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A Kriging Algorithm for Location Fingerprinting based on Received Signal Strength

Chao Liu, Aroland Kiring, Naveed Salman, Lyudmila Mihaylova, Iñaki Esnaola,
Department of Automatic Control and Systems Engineering, University of Sheffield, S1 3JD, UK
{cliu47, amkiring1, n.salman, l.s.mihaylova, esnaola}@sheffield.ac.uk

***Abstract*—Received signal strength (RSS) based location fingerprinting is a powerful wireless positioning technique. It estimates the target location by consulting a preliminary database and searching for the best matched RSS fingerprints. The construction and maintenance of a sufficient fingerprint database could be laborious and problematic. This paper proposes a new approach that utilizes the Kriging spatial interpolation algorithm to build complete fingerprint databases from sparsely collected measurements. The interpolation performance is analyzed over various extents of sparsity and number of measurements. The constructed fingerprint databases are utilized to locate a static target and the localization performances are analyzed. It is shown that the Kriging algorithm can be used to build RSS fingerprint databases of good accuracy based on sparsely collected measurements.**

***Index Terms*—Localization, received signal strength, fingerprinting, spatial interpolation, variogram, Kriging.**

I. INTRODUCTION

With prevalence of personal and industrial wireless devices, wireless positioning techniques have become an increasingly popular research topic in recent years. The global positioning system (GPS) is the most widely used technique in outdoor environments and its accuracy meets the requirement of most outdoor applications. However, the GPS performs poorly under non-line-of-sight conditions, such as in dense forests, high building area or indoor environments [1].

Wireless indoor positioning techniques have been researched in the past ten years and different algorithms have been proposed. One main type of positioning technique is based on computing geometric information between beacons and targets and utilizes classical geometric principles, e.g. triangulation and trilateration, to estimate locations. The triangulation algorithm requires measurement of the angle of arrival (AOA), while the trilateration algorithm requires measurement such as time of arrival (TOA) or received signal strength (RSS) with a known path loss model. Comprehensive surveys can be found in [2], [3]. In indoor environments, the reliability of these measurements suffers from the complex propagation conditions, such as none-line-of-sight, multipath and unknown power decay profile [1].

An alternative option is the fingerprinting technique. Instead of computing geometric information, the fingerprinting technique estimates the target location by consulting a pre-constructed database (e.g., a radio map). Each entry in the database consists of location information and a *fingerprint* of corresponding measurement pattern (e.g., RSS). The target location estimate is given by the best matching entry or merging several best matching entries.

However, offline construction of fingerprint databases is a laborious task. Intuitively, databases of more densely collected measurements could have more accuracy in positioning but the offline workload significantly increases. Moreover, in many cases there are locations that are restricted or inaccessible for data collection, such as office rooms and apartments. In addition, existing databases need maintenance since part of the data might get faulted or need to be updated to keep completeness and integrity. Thus, spatial interpolation algorithms can be utilized to build the fingerprint database.

Several interpolation approaches have been proposed for fingerprint database construction. In [4] and [5], measurements at unknown locations are interpolated using specific path loss models. The path loss models are calculated based on actual measurements at limited number of known locations and the floor layouts (e.g., wall configuration). It is necessary to emphasize that knowledge of beacon locations is a prerequisite for these approaches. In [6], high-resolution fingerprint databases are produced from low-resolution databases by merging the measurements from two adjacent reference points (RPs) and assigning the merging result to a new RP placed in the middle.

A recent paper [7] investigates spatial interpolation and extrapolation algorithms for construction of fingerprint databases. Lacking knowledge about the beacon locations, measurement at an unknown point is interpolated based on actual measurements in the surrounding. There are several interpolation algorithms considered in [7], these include *linear interpolation* based on Delaunay triangulation, the nearest neighbour (NN) and the inverse distance weighting (IDW) to name a couple. The results show that location accuracy is enhanced by utilizing constructed databases comparing to the incomplete database. In [8], an adaptive smoothing algorithm is employed with regard to the discontinuity of RSS as a result of walls. The complete fingerprint database is produced using IDW interpolation algorithm.

