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CELL DECOMPOSITION AND CLASSIFICATION OF DEFINABLE SETS IN p-OPTIMAL FIELDS

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Abstract. We prove that for p-optimal fields (a very large subclass of p-minimal fields containing all the known examples) a cell decomposition theorem follows from methods going back to Denef's paper [7]. We derive from it the existence of definable Skolem functions and strong p-minimality. Then we turn to strongly p-minimal fields satisfying the Extreme Value Property—a property which in particular holds in fields which are elementarily equivalent to a p-adic one. For such fields K, we prove that every definable subset of $K \times K^d$ whose fibers over K are inverse images by the valuation of subsets of the value group is semialgebraic. Combining the two we get a preparation theorem for definable functions on p-optimal fields satisfying the Extreme Value Property, from which it follows that infinite sets definable over such fields are in definable bijection iff they have the same dimension.

§1. Introduction. This paper is an attempt to continue the road opened by Haskell and Macpherson in [10] toward a p-adic version of o-minimality, by isolating large subclasses of p-minimal fields to which Denef's methods of [7] apply with striking efficiency.

Recall that a *p-adically closed* field is a field K elementarily equivalent in the language of rings to a *p-adic field*, that is a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers. For every a in K, $v(a)$ and $|a|$ denote the p-valuation of a and its norm. The norm is nothing but the valuation with a multiplicative notation so that $|0| = 0$, $|ab| = |a| \cdot |b|$, $|a + b| \leq \max(|a|, |b|)$ and of course $|a| \leq |b|$ if and only if $v(a) \geq v(b)$. The valuation ring of v is denoted by R, and we fix some π in R such that πR is the maximal ideal of R. We let $v(K)$ or $|K|$ denote the image of K by the valuation.

Throughout all this paper we consider a fixed expansion $(K, \mathcal{L})$ of a p-adically closed field K, that is an $\mathcal{L}$-structure extending the ring structure of K for some language $\mathcal{L}$ containing the language of rings. Except if otherwise specified, when we say that a set or a function is definable we always mean "definable in $\mathcal{L}$ with parameters in K". For sets and functions definable in the language of rings (with parameters in K as always), we use the term "semialgebraic" instead. Wherever it is convenient we will identify subsets of $K^m \times |K|^d$ with their inverse image in K^{m+d} by the valuation, thus saying for example that the former are definable, semialgebraic, and so on if the latter are so.

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$(K, \mathcal{L})$ is *p-minimal* if every definable subset of K is definable in the language of rings. It is *strongly p-minimal* (or P-minimal for short, as in [10]) if every elementarily equivalent $\mathcal{L}$-structure is p-minimal. When the distinction between the $\mathcal{L}$-structure and the ring structure of K is clear from the context, K itself is called a strongly p-minimal field.

Strong p-minimality was introduced by Haskell and Macpherson in [10]. Since their proofs make extensive use of the model-theoretic Compactness Theorem, very little is known on p-minimal fields without the "strong" assumption contrary to the situation in o-minimal expansions of real closed fields, where o-minimality already implies strong o-minimality. They also left open several questions, such as the existence of a cell decomposition.

Mourgues proved in [13] that a cell decomposition similar to the one of Denef in [7] holds for a strongly p-minimal field K if and only if it has *definable Skolem functions* ("definable selection" in [13]), that is if for every positive integers m, n and every definable subset S of K^{m+n} the coordinate projection of S onto K^m has a definable section. It is not known at the moment whether strongly p-minimal fields always have definable Skolem functions.

As Cluckers noted in [5], a preparation theorem for definable functions was lacking in [12]. This remark applies as well to [13]. Cluckers filled this lacuna for the classical analytic structure on K, and derived from his preparation theorem several important applications, for parametric integrals and classification of subanalytic sets up to definable bijection. The former gives the rationality of the Poincaré series of a restricted analytic function. It has been generalised recently to strongly p-minimal fields in [2], by means of a slightly different preparation theorem for definable functions. However this preparation theorem and the cell decomposition that it uses, are weaker than the original ones studied by Denef, Mourgues and Cluckers. In particular they do not imply the existence of definable Skolem functions, and neither the classification of definable sets up to definable bijection.

The aim of this paper is to address some of these questions by introducing another notion of minimality for expansions of p-adically closed fields, called "p-optimality" (see definition below) with the following properties:

1. It is *intrinsic* (that is its definition only involves the given structure, not those which are elementarily equivalent to it) *natural* and *general* enough to include all the known examples of p-minimal fields.
2. Nevertheless it implies strong p-minimality, the existence of definable Skolem functions, cell decomposition and (under a mild assumption which we will discuss in Remark 1.5) cell preparation, so that all the applications of [5] generalize to this context.

This paper is based on [10] and [7], with which the reader is expected to be familiar. We will also make extensive use of [4]. Moreover we borrowed ideas from papers of other authors, especially Raf Cluckers in [5]. The concept of p-optimal field seems to be new but appears implicitly in many papers on p-adic fields, especially [8] which has been a source of inspiration for us.

Defining p-optimal fields. By a celebrated theorem of Macintyre [11] (generalized to p-adically closed fields in [14]) when $K = \mathbf{Q}_p$ every semialgebraic subset of K^m

is a (finite) boolean combination of sets of the form

$$S = \{x \in K^m : f(x) \in P_N\} \tag{1}$$

with f a polynomial function, $N \geq 1$ an integer and

$$P_N = \{x \in K : \exists y \in K,\ x = y^N\}.$$

We define *d-basic functions* as m-ary functions for some m which are polynomial in the last d variables with as coefficients global definable functions in the $m - d$ first variables, and *d-basic sets (of power N)* as the sets of the same form as (1) with d-basic functions instead of polynomial[1] functions. When $d = 1$ we simply talk about basic functions and sets. We say that $(K, \mathcal{L})$ (or simply K for short) is *p-optimal* if every definable subset of K^m is a (finite) boolean combination of basic sets, for every m.

REMARK 1.1. By the argument of Lemma 2.1 in [7], the following subsets of K^m are d-basic, for every d-basic m-ary functions f, g.

$$\{x \in K^m : f(x) = 0\} \text{ and } \{x \in K^m : |g(x)| \leq |f(x)|\}.$$

Moreover, since $P_N^* = P_N \setminus \{0\}$ is a subgroup of finite index in K^*, the complement in K^m of a d-basic set is a finite union of d-basic sets. Hence every (finite) boolean combination of basic sets is the union of intersections of finitely many basic sets. All of them can be taken of the same power, because $P_{N'}^*$ is a subgroup of P_N^* of finite index for every N' which is divisible by N.

(Strong) p-minimality versus p-optimality. Note that p-optimal fields are *not assumed to be* strongly p-minimal. They are p-minimal because basic subsets of the affine line K are semialgebraic. Moreover it is difficult to imagine any proof of p-minimality which does not involve in a way or another a quantifier elimination result similar to Macintyre's Theorem. The condition defining p-optimality is actually very close to such kind of elimination. So close that we can expect it to be proved simultaneously in most cases, if not all, without additional effort. Although not surprising, it is then quite remarkable that every p-optimal field is strongly p-minimal. More precisely, recalling that $(K, \mathcal{L})$ is an expansion of a p-adically closed field we have (Theorem 3.2):

THEOREM 1.2. *The following are equivalent*:

1. $(K, \mathcal{L})$ *is p-optimal.*
2. *Denef's Cell Decomposition Theorem* 2.6 *holds in* $(K, \mathcal{L})$.
3. $(K, \mathcal{L})$ *is strongly p-minimal and has definable Skolem functions.*

Of course (3)$\Rightarrow$(2) follows from [13] (not the other implications, because Mourgues considers only strongly p-minimal fields). Since every known example of p-minimal field is strongly p-minimal and has definable Skolem functions, Theorem 1.2 shows that all of them are p-optimal.

