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Variational Coupling of Wave Slamming against Elastic Masts

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1 Introduction

We present a novel approach to fluid-structure interactions (FSI) that preserves energy and phase-space structure owing to the variatonal and Hamiltonian techniques used. We posit a variational principle (VP), for nonlinear potential-flow wave dynamics coupled to a nonlinear hyperelastic mast, and derive its linearization. Both linear and nonlinear formulations can then be discretized in a classical-mechanical VP, using finite element expansions.

2 Nonlinear Variational Formulation

Potential flow water waves: We consider water as an incompressible fluid with density ρ. The vector velocity field $\mathbf{u} = \mathbf{u}(x, y, z, t)$ has zero divergence, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, with spatial coordinates $\mathbf{x} = (x, y, z)^T$, and time coordinate t. Gravity acts in the negative z–direction and the associated acceleration of gravity is g. The velocity is expressed in terms of a scalar velocity potential $\phi = \phi(x, y, z, t)$ such that $\mathbf{u} = \boldsymbol{\nabla}\phi$. In a 3D domain $[0, L_x] \times [0, L_y] \times [0, h(x, y, t)]$ with solid walls at $x = 0$ and $x = L_x$, $y = 0$ and $y = L_y$, and the flat bottom $z = 0$, Luke's [4] VP for potential-flow water waves reads

$$
\begin{aligned}
0 =& \delta \int_0^T \int_0^{L_x} \int_0^{L_y} \int_0^{h(x,y,t)} -\rho\partial_t\phi \,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y - \mathcal{H}\,\mathrm{d}t \\
\equiv& \delta \int_0^T \int_0^{L_x} \int_0^{L_y} \int_0^{h(x,y,t)} -\rho\Big(\partial_t\phi + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + g(z - H_0)\Big)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\ \mathrm{d}t
\end{aligned}
\tag{1}
$$

for a fluid of constant depth and the single-valued free surface at $z = h(x, y, t)$. Here $h = h(x, y, t)$ is the water depth and H_0 the rest-state water level. The energy or Hamiltonian $\mathcal{H}$ consists of the sum of kinetic and potential energies. We use integration by parts in time together with Gauss' law with outward normal $\hat{\mathbf{n}} = (-\boldsymbol{\nabla}h, 1)^T/\sqrt{1 + |\boldsymbol{\nabla}h|^2}$ at the free surface. The passive and constant air pressure is denoted by p_a. Then, variation of (1) yields

$$
\begin{aligned}
0 =& \int_0^T \int_0^{L_x} \int_0^{L_y} \int_0^{h(x,y,t)} \rho\,\nabla^2\phi\,\delta\phi\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x - \int_{\partial\Omega_w} \rho\boldsymbol{\nabla}\phi\cdot\hat{\mathbf{n}}\,\delta\phi\,dS \\
&+ \int_0^{L_x}\int_0^{L_y} \rho\big(-\partial_z\phi + \partial_x\phi\partial_x h + \partial_y\phi\partial_y h + \partial_t h\big)|_{z=h}\delta\phi|_{z=h} + (p - p_a)_{z=h}\,\delta h\,\mathrm{d}y\,\mathrm{d}x\ \mathrm{d}t,
\end{aligned}
\tag{2}
$$

in which the pressure $p - p_a$ here acts as a shorthand placeholder for the Bernoulli expression $-\rho\big(\partial_t\phi + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + g(z - H_0)\big)$. The equations of motion emerge from the above relation (2), together with no-normal-flow boundary conditions $\boldsymbol{\nabla}\phi\cdot\hat{\mathbf{n}} = 0$ with outward normal $\hat{\mathbf{n}}$ at solid walls $\partial\Omega_w$, see [4].

Geometrically nonlinear elastic mast. We consider a nonlinear hyperelastic model for an elastic material in which the geometric nonlinearity of the displacements is also taken into account. The constitutive law is such that, after linearization, it satisfies a linear Hooke's law. The choice of this model is guided by our goal to couple the potential-flow water-wave model to a weakly nonlinear elastic model.

