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Analyticity and compactness of semigroups of composition operators

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Abstract

This paper provides a complete characterization of quasicontractive groups and analytic C_0-semigroups on Hardy and Dirichlet space on the unit disc with a prescribed generator of the form $Af = Gf'$. In the analytic case we also give a complete characterization of immediately compact semigroups. When the analyticity fails, we obtain sufficient conditions for compactness and membership in the trace class. Finally, we analyse the case where the unit disc is replaced by the right-half plane, where the results are drastically different.

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1 Introduction

Semigroups of composition operators acting on the Hardy space $H^2(\mathbb{D})$ or the Dirichlet space $\mathcal{D}$ have been extensively studied (see, for example, [3, 4, 6, 12, 20, 21]).

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These are associated with the notion of semiflow (φ_t) of analytic functions mapping the unit disc $\mathbb{D}$ to itself, and satisfying $\varphi_{s+t} = \varphi_s \circ \varphi_t$; here s and t lie either in $\mathbb{R}_+$ or in a sector of the complex plane. It is assumed that the mapping $(t, z) \mapsto \varphi_t(z)$ is jointly continuous. It follows that there exists an analytic function G on $\mathbb{D}$ such that

$$\frac{\partial \varphi_t}{\partial t} = G \circ \varphi_t.$$

A semiflow induces composition operators C_{φ_t} on $H^2(\mathbb{D})$ or $\mathcal{D}$, where $C_{\varphi_t} f = f \circ \varphi_t$. If it is strongly continuous, then it has a densely-defined generator A given by $Af = Gf'$, with G as above. Fuller details are given later.

In Section 2 we give a characterization of analytic semigroups in terms of the properties of G, using the complex Lumer–Phillips theorem [1] (this is appropriate, since the semigroup is quasicontractive, as explained below). In addition, we give a complete description of groups of composition operators in terms of the function G.

The theme of Section 3 is compactness, together with Hilbert–Schmidt and trace-class properties. For example, we give sufficient conditions on G for the semigroup to be immediately compact; these are necessary and sufficient (and equivalent to eventual compactness) when the semigroup is analytic. We give examples to illustrate the various possibilities involving the properties of immediate compactness and eventual compactness. Although most of our results are obtained in terms of the properties of G, we are also able to derive results on compactness from the semiflow model $\varphi_t(z) = h^{-1}(e^{-ct}h(z))$. In particular we are able to provide some answers to a question raised by Siskakis [21, Sec. 8] about how the behaviour of such semigroups depends on the properties of h.

Section 4 is concerned with analytic semigroups and groups of composition operators on the half-plane. Such operators are never compact.

2 Analytic semigroups and groups of composition operators

Definition 2.1. *Let $(\beta_n)_{n\geq 0}$ be a sequence of positive real numbers. Then $H^2(\beta)$ is the space of analytic functions*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

in the unit disc $\mathbb{D}$ that have finite norm

$$\|f\|_\beta = \left(\sum_{n=0}^{\infty} |c_n|^2 \beta_n^2 \right)^{1/2}.$$

The case $\beta_n = 1$ gives the usual Hardy space $H^2(\mathbb{D})$.
The case $\beta_0 = 1$ and $\beta_n = \sqrt{n}$ for $n \geq 1$ provides the Dirichlet space $\mathcal{D}$, which is included in $H^2(\mathbb{D})$.
The case $\beta_n = 1/\sqrt{n+1}$ produces the Bergman space, which contains $H^2(\mathbb{D})$.

2.1 General properties of semigroups

A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X is a mapping $T : \mathbb{R}_+ \to \mathcal{L}(X)$ satisfying

$$\begin{cases} & T(0) = I, \\ \forall t, s \geqslant 0, & T(t+s) = T(t) \circ T(s), \\ \forall x \in X, & \lim_{t\to 0} T(t)x = x. \end{cases}$$

A consequence of this definition is the existence of two scalars $w \geqslant 0$ and $M \geqslant 1$ such that for all $t \in \mathbb{R}_+$, $\|T(t)\| \leqslant Me^{wt}$. In particular, if $M = 1$, the semigroup T is said to be quasicontractive. If in addition $w = 0$, T is a contractive semigroup.

A C_0-semigroup T will be called analytic (or holomorphic) if there exists a sector $\Sigma_\theta = \{re^{i\alpha}, r \in \mathbb{R}_+, |\alpha| < \theta\}$ with $\theta \in (0, \frac{\pi}{2}]$ and an analytic mapping $\widetilde{T} : \Sigma_\theta \to \mathcal{L}(X)$ such that $\widetilde{T}$ is an extension of T and

$$\sup_{\xi \in \Sigma_\theta \cap \mathbb{D}} \|\widetilde{T}(\xi)\| < \infty.$$

In both cases, the generator of T (or $\widetilde{T}$) will be the linear operator A defined by

$$D(A) = \left\{ x \in X, \lim_{\mathbb{R} \ni t \to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and, for all $x \in D(A)$,

$$Ax = \lim_{\mathbb{R} \ni t \to 0} \frac{T(t)x - x}{t}.$$

Recall that an operator A is dissipative if $\mathrm{Re}\langle Ax, x \rangle \le 0$ for $x \in D(A)$. The classical Lumer–Phillips theorem asserts that A generates a contraction semigroup if and only if A is dissipative and $I - A$ is surjective (see, for example, [1, Thm. 3.4.5]).

The following extension of this to analytic semigroups is given in [2].

Proposition 2.2. *Let A be an operator on a complex Hilbert space H and let $\theta \in (0, \pi/2)$. The following are equivalent.*

(i) A generates an analytic C_0-semigroup which is contractive on the sector Σ_θ;

(ii) $e^{\pm i\theta} A$ is dissipative and $I - A$ is surjective.

From this we have the following corollary, which appears to be new.

Corollary 2.3. *Suppose that A is an operator on a Hilbert space and $\theta \in (0, \pi/2)$. If A and $\pm e^{i\theta} A$ generate quasicontractive semigroups, then A generates an analytic semigroup on the sector Σ_θ.*

Proof. There exist $\delta_1, \delta_2, \delta_3 \ge 0$ such that $A - \delta_1 I$, $e^{i\theta} A - \delta_2 I$ and $e^{-i\theta} A - \delta_3 I$ are all dissipative.

It follows that $A - \alpha I$, $e^{i\theta}(A - \alpha I)$ and $e^{-i\theta}(A - \alpha I)$ are all dissipative provided that $\alpha \ge \max\{\delta_1, \delta_2/\cos\theta, \delta_2/\cos\theta\}$. Moreover, $I - (A - \alpha I)$ is surjective, and so the result follows from Proposition 2.2.

□

2.2 An algebraic characterization of composition operators

The following characterization will be useful in order to show that an analytic semigroup consists of composition operators whenever its restriction to $\mathbb{R}_+$ has this property.

