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RATIONAL CURVES ON SMOOTH CUBIC HYPERSURFACES OVER FINITE FIELDS

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ABSTRACT. Let k be a finite field with characteristic exceeding 3. We prove that the space of rational curves of fixed degree on any smooth cubic hypersurface over k with dimension at least 11 is irreducible and of the expected dimension.

1. INTRODUCTION

The geometry of a variety is intimately linked to the geometry of the space of rational curves on it. Given a field k and a projective variety X defined over k , a natural object to study is the moduli space of k -rational curves on X . There are many results in the literature establishing the irreducibility of such mapping spaces, but most such statements are only proved for generic X , there being relatively few results which are valid for *all* X in a family. The aim of this paper is to prove such a result for all smooth cubic hypersurfaces of large enough dimension which are defined over a finite field of characteristic exceeding 3.

Suppose that $k = \mathbb{C}$ and $X \subset \mathbb{P}_{\mathbb{C}}^{n-1}$ is a smooth cubic hypersurface with $n \geq 6$. Let $\overline{\text{Mor}}_d(\mathbb{P}_{\mathbb{C}}^1, X)$ be the Kontsevich moduli space of rational curves of degree d on X . Then it has been shown by Coskun and Starr [2] that $\overline{\text{Mor}}_d(\mathbb{P}_{\mathbb{C}}^1, X)$ is irreducible and of the expected dimension $d(n-3) + n - 5$. We would like to prove a similar result when $k = \mathbb{F}_q$ is a finite field with q elements and $X \subset \mathbb{P}_{\mathbb{F}_q}^{n-1}$ is a smooth cubic hypersurface defined over it. Rather than working with $\overline{\text{Mor}}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)$, which corresponds to “unparametrized” maps, we will study the moduli space $\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)$ of actual maps (see §2 for its construction). The expected dimension of $\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)$ is

$$D(d, n) = d(n - 3) + n - 2, \tag{1.1}$$

since $\mathbb{P}_{\mathbb{F}_q}^1$ has automorphism group of dimension 3.

For a smooth cubic hypersurface $X \subset \mathbb{P}_{\mathbb{F}_q}^{n-1}$, the Lang–Tsen theorem (see [3, Thm. 3.6]) ensures that $X(\mathbb{F}_q(t)) \neq \emptyset$ as soon as $n \geq 10$, in which case X

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contains a rational curve defined over \mathbb{F}_q . One can go further if one enlarges the size of the finite field. Let $n \geq 4$. Then, according to Kollár [6, Example 7.6], there exists a constant c_n depending only on n such that for any $q > c_n$ and any point $x \in X(\mathbb{F}_q)$, the cubic hypersurface X contains a rational curve (of degree at most 216) which is defined over \mathbb{F}_q and passes through x .

Following a suggestion of Ellenberg and Venkatesh, Pugin developed an “algebraic circle method” in his 2011 Ph.D. thesis [7] to study the spaces $\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)$. Thus, when $n \geq 13$ and $X \subset \mathbb{P}_{\mathbb{F}_q}^{n-1}$ is the diagonal cubic hypersurface

$$a_1x_1^3 + \cdots + a_nx_n^3 = 0, \quad (\text{for } a_1, \dots, a_n \in \mathbb{F}_q^*),$$

he succeeds in showing that the associated moduli space $\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)$ is irreducible and of the expected dimension $D(d, n)$, provided that $\text{char}(\mathbb{F}_q) \neq 3$. Our main result extends Pugin’s result to non-diagonal hypersurfaces, as follows.

Theorem 1.1. *Let $\text{char}(\mathbb{F}_q) > 3$ and let $X \subset \mathbb{P}_{\mathbb{F}_q}^{n-1}$ be a smooth cubic hypersurface defined over \mathbb{F}_q , with $n \geq 13$. Then for each $d \geq 1$ the moduli space $\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)$ is irreducible and of dimension $D(d, n)$.*

Inspired by Pugin’s approach, our proof of this result rests on an estimate for $\#\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)(\mathbb{F}_q)$, as $q \rightarrow \infty$. The cardinality of \mathbb{F}_q -points on $\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)$ is roughly equal to the number of $\mathbb{F}_q(t)$ -points on X of degree d . We shall access the latter quantity through a function field version of the Hardy–Littlewood circle method. The traditional setting for this is a fixed finite field \mathbb{F}_q , with the goal being to understand the $\mathbb{F}_q(t)$ -points on X of degree d , as $d \rightarrow \infty$. In contrast to this, Theorem 1.1 requires us to handle any fixed $d \geq 1$, as $q \rightarrow \infty$. The key ingredients will be drawn from work of Lee [4] on a $\mathbb{F}_q(t)$ version of Birch’s work on systems of forms in many variables and our own recent contribution to the subject [1], which is specific to cubic forms. Perhaps the chief interest of Theorem 1.1 lies in the fact that a result in algebraic geometry can be proved using methods of analytic number theory.