Main other results. Remember that, identifying any subset of $K^m \times |K|^d$ with its inverse image in K^{m+d} by the valuation, we call the former definable, semialgebraic, d-basic, or basic, if the latter is so. Similarly a function from $X \subseteq K^m$ to $|K|^d$ is

[1] Note that a global function in m variables is m-basic if and only if it is polynomial, hence Macintyre's theorem can be rephrased as: every semialgebraic subset of K^m is m-basic.

definable or semialgebraic if its graph is so, in this broader sense. In Section 4 we will consider strongly p-minimal fields satisfying the following condition.

($*$) Every continuous definable function from a closed and bounded definable set $X \subseteq K$ to $|K| \setminus \{0\}$ attains a minimum value.

We call it the *Extreme Value Property*. Note that it is not at all a restrictive assumption: if $(K, \mathcal{L})$ is elementarily equivalent to $(K', \mathcal{L})$ for some p-adic field K' then the Extreme Value Property trivially holds true in K' (because its p-valuation ring is compact), and passes to K by elementary equivalence. It is proved in [3] (Theorem 6) that if $(K, \mathcal{L})$ is strongly p-minimal then the definable subsets of $|K^d|$ are semialgebraic. The following is a "relative" version of this result (Theorem 4.1 and Corollary 4.4).

THEOREM 1.3. *If $(K, \mathcal{L})$ is strongly p-minimal and satisfies the Extreme Value Property, then every definable set $S \subseteq K \times |K|^d$ is semialgebraic. If moreover K is p-optimal then every definable subset of $K^m \times |K|^d$ is a boolean combination of $(d + 1)$-basic sets.*

In Section 5 we derive from it a preparation Theorem 5.3 for definable functions, analogous to Theorem 2.8 in [5]. As an application we get (Theorem 5.6):

THEOREM 1.4. *Two infinite sets definable over a p-optimal field satisfying the Extreme Value Property are isomorphic*[2] *if and only if they have the same dimension.*

REMARK 1.5. As already mentioned the Extreme Value Property is not a strong assumption. In particular it holds true for every semialgebraic function in a p-adically closed field (by reduction to the p-adic case, with the same argument as above). Moreover the Cell Preparation Theorem 5.3 applied to any unary definable function f from a closed and bounded set $S \subseteq K$ to $K \setminus \{0\}$ gives that the function $|f| : S \to |K| \setminus \{0\}$ is semialgebraic, hence has a minimum value. So the Cell Preparation Theorem holds true in a p-optimal field if and only if it satisfies the Extreme Value Property.

Other terminology and notation. For convenience we will sometimes add to K one more element ∞, with the property that $|x| < |\infty|$ for every x in K. We also denote by ∞ any partial function with constant value ∞.

Topological notions refer to the topology of the p-valuation, or its image in $|K|$.

For every subset X of K we let $X^* = X \setminus \{0\}$. Note the difference between $R^* = R \setminus \{0\}$ and $R^\times =$ the set of units in R.

Recall that K^0 is a one-point set. When a tuple $a = (x, t)$ is given in K^{m+1} it is understood that $x = (x_1, \ldots, x_m)$ and t is the last coordinate. We let $\widehat{a} = x$ denote the projection of a onto K^m. Similarly, the projection of a subset S of K^{m+1} onto K^m is denoted by $\widehat{S}$.

We extend $|.|$ (or v) to K^m coordinatewise. That is, for every $x \in K^m$ we let:

$$\big|(x_1, \ldots, x_m)\big| = \big(|x_1|, \ldots, |x_m|\big).$$

For every $A \subseteq K^m$ we let $|A|$ denote the image of A by this extension of the valuation.

For every integer $e \geq 1$ let $\mathbf{U}_e = \{x \in K : x^e = 1\}$. Analogously to Landau's notation $\mathcal{O}(x^n)$ of calculus, we let $\mathcal{U}_{e,n}(x)$ denote *any* definable function in the

[2] Following [3] we call "isomorphism" the definable bijections.

multi-variable x with values in $(1+\pi^n R)\mathbf{U}_e$. So, given a family of functions f_i, g_i on the same domain X, we write that $f_i = \mathcal{U}_{e,n} g_i$ for every i, when there are definable functions $\omega_i : X \to R$ and $\chi_i : X \to \mathbf{U}_e$ such that for every x in X, $f_i(x) = \big(1+\pi^n\omega_i(x)\big)\chi_i(x)g_i(x)$. When $e=1$, $\mathcal{U}_{1,n}(x)$ is simply written $\mathcal{U}_n(x)$.

If K° is a finite extension of $\mathbf{Q}_p$ to which K is elementarily equivalent as a ring, and R° is the p-valuation ring of K°, then the following set is semialgebraic (see Lemma 2.1, point 4, in [8])

$$Q^\circ_{N,M} = \{0\} \cup \bigcup_{k\in\mathbf{Z}} \pi^{kN}(1+\pi^M R^\circ).$$

We let $Q_{N,M}$ denote the semialgebraic subset of K corresponding[3] by elementary equivalence to $Q^\circ_{N,M}$ in K. If $M > 2v(N)$, Hensel's lemma implies that $1+\pi^M R$ is contained in P_N^*. Note that in this case, $Q^*_{N,M}$ is a clopen subgroup of P^*_N with finite index. The next property also follows from Hensel's lemma (see for example Lemma 1 and Corollary 1 in [3]).

Lemma 1.6. *The function $x \mapsto x^e$ is a group endomorphism of $Q^*_{N_0,M_0}$. If $M_0 > v(e)$ this endomorphism is injective and its image is $Q^*_{eN_0,v(e)+M_0}$.*

In particular $x \mapsto x^N$ defines a continuous bijection from $Q_{1,v(N)+1}$ to $Q_{N,2v(N)+1}$. We let $x \mapsto x^{\frac{1}{N}}$ denote the reverse bijection.

§**2. Cell decomposition.** This section gives an overview of the techniques used in Denef's cell decomposition. We emphasize that they do not only apply to polynomial functions, as in [7], but also to basic functions. This allows us to extend Denef's cell decomposition of semialgebraic sets over p-adic fields to definable sets over p-optimal fields (Theorem 2.6).

The cells which usually appear in the literature on p-adic fields are nonempty subsets of K^{m+1} of the form:

$$\{(x,t) \in X \times K : |\nu(x)| \,\square_1\, |t - c(x)| \,\square_2\, |\mu(x)| \text{ and } t - c(x) \in \lambda G\} \tag{2}$$

where $X \subseteq K^m$ is a definable set, c, μ, ν are definable functions from X to K, $\square_1, \square_2$ are $\leq, <$ or no condition, $\lambda \in K$ and G is a semialgebraic subgroup of K^* with finite index. In this paper we will only consider the cases when G is K^* (Theorem 2.4), P^*_N (Theorem 2.6) or $Q^*_{N,M}$ (Theorem 5.3).

In its simplest form, Denef's Cell Decomposition Theorem asserts that every semialgebraic subset of K^m is the disjoint union of finitely many cells. It will be convenient to fix a few more conditions on our cells, but most of all we want to pay attention on *how the functions defining the output cells depend on the input data*.

So we define *presented cells* in K^{m+1} as tuples $A = (c_A, \nu_A, \mu_A, \lambda_A, G_A)$ with c_A a definable function on a nonempty domain $X \subseteq K^m$ with values in K, ν_A and μ_A either definable functions on X with values in K^* or constant functions on X with values 0 or ∞, λ_A an element of K and G_A a semialgebraic subgroup of K^* with finite index, such that for every $x \in X$ there is $t \in K$ such that:

$$|\nu_A(x)| \leq |t - c_A(x)| \leq |\mu_A(x)| \text{ and } t - c_A(x) \in \lambda_A G_A. \tag{3}$$

[3]For a more intrinsic definition of $Q_{N,M}$ inside K, see [6].