In [14, Thm. 5.1.13] it is shown that a bounded linear operator T on $H^2(\mathbb{D})$ is a composition operator if and only if, for the functions $e_n : z \mapsto z^n$, we have $Te_n = (Te_1)^n$ for all $n \in \mathbb{N}$. A similar characterization holds in the weighted Hardy space $H^2(\beta)$, with one supplementary condition.

Proposition 2.4. *Let $T : H^2(\beta) \to H^2(\beta)$ be a bounded linear operator. The operator T is a composition operator if and only if both $Te_1(\mathbb{D}) \subset \mathbb{D}$ and for all $n \in \mathbb{N}$, $Te_n = (Te_1)^n$.*

Proof. If $T = C_\varphi$ is a composition operator, then $Te_1 = \varphi : \mathbb{D} \to \mathbb{D}$ and $Te_n = \varphi^n = (Te_1)^n$ for all $n \in \mathbb{N}$.

Conversely, we note that $\varphi = Te_1 \in H^2(\beta)$. The function φ is analytic and maps $\mathbb{D}$ to $\mathbb{D}$. Besides, for every $n \in \mathbb{N}$, $Te_n = \varphi^n = C_\varphi e_n$. Thus, the linearity and the continuity of T and C_φ imply that $T = C_\varphi$. □

We require this for the following result, which applies in particular to the Hardy and Dirichlet spaces.

Corollary 2.5. *Let $T = (T(t))_{t\geq 0}$ be a C_0-semigroup of composition operators on $H^2(\beta)$, where $\beta_n = O(\sqrt{n})$. If T has an analytic extension to a sector Σ_θ, then for every $\xi \in \Sigma_\theta$, $T(\xi)$ is composition operator.*

Proof. Let $n \in \mathbb{N}$. We define $f_n : \Sigma_\theta \to H^2(\mathbb{D}),\ \xi \mapsto T(\xi)e_n - (T(\xi)e_1)^n$. As $T(t)$ is a composition operator for each $t \in \mathbb{R}_+$, the function f_n is zero on $\mathbb{R}_+$. Thus by analyticity of f_n on Σ_θ, $f_n \equiv 0$.

It remains to check that for all $\xi \in \Sigma_\theta$, $T(\xi)e_1(\mathbb{D}) \subset \mathbb{D}$. Supposing that this is not true, then there exists $\alpha \in \mathbb{D}$ such that $|T(\xi)e_1(\alpha)| \geq 1$. Since $T(\xi)e_1$ is analytic, then we can suppose that either we have $|T(\xi)e_1(\alpha)| > 1$ or that $T(\xi)e_1$ is a constant λ of modulus 1. In the first case $|T(\xi)e_n(\alpha)|/\|e_n\| = |T(\xi)e_1(\alpha)|^n/\|e_n\| \to \infty$, as $n \to \infty$ and this contradicts the boundedness of $T(\xi)$. In the second case, $T(\xi)$ maps e_n to λ^n (including $n = 0$), and thus, if it were bounded on $H^2(\beta)$, it would be given as the inner product with the function $\sum_{k=0}^\infty \lambda^k z^k/\beta_k^2$. However, this function does not lie in $H^2(\beta)$ as the square of its norm would be $\sum_{k=0}^\infty 1/\beta_k^2$, which diverges. □

A similar characterization holds for weighted composition operators.

Theorem 2.6. *Let $T : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be a linear and bounded operator. Assume that $Te_0 \not\equiv 0$ and $Te_0 \in H^\infty(\mathbb{D})$. Then, T is a weighted composition operator if and only if $(Te_0)^{n-1}Te_n = (Te_1)^n$ for all positive integers n.*

Proof. If T is the weighted composition operator $M_w C_\varphi$ defined by $Tf = wf \circ \varphi$, then it follows that $Te_0 = w$ and $Te_1 = w\varphi$. Therefore, for all $n \geq 1$, $(Te_0)^{n-1} Te_n = (Te_1)^n$ is satisfied.

Conversely, assume that for every $n \geq 1$, $(Te_0)^{n-1} Te_n = (Te_1)^n$. Let $w = Te_0 \in H^2(\mathbb{D})$. Since Te_0 is not identically zero, the set Z of its zeroes is discrete. For $z \in \mathbb{D} \backslash Z$, let $\varphi(z) = \frac{Te_1(z)}{Te_0(z)}$. It remains to check that $\varphi(\mathbb{D} \backslash Z) \subset \mathbb{D}$ for $z \in \mathbb{D} \backslash Z$.

Assume that there exists $z_0 \in \mathbb{D} \backslash Z$ such that $|\varphi(z_0)| > 1$. Then,

$$|w(z_0)||\varphi^n(z_0)| = |\langle Te_n, k_{z_0} \rangle| \leqslant \|T\| \|k_{z_0}\|,$$

which contradicts $|\varphi^n(z_0)| \to \infty$.

Assume now the existence of $z_0 \in \mathbb{D} \backslash Z$ such that $|\varphi(z_0)| = 1$. By the maximum principle $\varphi(z) = \lambda \in \mathbb{T}$ for every $z \in \mathbb{D} \backslash Z$. Thus, $Te_n = \lambda^n w = \lambda^n Te_0$ for all $n \in \mathbb{N}$. Hence we get

$$\|T^* Te_0\|^2 = \sum_{n \in \mathbb{N}} |\langle Te_0, Te_n \rangle|^2 = \sum_{n \in \mathbb{N}} |\lambda|^{2n} = \infty,$$

a contradiction. Thus we obtain $|\varphi(z)| < 1$ and $Te_n = w\varphi^n$ for all $n \in \mathbb{N}$. If $Te_0 \in H^\infty(\mathbb{D})$, then the continuity of $M_w C_\varphi$ implies that $T = M_w C_\varphi$. □

Corollary 2.7. *Let $T(t)$ be a C_0-semigroup of weighted composition operators on $H^2(\mathbb{D})$. If $T(t)$ has an analytic extension to a sector Σ_θ, then for every $\xi \in \Sigma_\theta$, $T(\xi)$ is a weighted composition operator.*

Proof. Fix $n \geq 1$. Define $f_n : \Sigma_\theta \to H^2(\mathbb{D})$ by

$$f_n(\xi) = (T(\xi)e_0)^{n-1} T(\xi)e_n - (T(\xi)e_1)^n.$$

Since $T(t)$ is a weighted composition operator for each $t > 0$, the function f_n vanishes on $\mathbb{R}^+$. Thus, the analyticity of f_n on Σ_θ implies that $f_n \equiv 0$.
If $Te_0 \equiv 0$, then $T(t)$ is trivial. Otherwise, the semigroup T being analytic, $\sup_{\mathbb{D} \cap \Sigma_\theta} \|T(\xi)\| < +\infty$. It follows that for all $\xi \in \mathbb{D} \cap \Sigma_\theta$, $\|T(\xi)e_0\| \leqslant M$; i.e., for all $\xi \in n\mathbb{D} \cap \Sigma_\theta$, $\|T(\xi)e_0\| \leqslant M^n$. Thus, for all $\xi \in \Sigma_\theta$, $T(\xi)e_0 \in H^\infty(\mathbb{D})$. The conclusion follows from Theorem 2.6. □

2.3 Quasicontractive analytic semigroups on the Hardy and Dirichlet space

In order to characterise quasicontractive analytic semigroups in terms of the associated function G we begin with the following result. Note that here and elsewhere we use [4, Thm. 3.9], which makes the hypothesis that $G \in H^2(\mathbb{D})$. This hypothesis ensures that the generator has dense domain, but is not necessary, as, for example the case $G(z) = -z/(z+1)$ shows: here $D(A)$ contains $(z+1)^2\mathbb{C}[z]$, which is dense in $H^2(\mathbb{D})$.