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2. FROM MODULI SPACES TO COUNTING

Let k be a field and let $X \subset \mathbb{P}_k^{n-1}$ be a hypersurface cut out by an equation $F = 0$, where $F \in k[x_1, \dots, x_n]$ is a homogeneous cubic polynomial. Let $G_d(k)$ be the set of all homogeneous polynomials in u, v of degree $d \geq 1$,

with coefficients in k . A *rational curve* on X is a non-constant morphism $f : \mathbb{P}_k^1 \rightarrow X$. A morphism of degree d is given by

$$f = (f_1(u, v), \dots, f_n(u, v)),$$

with $f_1, \dots, f_n \in G_d(k)$, with no non-constant common factor in $k[u, v]$, such that $F(f_1(u, v), \dots, f_n(u, v))$ is identically zero. Using the coefficients of f_1, \dots, f_n we can regard f as a point in $\mathbb{P}_k^{n(d+1)-1}$. The morphisms of degree d on X are parameterised by $\text{Mor}_d(\mathbb{P}_k^1, X)$, which is an open subvariety of $\mathbb{P}_k^{n(d+1)-1}$ cut out by a system of $3d+1$ equations of degree 3. This directly leads to the naive expectation that $\text{Mor}_d(\mathbb{P}_k^1, X)$ should have dimension

$$n(d+1) - 1 - (3d+1) = D(d, n),$$

in the notation of (1.1). The complement to $\text{Mor}_d(\mathbb{P}_k^1, X)$ in its closure is the set of (f_1, \dots, f_n) with a common zero. We can obtain explicit equations by noting that f_1, \dots, f_n have a common zero if and only if the resultant $\text{Res}(\sum_i \lambda_i f_i, \sum_j \mu_j f_j)$ is identically zero as a polynomial in λ_i, μ_j . This gives a system of equations of degree $2d$ in the coefficients of f_1, \dots, f_n .

Now let $k = \mathbb{F}_q$ with $\text{char}(\mathbb{F}_q) > 3$ in the above discussion. Assuming that $d \geq 1$ and $n \geq 13$ we need to show that $\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)$ is irreducible and of dimension $D(d, n)$. We note that $\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)$ is also defined over any finite extension \mathbb{F}_{q^ℓ} of \mathbb{F}_q . Following Pugin's approach [7], our proof of Theorem 1.1 relies on estimating $\#\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)(\mathbb{F}_{q^\ell})$, as $\ell \rightarrow \infty$. According to Kollár [5, Thm. II.1.2/3], all irreducible components of $\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)$ have dimension at least $D(d, n)$. Hence, in view of the Lang–Weil estimate, Theorem 1.1 is a direct consequence of the following result.

Theorem 2.1. *Let $\text{char}(\mathbb{F}_q) > 3$ and let $X \subset \mathbb{P}_{\mathbb{F}_q}^{n-1}$ be a smooth cubic hypersurface defined over \mathbb{F}_q , with $n \geq 13$. Then for each $d \geq 1$ we have*

$$\lim_{\ell \rightarrow \infty} q^{-\ell D(d, n)} \#\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)(\mathbb{F}_{q^\ell}) \leq 1.$$

We henceforth redefine q^ℓ to be q . Our proof of Theorem 2.1 is based on the Hardy–Littlewood circle method over the function field $\mathbb{F}_q(t)$, always under the assumption that $\text{char}(\mathbb{F}_q) > 3$. The main input comes from our previous work [1] and a straightforward adaptation of work due to Lee [4]. We will adhere to the notation described in [1, §2.1 and §2.2] without further comment.

Assume that $F(\mathbf{x}) = \sum_{\mathbf{i}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$, with variables $\mathbf{x} = (x_1, \dots, x_n)$ and coefficients $a_{\mathbf{i}} \in \mathbb{F}_q$. In particular the height H_F and discriminant Δ_F of F satisfy

$$H_F = \max_{\mathbf{i}} |a_{\mathbf{i}}| = 1 \quad \text{and} \quad |\Delta_F| = 1.$$

We will make frequent use of these facts in what follows. To establish Theorem 2.1 we work with the naive space

$$M_d = \{\mathbf{x} = (x_1, \dots, x_n) \in G_d(\mathbb{F}_q)^n \setminus \{\mathbf{0}\} : F(\mathbf{x}) = 0\},$$

which corresponds to the \mathbb{F}_q -points on the affine cone of $\text{Mor}_d(\mathbb{P}_{\mathbb{F}_q}^1, X)$. Let us set

$$E(d, n) = D(d, n) + 1 = (n - 3)(d + 1) + 2.$$

It will clearly suffice to show that

$$\lim_{q \rightarrow \infty} q^{-E(d, n)} \#M_d \leq 1, \quad (2.1)$$

for $n \geq 13$. We proceed by relating the counting function $\#M_d$ to the counting function that lies at the heart of our earlier investigation [1].