Of course the set of tuples $(x, t) \in X \times K$ satisfying (3) is a cell of K^{m+1} in the usual sense of (2). We call it the *underlying cellular set* of A. Abusing the notation we will most often also denote that set by A. The existence, for every $x \in X$, of t satisfying (3) simply means that X is exactly $\widehat{A}$. We call it the *base* of A. The function c_A is called its *center*, μ_A and ν_A its *boundaries*. We also speak of a *presented cell mod G* when $G_A = G$.

A presented cell A is said to be of *type* 0 if $\lambda_A = 0$, and of type 1 otherwise. Contrary to its center, boundaries, and modulo, the type of A only depends on its underlying set.

The word "cell" will usually refer to presented cells. However, for sake of simplicity, we will freely talk of disjoint cells, bounded cells, families of cells partitioning some set and so on, meaning that the underlying cellular sets of these (presented) cells have the corresponding properties. For instance, it is clear that every cellular set as in (2) is in that sense the disjoint union of finitely many (presented) cells mod G.

Lemma 2.1 (Denef). *Let S be a definable subset of K^{m+n}. Assume that there is an integer $\alpha \geq 1$ such that for every x in K^m the fiber*

$$S_x = \{y \in K^n : (x, y) \in S\}$$

has cardinality $\leq \alpha$. Then the coordinate projection of S on K^m has a definable section.

Proof. Identical to the proof of Lemma 7.1 in [7]. ⊣

Lemma 2.2 (Denef). *Let f be an $(m+1)$-ary basic function with variables $(x, t) = (x_1, \ldots, x_m, t)$. Let $n \geq 1$ be a fixed integer. Then there exists a finite partition of K^{m+1} into sets A of the form*

$$A = \bigcap_{j \in S} \bigcap_{l \in S_j} \{(x, t) \in K^{m+1} : x \in C \text{ and } |t - c_j(x)| \,\square_{j,l}\, |a_{j,l}(x)|\}$$

where S and S_j are finite index sets, C is a definable subset of K^m, and c_j, $a_{j,l}$ are definable functions from K^m to K, such that for all (x, t) in A we have

$$f(x, t) = \mathcal{U}_n(x, t) h(x) \prod_{j \in S} \left(t - c_j(x)\right)^{e_j}$$

with $h : K^m \to K$ a definable function and $e_j \in \mathbf{N}$.

It is sufficient to check it for every n large enough so we can assume that:

$$1 + \pi^n R \subseteq P_N \cap R^\times \tag{4}$$

Thus $\mathcal{U}_n(x, t)$ in the conclusion could be replaced by $u(x, t)^N$ with u a definable function from A to $R^\times$. This is indeed how this result is stated in Lemma 7.2 of [7]. However it is the above equivalent (but slightly more precise) form which appears in Denef's proof, and which we retain in this paper.

Proof. The proof is exactly the same as the one of Lemma 7.2 of [7]. Of course, Lemma 7.1 used in Denef's proof has to be replaced with the analogous Lemma 2.1. (Denef's result assumes that f is a polynomial, but the proof only uses that it's a polynomial in the last variable, so it also applies to basic f.) ⊣

Remark 2.3 (co-algebraic functions). A remarkable by-product of Denef's proof is that the functions c_j and $a_{j,l}$ in the conclusion of Lemma 2.2 belong to $\operatorname{coalg}(f)$, which we define now.

Given a basic function f, we say that a function $h : X \subseteq K^m \to K$ belongs to $\mathrm{coalg}(f)$ if there exists a finite partition of X into definable pieces H, on each of which the degree in t of $f(x,t)$ is constant, say e_H, and such that the following holds. If $e_H \leq 0$ then $h(x)$ is identically equal to 0 on H. Otherwise there is a family $(\xi_1, \dots, \xi_{r_H})$ of K-linearly independent elements in an algebraic closure of K and a family of definable functions $b_{i,j} : H \to K$ for $1 \leq i \leq e_H$ and $1 \leq j \leq r_H$, and $a_{e_H} : H \to K^*$ such that for every x in H

$$f(x,T) = a_{e_H}(x) \prod_{1 \leq i \leq e_H} \Bigg(T - \sum_{1 \leq j \leq r_H} b_{i,j}(x)\xi_j \Bigg)$$

and

$$h(x) = \sum_{1 \leq i \leq e_H} \sum_{1 \leq j \leq r_H} \alpha_{i,j} b_{i,j}(x)$$

with the $\alpha_{i,j}$'s in K. If $\mathcal{F}$ is any family of basic functions we let $\mathrm{coalg}(\mathcal{F})$ denote the set of linear combinations of functions in $\mathrm{coalg}(f)$ for f in $\mathcal{F}$.

Theorem 2.4 (Denef). *Let $\mathcal{F}$ be a finite family of $(m+1)$-ary basic functions. Let $n \geq 1$ be a fixed integer. Then there exists a finite partition of K^{m+1} into presented cells H mod K^* such that the center and boundaries of H belong to* $\mathrm{coalg}(\mathcal{F}) \cup \{\infty\}$ *and for every (x,t) in H and every f in $\mathcal{F}$*

$$f(x,t) = \mathcal{U}_n(x,t) h_{f,H}(x) \big(t - c_H(x)\big)^{\alpha_{f,H}} \tag{5}$$

with $h_{f,H} : \widehat{H} \to K$ a definable function and $\alpha_{f,H} \in \mathbf{N}$.

Proof. Follow the proof of Theorem 7.3 in [7], using once again basic functions instead of polynomial functions. ⊣

Given two families $\mathcal{A}, \mathcal{B}$ of subsets of K^m, recall that $\mathcal{B}$ *refines* $\mathcal{A}$ if $\mathcal{B}$ is a partition of $\bigcup \mathcal{A}$ such that every A in $\mathcal{A}$ which meets some B in $\mathcal{B}$ contains it.

Corollary 2.5 (Denef). *Let $\mathcal{F}$ be a finite family of m-ary basic functions, $N \geq 1$ an integer and $\mathcal{A}$ a family of boolean combinations of subsets of K^m defined by $f(x) \in P_N$ with f in $\mathcal{F}$. Then there exists a finite family $\mathcal{H}$ of cells mod P_N^* with center and boundaries in* $\mathrm{coalg}(\mathcal{F})$ *which refines $\mathcal{A}$.*

Proof. Theorem 2.4 applies to $\mathcal{F}$ with $n > 2v(N)$, so that $1 + \pi^n R \subseteq P_N$. It gives a partition of K^m into presented cells B mod K^*. Every such cell B is the disjoint union of finitely many presented cells H mod P_N^*, whose centers and boundaries are the restrictions to $\widehat{H}$ of the center and boundaries of B (hence belong to $\mathrm{coalg}(\mathcal{F})$), on which $h_{f,B}(x)P_N^*$ and $(t - c_B(x))P_N^*$ are constant, simultaneously for every f in $\mathcal{F}$. Thus every A in $\mathcal{A}$ either contains H or is disjoint from H by (5) and our choice of n, which proves the result. ⊣

The following simpler statement, which follows directly from Corollary 2.5 by p-optimality, is sufficient in most cases.

Theorem 2.6 (Denef's cell decomposition). *If $(K, \mathcal{L})$ is p-optimal, then for every finite family $\mathcal{A}$ of definable subsets of K^m there is for some N a finite family of presented cells mod P_N^* refining $\mathcal{A}$.*

Remark 2.7. It has been proved in [1] that every definable function in a strongly p-minimal field is piecewise continuous. We will show in the next section that

p-optimal fields are strongly p-optimal. Thus the boundaries and centers of the cells in the above cell decompositions can be chosen continuous by refining appropriately a given cell decomposition.