Theorem 2.8. *Let $G : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that the operator A defined by $Af(z) = G(z)f'(z)$ has dense domain $D(A) \subset H^2(\mathbb{D})$ (resp. $D(A) \subset \mathcal{D}$). Then the following are equivalent:*

1. *The operator A generates a quasicontractive analytic semigroup on $H^2(\mathbb{D})$ (resp. $\mathcal{D}$).*
2. *The operator A generates an analytic semigroup of composition operators on $H^2(\mathbb{D})$ (resp. $\mathcal{D}$).*
3. *There exists $\theta \in (0, \frac{\pi}{2})$ such that the operators $e^{i\theta}A$, $e^{-i\theta}A$ and A generate C_0-semigroups of composition operators on $H^2(\mathbb{D})$ (resp. $\mathcal{D}$).*
4. *There exists $\theta \in (0, \frac{\pi}{2})$ such that*

$$\sup \left\{ \mathrm{Re}\langle e^{\pm i\theta} Af, f\rangle : f \in D(A), \|f\| = 1 \right\} < \infty,$$

   and $\lambda > 0$ such that

$$(A - \lambda I)D(A) = H^2(\mathbb{D}) \ (resp.\ \mathcal{D}).$$

Proof.

1.⇒ 2. We denote by $(T(t))$ the semigroup on the sector Σ_θ generated by the operator A. From [4, Theorem 3.9], the restriction to $\mathbb{R}_+$ of this semigroup is a semigroup of composition operators. Now by Corollary 2.5 it follows that $(T(t))$ consists of composition operators.

2.⇒ 3. This is immediate.

3.$\Rightarrow$ 4. By [4, Theorem 3.9] any C_0-semigroup of composition operators is quasicontractive. The result now follows from the Lumer–Phillips theorem.

4.$\Rightarrow$ 1. This follows from an obvious corollary of Proposition 2.2.

$\square$

If G generates a semiflow of analytic functions on $\mathbb{D}$, then G has an expression of the form $G(z) = (\alpha - z)(1 - \overline{\alpha} z)F(z)$, where $\alpha \in \overline{\mathbb{D}}$ and $F : \mathbb{D} \to \mathbb{C}_+$ is holomorphic (see [6]). In particular, G has radial limits almost everywhere on $\mathbb{T}$, since F is the composition of a Möbius mapping and a function in $H^\infty(\mathbb{D})$. Note that this applies to every semigroup of composition operators, independently of the underlying Hilbert function space, since it is associated with a semiflow. In [4] it is shown that $A : f \mapsto Gf'$ generates a C_0-semigroup of composition operators on $H^2(\mathbb{D})$ or $\mathcal{D}$ if and only if $\operatorname{ess\,sup}_{z\in\mathbb{T}} \operatorname{Re} \overline{z} G(z) \le 0$. As before, it is not necessary to assume that $G \in H^2(\mathbb{D})$. This can now be applied to give easy conditions on the same operator A for the case of analytic semigroups.

Corollary 2.9. *Let $G : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that the operator A defined by $Af(z) = G(z)f'(z)$ has dense domain $D(A) \subset H^2(\mathbb{D})$ (resp. $D(A) \subset \mathcal{D}$). The operator A generates an analytic semigroup of composition operators on $H^2(\mathbb{D})$ (resp. $\mathcal{D}$) if and only if there exists $\theta \in (0, \frac{\pi}{2})$ such that for all $\alpha \in \{-\theta, 0, \theta\}$*

$$\operatorname*{ess\,sup}_{z\in\mathbb{T}} \operatorname{Re}(e^{i\alpha}\overline{z}G(z)) \le 0.$$

Geometrically, this condition says that the image of $\mathbb{T}$ under $z \mapsto \overline{z}G(z)$ is contained in a sector $-\Sigma_{(\frac{\pi}{2}-\theta)}$ in the left half-plane.

2.4 Groups of composition operators

The following remark enables one to characterize groups of composition operators on $H^2(\mathbb{D})$ and $\mathcal{D}$.

Proposition 2.10. *Let $G : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that the operator A defined by $Af(z) = G(z)f'(z)$ has dense domain $D(A) \subset H^2(\mathbb{D})$ (resp. $D(A) \subset \mathcal{D}$). The following are equivalent.*

1. *The operator A generates a C_0-group of composition operators.*

2. $\mathrm{Re}(\overline{z}G(z)) = 0$ *almost everywhere on* $\mathbb{T}$.

Proof. The result follows directly from [4, Thm. 3.9] since A generates a C_0-group of composition operators if and only if both A and $-A$ generate C_0-semigroups of composition operators. □

Corollary 2.11. *The only analytic group of composition operators on $H^2(\mathbb{D})$ or $\mathcal{D}$ is the trivial semigroup.*

Proof. This follows immediately from Corollary 2.9 and Proposition 2.10. □

This corollary can be seen as a consequence of more general results: an analytic group is norm-continuous at 0, and so its generator is bounded (see [22]). However, a non-trivial group of composition operators never has a bounded generator.

Example 2.12. *The semigroup (C_{φ_t}) of composition operators is a group if and only if for one (and thus any) $t_0 \in \mathbb{R}_+$, the operator $C_{\varphi_{t_0}}$ is invertible: thus, if and only if (φ_t) is a group of automorphisms. In that case, the group will satisfy $\varphi_t^{-1} = \varphi_{-t}$. Considering the automorphism semigroup given by*

$$\varphi_t(z) = \frac{z + \tanh t}{1 + z \tanh t}$$

with generator $G(z) = 1 - z^2$, we see that $\overline{z}G(z) = -2i\,\mathrm{Im}(z) \in i\mathbb{R}$. Thus the given condition is satisfied.

The easy example $G(z) = z(z-1)$ shows that it is possible for a C_0-semigroup of composition operators to be neither analytic nor a group.