Let $w : K_\infty^n \rightarrow \{0, 1\}$ be given by $w(\mathbf{x}) = \prod_{1 \leq i \leq n} w_\infty(x_i)$, where

$$w_\infty(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Putting $P = t^{d+1}$, we then have $\#M_d \leq N(P)$, where

$$N(P) = \sum_{\substack{\mathbf{x} \in \mathcal{O}^n \\ F(\mathbf{x})=0}} w(\mathbf{x}/P). \quad (2.2)$$

It follows from [1, Eq. (4.1)] that for any $Q \geq 1$ we have

$$N(P) = \sum_{\substack{r \in \mathcal{O} \\ |r| \leq \widehat{Q} \\ r \text{ monic}}} \sum_{|a| < |r|}^* \int_{|\theta| < |r|^{-1} \widehat{Q}^{-1}} S\left(\frac{a}{r} + \theta\right) d\theta, \quad (2.3)$$

where \sum^* means that the sum is taken over residue classes $|a| < |r|$ for which $(a, r) = 1$, and where

$$S(\alpha) = \sum_{\mathbf{x} \in \mathcal{O}^n} \psi(\alpha F(\mathbf{x})) w(\mathbf{x}/P), \quad (2.4)$$

for any $\alpha \in \mathbb{T}$. We will work with the choice $Q = 3(d+1)/2$, so that $\widehat{Q} = |P|^{3/2}$.

We henceforth set

$$\delta = \frac{3}{d+1}.$$

Let $A(P)$ denote the contribution to $N(P)$ in (2.3) from values of r, θ such that either $|\theta| < \widehat{Q}^{-4}$, or else $r = 1$ and $|\theta| < |P|^{-3+\delta}$.

Lemma 2.2. *We have $\lim_{q \rightarrow \infty} q^{-E(d, n)} A(P) = 1$.*

Proof. Let us put $A_1(P)$ for the contribution from $r = 1$ and $|\theta| < |P|^{-3+\delta}$, and $A_2(P)$ for the remaining contribution. Taking the trivial bound $|S(\alpha)| \leq |P|^n$, it is easy to check that $\lim_{q \rightarrow \infty} q^{-E(d,n)} A_2(P) = 0$ and so our attention shifts to $A_1(P)$. For this we invoke [1, Lemma 2.2], which gives

$$\begin{aligned} A_1(P) &= \int_{|\theta| < |P|^{-3+\delta}} S(\theta) d\theta \\ &= |P|^{-3+\delta} \# \{ \mathbf{x} \in \mathcal{O}^n : |\mathbf{x}| < |P|, |F(\mathbf{x})| < |P|^{3-\delta} \}. \end{aligned}$$

Note that our choice of δ implies that $|P|^{3-\delta} = q^{3(d+1)-3} = q^{3d}$ and so this result is applicable since $3d$ is an integer. Any \mathbf{x} to be counted is an n -tuple of polynomials with j th component $x_j = a_{0,j}t^d + \cdots + a_{d,j}$ for coefficients $a_{i,j} \in \mathbb{F}_q$. The condition $|F(\mathbf{x})| < |P|^{3-\delta}$ is therefore equivalent to the condition $F(a_{0,1}, \dots, a_{0,n}) = 0$. Since F is non-singular it is certainly absolutely irreducible over \mathbb{F}_q . Thus the Lang–Weil estimate implies that the total number of available \mathbf{x} is $q^{dn+n-1}(1+O_n(q^{-1/2}))$, where the implied constant depends only on n . Thus

$$A_1(P) = q^{-3d+dn+n-1}(1+O_n(q^{-1/2})),$$

from which the statement of the lemma follows. \square

Let us put $B(P)$ for the contribution to $N(P)$ in (2.3) from values of r, θ with $|\theta| \geq \widehat{Q}^{-4}$, such that either $|r| > 1$, or else $r = 1$ and $|\theta| \geq |P|^{-3+\delta}$. The remainder of this paper is devoted to a proof of the following result.

Lemma 2.3. *We have $\lim_{q \rightarrow \infty} q^{-E(d,n)} B(P) = 0$ for $n \geq 13$.*

Recalling that $\#M_d \leq A(P) + B(P)$, we see that (2.1) follows from Lemmas 2.2 and 2.3. Thus it remains to prove Lemma 2.3 in order to complete the proof of Theorem 2.1.