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We first model the positions $\mathbf{X} = \mathbf{X}(a,b,c,t) = (X,Y,Z)^T = (X_1,X_2,X_3)^T$ of an infinitesimal 3D element of solid material as a function of Lagrangian coordinates $\mathbf{a} = (a,b,c)^T = (a_1,a_2,a_3)^T$ and time t. At time $t=0$ we take $\tilde{\mathbf{X}}(\mathbf{a},0) = \mathbf{a}$. The displacements $\tilde{\mathbf{X}}$ follow from the positions as $\tilde{\mathbf{X}} = \mathbf{X} - \mathbf{a}$. The velocity of the displacements is $\partial_t \tilde{\mathbf{X}} = \mathbf{U} = (U,V,W)^T = (U_1,U_2,U_3)^T$, where the displacement velocity $\mathbf{U} = \mathbf{U}(\mathbf{a},t)$ is again a function of Lagrangian coordinates $\mathbf{a}$ and time t. The variational formulation of the elastic material is close to the variational formulation of a linear elastic solid obeying Hooke's law, but with one difference: the material is Lagrangian with finite, rather than infinitesimal, displacements. The variational formulation then consists of the kinetic and potential energies in the Lagrangian framework. In the linear case, $\mathbf{X} = \mathbf{a} + \tilde{\mathbf{X}}(\mathbf{a},t) \approx \mathbf{a} + \tilde{\mathbf{X}}(\mathbf{x},t)$ since we take $\mathbf{a} = \mathbf{x}$ for small $\tilde{\mathbf{X}}$. The VP for the hyperelastic model is then as follows [2]

$$0 = \delta \int_0^T \iiint_{\Omega_0} \rho_0 \mathbf{U} \cdot \partial_t \mathbf{X} - \frac{1}{2}\rho_0 |\mathbf{U}|^2 - \rho_0 g Z - \frac{1}{2}\lambda [\mathrm{tr}(\mathbf{E})]^2 - \mu\, \mathrm{tr}(\mathbf{E}^2)\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}c\, \mathrm{d}t, \tag{3}$$

with $\rho_0 = \rho_0(\mathbf{a})$ and the Green-Lagrangian strain tensor $E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij}) = E_{ji}$ with $F_{ij} = \partial X_i / \partial a_j$. Evaluation of the variation in (3) yields

$$\begin{aligned} 0 = \delta \int_0^T \iiint_{\Omega_0} & \rho_0 \big(\partial_t \mathbf{X} - \mathbf{U}\big) \cdot \delta \mathbf{U} - \rho_0 \partial_t \mathbf{U} \cdot \delta \mathbf{X} - \rho_0 \delta_{l3} \delta X_l \\ & + \partial_{a_i} \big(\lambda \mathrm{tr}(\mathbf{E}) F_{li} + 2\mu\, E_{ki} F_{lk}\big) \delta X_l \, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}c \\ & - \iint_{\partial\Omega_0} n_i \big(\lambda\, \mathrm{tr}(\mathbf{E}) F_{li} + 2\mu\, E_{ki} F_{lk}\big) \delta X_l \, \mathrm{d}S\, \mathrm{d}t\,, \end{aligned} \tag{4}$$

in which we have used the temporal end-point conditions $\delta \mathbf{X}(0) = \delta \mathbf{X}(T) = 0$.