3 Compactness of semigroups

3.1 Immediate and eventual compactness

We recall that a semigroup $(T(t))_{t\geq 0}$ is said to be *immediately compact* if the operators $T(t)$ are compact for all $t > 0$. A semigroup $(T(t))_{t\geq 0}$ is said to be *eventually compact* if there exists $t_0 > 0$ such that $T(t)$ is compact for all $t \geq t_0$. Similar definitions hold for immediately/eventually Hilbert–Schmidt and trace-class.

We begin with an elementary observation.

Proposition 3.1. *Suppose that for some $t_0 > 0$ one has $|\varphi_{t_0}(\xi)| = 1$ on a set of positive measure; then $C_{\varphi_{t_0}}$ is not compact on $H^2(\mathbb{D})$ or $\mathcal{D}$, and so the semigroup $(C_{\varphi_t})_{t\geq 0}$ is not immediately compact.*

Proof. For the Hardy space, this follows since the weakly null sequence $(e_n)_{n\geq 0}$ with $e_n(z) = z^n$ is mapped into $(\varphi_{t_0}^n)$, which does not converge to 0 in norm. For the Dirichlet space the result follows from [9, Ex. 6.3]. $\square$

A slightly stronger result can be shown for the Hardy space, using the following theorem [16, Chap. 2, Thm 3.3], which links immediate compactness with continuity in norm.

Theorem 3.2. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup and let A be its infinitesimal generator. Then $(T(t))_{t\geq 0}$ is immediately compact if and only if*
(i) the resolvent $R(\lambda, A)$ is compact for all (or for one) $\lambda \in \mathbb{C} \setminus \sigma(A)$, and
(ii) $\lim_{s\to t} \|T(s) - T(t)\| = 0$ for all $t > 0$.

Combining this with the following result due to Berkson [5], we see that, under the hypotheses of of Proposition 3.1, in the case of the Hardy space, the semigroup is not norm-continuous and hence not immediately compact.

Theorem 3.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. If $m\{\xi \in \mathbb{T} : |\varphi(\xi)| = 1\} = \delta > 0$, then considered as operators on $H^2(\mathbb{D})$, we have $\|C_\varphi - C_\psi\| \geq \sqrt{\delta/2}$ for all $\psi : \mathbb{D} \to \mathbb{D}$ analytic with $\psi \neq \varphi$.*

We shall now give a sufficient condition for immediate compactness of a semigroup of composition operators, in terms of the associated function G. First, we recall a classical necessary and sufficient condition for compactness of a composition operator C_φ in the case when φ is univalent [8, pp. 132, 139].

Theorem 3.4. *For $\varphi : \mathbb{D} \to \mathbb{D}$ analytic and univalent, the composition operator C_φ is compact on $H^2(\mathbb{D})$ if and only if*

$$\lim_{z\to\xi} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0$$

for all $\xi \in \mathbb{T}$.

The following proposition collects together standard results on Hilbert–Schmidt and trace-class composition operators.

Proposition 3.5. *For $\varphi : \mathbb{D} \to \mathbb{D}$ analytic with $\|\varphi\|_\infty < 1$, the composition operator C_φ is trace-class on $H^2(\mathbb{D})$ [8, p. 149]; if in addition $\varphi \in \mathcal{D}$, then C_φ is Hilbert–Schmidt on $\mathcal{D}$ [9, Cor. 6.3.3].*

Siskakis [21] has given sufficient conditions for compactness of the resolvent operator $R(\lambda, A)$ (which is a necessary condition for the immediate compactness of the semigroup), in the case $G(z) = -zF(z)$, although they are not necessary, as the case $G(z) = -z$ illustrates.

Theorem 3.6. *Let $\delta > 0$; suppose that there exists $\epsilon > 0$ such that*

$$\mathrm{Re}(\overline{z}G(z)) \leq -\delta \qquad \textit{for all} \quad z \quad \textit{with} \quad 1-\epsilon < |z| < 1.$$

Then A, defined by $Af = Gf'$, generates an immediately compact semigroup of composition operators on $H^2(\mathbb{D})$ and $\mathcal{D}$. Indeed the semigroup is immediately trace-class.

Proof. For $t > 0$ and $z \in \mathbb{D}$ we have

$$\begin{aligned} \frac{\partial}{\partial t}|\varphi_t(z)|^2 &= 2\,\mathrm{Re}\left(\overline{\varphi_t(z)}\frac{\partial}{\partial t}\varphi_t(z)\right) \\ &= 2\,\mathrm{Re}\left(\overline{\varphi_t(z)}G(\varphi_t(z))\right) \end{aligned}$$

Hence

$$\frac{\partial}{\partial t}|\varphi_t(z)|^2 \leq -2\delta$$

whenever $1-\epsilon < |\varphi_t(z)| < 1$. Also, by compactness, there exists an $M > 0$ such that

$$\frac{\partial}{\partial t}|\varphi_t(z)|^2 \leq M$$

whenever $|\varphi_t(z)| \leq 1-\epsilon$.

Choose $t_0 > 0$ such that

$$a := (1-\epsilon)^2 + t_0 M < 1.$$

For a fixed $z \in \mathbb{D}$ we consider $\varphi_t(z)$ over the interval $[0, t_0]$. Suppose first that $|\varphi_t(z)| > (1-\epsilon)$ for all $t \in [0, t_0]$. Then $|\varphi_t(z)|^2 \leq |z|^2 - 2\delta t$ for all $t \in [0, t_0]$.

Otherwise let

$$t_1 = \inf\left\{t \in [0, t_0] : |\varphi_t(z)| \leq 1-\epsilon\right\}.$$

Then $|\varphi_t(z)|^2 \le a$ for all $t \in [t_1, t_0]$ and $|\varphi_t(z)|^2 \le |z|^2 - 2\delta t$ for all $t \in [0, t_1]$.

Thus $\|\varphi_t\|_\infty^2 \le \max\{1 - 2\delta t, a\} < 1$ for $0 < t \le t_0$ and hence C_{φ_t} is Hilbert–Schmidt for these t, by Proposition 3.5, and hence trace-class for all $t > 0$ since $C_{\varphi_t} = C^2_{\varphi_{t/2}}$. □

Corollary 3.7. *Let $\eta > 0$; suppose that* $\operatorname{ess\,sup}_{z\in\mathbb{T}} \operatorname{Re}(\overline{z}G(z)) \le -\eta$, *and that* $\operatorname{Re} G'$ *is bounded on* $\mathbb{D}$. *Then A, defined by $Af = Gf'$, generates an immediately trace-class semigroup of composition operators on $H^2(\mathbb{D})$ and $\mathcal{D}$.*

Proof. Define $K(z) := G(z) + \eta z$, so that $\operatorname{ess\,sup}_{z\in\mathbb{T}} \operatorname{Re}(\overline{z}K(z)) \le 0$. By [4, Thm. 4.3], it follows that

$$2\operatorname{Re}(\overline{z}K(z)) + (1 - |z|^2)\operatorname{Re} K'(z) \le 0 \qquad (z \in \mathbb{D}),$$

and hence

$$2\operatorname{Re}(\overline{z}G(z)) + (1 - |z|^2)\operatorname{Re} G'(z) \le -\eta(1 + |z|^2) \le -\eta \qquad (z \in \mathbb{D}).$$

Now if $\|\operatorname{Re} G'(z)\|_\infty \le M$, then we have $\operatorname{Re}(\overline{z}G(z)) \le -\eta$ whenever $|z| \ge 1 - \dfrac{\eta}{2M}$. The result now follows from Theorem 3.6. □

Easy examples of the above are $G(z) = -z$ and $G(z) = z(z^2 - 2)$. However, Siskakis [21] gives the example $G(z) = (1 - z)\log(1 - z)$, where the semigroup is immediately compact while $\operatorname{ess\,sup} \operatorname{Re} \overline{z}G(z) = 0$ on $\mathbb{T}$.