In our analysis of $B(P)$ it will be convenient to sort the sum according to the size of $|r|$ and $|\theta|$. Consequently, we let $S(d)$ denote the set of $(Y, \Theta) \in \mathbb{Z}^2$ such that

$$0 \leq Y \leq Q \quad \text{and} \quad -4Q \leq \Theta < -(Y + Q),$$

with either $Y \geq 1$, or else $Y = 0$ and $\widehat{\Theta} \geq |P|^{-3+\delta}$. In particular it is clear that $\#S(d) \leq 7(d+1) = c_d$, say. We then have

$$B(P) \leq \sum_{(Y, \Theta) \in S(d)} |N(P, Y, \Theta)| \leq c_d \max_{(Y, \Theta) \in S(d)} |N(P, Y, \Theta)|,$$

where

$$N(P, Y, \Theta) = \sum_{\substack{r \in \mathcal{O} \\ |r| = \widehat{Y} \\ r \text{ monic}}} \sum_{|a| < |r|}^* \int_{|\theta| = \widehat{\Theta}} S\left(\frac{a}{r} + \theta\right) d\theta. \quad (2.5)$$

We will use two basic methods for analysing $N(P, Y, \Theta)$.

Let

$$S_1(d) = \{(Y, \Theta) \in S(d) : Y \geq 1 \text{ and } \Theta \leq (n/6 - 4/3)Y - 2Q\}.$$

For (Y, Θ) belonging to this set we will apply our previous work [1], which is founded on Poisson summation. This is the object of §3. Alternatively, in §4, we will use a function field version of Weyl differencing to handle (Y, Θ) belonging to the set

$$S_2(d) = \{(Y, \Theta) \in S(d) : \text{If } Y \geq 1 \text{ then } \Theta > (n/6 - 4/3)Y - 2Q\}.$$

This part of the argument is essentially due to Lee [4]. It will be convenient to set

$$B_i(P) = \max_{(Y, \Theta) \in S_i(d)} |N(P, Y, \Theta)|, \quad \text{for } i = 1, 2,$$

so that $B(P) \leq c_d \{B_1(P) + B_2(P)\}$. Assuming that $n \geq 13$, it now suffices to show that $\lim_{q \rightarrow \infty} q^{-E(d, n)} B_i(P) = 0$ for $i = 1, 2$.

3. POISSON SUMMATION

The counting function (2.2) is equal to the counting function $N(P)$ considered in [1, §4] with $M = 1$ and $\mathbf{b} = \mathbf{0}$. (Equivalently this is [1, Eq. (7.4)] with $M = 1$, $\mathbf{b} = \mathbf{0}$, $L = 0$ and $\mathbf{x}_0 = \mathbf{0}$.) Throughout this section we shall assume that the cubic form F has $n \geq 13$ variables. The main part of [1] is actually concerned with non-singular cubic forms in only $n \geq 8$ variables. Intrinsic to the success of this endeavour is the choice of counting function, in which $\mathbb{F}_q(t)$ -solutions are singled out for consideration if they are sufficiently close to a conveniently chosen solution over K_∞ . The fact that we must consider all $\mathbb{F}_q(t)$ -solutions in (2.2) directly accounts for this loss of precision.

Let $J(\Theta) = \max\{1, \widehat{\Theta}|P|^3\}$. Appealing to [1, Lemma 7.2], we find that

$$N(P, Y, \Theta) = |P|^n \sum_{\substack{r \in \mathcal{O} \\ |r| = \widehat{Y} \\ r \text{ monic}}} |r|^{-n} \int_{|\theta| = \widehat{\Theta}} \sum_{\substack{\mathbf{c} \in \mathcal{O}^n \\ |\mathbf{c}| \leq \widehat{C}}} S_r(\mathbf{c}) I_r(\theta; \mathbf{c}) d\theta,$$

where $\widehat{C} = \widehat{Y}|P|^{-1}J(\Theta)$ and

$$S_r(\mathbf{c}) = \sum_{|a| < |r|}^* \sum_{\substack{\mathbf{y} \in \mathcal{O}^n \\ |\mathbf{y}| < |r|}} \psi \left(\frac{aF(\mathbf{y}) - \mathbf{c} \cdot \mathbf{y}}{r} \right),$$

$$I_r(\theta; \mathbf{c}) = \int_{K_\infty^n} w(\mathbf{x}) \psi \left(\theta P^3 F(\mathbf{x}) + \frac{P\mathbf{c} \cdot \mathbf{x}}{r} \right) d\mathbf{x}.$$