Given the arbitrariness of the respective variations, the resulting equations of motion become

$$\delta \mathbf{U}: \quad \partial_t \mathbf{X} = \mathbf{U} \qquad \text{in} \quad \Omega_0 \tag{5a}$$

$$\delta X_l: \quad \rho_0 \partial_t U_l = -\rho_0 g \delta_{3l} + \partial_{a_i} \big(\lambda \mathrm{tr}(\mathbf{E}) F_{li} + 2\mu E_{ki} F_{lk}\big) = -\rho_0 g \delta_{3l} + \partial_{a_i} T_{li} \qquad \text{in} \quad \Omega_0 \tag{5b}$$

$$\delta X_l: \quad 0 = n_i \big(\lambda\, \mathrm{tr}(\mathbf{E}) F_{li} + 2\mu\, E_{ki} F_{lk}\big) = n_i T_{li} \qquad \text{on} \quad \partial\Omega_0 \tag{5c}$$

with stress tensor $T_{li} = \lambda\, \mathrm{tr}(\mathbf{E}) F_{li} + 2\mu\, E_{ki} F_{lk}$.

Linearized elastic dynamics. Given that $\mathbf{X} = \mathbf{a} + \tilde{\mathbf{X}}$, we find that [3]

$$\mathbf{E} = \frac{1}{2}\Big(\big(\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{a}}\big)^T + \big(\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{a}}\big) \Big) + \frac{1}{2}\big(\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{a}}\big)^T \cdot \big(\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{a}}\big). \tag{6}$$

The linearization entails that $\mathbf{a} = \mathbf{x}$ such that $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}(\mathbf{x},t)$. To be precise, we define $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)^T = \mathbf{a} - \mathbf{x}$, Taylor expand around $\mathbf{a} = \mathbf{x}$ and use Taylor's remainder theorem to yield

$$\mathbf{X}(\mathbf{a},t) = \mathbf{a} + \tilde{\mathbf{X}}(\mathbf{a},t) = \mathbf{x} + \boldsymbol{\epsilon} + \tilde{\mathbf{X}}(\mathbf{x},t) + \boldsymbol{\epsilon}^T \frac{\partial \tilde{\mathbf{X}}}{\partial (\mathbf{x} + \boldsymbol{\epsilon})}\Big|_{\mathbf{x}+\boldsymbol{\epsilon} = \boldsymbol{\zeta}} \tag{7}$$

for $|\mathbf{a}| \leq \boldsymbol{\zeta} \leq |\mathbf{x}|$. Hence, we find that

$$\frac{\partial \mathbf{X}(\mathbf{a},t)}{\partial \mathbf{a}} = \mathbf{I} + \frac{\partial \tilde{\mathbf{X}}(\mathbf{a},t)}{\partial \mathbf{a}} = \mathbf{I} + \frac{\partial \tilde{\mathbf{X}}(\mathbf{x},t)}{\partial \mathbf{x}} + \mathcal{O}(\boldsymbol{\epsilon}) \quad \text{and} \quad \frac{\partial \tilde{\mathbf{X}}(\mathbf{a},t)}{\partial \mathbf{a}} = \frac{\partial \tilde{\mathbf{X}}(\mathbf{x},t)}{\partial \mathbf{x}} + \mathcal{O}(\boldsymbol{\epsilon}). \tag{8}$$

Consequently, the linearized version $\mathbf{e}$ of $\mathbf{E}$ is [3]

$$\mathbf{e} = \frac{1}{2}\Big(\big(\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{x}}\big)^T + \big(\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{x}}\big) \Big) \quad \text{or} \quad e_{ij} = \frac{1}{2}\big(\frac{\partial X_j}{\partial x_i} + \frac{\partial X_i}{\partial x_j}\big). \tag{9}$$

Moreover, $\mathrm{tr}(\mathbf{E})^2 = E_{ii}E_{jj} \approx e_{ii}e_{jj}$ and $\mathrm{tr}(\mathbf{E}\cdot\mathbf{E}) = E_{ij}^2 \approx e_{ij}^2$, whence the standard VP for linear elastodynamics emerges:

$$0 = \delta \int_0^T \iiint_{\Omega_0} \rho_0 \mathbf{U} \cdot \partial_t \tilde{\mathbf{X}} - \frac{1}{2}\rho_0 |\mathbf{U}|^2 - \rho_0 g Z - \frac{1}{2}\lambda e_{ii} e_{jj} - \mu e_{ij}^2 \, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}t. \tag{10}$$

In the limit of small displacements, the following approximations hold

$$\mathrm{tr}(\mathbf{E}) F_{li} = E_{jj} F_{li} \approx e_{jj} \delta_{li}, \quad E_{ki} F_{lk} \approx e_{ik} \delta_{lk} = e_{il}. \tag{11}$$

Either by linearizing (5) or taking the variation of (10), the linearized equations of motion emerge.