Remark 3.8. Note that all the examples of immediately compact semigroups have the Denjoy–Wolff point of φ_t in the open disc $\mathbb{D}$. This is always the case: for let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic, such that C_φ is a compact composition operator on $H^2(\mathbb{D})$. Then the Denjoy–Wolff point of φ lies in $\mathbb{D}$, since if φ has its Denjoy–Wolff point on $\mathbb{T}$, then φ has an angular derivative there, of modulus at most 1. But this contradicts the compactness of C_φ, by [8, Cor. 3.14].

3.2 Applications of the semiflow model

In this section we work with an immediately compact semigroup $(C_{\varphi_t})_{t\ge 0}$ acting on $H^2(\mathbb{D})$ or $\mathcal{D}$. As in Remark 3.8 we know that the Denjoy–Wolff point of the semiflow $(\varphi_t)_{t\geq 0}$ lies in $\mathbb{D}$, and by conjugating by the automorphism b_α, where

$$b_\alpha(z) := \frac{\alpha - z}{1 - \overline{\alpha}z},$$

we may suppose without loss of generality that $\alpha = 0$. In this case there is a semiflow model

$$\varphi_t(z) = h^{-1}(e^{-ct}h(z)),$$

where $c \in \mathbb{C}$ with $\operatorname{Re} c \geq 0$, and $h : \mathbb{D} \to \Omega$ is a conformal bijection between $\mathbb{D}$ and a domain $\Omega \subset \mathbb{C}$, with $h(0) = 0$ and Ω is spiral-like or star-like (if c is real), in the sense that

$$e^{-ct}w \in \Omega \quad \text{for all} \quad w \in \Omega \quad \text{and} \quad t \geq 0.$$

For more details we refer to [20, 21].

Even in the case when $(C_{\varphi_t})_{t\geq 0}$ is only eventually compact, we have $\operatorname{Re} c > 0$. Indeed, if $c = i\theta$ with $\theta \in \mathbb{R}$, then there exist arbitrarily large $t > 0$ such that $\theta t \in 2\pi\mathbb{Z}$, and then C_{φ_t} is the identity mapping, and hence not compact.

Lemma 3.9. *Let $(\varphi_t)_{t\geq 0}$ be a semiflow on $\mathbb{D}$ with Denjoy–Wolff point 0. Then the following are equivalent:*

1. *There is a $t_0 > 0$ with $\|\varphi_{t_0}\|_\infty < 1$;*
2. *There is a $t_0 > 0$ with $\|\varphi_t\|_\infty < 1$ for all $t \geq t_0$;*
3. *In the semiflow model for $(\varphi_t)_{t\geq 0}$, $\operatorname{Re} c > 0$, and the domain Ω is bounded.*

Proof. 1. $\Rightarrow$ 2. This follows since $\varphi_t(z) = \varphi_{t_0}(\varphi_{t-t_0}(z))$ for all $t \geq t_0$.
2. $\Rightarrow$ 3. Since φ_t is not the identity mapping on $\mathbb{D}$ for all $t \geq t_0$, we have that $\operatorname{Re} c > 0$ by the argument above. Assume that Ω is unbounded; then there is a sequence $(z_n)_n$ in Ω with $|z_n| \to \infty$; clearly also $|e^{-ct}z_n| \to \infty$ for each fixed $t \geq 0$. Since $\|\varphi_{t_0}\|_\infty < 1$, there exists a subsequence $(z_{n_k})_k$ of $(z_n)_n$ such that $(h^{-1}(e^{-ct_0}z_{n_k}))_k$ tends to $\xi \in \mathbb{D}$. Therefore $e^{-ct_0}z_{n_k} \to h(\xi)$, a contradiction since $(z_{n_k})_k$ is unbounded.
3. $\Rightarrow$ 1. If $M = \sup\{|z| : z \in \Omega\} < \infty$ and $\operatorname{Re} c > 0$, then for all $\epsilon > 0$ we have

$$|e^{-ct}h(z)| \leq e^{-(\operatorname{Re} c)t}M < \epsilon$$

for t sufficiently large. Then, since $h(0) = 0$, h^{-1} is continuous, and $\varphi_t(z) = h^{-1}(e^{-ct}h(z))$, it follows that $\|\varphi_t\|_\infty < 1$ for sufficiently large t. □

We recall that a topological space is locally connected if every point has a neighbourhood base of connected open sets.

Theorem 3.10. *Let $(C_{\varphi_t})_{t\geq 0}$ be an immediately compact semigroup on $H^2(\mathbb{D})$ or $\mathcal{D}$, such that in the semiflow model $\partial\Omega$ is locally connected. Then the following conditions are equivalent:*
1. There exists a $t_0 > 0$ such that $\|\varphi_{t_0}\|_\infty < 1$;
2. For all $t > 0$ one has $\|\varphi_t\|_\infty < 1$.
Therefore, if there exists a $t_0 > 0$ such that $\|\varphi_{t_0}\|_\infty < 1$, then $(C_{\varphi_t})_{t\geq 0}$ is immediately trace-class.

Proof. The only thing to prove is that 1. $\Rightarrow$ 2.

We work with the semiflow model with Denjoy-Wolff point 0, so that

$$\varphi_t(z) = h^{-1}(e^{-ct}h(z)), \tag{1}$$

with $h : \mathbb{D} \to \Omega$ and c as above. Assume that for some $t_1 > 0$ we have $\|\varphi_{t_1}\|_\infty = 1$. Then there is a sequence $(z_n)_n$ in $\mathbb{D}$ with $h^{-1}(e^{-ct_1}h(z_n)) \to e^{i\theta} \in \mathbb{T}$.

Since, by Lemma 3.9, Ω is bounded, there is a subsequence of $(h(z_n))_n$ converging to a point $\xi_1 \in \overline{\Omega}$. Moreover ξ_1 lies in $\partial\Omega$, as otherwise $e^{-ct_1}\xi_1 \in \Omega$, so $h^{-1}(e^{-ct_1}\xi_1) \in \mathbb{D}$, which is a contradiction. We also have $e^{-ct_1}\xi_1 \in \partial\Omega$.