This paper proposes using the Kriging algorithm to build fingerprint databases with sparsely collected measurements.The Kriging algorithm has been widely used in geostatistics principle for spatial interpolation, but is not broadly used in wireless network area. First, a complete fingerprint database is simulated with an indoor log-distance path loss model [9]. Then, parts of fingerprints are removed randomly according to a *sparsity* parameter. Afterwards, the database is reconstructed utilizing the interpolation algorithms and compared to the original database. The interpolation results are studied. At last, localization of a static target is tested using the reconstructed databases and the accuracy is investigated.

The paper is organized as follows. Section II introduces the proposed location fingerprinting technique. Section III presents details on the geostatistics tools and the Kriging algorithm considered in this paper. The simulation results are illustrated in section IV. Finally, the conclusions and future directions are drawn in section V.

II. RSS Fingerprinting Positioning

The fingerprinting technique comprises two phases. In the first phase, a fingerprint database is built offline. In [1], [7], the RSS measurements are collected at pre-determined RPs over multiple times. The mean of RSS value received from the j^{th} beacon at i^{th} RP, denoted by $\bar{z}_i^j$, are stored in the form of a pattern. Thus, the entry of the i^{th} RP and corresponding RSS fingerprint is denoted as

$$\{\mathbf{x}_i, \bar{\mathbf{z}}_i\}$$

where $\mathbf{x}_i = [x_i, y_i]^T$ is a location vector of i^{th} RP in two dimension, $\bar{\mathbf{z}}_i = [\bar{z}_i^1, \bar{z}_i^2, \cdots, \bar{z}_i^n]^T$ is a vector of mean RSS values from different beacons, n is the number of beacons.

Given a complete database of the area, the fingerprinting technique estimates the target location by searching for the best matching fingerprints in the database. There are different frameworks of matching the target measurements $\mathbf{z}_t \triangleq [z_t^1, z_t^2, ..., z_t^n]^T$ with the measurement fingerprint $\bar{\mathbf{z}}_i$ based on different norms, such as ℓ^1 norm, also known as Manhattan distance, the infinity-norm and Mahalanobis-norm [1]. The one used in this paper is the ℓ^2 norm, also known as Euclidean distance, that is,

$$d(\mathbf{z}_t, \bar{\mathbf{z}}_i) = \|\mathbf{z}_t - \bar{\mathbf{z}}_i\| = \sqrt{\sum_{j=1}^{n} \left(z_t^j - \bar{z}_i^j\right)^2}. \tag{1}$$

By matching the target measurement to the fingerprints, the target location can be estimated as a weighted sum of location of K best matching RPs, that is,

$$\mathbf{x}_t^* = \sum_{i=1}^{K} w_i \mathbf{x}_i, \tag{2}$$

where $\mathbf{x}_t^*$ is the target location estimate, w_i are weights of the RPs, which can be simply calculated as

$$w_i = \frac{\frac{1}{d(\mathbf{z}_t, \bar{\mathbf{z}}_i)}}{\sum_{i=1}^{K} \frac{1}{d(\mathbf{z}_t, \bar{\mathbf{z}}_i)}}. \tag{3}$$

III. Variogram and Ordinary Kriging

In the scenario where the beacon locations are unknown, distances between the beacons and a target location cannot be computed. As a consequence, specific path loss model is difficult to derive and the measurements at unknown locations cannot be predicted using the path loss model. Instead, the *geostatistics* approach of modelling the spatial correlation as *variogram* is an alternative solution.

A. Variogram

The idea of modelling the spatial correlation as a variogram is initiated by Matheron in [10]. Given an area of interest $\mathcal{G} \subset \mathrm{R}^n$, the mean of RSS value at a location $\mathbf{x}_i$ is considered as a *random variable* (RV) $\bar{Z}_i$. Then, the mean of RSS values over the area can be represented by a *random field* (RF), which is a collection of spatial RVs, $\{\bar{Z}_i | \, \mathbf{x}_i \in \mathcal{G}\}$. However, for each RV $\bar{Z}_i$, there is only one observation, which is the sample mean value of RSS measurement, $\bar{z}_i$. To characterize the RF, the assumption of stationarity, i.e., *intrinsic stationarity*, is required.