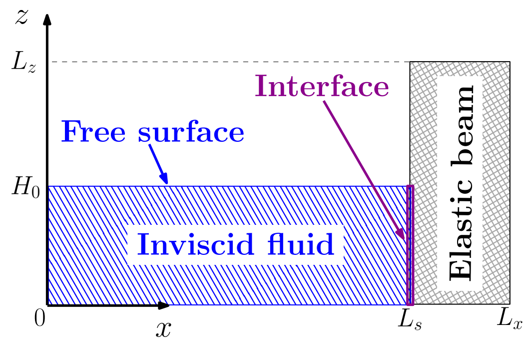

Figure 1: Geometry of the linearized or rest system: fluid (hatched) and elastic beam (cross-hatched).

2.1 Coupled Model

The domain occupied by the fluid is denoted by Ω and the domain occupied by the hyperelastic material by Ω_0. For simplicity we consider a block of hyperelastic material. The interface between the fluid and solid domain is parameterized by $\mathbf{X}_s = \mathbf{X}(L_s, b, c, t)$ and, at rest, $\mathbf{X} = \mathbf{a}$ for Cartesian $a \in [L_s, L_x],\ b \in [0, L_y],\ c \in [0, L_z]$, while the fluid domain at rest is $x \in [0, L_s],\ y \in [0, L_y],\ z \in [0, H_0]$. The (outward-from-fluid) normal at this interface $\mathbf{X}(L_s, b, c, t)$ with $b \in [0, L_y],\ c \in [0, L_z]$ is $\hat{\mathbf{n}} = \partial_b \mathbf{X} \times \partial_c \mathbf{X} / |\partial_b \mathbf{X} \times \partial_c \mathbf{X}|$. The nonlinear hyperelastic material is assumed to be stiff and nearly linear, such that at the interface $X \approx L_s,\ Y \approx b$ and $Z \approx c$, whence $\hat{\mathbf{n}} \approx (1, 0, 0)^T$. A sketch of this linearized domain or domain at rest is given in Fig. 1. This confirms that our expression is the outward normal to the fluid domain at the fluid-structure interface.

We assume that the elastic material is sufficiently stiff so that the fluid and elastic domains are

$$\Omega\colon z \in (0, h(x,y,t)),\ y \in (0, L_y),\ x \in (0, x_s(y,z,t));\quad \Omega_0\colon a \in (L_s, L_x),\ b \in (0, L_y),\ c \in (0, L_z), \tag{12}$$

where we remark that this is an implicit description of the fluid domain, because the waterline height z at the fluid-beam interface is defined by $z = h(x_s(y,z,t), y, t)$. We therefore introduce a new horizontal coordinate $\chi = L_s x / x_s(y,z,t)$ such that $\Omega:\ \chi \in (0, L_s),\ y \in (0, L_y),\ z \in (0, h(\chi, y, t))$. Alternatively, we can introduce $x_s(y,z,t) \circ \mathbf{X}_s(L_s, b, c, t) = X_s(L_s, b, c, t)$ as an unknown and use a Lagrange multiplier $\gamma = \gamma(b, c, t)$ to equate $x_s\big(y = Y(L_s, b, c, t), z = Z(L_s, b, c, t)\big)$ to $X(L_s, b, c, t)$.