It will be convenient to put $\gamma = \theta P^3$ in $I_r(\theta; \mathbf{c})$. The definition of w implies that the integral is over \mathbb{T}^n , whence an application of [1, Lemma 2.7] shows that

$$|I_r(\theta; \mathbf{c})| \leq \text{meas}\{\mathbf{x} \in \mathbb{T}^n : |\gamma \nabla F(\mathbf{x}) + r^{-1} P \mathbf{c}| \leq \max\{1, |\gamma|^{1/2}\}\} = J(\Theta)^{1/2}.$$

The exponential sum $S_r(\mathbf{c})$ is a multiplicative function of r by [1, Lemma 4.5]. We will adopt the notation conceived in [1, Definition 4.6], so that associated to any $r \in \mathcal{O}$ and $i \in \mathbb{Z}_{>0}$ are the elements

$$b_i = \prod_{\varpi^i \parallel r} \varpi^i \quad \text{and} \quad r_i = \prod_{\substack{\varpi^e \parallel r \\ e \geq i}} \varpi^e.$$

Applying [1, Lemma 5.1], we therefore find that there exists a constant $A_n > 0$ depending only on n such that

$$\sum_{\substack{\mathbf{c} \in \mathcal{O}^n \\ |\mathbf{c}| \leq \widehat{C}}} |S_r(\mathbf{c}) I_r(\theta; \mathbf{c})| \leq A_n^{\omega(b_1 b_2)} |b_1 b_2|^{n/2+1} \int_{\mathbb{T}^n} \sum_{\mathbf{c} \in \mathcal{C}(\mathbf{x})} |S_{r_3}(\mathbf{c})| d\mathbf{x},$$

where

$$\mathcal{C}(\mathbf{x}) = \left\{ \mathbf{c} \in \mathcal{O}^n : |\mathbf{c} + r \theta P^2 \nabla F(\mathbf{x})| \leq |P|^{-1} \widehat{Y} J(\Theta)^{1/2} \right\}.$$

It now follows from [1, Lemma 6.4] that for any $\varepsilon > 0$ there is a constant $c_{n,\varepsilon} > 0$, depending only on n and ε , such that

$$\sum_{\mathbf{c} \in \mathcal{C}(\mathbf{x})} |S_{r_3}(\mathbf{c})| \leq c_{n,\varepsilon} |r_3|^{n/2+1+\varepsilon} \left(|r_3|^{n/3} + \frac{\widehat{Y}^n J(\Theta)^{n/2}}{|P|^n} \right).$$

According to [1, Lemma 2.2] we have

$$\int_{|\theta|=\widehat{\Theta}} d\theta = \widehat{\Theta + 1} - \widehat{\Theta} \leq \widehat{\Theta + 1}.$$

Hence, on integrating trivially over \mathbf{x} and then over θ , we deduce the existence of a constant $c_{n,\varepsilon} > 0$ such that

$$\begin{aligned} & \frac{|P|^n}{|r|^n} \int_{|\theta|=\widehat{\Theta}} \sum_{\substack{\mathbf{c} \in \mathcal{O}^n \\ |\mathbf{c}| \leq \widehat{C}}} |S_r(\mathbf{c}) I_r(\theta; \mathbf{c})| d\theta \\ & \leq c_{n,\varepsilon} \widehat{Y}^{n/2+1+\varepsilon} \widehat{\Theta + 1} \left(\frac{|r_3|^{n/3} |P|^n}{\widehat{Y}^n} + J(\Theta)^{n/2} \right). \end{aligned}$$

It remains to sum this over all monic $r \in \mathcal{O}$ such that $|r| = \widehat{Y}$, of which there are precisely \widehat{Y} . For this we note that

$$\sum_{\substack{r \in \mathcal{O} \\ |r| = \widehat{Y} \\ r \text{ monic}}} |r_3|^{n/3} \leq \widehat{Y}^{n/3} \sum_{\substack{r = b_1 b_2 r_3 \in \mathcal{O} \\ |r| = \widehat{Y} \\ r \text{ monic}}} \frac{1}{|b_1 b_2|^{n/3}} \leq c_n \widehat{Y}^{n/3+1/3},$$

for an appropriate constant $c_n > 0$ such that there are at most $c_n \widehat{Y}^{1/3}$ values of $|r_3| \leq \widehat{Y}$. Recalling that $Y \leq Q$ and $\Theta < -(Y + Q)$, we easily deduce that

$$J(\Theta)^{n/2} \leq \max \left\{ 1, \frac{|P|^3}{\widehat{Y}\widehat{Q}} \right\}^{n/2} = \frac{\widehat{Q}^{n/2}}{\widehat{Y}^{n/2}}.$$