By constructing a new variable, which is the difference between two neighbour points $\delta_{i,j} = \bar{Z}_i - \bar{Z}_j$, the intrinsic stationarity implies that the mean of $\delta_{i,j}$ is zero in the local neighbourhood and the variance of $\delta_{i,j}$ depends only on the separation distance $\|\mathbf{x}_i - \mathbf{x}_j\|$, that is,

$$\begin{aligned} &\mathbb{E}[\delta_{i,j}] = 0, \\ &\mathrm{Var}(\delta_{i,j}) = 2\gamma(\|\mathbf{x}_i - \mathbf{x}_j\|) = 2\gamma(h), \end{aligned} \tag{4}$$

where $\mathbb{E}[\cdot]$ is the mathematical expectation, $\mathrm{Var}(\cdot)$ is the variance, $\gamma(\cdot)$ is the *variogram* function and $h = \|\mathbf{x}_i - \mathbf{x}_j\|$, is the *lag* which represents the separation distance between $\mathbf{x}_i$ and $\mathbf{x}_j$.

As shown in the equation (4), the variogram $\gamma(h)$ equals to the semi-variance of the difference $\delta_{i,j}$, that is,

$$\gamma(h) = \frac{1}{2}\mathrm{Var}(\delta_{i,j}) = \frac{1}{2}\mathbb{E}[\bar{Z}_i - \bar{Z}_j]^2. \tag{5}$$

To obtain a variogram, the *empirical variogram*, $\hat{\gamma}(h)$, is calculated first as follows,

$$\hat{\gamma}(h) = \frac{1}{2} \times \frac{1}{N(h)} \sum_{i=1}^{N(h)} \left(\bar{z}_i - \bar{z}_{i+h}\right)^2, \tag{6}$$

where $N(h)$ is the number of pairs of observations separated by lag h, $\bar{z}_{i+h}$ is the observation at a distance h from $\mathbf{x}_i$.

The empirical variogram contains values at a limited number of h. To estimate the measurements at unknown locations, access to the value of h between the scattered points in the empirical variogram is required. Hence, a mathematical model is selected to be fitted in the empirical variogram. This model is frequently chosen from *spherical* model , *exponential* model , *Gaussian model* , *power* model and *linear* model [11].

B. Ordinary Kriging

Once the variogram is obtained, values at unknown locations can be estimated based on known data points. Mathematically, this problem can be regarded as a spatial interpolation problem [12].

Kriging refers to a group of least-squared based interpolation algorithms. It is named after Danie G. Krige, who developed empirical statistical algorithms to predict ore grades from spatial correlated sample data in the gold mines of South Africa [13].

Kriging estimates the value at an unknown location $\mathbf{x}_u$ as a weighted sum of k known neighbour data points, that is,

$$\bar{z}_u^* = \sum_{i=1}^{k} \lambda_i \bar{z}_i, \quad u = 1, ..., U \tag{7}$$

where $\bar{z}_i$ is the neighbour data point, λ_i is the neighbour weight, called *Kriging weight*, U is the number of unknown locations. The Kriging weights are derived through minimising the estimator error variance, that is,

$$\min_{\lambda_i \in R} \quad \mathrm{Var}(\bar{z}_u^* - \bar{z}_u), \tag{8}$$

under the unbiasedness constraint, given by,

$$\mathbb{E}[\bar{z}_u^* - \bar{z}_u] = 0. \tag{9}$$

The Kriging algorithm used in this paper is the ordinary Kriging. Assuming the intrinsic stationarity and utilizing Lagrange multiplier optimization algorithm to minimize the estimator error variance (8) under the unbiasedness constraint (9), the Kriging weights λ_i in (7) can be calculated as:

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \\ L \end{pmatrix} = \begin{pmatrix} \gamma_{1,1} & \dots & \gamma_{1,k} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \gamma_{k,1} & \dots & \gamma_{k,k} & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_{1,u} \\ \vdots \\ \gamma_{k,u} \\ 1 \end{pmatrix}. \tag{10}$$

where L is the introduced Lagrange multiplier, $\gamma_{i,j}$ is the variogram value between the i^{th} and j^{th} neighbour data points, $\gamma_{i,u}$ is the variogram value between the i^{th} neighbour data point and the interpolation point. The derivation of the Kriging weights is given in the Appendix.