It follows that the arc $\{e^{-ct}\xi_1 : 0 \leq t \leq t_1\}$ is contained in $\partial\Omega$, and since $\partial\Omega$ is locally connected, the mapping h extends continuously to $\overline{\mathbb{D}}$, and maps $\mathbb{T}$ onto $\partial\Omega$ [18, Thm. 2.1, p. 20]. Thus $h^{-1}\{e^{-ct}\xi_1 : 0 \leq t \leq t_1/2\}$ is a subset of $\mathbb{T}$ of positive measure, on which $|\varphi_{t_1/2}(z)| = 1$. Thus $C_{\varphi_{t_1/2}}$ is not compact by Proposition 3.1, which is a contradiction. $\square$

We shall use similar methods to study the compactness of analytic semigroups, which is the subject of the next subsection.

3.3 Compact analytic semigroups

In the particular case of analytic semigroups, the compactness is equivalent to the compactness of the resolvent, by Theorem 3.2, since the analyticity implies the uniform continuity [11, p. 109].

Remark 3.11. For an analytic semigroup $(T(t))_{t\geq 0}$, being eventually compact is equivalent to be immediately compact. Indeed, consider Q the quotient map from the linear and bounded operators on a Hilbert space $\mathcal{L}(\mathcal{H})$, onto the Calkin algebra (the quotient of $\mathcal{L}(\mathcal{H})$ by the compact operators). Then $(QT(t))_{t\geq 0}$ is an analytic semigroup which vanishes for $t > 0$ large enough, and therefore vanishes identically (see [17], where this observation is attributed to W. Arendt).

Before stating the complete characterization of compact and analytic semigroups of composition operators on $H^2(\mathbb{D})$ in terms of properties of its generator, we need the following key lemma which appears in the proof of Theorem 6.1 of [21].

Lemma 3.12. *Let $(\psi_t)_{t\geq 0}$ be a semiflow of analytic functions from $\mathbb{D}$ to $\mathbb{D}$, with common Denjoy–Wolff fixed point 0, and denote by G its infinitesimal generator. Then the resolvent operator of the semigroup of composition operators $(C_{\psi_t})_{t\geq 0}$ on $H^2(\mathbb{D})$ is compact if and only if*

$$\forall \xi \in \mathbb{T}, \lim_{z\to\xi, z\in\mathbb{D}} \left|\frac{G(z)}{z-\xi}\right| = \infty.$$

Theorem 3.13. *Let $G : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that the operator A defined by $Af(z) = G(z)f'(z)$ with dense domain $D(A) \subset H^2(\mathbb{D})$ generates an analytic semigroup $(T(t))_{t\geq 0}$ of composition operators. Then the following assertions are equivalent:*

1. *$(T(t))_{t\geq 0}$ is immediately compact;*
2. *$(T(t))_{t\geq 0}$ is eventually compact;*
3. *$\forall \xi \in \mathbb{T}$, $\lim_{z\in\mathbb{D}, z\to\xi} \left|\frac{G(z)}{z-\xi}\right| = \infty$.*

Proof. The equivalence between 1. and 2. is given in Remark 3.11.

We now show that 1. implies 3. Suppose that $T(t) = C_{\varphi_t}$, where all φ_t have a common Denjoy–Wolff fixed point $\alpha \in \overline{\mathbb{D}}$. Assume from now that $(T(t))_{t\geq 0}$ is immediately compact. As in Remark 3.8, it follows that $\alpha \in \mathbb{D}$. In order to use Lemma 3.12 we will consider another semigroup with Denjoy–Wolff point 0. To that aim, consider the automorphism $b_\alpha(z) := \frac{\alpha - z}{1-\overline{\alpha}z}$ and $\psi_t(z) := b_\alpha \circ \varphi_t \circ b_\alpha$. Since C_{b_α} is invertible (equal to its inverse), and since $C_{\psi_t} = C_{b_\alpha} C_{\varphi_t} C_{b_\alpha}$, it is clear that $(T(t))_{t\geq 0}$ is immediately compact if and only if $(C_{\psi_t})_{t\geq 0}$ is immediately compact.

Denote by $\widetilde{G}$ (resp. G) the generator of the semiflow $(\psi_t)_{t\geq 0}$ (resp. $(\varphi_t)_{t\geq 0}$). By [6], $G(\alpha) = 0$. Moreover, since $b_\alpha \circ \psi_t = \varphi_t \circ b_\alpha$, we get

$$\frac{|\alpha|^2 - 1}{(1-\overline{\alpha}\psi_t(z))^2} \frac{\partial \psi_t}{\partial t}(z) = \frac{\partial \varphi_t}{\partial t}(b_\alpha(z)).$$

Taking the limit as t tends to 0, we get:

$$\frac{|\alpha|^2 - 1}{(1 - \overline{\alpha} z)^2} \widetilde{G}(z) = G(b_\alpha(z)),$$

and thus

$$G(z) = \frac{|\alpha|^2 - 1}{(1 - \overline{\alpha} b_\alpha(z))^2} \widetilde{G}(b_\alpha(z)).$$

It follows that

$$\frac{(1 - |\alpha|^2)}{4} |\widetilde{G}(b_\alpha(z))| \leq |G(z)| \leq \frac{(1 + |\alpha|)}{(1 - |\alpha|)} |\widetilde{G}(b_\alpha(z))|,$$

and then

$$\lim_{z \to \xi} \left| \frac{G(z)}{z - \xi} \right| = \infty \quad \Longleftrightarrow \quad \lim_{z \to \xi} \left| \frac{\widetilde{G}(b_\alpha(z))}{z - \xi} \right| = \infty.$$

Note that

$$b_\alpha(z) - b_\alpha(\xi) = (z - \xi) \frac{|\alpha|^2 - 1}{(1 - \overline{\alpha} z)(1 - \overline{\alpha} \xi)},$$

with

$$\frac{1 - |\alpha|}{1 + |\alpha|} \leq \left| \frac{(1 - \overline{\alpha} z)(1 - \overline{\alpha} \xi)}{|\alpha|^2 - 1} \right| \leq \frac{4}{1 - |\alpha|^2}.$$

Therefore we get

$$\lim_{z \to \xi} \left| \frac{G(z)}{z - \xi} \right| = \infty \quad \Longleftrightarrow \quad \lim_{z \to \xi} \left| \frac{\widetilde{G}(b_\alpha(z))}{b_\alpha(z) - b_\alpha(\xi)} \right| = \infty.$$

Using Lemma 3.12 and since $b_\alpha(\xi) \in \mathbb{T}$, it follows that 1. implies 3..