As the coupled fluid-structure VP in $\{x, y, z, t\}$–coordinates, we take the sum of the two VPs

$$\begin{aligned}
0 =&\delta \iiint_\Omega -\rho\Big(\partial_t\phi + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + g(z - H_0)\Big)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_0^{L_y}\int_0^{L_z} \gamma\Big(x_s\big(Y(L_s,b,c,t), Z(L_s,b,c,t), t\big) - X(L_s,b,c,t)\Big)\,\mathrm{d}b\,\mathrm{d}c \\
&+ \iiint_{\Omega_0} \rho_0 \mathbf{U}\cdot\partial_t\mathbf{X} - \frac{1}{2}\rho_0|\mathbf{U}|^2 - \rho_0 g Z - \frac{1}{2}\lambda[\mathrm{tr}(\mathbf{E})]^2 - \mu\,\mathrm{tr}(\mathbf{E}^2)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}t.
\end{aligned} \tag{13}$$

For non-breaking waves, a coordinate change then becomes suitable, from coordinates $\{x, y, z, t\}$ to $\{\chi = L_s \frac{x}{x_s(y,z,t)}, y, z, t\}$ and fluid domain Ω. In these new coordinates, using transformation formulae, (13) becomes

$$\begin{aligned}
0 =&\delta \int_0^T\int_0^{L_s}\int_0^{L_y}\int_0^{h(\chi,y,t)} -\rho\Big(\frac{x_s}{L_s}\partial_t\phi - \frac{\chi}{L_s}\partial_t x_s \partial_\chi\phi \\
&+ \frac{1}{2}\frac{L_s}{x_s}(\partial_\chi\phi)^2 + \frac{1}{2}\frac{x_s}{L_s}\left(\partial_y\phi - \frac{\chi}{x_s}\partial_y x_s\partial_\chi\phi\right)^2 \\
&+ \frac{1}{2}\frac{x_s}{L_s}\left(\partial_z\phi - \frac{\chi}{x_s}\partial_z x_s\partial_\chi\phi\right)^2 + \frac{x_s}{L_s}g(z - H_0)\Big)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\chi \\
&+ \int_0^{L_y}\int_0^{L_z} \rho\gamma\Big(x_s\big(Y(L_s,b,c,t),\ Z(L_s,b,c,t), t\big) - X(L_s,b,c,t)\Big)\,\mathrm{d}b\,\mathrm{d}c \\
&+ \int_{L_s}^{L_x}\int_0^{L_y}\int_0^{L_z} \rho_0\mathbf{U}\cdot\partial_t\mathbf{X} - \frac{1}{2}\rho_0|\mathbf{U}|^2 - \rho_0 g Z - \frac{1}{2}\lambda[\mathrm{tr}(\mathbf{E})]^2 - \mu\,\mathrm{tr}(\mathbf{E}^2)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}t.
\end{aligned} \tag{14}$$

3 Linearized Wave-Beam Dynamics for FSI

We linearize (14) around a state of rest. Small-amplitude perturbations around this rest state are denoted by tilded variables and introduced as follows

$$x_s = L_s + \tilde{x}_s,\ \phi = 0 + \phi,\ h = H_0 + \eta,\ \mathbf{X} = \mathbf{x} + \tilde{\mathbf{X}},\ \mathbf{U} = \mathbf{0} + \tilde{\mathbf{U}}\,. \tag{15}$$