Hence there is a constant $c_{n,\varepsilon} > 0$ such that

$$|N(P, Y, \Theta)| \leq c_{n,\varepsilon} \widehat{Y}^{n/2+1+\varepsilon} \widehat{\Theta + 1} \left(\frac{\widehat{Y}^{n/3+1/3} |P|^n}{\widehat{Y}^n} + \frac{\widehat{Q}^{n/2}}{\widehat{Y}^{n/2-1}} \right),$$

whence in fact

$$|N(P, Y, \Theta)| \leq c_{n,\varepsilon} \widehat{\Theta + 1} \left\{ \frac{|P|^n}{\widehat{Y}^{n/6-4/3-\varepsilon}} + \widehat{Y}^2 \widehat{Q}^{n/2+\varepsilon} \right\}.$$

Taking $\widehat{\Theta + 1} \leq \widehat{Y}^{-1} \widehat{Q}^{-1}$ we see that the second term is at most

$$c_{n,\varepsilon} \widehat{Y} \widehat{Q}^{n/2-1+\varepsilon} \leq c_{n,\varepsilon} \widehat{Q}^{n/2+\varepsilon} \leq c_{n,\varepsilon} |P|^{3n/4+2\varepsilon}.$$

But we also have $\widehat{\Theta + 1} \leq q \widehat{Y}^{n/6-4/3} / \widehat{Q}^2$ for any $(Y, \Theta) \in S_1(d)$, whence

$$B_1(P) \leq c_{n,\varepsilon} \{ q |P|^{n-3+2\varepsilon} + |P|^{3n/4+2\varepsilon} \}.$$

Assuming that $\varepsilon > 0$ is taken to be sufficiently small in term of d , it easily follows that $\lim_{q \rightarrow \infty} q^{-E(d,n)} B_1(P) = 0$ for $n \geq 13$.

4. WEYL DIFFERENCING

The goal of this section is to show that $\lim_{q \rightarrow \infty} q^{-E(d,n)} B_2(P) = 0$ for $n \geq 13$. Our starting point is an analysis of the exponential sum (2.4), for which we will use the function field version of Birch's Weyl differencing that was worked out by Lee [4]. Our task is to make the dependence on q completely explicit, but the argument is very standard and so we shall be brief where possible. Since we are only concerned with cubic forms one needs to take $R = 1$ and $d = 3$ in Lee's work [4, §3]. As usual we will assume that $\text{char}(\mathbb{F}_q) > 3$.

Define the Hessian matrix

$$\mathbf{H}(\mathbf{x}) = \left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq n}$$

that is associated to our cubic form F . For any $\beta = \sum_{-\infty < i \leq N} b_i t^i \in K_\infty$, we let $\|\beta\| = |\sum_{-\infty < i < 0} b_i t^i|$. Beginning with an application of [4, Cor. 3.3], it follows that

$$|S(\alpha)|^4 \leq |P|^{2n} \# \{ \mathbf{u}, \mathbf{v} \in \mathcal{O}^n : |\mathbf{u}|, |\mathbf{v}| < |P|, \|\alpha \mathbf{H}(\mathbf{u}) \mathbf{v}\| < |P|^{-1} \}.$$

for any $\alpha \in \mathbb{T}$. We are only interested in values of α with rational approximation $\alpha = a/r + \theta$, where $|r| = \widehat{Y}$ and $|\theta| = \widehat{\Theta}$ for $(Y, \Theta) \in S_2(d)$. We recall here, for the sake of convenience, that this means

$$1 \leq \widehat{Y} \leq \widehat{Q} \quad \text{and} \quad \widehat{\Theta} < \frac{1}{\widehat{Y}\widehat{Q}},$$

with either $\widehat{Y} \geq q$ and $\widehat{\Theta} > \widehat{Y}^{n/6-4/3}/\widehat{Q}^2$, or else $\widehat{Y} = 1$ and $\widehat{\Theta} \geq |P|^{-3+\delta}$. In either case we therefore have $\widehat{\Theta} > \widehat{Y}^{n/6-4/3}/\widehat{Q}^2$. We note that $S_2(d)$ is non-empty only when $\widehat{Y} < |P|^{9/(n-2)}$, which we now assume.