For the implication 3. implies 1., we see from [7, Thm. 1], that the Denjoy–Wolff point of the semigroup must belong to the unit disc, as otherwise there is $\tau \in \partial\mathbb{D}$ such that the angular limit of $\frac{G(z)}{z-\tau}$, as $z \to \tau$, is zero. The conclusion now follows along the same lines as the previous implication. $\square$

Using the semiflow model, we have the following result.

Theorem 3.14. *Let $(C_{\varphi_t})_{t\geq 0}$ be an immediately compact analytic semigroup on $H^2(\mathbb{D})$ or $\mathcal{D}$. Then the following conditions are equivalent:*
1. There exists a $t_0 > 0$ such that $\|\varphi_{t_0}\|_\infty < 1$;
2. For all $t > 0$ one has $\|\varphi_t\|_\infty < 1$.
Therefore, if there exists a $t_0 > 0$ such that $\|\varphi_{t_0}\|_\infty < 1$, then $(C_{\varphi_t})_{t\geq 0}$ is immediately trace-class.

Proof. Note that in the semiflow model, the semigroup is represented by (1) for all $t \in \Sigma_\alpha$: this is the correct extension, by Corollary 2.5 and the isolated zeroes result for analytic functions.

Following the proof of Theorem 3.10, we take a $t_1 > 0$ such that $\|\varphi_{t_1}\|_\infty = 1$. This implies that there exists $\xi_1 \in \partial\Omega$ such that $e^{-cu}\xi_1 \in \partial\Omega$ for all u in the triangle $\Sigma_\alpha \cap \{z \in \mathbb{C} : \operatorname{Re} z < t_1\}$ (cf. Lemma 3.9). This is a contradiction since a point in $\partial\Omega$ cannot have a neighbourhood consisting of points of $\partial\Omega$. □

3.4 Examples

In Remark 3.11 we saw that whenever a semigroup is analytic, immediate compactness is equivalent to eventual compactness. This is not true in general, and here is an explicit example showing this, based on an idea in [21, Sec. 3].

Example 3.15. *Let h be the Riemann map from $\mathbb{D}$ onto the starlike region*

$$\Omega := \mathbb{D} \cup \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < 2 \text{ and } 0 < \operatorname{Im}(z) < 1\},$$

with $h(0) = 0$. Since $\partial\Omega$ is a Jordan curve, the Carathéodory theorem [18, Thm 2.6, p. 24] implies that h extends continuously to $\partial\mathbb{D}$.

Let $\varphi_t(z) = h^{-1}(e^{-t}h(z))$. Note that for $0 < t < \log 2$, $\varphi_t(\mathbb{T})$ intersects $\mathbb{T}$ on a set of positive measure, and thus, C_{φ_t} is not compact by Proposition 3.1. Moreover, for $t > \log 2$, $\|\varphi_t\|_\infty < 1$, and therefore C_{φ_t} is compact (actually trace-class). Figure 1 represents the image of φ_t for different values of t.

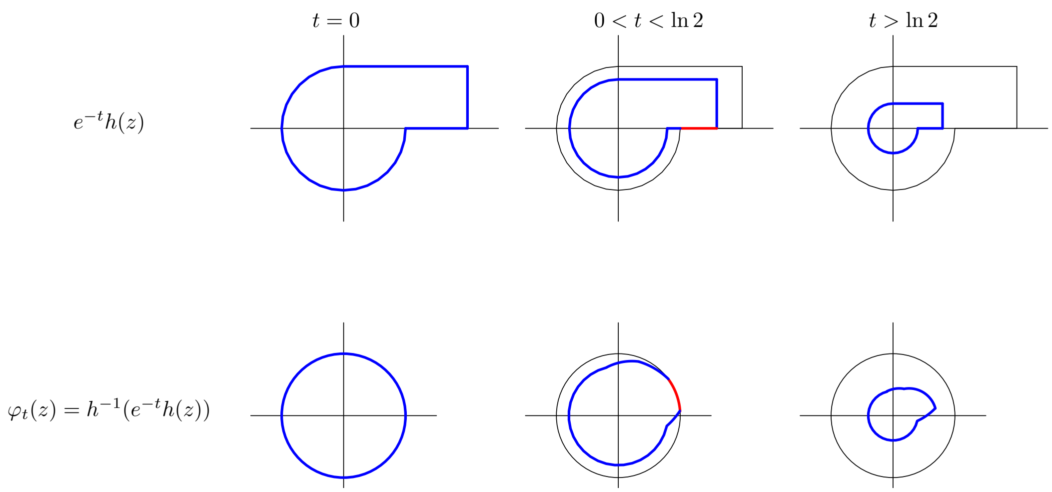

Figure 1: an example of eventual but not immediate compactness

It is of interest to consider the relation between immediate compactness and analyticity for a C_0-semigroup of composition operators: this is because compactness of a semigroup $(T(t))_{t\geq 0}$ is implied by compactness of the resolvent together with norm-continuity at all points $t>0$, as in Theorem 3.2.

Example 3.16. *Consider*

$$G(z)=\frac{2z}{z-1},$$

Now the image of the unit circle under $\overline{z}G(z)$ *is the line* $\{z\in\mathbb{C}:\operatorname{Re} z=-1\}$, *and so the operator* $A:f\mapsto Gf'$ *generates a non-analytic* C_0*-semigroup of composition operators* $(C_{\varphi_t})_{t\geq 0}$ *on* $H^2(\mathbb{D})$. *On the other hand, it can be shown that* C_{φ_t} *is compact – even trace-class – for each* $t>0$. *For we have the equation*

$$\varphi_t(z)e^{-\varphi_t(z)}=e^{-2t}ze^{-z}.$$

Now the function $z\mapsto ze^{-z}$ *is injective on* $\overline{\mathbb{D}}$; *this follows from the argument principle, for the image of* $\mathbb{T}$ *is easily seen to be a simple Jordan curve. It follows that* $\|\varphi_t\|_\infty<1$ *for all* $t>0$, *and so* C_{φ_t} *is trace-class.*

Example 3.17. *The semigroup corresponding to* $G(z) = (1-z)^2$ *is analytic but not immediately compact. For*

$$\varphi_t(z) = \frac{(1-t)z+t}{-tz+1+t}$$

(note that the formula given in [21] contains a misprint); the Denjoy–Wolff point is 1*, so the semigroup cannot be immediately compact.*

The analyticity follows on calculating $\overline{z}G(z)$ *for* $z = e^{i\theta}$*. We obtain* $-4\sin^2(\theta/2)$*, which gives the result by Corollary 2.9.*

Example 3.18. *Let* φ *be the Riemann map from* $\mathbb{D}$ *onto the semi-disc defined by* $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0, |z-1/2| < 1/2\}$ *which fixes* 1*. Lotto [13] proved that* C_φ *is compact but not Hilbert–Schmidt. Moreover, Lotto gave an explicit formula for* φ*, namely:*

$$\varphi(z) = \frac{1}{1-ig(z)}, \quad \textit{where} \quad g(z) = \sqrt{i\frac{1-z}{1+z}}.$$