§3. From p-optimality to strong p-minimality with Skolem functions.

LEMMA 3.1. *Assume that Denef's Cell Decomposition Theorem* 2.6 *holds true for* $(K, \mathcal{L})$. *Then it has definable Skolem functions.*

The proof is taken from the appendix of [9]. It is similar to proposition 4.1 in [13] except that we do not assume strong p-minimality (nor any continuity in the boundaries of the cells).

PROOF. By a straightforward induction it suffices to prove that for every definable subset A of K^{m+1} the coordinate projection of A onto $\widehat{A}$ has a definable section. If A is a union of finitely many definable sets B and if a definable section $\sigma_B : \widehat{B} \to B$ has been found for each projection of B onto $\widehat{B}$ we are done. Thus, by cell decomposition, we can assume that A is a presented cell mod P_N^* for some N. We deal with the case when $A = (c_A, \nu_A, \mu_A, \lambda_A)$ is of type 1 and $\nu_A \neq 0$ or $\mu_A \neq \infty$, the other cases being trivial.

If $\nu_A \neq 0$, as P_N^* is a definable subgroup of $K^\times$ with finite index, there is a partition of $\widehat{A}$ into finitely many definable pieces X on each of which ν_A/λ_A has constant residue class modulo P_N^*. Again it suffices to prove the result for each piece $A \cap (X \times K)$ of A. So we can assume that $X = \widehat{A}$, that is $\nu_A(x)/\lambda_A \in aP_N^*$ for some constant $a \in K^\times$ and every $x \in \widehat{A}$. Moreover we can choose a so that $v(a)$ is a nonnegative integer $k < N$. Let $\tau : x \in \widehat{A} \to c_A(x) + \nu_A(x)/a$. If $(x, \tau(x)) \in A$ for every $x \in \widehat{A}$ we are done, since $\sigma : x \in \widehat{A} \mapsto (x, \tau(x))$ is then a definable section of the coordinate projection of A onto $\widehat{A}$. So let us prove this.

Since $\tau(x) - c_A(x) = \lambda_A(\nu_A(x)/(a\lambda_A))$, it belongs to $\lambda_A P_N^\times$ by construction. Obviously we also have $|\nu_A(x)| \leq |\nu_A(x)/a|$ because $a \in R$, and thus $|\nu_A(x)| \leq |\tau(x) - c_A(x)|$. It remains to check that $|\tau(x) - c_A(x)| \leq |\mu_A(x)|$, that is $|\nu_A(x)/a| \leq |\mu_A(x)|$. Pick any $t \in K^\times$ such that $(x, t) \in A$. We have $|\nu_A(x)| \leq |t - c_A(x)| \leq |\mu_A(x)|$, so it suffices to check that $|\nu_A(x)/a| \leq |t - c_A(x)|$, that is $v(\nu_A(x)) - k \geq v(t - c_A(x))$. Let $\delta = (t - c_A(x))/\lambda_A$, since $(x, t) \in A$ we have $v(\nu_A(x)/\lambda_A) \geq v(\delta)$ and $v(\delta) \in v(P_N^*) = N\mathcal{Z}$. By construction we also have $v(\nu_A(x)/\lambda_A) \in v(aP_N^*) = k + N\mathcal{Z}$. Altogether, since $0 \leq k < N$, this implies that $v(\nu_A(x)/\lambda_A) \geq v(\delta) + k$. So $v(\nu_A(x)) - k \geq v(\delta) + v(\lambda_A) = v(t - c_A(x))$, which finishes the proof in this case. If $\nu_A = 0$ and $\mu_A \neq \infty$ a similar argument on μ_A gives the conclusion. ⊣

THEOREM 3.2. *The following are equivalent*:

1. $(K, \mathcal{L})$ *is p-optimal.*
2. *Denef's cell decomposition Theorem* 2.6 *holds in* $(K, \mathcal{L})$.
3. $(K, \mathcal{L})$ *is strongly p-minimal and has definable Skolem function.*

PROOF. (1)⇒(2) is Theorem 2.6. Let us prove that (2)⇒(3). By Lemma 3.1 it only remains to derive strong p-minimality from the Cell Decomposition Theorem 2.6.

Let $\Phi(\xi, \sigma)$ be a parameter-free formula with $m + 1$ variables. It defines a subset S of K^{m+1} which splits into finitely many cells C mod P_N^* for some N. Let $\mathcal{C}$ be the family of these cells, and $X_1, \ldots, X_r$ a finite partition of $\widehat{S}$ refining the $\widehat{C}$'s for

$C \in \mathcal{C}$. For each $i \le r$ let $\theta_i(\alpha_i, \xi)$ be a parameter-free formula in $n_i + m$ variables and $a_i \in K^{n_i}$ such that

$$X_i = \{x \in K^m : K \models \theta_i(a_i, x)\}.$$

Let $\Theta(\alpha_1, \dots, \alpha_r)$ be the parameter-free formula in $n_1 + \cdots + n_r$ variables saying that, given any values a_i' of the parameters α_i, the formulas $\theta_i(a_i', \xi)$ define a partition of $\widehat{S}$. In particular we have $K \models \Theta(a_1, \dots, a_r)$.

Let $\mathcal{C}_i$ be the family of all the cells $C \cap (X_i \times K)$ for $C \in \mathcal{C}$. This is a finite partition of $S \cap (X_i \times K)$ into cells mod P_N^*, which consists of k_0^i cells of type 0, k_1^i cells D of type 1 with $\mu_D \neq \infty$, and k_∞^i cells D of type 1 with $\mu_D = \infty$. We let $k^i = (k_0^i, k_1^i, k_\infty^i)$. For every $x \in X_i$, the fiber $S_x = \{t \in K : (x, t) \in S\}$ is the disjoint union of the fibers C_x for $C \in \mathcal{C}_i$, each of which is of the same type as C. Given a tuple $k = (k_0, k_1, k_\infty)$ it is an easy exercise to write a parameter-free formula $\Psi_{k,N}(\xi)$ in m free variables saying that, given any value x' of the parameter ξ, the set of points t' in K such that $K \models \Phi(x', t')$ is the disjoint union of k_0 cells mod P_N^* of type 0, k_1 cells D' mod P_N^* of type 1 with $\mu_{D'} \neq \infty$, and k_∞ cells D' mod P_N^* of type 1 with $\mu_{D'} = \infty$. By construction we have

$$K \models \exists \alpha_1, \dots, \alpha_r\, \Theta(\alpha_1, \dots, \alpha_r) \wedge \bigwedge_{i \le r} \forall \xi \left[\theta_i(\alpha_i, \xi) \to \Psi_{k^i,N}(\xi)\right].$$

This formula is satisfied in every $\tilde{K} \equiv K$. So there are $\tilde{a}_i$ in $\tilde{K}^{n_i}$ for $i \le r$ such that the sets

$$\tilde{X}_i = \{\tilde{x} \in \tilde{K}^m : \tilde{K} \models \theta_i(\tilde{a}_i, \tilde{x})\}$$

form a partition of $\{\tilde{x} \in \tilde{K}^m : \exists \tilde{t} \in \tilde{K},\ \tilde{K} \models \Phi(\tilde{x}, \tilde{t})\}$, and for every $\tilde{x} \in \tilde{X}_i$ the set of $\tilde{t} \in \tilde{K}$ such that $\tilde{K} \models \theta_i(\tilde{x}, \tilde{t})$ is the disjoint union of $k_0^i + k_1^i + k_\infty^i$ cells of $\tilde{K}$. In particular the formula $\Phi(\tilde{x}, \tau)$ defines a semialgebraic subset of $\widetilde{K}$, whatever is the value of the parameter $\tilde{x}$ in $\tilde{K}^m$. This being true for every formula Φ, it follows that $\tilde{K}$ is p-minimal hence that K is strongly p-minimal.