The next stage in the analysis of $S(\alpha)$ is a double application of the function field analogue of Davenport's "shrinking lemma", as proved in [4, Lemma 3.4]. Let $\Gamma = (\gamma_{ij})$ be a symmetric $n \times n$ matrix with entries in K_∞ . For $1 \leq i \leq n$ we introduce the linear forms

$$L_i(u_1, \dots, u_n) = \sum_{j=1}^n \gamma_{ij} u_j. \quad (4.1)$$

Next, for given real numbers a, Z , we let $N(a, Z)$ denote the number of vectors $(u_1, \dots, u_{2n}) \in \mathcal{O}^{2n}$ such that

$$|u_j| < \widehat{a}\widehat{Z} \quad \text{and} \quad |L_j(u_1, \dots, u_n) + u_{j+n}| < \frac{\widehat{Z}}{\widehat{a}} \quad \text{for } 1 \leq j \leq n.$$

In due course we will adapt the argument of [4, Lemma 3.4] to show that for any $a, Z_1, Z_2 \in \mathbb{R}$ with $Z_1 \leq Z_2 \leq 0$, we have

$$\frac{N(a, Z_1)}{N(a, Z_2)} \geq \widehat{K}^n, \quad (4.2)$$

where $K = [Z_1 - \{a\}] - [Z_2 + \{a\}]$ and $\{a\}$ denotes the fractional part of a .

Taking this on faith for the moment, let Z be such that

$$\widehat{Z} = \sqrt{\widehat{Y}\widehat{\Theta}|P|}.$$

Our assumptions on Y, Θ easily imply that $\widehat{Z} \leq 1$ and $Z \in \frac{1}{2}\mathbb{Z}$. We may therefore apply the shrinking lemma first with $(\widehat{a}, \widehat{Z}_1, \widehat{Z}_2) = (|P|, \widehat{Z}, 1)$. This allows us to take $K \geq Z_1$ in (4.2). Next we apply the lemma a second time with $(\widehat{a}, \widehat{Z}_1, \widehat{Z}_2) = (\widehat{Z}^{-1/2}|P|, \widehat{Z}^{3/2}, \widehat{Z}^{1/2})$. We may write $Z/2 = N + k/4$ for some integer N and $k \in \{0, 1, 2, 3\}$. Thus

$$[Z_1 - \{a\}] - [Z_2 + \{a\}] = (3N + k) - N = 2N + k \geq Z_1 - Z_2.$$

This therefore implies

$$|S(\alpha)|^4 \leq \frac{|P|^{2n}}{\widehat{Z}^{2n}} \# \left\{ \mathbf{u}, \mathbf{v} \in \mathcal{O}^n : |\mathbf{u}|, |\mathbf{v}| < \widehat{Z}|P|, \|\alpha \mathbf{H}(\mathbf{u})\mathbf{v}\| < \widehat{Z}^2|P|^{-1} \right\}.$$

The next step is an application of the function field analogue of Heath-Brown's Diophantine approximation lemma, as worked out in [4, Lemma 3.6]. Noting that $|\mathbf{H}(\mathbf{u})\mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$, we shall apply this with $\widehat{M} = (\widehat{Z}|P|)^2$ and $\widehat{Y}_0 = \widehat{Z}^{-2}|P|$. (In order to avoid a clash of notation we let Y_0 denote the parameter Y that features in [4, Lemma 3.6].) This result allows us to conclude that $\mathbf{H}(\mathbf{u})\mathbf{v} = \mathbf{0}$ provided that $\widehat{Y}_0 > |r|$ and $\widehat{M}^{-1} > |r\theta| \geq \widehat{Y}_0^{-1}$. Since $|r| = \widehat{Y}$ and $|\theta| = \widehat{\Theta}$ for $(Y, \Theta) \in S_2(d)$ it is easy to check that our choice of Z ensures that all of these inequalities are satisfied. Hence

$$|S(\alpha)|^4 \leq \frac{|P|^{2n}}{\widehat{Z}^{2n}} \# \left\{ \mathbf{u}, \mathbf{v} \in \mathcal{O}^n : |\mathbf{u}|, |\mathbf{v}| < \widehat{Z}|P|, \mathbf{H}(\mathbf{u})\mathbf{v} = \mathbf{0} \right\}.$$

The proof of [1, Lemma 6.5] directly yields the existence of a constant $c_n > 0$ such that the remaining cardinality is bounded by $c_n(\widehat{Z}|P|)^n$. In conclusion we have shown that

$$|S(\alpha)| \leq \frac{c_n |P|^n}{(\widehat{Y}\widehat{\Theta}|P|^3)^{n/8}}.$$

Turning now to the estimation of $N(P, Y, \Theta)$, it follows from (2.5) that

$$\begin{aligned} B_2(P) &\leq c_n \max_{(Y, \Theta) \in S_2(d)} \frac{\widehat{Y}^2 \widehat{\Theta} + 1 |P|^n}{(\widehat{Y}\widehat{\Theta}|P|^3)^{n/8}} \\ &= c_n q \max_{(Y, \Theta) \in S_2(d)} \widehat{Y}^{2-n/8} \widehat{\Theta}^{1-n/8} |P|^{5n/8}. \end{aligned}$$

