunctions, as we first learned from Dominic Verity.

Remark 4.3. Given an awfs (L, R) on a category A, we get a dual awfs $(R^\bullet, L^\bullet)$ on $A^\bullet$. Note that

> L is a comonad on $A\ \ $, so $L^\bullet$ is a monad on $(A\ \ )^\bullet$, and
>
> R is a monad on $A\ \ $, so $R^\bullet$ is a comonad on $(A\ \ )^\bullet$.

Also, given awfs (L_1, R_1) on A_1 and (L_2, R_2) on A_2 we get an awfs

$$(L_1 \times L_2, R_1 \times R_2)$$

on $A_1 \times A_2$.

Definition 4.4. Suppose we have for each $0 \le i \le n$ a category A_i equipped with an awfs (L_i, R_i). Then an n**-variable adjunction of awfs**

$$A_1 \times \cdots \times A_n \longrightarrow A_0^\bullet$$

is given by the following.

> A functor $F_0 : A_1 \times \cdots \times A_n \longrightarrow A_0^\bullet$ equipped with n-variable *right* adjoints $F_1, \ldots, F_n$, and
>
> For each i a natural transformation λ_i as shown below
>
> $$\begin{tikzcd} A_{i+1} \times \cdots \times A_{i-1} \arrow[r, "L_{i+1} \times \cdots \times L_{i-1}"] \arrow[d, "\hat{F}_i"'] \arrow[dr, phantom, "\swArrow \lambda_i"] & A_{i+1} \times \cdots \times A_{i-1} \arrow[d, "\hat{F}_i"] \\ A_i^\bullet \arrow[r, "R_i^\bullet"'] & A_i^\bullet \end{tikzcd}$$
>
> making $(\hat{F}_i, \lambda_i)$ into a colax comonad map
>
> $$L_{i+1} \times \cdots \times L_{i-1} \longrightarrow R_i^\bullet.$$
>
> As in the 1-variable case, such a λ_i is completely determined by a natural transformation

$$\begin{tikzcd}
A_{i+1} \times \cdots \times A_{i-1} \arrow[r, "E_{i+1} \times \cdots \times E_{i-1}"] \arrow[d, "\hat{F}_i"'] \arrow[dr, phantom, "\swarrow \alpha_i"] & A_{i+1} \times \cdots \times A_{i-1} \arrow[d, "F_i"] \\
A_i^\bullet \arrow[r, "E_i^\bullet"'] & A_i^\bullet
\end{tikzcd}$$

and we require the α_i to be parametrised mates.

Example 4.5. An algebraic, or perhaps constructive, encoding of the classical result that the simplicial hom-space from a simplicial set A to a Kan complex X is again a Kan complex is that the tensor-hom 2-variable adjunction is a 2-variable adjunction of awfs. This example is prototypical, so we explain it further. The sets of maps

$$I = \{\partial\Delta^n \longrightarrow \Delta^n \mid n \geq 0\}$$

and

$$J = \{\Lambda^n_k \longrightarrow \Delta^n \mid n \geq 1, 0 \leq k \leq n\}$$

generate two awfs (C, F_t) and (C_t, F) on **sSet** by Garner's algebraic small object argument [3]. A simplicial set X is a **Kan complex** if the unique map $X \longrightarrow \Delta^0$ satisfies the right lifting property with respect to J.

The sets I and J determine the cofibrations and fibrations in Quillen's model structure on **sSet**, which is a monoidal algebraic model category. The key technical step in the proof of this fact is that the 2-variable morphism

$$\begin{array}{ccc}
\mathbf{sSet} \times \mathbf{sSet} & \xrightarrow{-\hat{\times}-} & \mathbf{sSet} \\
(C, F_t) \times (C_t, F) & \longmapsto & (C_t, F)
\end{array}$$

induced from the cartesian product is part of a 2-variable adjunction of awfs.

The modern proof of the non-algebraic version of this result makes use of the closure properties of left classes of weak factorisation systems and is non-constructive; see [6]. This argument does not suffice to prove the algebraic statement. However, the classical constructive proof does suffice: the proof given in [21, Theorem 6.9] explicitly constructs the required lifts of $\mathbf{Hom}(A, X) \longrightarrow \Delta^0$ against J, supposing that similar lifts for $X \to \Delta^0$ are given. By the main result of [24], this argument shows that the 2-variable right adjoint

$$\begin{array}{ccc}
(\mathbf{sSet})^\bullet \times \mathbf{sSet} & \xrightarrow{\hat{\mathbf{Hom}}} & \mathbf{sSet} \\
(F_t, C) \times (C_t, F) & \longmapsto & (C_t, F)
\end{array}$$

defines a 2-variable adjunction of awfs. By our main theorem (Theorem 3.10) this is equivalent to the desired statement. See [24] for more details.

An important corollary of our main theorem in this context is the following result.

Theorem 4.6. *Multivariable adjunctions of awfs compose to yield new multivariable adjunctions of awfs.*

Proof. Multivariable colax comonad morphisms compose multicategorically. Using the notation of Definition 4.4, the composite is obtained by composing the F_i and the λ_i in the obvious way.

Now, by the relationship between the λ_i and the α_i, the composite of the λ_i is determined by the multicategorical composite of the α_i. So we check that these composites satisfy the mate condition required by the definition. This follows from Theorem 3.10. $\square$

While only 2-variable adjunctions of awfs are required to make the definition of a monoidal algebraic model category, the higher arity versions are useful in the following way. Enriched categories, functors, adjunctions, and 2-variable adjunctions over a closed symmetric monoidal category $\mathcal{V}$ can be encoded by an *a priori* unenriched tensor/hom/cotensor 2-variable adjunction together with coherence isomorphisms. These are isomorphisms between various composite 2-, 3- and 4-variable functors [25]. There are many equivalent ways to encode this data having to do with choices of left and right adjoints. Our main result allows a seamless translation between these equivalent formulations. Related considerations arise in homotopy theory where these arguments may be used to prove that the total derived functor of a $\mathcal{V}$-functor between $\mathcal{V}$-model categories admits a canonical enrichment over the homotopy category of $\mathcal{V}$.

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