Since $(z^n)_n$ *is an orthonormal basis of* $H^2(\mathbb{D})$*, it follows that a composition operator* C_ψ *on* $H^2(\mathbb{D})$ *is Hilbert–Schmidt if and only if* $\int_0^{2\pi} \frac{1}{1-|\psi(e^{it})|^2} dt < \infty$ *(see [19]). It is then possible to check that* $C_{\varphi\circ\varphi}$ *is Hilbert-Schmidt, providing an example of a* discrete *immediately compact semigroup that is not immediately Hilbert–Schmidt, but is eventually Hilbert–Schmidt.*

4 Composition semigroups on the half-plane

4.1 Quasicontractive C_0-semigroups

Let $\mathbb{C}_+$ denote the right half-plane in $\mathbb{C}$. Berkson and Porta [6] gave the following criterion for an analytic function G to generate a one-parameter semigroup of analytic mappings from $\mathbb{C}_+$ into itself, namely, solutions to the initial value problem

$$\frac{\partial \varphi_t(z)}{\partial t} = G(\varphi_t(z)), \qquad \varphi_0(z) = z, \tag{2}$$

namely the condition

$$x\frac{d(\operatorname{Re} G)}{\partial x} \leq \operatorname{Re} G \quad \text{on } \mathbb{C}_+, \tag{3}$$

where as usual $x = \operatorname{Re} z$. Note that this does not automatically yield a C_0-semigroup of bounded composition operators, since not all composition operators C_φ are bounded on $H^2(\mathbb{C}_+)$. In fact the norm of such a composition operator is finite if and only if the non-tangential limit $\angle \lim_{z\to\infty} \varphi(z)/z$ exists and is non-zero; we denote this by $\varphi'(\infty)$ (it is positive), and in this case $\|C_\varphi\| = \varphi'(\infty)^{-1/2}$. See [10] for more details.

In fact, from the proof of [6, Thm. 2.13] one sees that if the operator A gven by $Af = Gf'$ generates a semigroup and (3) is satisfied, then the semigroup consists of composition operators on $H^2(\mathbb{C}_+)$. Moreover Arvanitidis [3] used the results of [7] to show that a necessary and sufficient condition for these composition operators to be bounded is that the non-tangential limit $\angle \lim_{z\to\infty} G(z)/z$ exists: if it has the value δ, then $\|C_{\varphi_t}\| = e^{-\delta t/2}$, and so the semigroup is quasicontractive. We may summarize the results above as follows.

Theorem 4.1. *For an operator A given by $Af = Gf'$ on $D(A) \subset H^2(\mathbb{C}_+)$, the following are equivalent.*
(i) A generates a quasi-contractive C_0-semigroup of bounded composition operators on $H^2(\mathbb{C}_+)$;
(ii) Condition (3) holds and $\angle \lim_{z\to\infty} G(z)/z$ exists.

A necessary condition for generation of a (quasi)contractive semigroup is given by the following result:

Theorem 4.2. *For an operator A given by $Af = Gf'$ on $D(A) \subset H^2(\mathbb{C}_+)$, if A generates a quasicontractive C_0-semigroup, then*

$$\inf_{w\in\mathbb{C}_+} \frac{\operatorname{Re} G(w)}{\operatorname{Re} w} > -\infty.$$

If the semigroup is contractive, then $(\operatorname{Re} G(w))/(\operatorname{Re} w) \geq 0$ for $w \in \mathbb{C}_+$.

Proof. Let k_w be the reproducing kernel for $H^2(\mathbb{C}_+)$ given by

$$k_w(z) = \frac{1}{2\pi}\frac{1}{z+\overline{w}} \qquad (z, w \in \mathbb{C}_+)$$

(cf. [15, p. 8]). Then

$$\operatorname{Re} \frac{\langle Ak_w, k_w\rangle}{\langle k_w, k_w\rangle} = -\frac{\operatorname{Re} G(w)}{2\operatorname{Re} w},$$

and the result follows immediately from the Lumer–Phillips theorem. □

For analytic semigroups, we have the following necessary condition.

Proposition 4.3. *Suppose that $A : f \to Gf'$ generates an analytic semigroup on $H^2(\mathbb{C}_+)$. Write $G = u + iv$ where u and v are real functions and similarly $z = x + iy$. Then there is an α with $0 < \alpha < \pi/2$ such that, for every fixed $y \in \mathbb{R}$, $(u\cos\theta + v\sin\theta)/x$ is a decreasing function of x for all $\theta \in [-\alpha, \alpha]$.*

Proof. Note that the criterion (3) may be rewritten as

$$\frac{\partial}{\partial x}\left(\frac{u}{x}\right) \leq 0.$$

The result now follows immediately on applying this to the functions $Ge^{-i\theta}$, which generate C_0-semigroups. $\square$

4.2 Groups of composition operators on the half-plane

It turns out that there are very few groups of composition operators on the half-plane.

Proposition 4.4. *Suppose that $A : f \to Gf'$ generates a C_0 quasicontractive group of bounded composition operators on $H^2(\mathbb{C}_+)$. Then $G(z) = pz + iq$ for some real p and q, and hence, for $t \in \mathbb{R}, z \in \mathbb{C}_+$, we get*

$$\varphi_t(z) = ze^{pt} + \frac{iq}{p}\left(e^{pt} - 1\right) \tag{4}$$

if $p \neq 0$, and

$$\varphi_t(z) = z + iqt \tag{5}$$

if $p = 0$.

Proof. Once again we begin with condition (3), applying it to G and $-G$, to obtain

$$x\frac{\partial u}{\partial x} = u.$$

The solution to this is $u = F(y)x$ for some smooth real function of y. The Cauchy–Riemann equations imply that $\dfrac{\partial v}{\partial y} = F(y)$; that is, $v = \int F\, dy + E(x)$ for some function E. Likewise,

$$\frac{\partial u}{\partial y} = F'(y)x = -E'(x),$$

and thus $F(y) = ay + b$ and $E(x) = -ax^2/2 + c$ for some real constants a, b and c. We conclude that $G(z) = -iaz^2/2 + bz - ic$. Now Theorem 4.1 implies that $a = 0$ and the result for G follows. It is now clear from (2) that φ_t is as given in (4) or (5). □

Example 4.5. *Taking $G(z) = pz + iq$ with $p, q \in \mathbb{R}$ and $q \neq 0$, we obtain a group which is not analytic.*

Moreover, taking $G(z) = 1 - z$ (so that $\varphi_t(z) = e^{-t}z + 1 - e^{-t}$), we get an example of C_0-semigroup which is neither a group nor analytic.

Remark 4.6. The question of characterizing the compact semigroups is not relevant, since no composition operator on the Hardy space of the half-plane is compact [10, Cor. 3.3].

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