Note that the exponent of $\widehat{\Theta}$ is negative for $n \geq 13$. Let $(Y, \Theta) \in S_2(d)$. Taking $\widehat{\Theta} > \widehat{Y}^{n/6-4/3}/\widehat{Q}^2$, we get

$$\widehat{Y}^{2-n/8} \widehat{\Theta}^{1-n/8} |P|^{5n/8} < \frac{\widehat{Y}^{2-n/8} |P|^{n-3}}{\widehat{Y}^{(n/8-1)(n/6-4/3)}} \leq |P|^{n-3},$$

since $\widehat{Y} \geq 1$ and $(2 - n/8) - (n/8 - 1)(n/6 - 4/3) \leq 0$ for $n \geq 13$. Hence $\lim_{q \rightarrow \infty} q^{-E(d,n)} B_2(P) = 0$ for $n \geq 13$.

Our final task is to show that (4.2) holds with $K = \lceil Z_1 - \{a\} \rceil - \lceil Z_2 + \{a\} \rceil$. The argument is based on the geometry of numbers. Every matrix corresponds to an \mathcal{O} -lattice spanned by its columns. We will abuse notation and identify a matrix with its corresponding lattice. Given a lattice \mathbf{M} , the adjoint lattice Λ is defined to satisfy $\Lambda^T \mathbf{M} = I$. Let $\Gamma = (\gamma_{ij})$ be a symmetric $n \times n$ matrix

with entries in K_∞ . Given any integer m , we define the special lattice

$$\mathbf{M}_m = \begin{pmatrix} t^{-m}I_n & 0 \\ t^m\Gamma & t^mI_n \end{pmatrix},$$

with corresponding adjoint lattice

$$\mathbf{\Lambda}_m = \begin{pmatrix} t^mI_n & -t^m\Gamma \\ 0 & t^{-m}I_n \end{pmatrix}.$$

Let $\widehat{R}_1, \dots, \widehat{R}_{2n}$ denote the successive minima of the lattice corresponding to \mathbf{M}_m and note that the lattices \mathbf{M}_m and $\mathbf{\Lambda}_m$ can be identified with one another. It follows from [4, Lemma B.6] that $R_\nu + R_{2n-\nu+1} = 0$ for each $1 \leq \nu \leq 2n$. Let $L_i(u_1, \dots, u_n)$ be the linear forms (4.1) for $1 \leq i \leq n$. Then for any real number Z , it is easy to see that

$$N(m, Z) = \{\mathbf{x} \in \mathbf{M}_m : |\mathbf{x}| < \widehat{Z}\},$$

in the notation of (4.2). We denote the right hand side by $M_m(Z)$ and proceed to establish the following inequality.

Lemma 4.1. *Let $m, Z_1, Z_2 \in \mathbb{Z}$ such that $Z_1 \leq Z_2 \leq 0$. Then we have*

$$\frac{M_m(Z_1)}{M_m(Z_2)} \geq \left(\frac{\widehat{Z}_1}{\widehat{Z}_2}\right)^n.$$

Proof. Let $1 \leq \mu, \nu \leq 2n$ be such that $R_\mu < Z_1 \leq R_{\mu+1}$ and $R_\nu < Z_2 \leq R_{\nu+1}$. Since R_j is a non-decreasing sequence which satisfies $R_j + R_{2n-j+1} = 0$, we must have $0 \leq R_{n+1}$, whence in fact $\mu \leq \nu \leq n$. It follows from [4, Lemma B.5] that

$$\frac{M_m(Z_1)}{M_m(Z_2)} = \begin{cases} 1 & \text{if } Z_1, Z_2 < R_1, \\ \left(\prod_{j=1}^\nu \widehat{R}_j / \widehat{Z}_1\right) (\widehat{Z}_1 / \widehat{Z}_2)^\nu & \text{if } Z_1 < R_1 \leq Z_2, \\ \left(\prod_{j=\mu+1}^\nu \widehat{R}_j / \widehat{Z}_1\right) (\widehat{Z}_1 / \widehat{Z}_2)^\nu & \text{if } R_1 \leq Z_1 \leq Z_2, \end{cases}$$

The statement of the lemma is now obvious. \square

Now let $a \in \mathbb{R}$ and put $m = \lfloor a \rfloor$. For any real number Z it is clear that

$$M_m(Z - \{a\}) \leq N(a, Z) \leq M_m(Z + \{a\}).$$

Lemma 4.1 therefore yields (4.2) with $K = \lceil Z_1 - \{a\} \rceil - \lceil Z_2 + \{a\} \rceil$, as required.

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