Kriging has certain advantages over some other spatial interpolation algorithms. First, Kriging gives the best unbiased estimate with minimized variance. Second, the distance between neighbour points are considered in the calculation of Kriging weights. It results in less weights for the points within a data cluster than isolated data points. Therefore, the effects of redundant information contained in a data cluster are weakened [14], [15].

Referring to the construction of fingerprint databases, the unknown values in RSS measurement fingerprints can be interpolated based on sparsely-collected data and variogram models. Therefore, the offline workload of constructing a fingerprint databases can be reduced significantly.

IV. Simulation Evaluation

This section demonstrates the simulation results of the construction of fingerprint databases and localization of a target using constructed databases. First, the signal propagation is simulated over an area of 150m $\times$ 150m from five beacons utilizing the log-distance path-loss model in [9]:

$$z_i^j = Z_t - L_0 - 10\beta \log_{10} \|\mathbf{x}_i - \mathbf{x}^j\| + \chi_\sigma \quad \text{(in dB)} \tag{11}$$

where z_i^j is the RSS value received from the j^{th} beacon at the i^{th} RP, Z_t is transmitted signal powers, L_0 represents the path loss at a relatively short distance away from the beacon, β is the path loss exponent, $\mathbf{x}^j$ is the location of the j^{th} beacon, $\|\mathbf{x}_i - \mathbf{x}^j\|$ is the distance between the i^{th} RP and the j^{th} beacon, χ_σ representing the shadowing noise, which is modelled as a zero-mean Gaussian RV with the variance $\sigma^2 = 5$.

In practice the RSS is measured sufficiently over multiple times at a location and the values in RSS fingerprints are computed as the means in order to eliminate the noise effect. In the experiment, the values in the fingerprint are computed as the averages of 1000 RSS measurements for each RP.

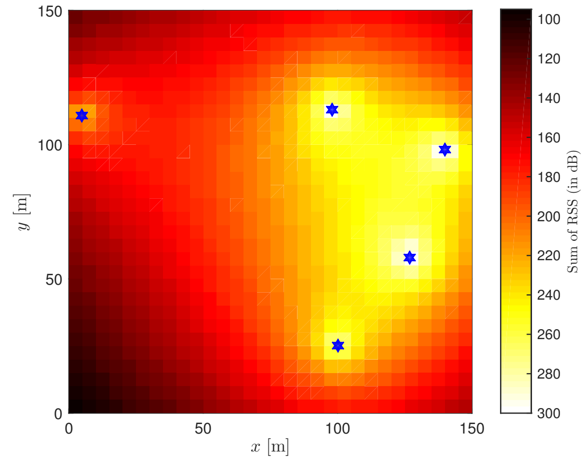

Fig. 1. The original RSS fingerprint map. The RSS values are the sum of values in RSS fingerprints.

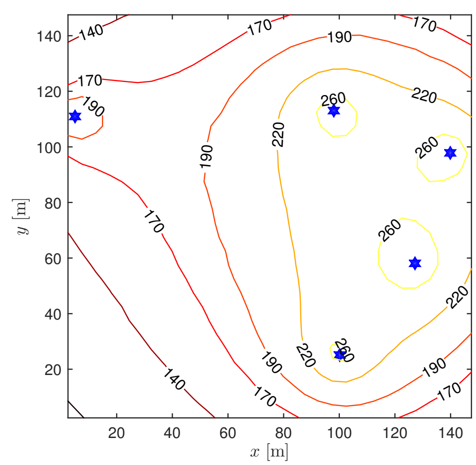

Fig. 2. The the original RSS fingerprint contour map. The RSS values are the sum of values in RSS fingerprints.