After linearizing (14), we obtain the VP

$$\begin{aligned}
0 =& \delta \int_0^T \int_0^{L_y} \int_0^{H_0} \rho \phi_w \partial_t \tilde{x}_s \,\mathrm{d}y\,\mathrm{d}z - \int_0^{L_s} \int_0^{L_y} \int_0^{H_0} \rho\Big(\frac{1}{2}(\partial_\chi \phi)^2 + \frac{1}{2}(\partial_y \phi)^2 + \frac{1}{2}(\partial_z \phi)^2\Big)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\chi \\
&+ \int_0^{L_y} \int_0^{L_s} \rho \phi_s \partial_t \eta - \frac{1}{2}\rho g \eta^2 \,\mathrm{d}y\,\mathrm{d}\chi + \int_0^{L_y} \int_0^{H_0} \rho\gamma\Big(\tilde{x}_s(y,z,t) - \tilde{X}(L_s,y,z,t)\Big)\,\mathrm{d}y\,\mathrm{d}z \\
&+ \int_{L_s}^{L_x} \int_0^{L_y} \int_0^{L_z} \rho_0 \tilde{\mathbf{U}} \cdot \partial_t \tilde{\mathbf{X}} - \frac{1}{2}\rho_0 |\tilde{\mathbf{U}}|^2 - \frac{1}{2}\lambda e_{ii} e_{jj} - \mu e_{ij}^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t,
\end{aligned} \tag{16}$$

which, being the variation of a quadratic form, yields the dynamics linearized around a state of rest with $\phi_w = \phi(L_s, y, z, t)$ and $\phi_s = \phi(\chi, y, H_0, t)$. After using $\delta\tilde{\mathbf{X}}(\mathbf{x}, 0) = \delta\tilde{\mathbf{X}}(\mathbf{x}, T) = 0$ and $\delta\eta(\chi, y, 0) = \delta\eta(\chi, y, T) = 0$, the variation in (16) yields

$$\delta\phi_w: \quad \rho\partial_t \tilde{x}_s = \rho\partial_\chi \phi \quad \text{at} \quad \chi = L_s, \quad \delta\tilde{x}_s: \quad \rho\partial_t \phi_w = \gamma \quad \text{at} \quad \chi = L_s \tag{17a}$$

$$\delta\gamma: \quad \tilde{x}_s(y,z,t) = \tilde{X}(L_s,y,z,t), \quad \delta\tilde{X}_j(L_s,y,z,t): \quad \gamma\delta_{1j} = T_{1j} \quad \text{at} \quad x = L_s \tag{17b}$$

$$\delta\phi_s: \quad \partial_t \eta = \partial_z \phi \quad \text{at} \quad z = H_0, \quad \delta\eta: \quad \partial_t \phi_s = -g\eta \quad \text{at} \quad z = H_0 \tag{17c}$$

$$\delta\phi: \quad (\partial_{\chi\chi} + \partial_{yy} + \partial_{zz})\phi = 0 \quad \text{in} \quad \bar{\Omega} \tag{17d}$$

$$\delta\tilde{\mathbf{U}}: \quad \partial_t \tilde{\mathbf{X}} = \tilde{\mathbf{U}} \quad \text{in} \quad \bar{\Omega}_0, \quad \delta\tilde{\mathbf{X}}: \quad \partial_t \tilde{\mathbf{U}}_j = \boldsymbol{\nabla}_k T_{jk} \quad \text{in} \quad \bar{\Omega}_0 \tag{17e}$$

with $\bar{\Omega}_0\colon x \in [L_s, L_x]$, $y \in [0, L_y]$, $z \in [0, L_z]$ and $\bar{\Omega}\colon \chi \in [0, L_s]$, $y \in [0, L_y]$, $z \in [0, H_0]$. Note that we replaced Lagrangian coordinates (a, b, c) by (x, y, z) in the linearization but kept coordinates (χ, y, z) in the fluid domain.

4 Discussion and Conclusion

After elimination of the Lagrange multiplier γ, the system (17) of linearized water-wave dynamics coupled to an elastic beam, *i.e.*, a system of linearized fluid-structure interaction (FSI) equations, is equivalent to the FSI with *ad hoc* coupling derived in [5], in which the linear equations are discretized using dis/continuous variational finite element methods, employing techniques from [1], leading to fully coupled and stable FSI with overall energy conservation, *i.e.*, without any energy loss between the subsystems. We shall also present these results. The numerical extension of these FSI to the nonlinear realm is planned as future research.

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