Finally let us prove that (3)⇒(1). Let S be a definable subset of K^{m+1}, and S' the corresponding definable set in an elementary extension K' of K. For every x' in K'^m let $S'_{x'}$ denote the fiber of $\widehat{S}'$ over x':

$$S'_{x'} = \left\{t' \in K' : (x', t') \in S'\right\}.$$

For every x' in $\widehat{S}'$ the p-minimality of K' and Macintyre's Theorem (see Footnote 1) give a tuple $z'_{x'}$ of coefficients of a description of $S'_{x'}$ as a boolean combination of basic sets. The model-theoretic Compactness Theorem then gives definable subsets $X_1, \dots, X_q$ partitioning K^m and for every $i \le q$ an $\mathcal{L}$–formula $\varphi_i(x, t, z)$ with $m + 1 + n_i$ free variables which is a boolean combination of formulas of the form $f(x, t, z) \in P_N$ with $f \in \mathbf{Z}[x, t, z]$, such that for every x in X_i there is a list of coefficients z_x such that

$$S_x = \left\{t \in K : K \models \varphi(x, t, z_x)\right\}.$$

In other words, for every x in X_i

$$K \models \exists z\ \forall t\ \big((x, t) \in S \leftrightarrow \varphi_i(x, t, z)\big).$$

Our assumption (3) then gives for each $i \le q$ a definable function $\zeta_i : X_i \to K^{n_i}$ such that for every $x \in X_i$

$$K \models \forall t\ \left[(x, t) \in S \leftrightarrow \varphi_i\big(x, t, \zeta_i(x)\big)\right].$$

Let $B_i = \{(x,t) \in K^{m+1} : K \models \varphi_i(x,t,\zeta_i(x))\}$. By construction this is a boolean combination of basic subsets of K^{m+1}, hence so is $C_i = B_i \cap (X_i \times K)$. The conclusion follows, since S is the union of these C_i's. ⊣

§4. **Relative p-minimality.** The aim of this section is to prove the following result. It may be called "relative p-minimality".

THEOREM 4.1. *Assume that $(K,\mathcal{L})$ is strongly p-minimal and satisfies the Extreme Value Property. Then every definable set $S \subseteq K \times |K|^d$ is semialgebraic, for every d.*

We need to state a few preliminary results and to introduce some notation. For every $a \in K$ and $r \in |K^*|$ we let

$$B(a,r) = \{y \in K : |x-y| < r\}$$

denote the ball of center a and radius r.

FACT 4.2. *For every definable set $S \subseteq K^m \times |K|^d$, if $A \subseteq K^m$ is the image of the coordinate projection of S onto K^m, there is a definable function $\sigma : A \to |K|^d$ such that $(x,\sigma(x)) \in S$ for every $x \in A$.*

PROOF. By p-minimality, the value group $v(K^*)$ is simply a $\mathbf{Z}$-group. Every nonempty definable subset of a $\mathbf{Z}$-group which is bounded above (resp. below) has a largest (resp. smallest) element. The conclusion easily follows if $d = 1$, and for $d \geq 1$, it is a straightforward induction. ⊣

Beware that σ in Fact 4.2 *is not a Skolem function* over K, because its codomain is in $|K|$. The next Lemma shows that this can be fixed, in a strong sense.

LEMMA 4.3. *Assume that $(K,\mathcal{L})$ is strongly p-minimal and satisfies the Extreme Value Property. Then every definable function $f : X \subseteq K \to |K|^d$ is semialgebraic. In particular there is a semialgebraic function $\tilde{f} : X \to K^d$ such that $f = |\tilde{f}|$.*

For every $r \in |K^*|$ we let r^+ denote the element of $|K^*|$ immediately greater than r.

PROOF. If $f = (f_1,\dots,f_d)$ it suffices to prove the result separately for each f_i, hence we can assume that $d = 1$. Given a finite partition of X into definable pieces Y it suffices to prove the result for the restriction of f to each Y separately. Thus by splitting X into $f^{-1}(\{0\})$ and $X \setminus f^{-1}(\{0\})$ we can assume that $f(X) \subseteq |K^*|$. By Theorem 3.3 and Remark 3.4 in [10] there is a definable open set U contained in X such that $X \setminus U$ is finite and f is continuous on U. By throwing away a finite set if necessary, we can therefore assume that f is continuous and X is open in K. Finally we can assume that f is not constant on X, otherwise the result is trivial.

For every $a \in X$ the set of $r \in |K^*|$ such that $B(a,r) \subseteq X$ and f is constant on this ball is definable, nonempty and bounded above (otherwise $X = K$ and f is constant, which we have excluded) hence by Fact 4.2 it has a maximum element $\rho(a)$. We are claiming that the following set

$$S = \{a \in X : \forall b \in B(a,\rho(a)^+) \cap X,\ f(a) \leq f(b)\}$$

has the property that for every ball $B \subseteq X$ on which f is nonconstant, B intersects both S and $X \setminus S$. Indeed let $B = B(c,r)$ be any such ball. The function ρ is definable, so the Extreme Value Property gives $a_0 \in B$ such that $\rho(a_0) = \min_{b\in B} \rho(b)$. Since f is nonconstant on B, necessarily $\rho(a_0) < r$ hence $B(b,\rho(a_0)^+) \subseteq B$ for every $b \in B$.

By construction f is nonconstant on $B(a_0, \rho(a_0)^+)$. The latter is the disjoint union of $B(a_0, \rho(a_0))$ and finitely many balls $B(a_i, \rho(a_0))$ for $1 \leq i \leq n$ (where $n+1 \geq 2$ is the cardinality of the residue field). By minimality of $\rho(a_0)$, f is constant on each $B(a_i, \rho(a_0))$ hence there are $i \neq j$ between 0 and n such that

$$\forall b \in B(a_0, \rho(a_0)^+),\ f(a_i) \leq f(b) \leq f(a_j). \tag{6}$$

Moreover f is nonconstant on the union of $B(a_k, \rho(a_0))$ for $0 \leq k \leq n$ hence $f(a_i) < f(a_j)$. It follows that $\rho(a_i) = \rho(a_j) = \rho(a_0)$ and hence $a_i \in S$ and $a_j \notin S$ by (6), which proves our claim.

X and S are definable subsets of K, hence semialgebraic by p-minimality. Thus there exists a partition $\mathcal{A}$ of X into finitely many cells mod $Q^*_{N,M}$ for some N, M such that S is also the union of the cells in $\mathcal{A}$ that it contains. Every cell $A \in \mathcal{A}$ can be presented as the set of elements $t \in K$ such that

$$|\nu_A| \leq |t - c_A| \leq |\mu_A| \text{ and } t - c_A \in \lambda_A Q^*_{N,M}.$$

We are claiming that $f(t)$ only depends on $|t - c_A|$ as t ranges over A. If $\lambda_A = 0$ then A is reduced to a point, hence f is constant on A. Otherwise $\lambda_A \neq 0$ and for every $a \in K^\times$, we have to prove that f is constant on the set B_a of $t \in A$ such that $|t - c_A| = |a|$. We can assume that B_a is nonempty, hence $|a| = |t_a - c_A|$ for some $t_a \in A$. Then $|\nu_A| \leq |a| \leq |\mu_A|$, hence $t \in B_a$ if and only if $|t - c_A| = |a|$ and $t - c_A \in \lambda_A Q^*_{N,M}$, that is $B_a = aR^\times \cap \lambda_A Q^*_{N,M}$. Pick any $b \in B_a$, then $bR^\times = aR^\times$ and $bQ^*_{N,M} = \lambda_A Q^*_{N,M}$ hence

$$B_a = aR^\times \cap aQ^*_{N,M} = a(R^\times \cap Q^*_{N,M}) = a(1 + \pi^M R).$$

In particular B_a is a ball. So by construction of $\mathcal{A}$, A is either contained in S or in $X \setminus S$ hence so is B. But then, by construction of S, f is constant on B. This proves our claim.