The area is divided into 30 $\times$ 30 grids. The complete RSS fingerprint database (i.e., RSS map) consists of the coordinates of all 900 grid center points and corresponding RSS pattern of five beacons. The original RSS map and the beacon locations is shown in Fig. 1 and Fig. 2.

Next, part of the fingerprint database is removed according to a *sparsity* parameter ρ. The removal process is a Bernoulli process with the probability that a fingerprint would be removed is $1 - \rho$, the probability that a fingerprint would be retained is equal to ρ. Fig. 3 illustrates a sparsely-collected fingerprints map with $\rho = 0.15$.

Fig. 4 presents the empirical variogram and fitting result of a beacon. The fitted curve demonstrates the spatial correlation model of RSS and is used to estimate the RSS at an unknown location. As shown, the value of empirical variogram, which is the scatter plot in Fig. 4, increases with h. According to

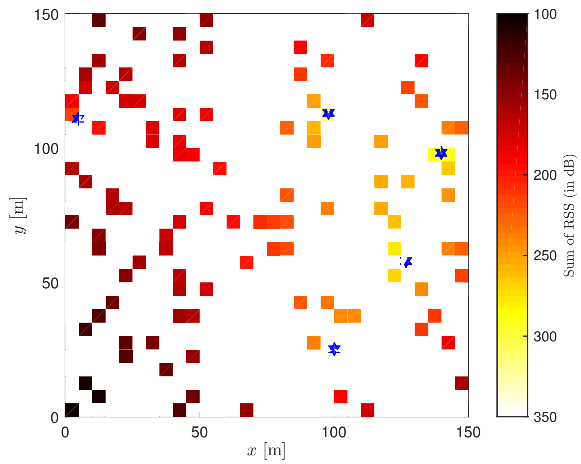

Fig. 3. RSS map of sparsely-collected measurements. $\rho = 0.15$.

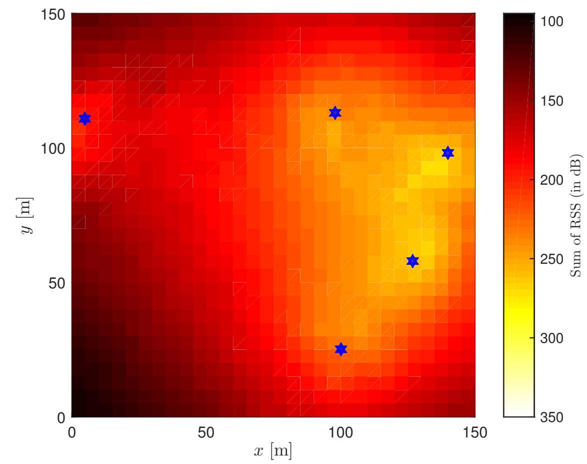

Fig. 5. Kriging interpolated RSS map with measurement sparsity $\rho = 0.15$.

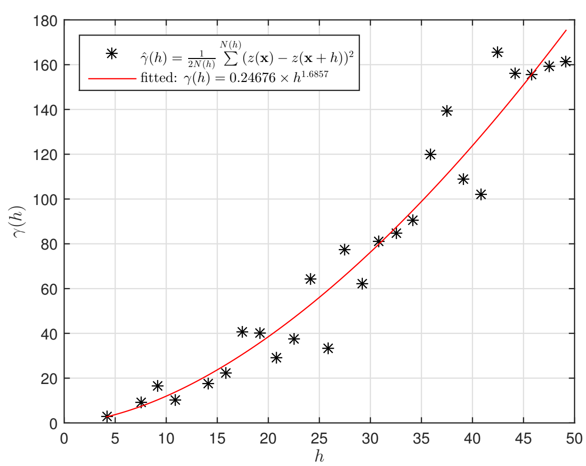

Fig. 4. Empirical variogram and fitting result.