Now pick any $A \in \mathcal{A}$ and translate it by c_A. The result is a cell A' mod $Q^*_{N,M}$ centered at 0 on which $f(t)$ only depends on $|t|$. Thus the graph of the restriction $f_{|A}$ of f to A is the intersection with $\lambda_A Q^*_{N,M}$ of the pre-image by the valuation of a definable function $\theta : |A'| \to |K|$. By Theorem 6 in [4] it follows that $f_{|A}$ is semialgebraic, hence so is f. The last point immediately follows from the existence of definable Skolem functions for semialgebraic sets (see for example [15]). ⊣

As already mentioned in the introduction, Theorem 4.1 is a "relative" version of Theorem 6 in [4]. Since our proof heavily depends on the main results of [4] it is more convenient here to use additive notation for the value group, so let $G = v(K^*)$. Theorem 6 in [4] actually says that for every definable set $S \subseteq (K^*)^d$, with $(K, \mathcal{L})$ a strongly p-minimal expansion of a p-adically closed field, the image of S in G^d by the valuation is definable in Presburger language

$$\mathcal{L}_{Pres} = \{0, 1, +, \leq, (\equiv_n)_{n>0}\}$$

where $\equiv_n$ is interpreted in G as the binary congruence relation modulo the integer n.

It follows from Theorem 1 in [4] and Remark (iii) just above it that every subset of G^d definable in the language $\mathcal{L}_{Pres}$ is the union of finitely many disjoint sets defined by the conjunction for $1 \leq i \leq d$ of conditions (E_i) of the form

$$\zeta_i + \sum_{1 \leq j < i} a_{i,j} \frac{X_j - c_j}{n_j} \,\square_{i,1}\, X_i \,\square_{i,2}\, \zeta'_i + \sum_{1 \leq j < i} a'_{i,j} \frac{X_j - c_j}{n_j} \text{ and } X_i \equiv c_i\ [n_i]$$

with every $\zeta_i, \zeta'_i \in G$, $a_{i,j}, a'_{i,j}, c_i, n_i \in \mathbf{Z}$, $0 \leq c_i < n_i$ and $\Box_{i,1}, \Box_{i,2}$ being either $\leq$ or no condition. Let λ be the list of all these integers and symbols. Let Λ_d denote the set of lists λ of this sort. The conjunction of the above conditions (E_i) for $1 \leq i \leq d$ is expressed by a formula $\varphi_\lambda(X, \zeta)$ with free variables $X = (X_1, \dots, X_d)$ and parameters $\zeta = (\zeta_1, \dots, \zeta_d, \zeta'_1, \dots, \zeta'_d)$. We let $\varphi_\lambda(X, Z)$ be the corresponding parameter-free formula in $\mathcal{L}_{Pres}$ with $d + 2d$ free variables.

With these results in mind we can turn to the proof of Theorem 4.1.

Proof. Let S be a definable[4] subset of $K \times G^d$. For every $x \in K$ the fiber $S_x = \{\tau \in G^d : (x, \tau) \in S\}$ is definable in $\mathcal{L}_{Pres}$ by Theorem 6 in [4]. Hence there is a finite set of elements $\lambda_1, \dots, \lambda_r \in \Lambda_d$ and parameters $\gamma_k \in G^{2d}$ such that the sets $C_{\lambda_k}(\gamma_k)$, defined as the set of elements $\tau \in G^d$ such that $G \models \varphi_{\lambda_k}(\tau, \gamma_k)$, form a partition of S_x. These formulas $\varphi_{\lambda_k}(T, Z)$ easily translate into formulas $\psi_{\lambda_k}(T, Z)$ in the language of rings such that for every $t \in K^d$ and every $z \in K^{2d}$, $K \models \psi_{\lambda_k}(t, z)$ if and only if $G \models \varphi_{\lambda_k}(v(t), v(z))$.

By strong p-minimality the same holds true in every $(K', \mathcal{L}) \equiv (K, \mathcal{L})$. Hence by the model-theoretic Compactness Theorem there is a partition of K into finitely many definable sets $A_1, \dots, A_s$ and for each $l \leq s$ a finite set of indexes $\lambda_{1,l}, \dots, \lambda_{r_l,l} \in \Lambda_d$ such that for every $x \in A_l$ there are parameters $\zeta_{x,k,l} \in G^{2d}$ such that S_x is partitioned by the sets $C_{\lambda_{k,l}}(\zeta_{x,k,l})$ for $k \leq r_l$. By Fact 4.2 there are definable functions $\zeta_{k,l}$ from A_l to G^{2d} such that for every $x \in A_l$ the sets $C_{\lambda_{k,l}}(\zeta_{k,l}(x))$ for $k \leq r_l$ form a partition of S_x. By Lemma 4.3 and the Extreme Value Property there are semialgebraic functions $\tilde{z}_{k,l}$ from A_k to K^{2d} such that $\zeta_{k,l} = |\tilde{z}_{k,l}|$ (that is $\zeta_{k,l} = v \circ \tilde{z}_{k,l}$ with additive notation).

By the above construction $v^{-1}(S)$ is the disjoint union for $l \leq s$ and $k \leq r_l$ of the sets $B_{k,l}$ of tuples $(x, t) \in A_k \times K^d$ such that $K \models \psi_{\lambda_{k,l}}(t, \tilde{z}_{k,l}(x))$. These sets are semialgebraic because $\psi_{\lambda_{k,l}}(T, Z)$ is a formula in the language of rings and $\tilde{z}_{k,l}$ a semialgebraic function. Thus $v^{-1}(S)$ itself is semialgebraic, hence so is S by definition. ⊣

Corollary 4.4. *Assume that K is p-optimal and satisfies the Extreme Value Property. Then every definable subset of $K^m \times |K|^d$ is a boolean combination of $(d + 1)$-basic sets.*

Proof. If $m = 1$ the conclusion follows from Theorem 4.1 and Macintyre's Theorem (see Footnote 1). Assume that it has been proved for $m \geq 1$ and let S be a definable subset of K^{m+1+d} which is the pre-image by the valuation of a subset of $K^{m+1} \times |K|^d$. Let S' be the corresponding definable set over an elementary extension K' of K. For every x' in K'^m let $S'_{x'}$ denote the fiber of S' over x':

$$S'_{x'} = \left\{(t', z') \in K' \times K'^d : (x', t', z') \in S'\right\}.$$

This set $S'_{x'}$ is obviously the inverse image in $K' \times K'^d$ by the valuation of a subset of $K' \times |K'|^d$. Note that K' is strongly p-minimal and satisfies the Extreme Value Property, because these two properties are preserved by elementary equivalence. Thus Theorem 4.1 applies in K' and gives a tuple $a'_{x'}$ of coefficients of a description of $S'_{x'}$ as a boolean combination of $(d+1)$-basic subsets of K'^{d+1}. The model-theoretic

[4]Recall that in this context, "definable" means that the inverse image of S by the valuation is definable in $K \times K^d$.

Compactness Theorem then gives definable subsets $A_1, \dots, A_q$ partitioning K^m, and for every $i \le q$ an $\mathcal{L}$–formula $\varphi_i(\alpha, \tau, \zeta)$ with $n_i + 1 + d$ free variables which is a boolean combination of formulas of the form $f(\alpha, \tau, \zeta) \in P_N$ with $f \in \mathbf{Z}[\alpha, \tau, \zeta]$, such that for every x in A_i there is a list of coefficients a_x such that

$$S_x = \big\{(t, z) \in K \times K^d : K \models \varphi(a_x, t, z)\big\}.$$

In other words, for every x in A_i

$$K \models \exists a\ \forall t, z\ \big((x, t, z) \in S \leftrightarrow \varphi_i(a, t, z)\big).$$

By Theorem 3.2, K has definable Skolem functions, hence for each $i \le q$ there is a definable function $\sigma_i : A_i \to K^{n_i}$ such that for every $x \in A_i$

$$K \models \forall t, z\ \big[(x, t, z) \in S \leftrightarrow \varphi_i\big(\sigma_i(x), t, z\big)\big].$$

Let $B_i = \{(x, t, z) \in K^{m+1+d} : K \models \varphi_i(\sigma_i(x), t, z)\}$. By construction, this is a boolean combination of $(d + 1)$-basic subsets of K^{m+1+d}. On the other hand, $A_i \times K^{d+1}$ is obviously a $(d + 1)$-basic subset of K^{m+1+d}. Indeed, if $c_i(x)$ denotes the indicator function of A_i, then $h_i(x, t, z) = c_i(x) - 1$ is $(d + 1)$-basic and we have

$$A_i \times K^{d+1} = \big\{(x, t, z) \in K^{m+1+d} : h_i(x, t, z) = 0\big\}$$

which is a $(d + 1)$-basic set by Remark 1.1. The conclusion follows, since S is the union of the sets $B_i \cap (A_i \times K^{d+1})$. ⊣

§**5. Cell preparation.** The main result of this section is the Cell Preparation Theorem 5.3 for definable functions. We derive from it our last main result, Theorem 5.6, which classifies up to definable bijections the definable sets over any p-optimal field satisfying the Extreme Value Property.

Lemma 5.1 (Denef). *Assume that K is p-optimal and satisfies the Extreme Value Property. Then for every definable function $f : X \subseteq K^m \to K$ there is an integer $e \ge 1$ and a partition $\mathcal{A}$ of X into definable sets A such that for every x in A*

$$\big|f(x)\big|^e = \left|\frac{p_A(x)}{q_A(x)}\right|$$

with p_A, q_A a pair of basic functions such that $q_A(x) \neq 0$ for every x in A.

Proof. By Corollary 4.4, the set $S = \{(x, t) \in K^m \times K : |t| = |f(x)|\}$ is a boolean combination of 2-basic subsets of K^{m+1}. The proof of Denef's Theorem 6.3 in [7] then applies word-for-word, with basic functions instead of polynomial functions. It gives a partition of X into finitely many definable pieces A, on each of which $|f|^e = |p_A/q_A|$ for some 1-basic functions such that $q_A(x) \neq 0$ for every x in A. ⊣

Note that, in the above proof, if S is a boolean combination of $(d + 1)$-basic sets then Denef's proof of Theorem 6.3 also goes through and the resulting functions p_A, q_A are d-basic. In particular, it is not sufficient to know that S is a boolean combination of basic sets (as it would follow directly from p-optimality), because Denef's argument then would yield functions p_A, q_A which are only 0-basic, that is just definable, without providing any gain. So, contrary to what happened in Section 2 with the Cell Decomposition, the generalization of Denef's Cell Preparation to p-optimal fields is not at all straightforward: all the results of the previous section leading to Corollary 4.4 seem to be mandatory here, in order to ensure that S is a boolean combination of 2-basic sets.

Remark 5.2. Given an integer $n_0 \geq 1$, the set $1 + \pi^{n_0} R$ is a definable subgroup of $R^\times$ with finite index. Thus in Lemma 5.1 we can always assume, refining if necessary the partition of X (but keeping the same integer e independently of n_0), that for every x in A

$$f(x)^e = \mathcal{U}_{n_0}(x) \frac{p_A(x)}{q_A(x)}.$$

Theorem 5.3 (Cell preparation). *Assume that K is p-optimal and satisfies the Extreme Value Property. Let $(\theta_i : A_i \subseteq K^{m+1} \to K)_{i \in I}$ be a finite family of definable functions. Then there exists an integer $e \geq 1$ and, for every $n \in \mathbf{N}^*$, a pair of integers M, N and a finite family $\mathcal{H}$ of presented cells mod $Q^*_{N,M}$ such that $M > 2v(e)$, e divides N, $\mathcal{H}$ refines $(A_i)_{i \in I}$, and for every $(x,t) \in H$,*

$$\theta_i(x,t) = \mathcal{U}_{e,n}(x,t) h(x) \left[\lambda_H^{-1} \left(t - c_H(x) \right) \right]^{\frac{\alpha}{e}} \tag{7}$$

for every $i \in I$ and every $H \in \mathcal{H}$ contained in A_i, with $h : \widehat{H} \to K$ a continuous definable function and $\alpha \in \mathbf{Z}$ (both depending on i and H)[5].

Remark 5.4. Remark 2.7 applies to the above theorem as well, so the center and boundaries of every cell in $\mathcal{H}$ can be chosen to be continuous.

Proof. For each i let e_i be an integer, $\mathcal{A}_i$ a partition of A_i and $\mathcal{F}_i$ a family of basic functions, all given by Lemma 5.1 applied to θ_i. By replacing each e_i with a common multiple[6] we can assume that all of them are equal to some integer $e \geq 1$. Given an integer $n \geq 1$ from the theorem, we set $n_0 = n + v(e)$ and we refine the partition $\mathcal{A}_i$ as in Remark 5.2.

Let $\mathcal{A}$ be a finite family of definable sets refining $\bigcup_{i \in I} \mathcal{A}_i$. We can assume that each of them is a boolean combination of basic sets of the same power N, with N a multiple of e. For every A in $\mathcal{A}$, every $i \in I$ such that A_i contains A and every (x,t) in A we have

$$\theta_i(x,t)^e = \mathcal{U}_{n_0}(x,t) \frac{p_{i,A}(x,t)}{q_{i,A}(x,t)} \tag{8}$$

with $p_{i,A}$ and $q_{i,A}$ a pair of basic functions such that $q_{i,A}(x,t) \neq 0$ on A.

For each A in $\mathcal{A}$ let $\mathcal{F}_A$ be the set of basic functions involved in a description of A as a boolean combination of basic sets of power N. Theorem 2.4 applies to the family $\mathcal{F}$ of all the basic functions $p_{i,A}$, $q_{i,A}$ and the functions in $\mathcal{F}_A$, for all A's and i's. It gives a partition of K^{m+1} into finitely many presented cells B mod K^* such that for every f in $\mathcal{F}$ and every (x,t) in B

$$f(x,t) = \mathcal{U}_M(x,t) h_{f,B}(x) \left(t - c_B(x) \right)^{\beta_{f,B}} \tag{9}$$

with $M = n_0 + 2v(N)$, $h_{f,B} : \widehat{B} \to K$ a definable function and $\beta_{f,B}$ a positive integer.

[5] If H is of type 0 then it is understood that $\alpha = 0$ and we use the conventions that in this case $\lambda_H^{-1} = 0$ and $0^0 = 1$.

[6] Note that we can require e to be divisible as well by any given integer N_0 if needed.

Partitioning $\widehat{B}$ if necessary, we can assume that the cosets $h_{f,B}(x)Q^*_{N,M}$ are constant on $\widehat{B}$. Since $\mathcal{U}_M(x,t) \in 1+\pi^M R \subseteq Q^*_{N,M}$, by (9) $f(x,t)Q^*_{N,M}$ only depends on $(t-c_B(x))Q^*_{N,M}$. Hence B can be partitioned into cells H mod $Q^*_{N,M}$ such that $\widehat{H} = \widehat{B}$, $c_H = c_B$ and $f(x,t)Q^*_{N,M}$ is constant on H, for every f in $\mathcal{F}$. *A fortiori*[7] $f(x,t)P^*_N$ is constant on H for every f in $\mathcal{F}$, hence each A in $\mathcal{A}$ either contains H or is disjoint from H. So the family $\mathcal{H}$ of all those cells H that are contained in $\bigcup \mathcal{A}$ refines $\mathcal{A}$, hence refines $\{A_i : i \in I\}$ as well.

For every cell H in $\mathcal{H}$ there is a unique cell B as above containing H. For every $i \in I$ such that H is contained in A_i, the unique A in $\mathcal{A}$ containing B is also contained in A_i. By (9) applied to $f = p_{i,A}$ and to $f = q_{i,A}$, and by (8) we have for every $(x,t) \in H$

$$\theta_i(x,t)^e = \mathcal{U}_{n_0}(x,t)\frac{\mathcal{U}_M(x,t)h_{p_{i,A},B}(x)\big(t-c_B(x)\big)^{\beta_{p_{i,A},B}}}{\mathcal{U}_M(x,t)h_{q_{i,A},B}(x)\big(t-c_B(x)\big)^{\beta_{q_{i,A},B}}}. \tag{10}$$

The $\mathcal{U}_{n_0}$ and $\mathcal{U}_M$ factors simplify to a single $\mathcal{U}_{n_0}$ since $M \geq n_0$. By construction $c_H = c_B$ and $\widehat{H} = \widehat{B}$. So, for every (x,t) in H we get

$$\theta_i(x,t)^e = \mathcal{U}_{n_0}(x,t)g(x)\big[\lambda_H^{-1}\big(t-c_H(x)\big)\big]^{\alpha} \tag{11}$$

with $g : \widehat{H} \to K$ a definable function and $\alpha \in \mathbf{Z}$ (both depending on i and H). Since $n_0 > 2v(e)$, $(\mathcal{U}_{n_0}(x,t))^{\frac{1}{e}}$ is well defined and takes values in $1+\pi^{n_0-v(e)}$ by Lemma 1.6, that is $\mathcal{U}_{n_0} = \mathcal{U}^e_{n_0-v(e)}$. We have $n_0 - v(e) = n + v(e) \geq n$, hence *a fortiori* $\mathcal{U}_{n_0} = \mathcal{U}^e_n$. So (11) becomes

$$\theta_i(x,t)^e = \mathcal{U}_n(x,t)^e g(x)\Big(\big[\lambda_H^{-1}\big(t-c_B(x)\big)\big]^{\frac{\alpha}{e}}\Big)^e. \tag{12}$$

This implies that g takes values in P_e, hence $g = h^e$ for some definable function $h : \widehat{H} \to K$, from which (7) follows. ⊣

Corollary 5.5. *Suppose that K is p-optimal and satisfies the Extreme Value Property. Let $(\theta_i : A \subseteq K^m \to K)_{i\in I}$ be a finite family of definable functions with the same domain. Then for every integer $n \geq 1$, there exists an integer e, a semialgebraic set $\tilde{A} \subseteq K^m$ and a definable bijection $\varphi : \tilde{A} \to A$ such that for every $i \in I$ and every x in $\tilde{A}$*

$$\theta_i \circ \varphi(x) = \mathcal{U}_{e,n}(x)\tilde{\theta}_i(x)$$

with $\tilde{\theta}_i : \tilde{A} \subseteq K^m \to K$ semialgebraic functions.

Proof. The proof goes by induction on m. Let us assume that it has been proved for some $m \geq 0$ (it is trivial for $m = 0$) and that a finite family $(\theta_i)_{i\in I}$ of definable functions is given with domain $A \subseteq K^{m+1}$. If A is a disjoint union of sets B, it suffices to prove the result for the restrictions of the θ_i's to B. So, for any given integer $n \geq 1$, by Theorem 5.3 we are reduced to the case when A is a presented cell mod $Q^*_{N,M}$ for some N, M such that for some $e_0 \geq 1$ dividing N, $M > 2v(e_0)$ and for every $i \in I$ and every (x,t) in A

$$\theta_i(x,t) = \mathcal{U}_{e_0,n}(x,t)h_i(x)\big[\lambda_A^{-1}\big(t-c_A(x)\big)\big]^{\frac{\alpha_i}{e_0}} \tag{13}$$

with $h_i : \widehat{A} \to K$ a definable function and $\alpha_i \in \mathbf{Z}$.

[7]Recall that $M = n_0 + 2v(M) > 2v(M)$ hence $Q_{N,M} \subseteq P_N$ by Hensel's Lemma.

Let $e_1 \geq 1$ be an integer, $Y \subseteq K^m$ a semialgebraic set, $\psi : Y \to \widehat{A}$ a definable bijection, $\tilde{f} : Y \to K$ a semialgebraic function for each f in $\mathcal{F}$, all of this given by the induction hypothesis applied to $\mathcal{F} = \{\mu_A, \nu_A\} \cup \{h_i\}_{i \in I}$. Let $\tilde{A}$ be the set of $(y, s) \in Y \times K$ such that

$$|\tilde{\nu}_A(y)| \leq |s| \leq |\tilde{\mu}_A(x)| \text{ and } s \in \lambda_A Q^*_{N,M}.$$

Then $\varphi : (y, s) \mapsto (\psi(y), c_A(\psi(y)) + s)$ defines a bijection from $\tilde{A}$ to A. For every $i \in I$ and every $(y, s) \in \tilde{A}$ we have

$$\theta_i \circ \varphi(y, s) = \mathcal{U}_{e_0,n}(y, s)\mathcal{U}_{e_1,n}(y, s)\tilde{h}_i(y)(\lambda_A^{-1} s)^{\frac{\alpha_i}{e_0}}.$$

The first two factors can be replaced by $\mathcal{U}_{e,n}$ with e any common multiple of e_0 and e_1. Since $\tilde{\theta} : (y, s) \mapsto \tilde{h}_i(y)(\lambda_A^{-1} s)^{\frac{\alpha_i}{e_0}}$ is a semialgebraic function on $\tilde{A}$ the conclusion follows. ⊣

Theorem 5.3 and Corollary 5.5 are exactly analogous to Theorems 2.8 and 3.1 in [5], except that we obtain a slightly more precise equality of functions mod $(1 + \pi^n R)\mathbf{U}_e$ instead of equality of their norm (which is the same as equality of functions mod $R^\times$). Thus all the applications that are derived from these theorems in [5] for the classical analytic structure remain valid in every p-optimal field which satisfies the Extreme Value Property, with exactly the same proofs as in [5]. As already mentioned in the introduction some of these applications, which concern the constructibility of functions defined by parametric integrals and give the rationality of Poincaré series attached to definable functions, have already been generalised to strongly p-minimal fields in [2]. The other main application of Theorems 2.8 and 3.1 in [5] is the classification of subanalytic sets up to subanalytic bijections (Theorem 3.2 in [5]). It is not known at the moment if it holds true for strongly p-minimal fields.

THEOREM 5.6. *Assume that K is p-optimal and satisfies the Extreme Value Property. Then there exists a definable bijection between two infinite definable sets $A \subseteq K^m$ and $B \subseteq K^n$ if and only if they have the same dimension.*

PROOF. If there is a definable bijection (an "isomorphism") between A and B they have the same dimension by Corollary 6.4 in [10]. Conversely, if A and B have the same dimension d, then by Corollary 5.5 they are isomorphic to infinite semialgebraic sets $\tilde{A}$ and $\tilde{B}$ respectively, both of which have dimension d, by Corollary 6.4 in [10] again. Then $\tilde{A}$ and $\tilde{B}$ are semialgebraically isomorphic by the main result of [3], hence A and B are isomorphic. ⊣

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