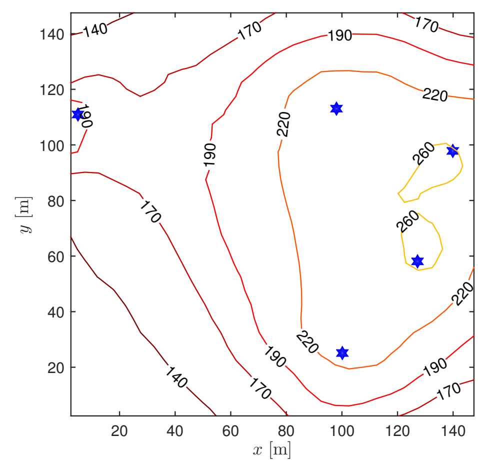

Fig. 6. Kriging interpolated RSS contour map with measurement sparsity $\rho = 0.15$.

[11], it infers that there is an obvious *trend* (general spatial variation of the mean value) of RSS distribution in the area. Compared with more widely used fitting functions, e.g. the spherical and exponential function, it is suggested to select a power model, that is,

$$\gamma(h) = \begin{cases} 0 & , h = 0 \\ \alpha \cdot h^{\beta} & , h \geq 0 \end{cases} \tag{12}$$

where α and β are the fitting parameters with a strict constraint that $0 < \beta < 2$. As indicated in the figures, the power function is well fitted.

Based on the sparse-collected measurements as shown in Fig. 2, a Kriging constructed database is produced and demonstrated in Fig. 5 and Fig. 6. As evident, the Kriging algorithm retrieved the missing RSS fingerprints accurately with acceptable interpolation errors.

Fig. 7 demonstrates interpolation error statistics of the NN, IDW based on 5 nearest neighbours and Kriging algorithms with respect to the sparsity. There are 5 beacons and correspondingly 5 entries in the RSS fingerprint of a RP. The interpolation error for each unknown RP is computed as the root-mean-square error (RMSE) of the RSS fingerprint by comparing the interpolated fingerprint to the original one. The interpolation error for each sparsity is computed as the mean of RMSE for all unknown RPs, that is,

$$\epsilon = \frac{\sum_u^U \|\bar{\mathbf{z}}_u^* - \bar{\mathbf{z}}_u\|}{U} \tag{13}$$

where the $\bar{\mathbf{z}}_u^*$ is the interpolated RSS fingerprint, $\bar{\mathbf{z}}_u$ is the original RSS fingerprint, $\|\cdot\|$ is the Euclidean distance operator in equation (1) which can be used to compute the RMSE.

As shown in the figure, the NN interpolation error is the highest one for all sparsity values and remains at the same level. The IDW interpolation error decrease slightly and has the lowest values when the sparsity of the measurement is between 0.05 to 0.06. The Kriging interpolation error decreases dramatically between this sparsity range and is higher than the IDW algorithm. The reason is that the extremely sparse measurements are not sufficient for building a good variogram model, which can lead to significant interpolation error at some

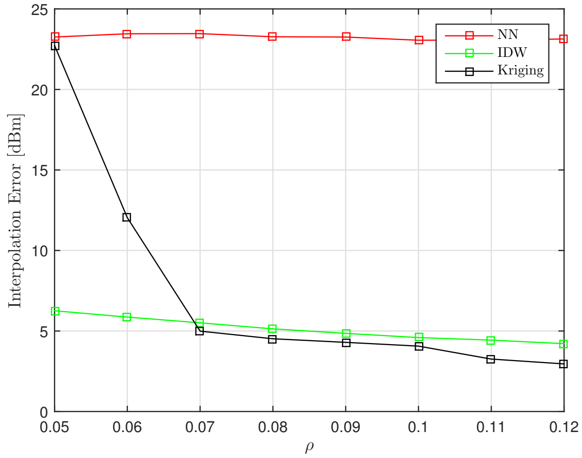

Fig. 7. Comparison of interpolation error of the NN, IDW and Kriging algorithms with respect to sparsity.

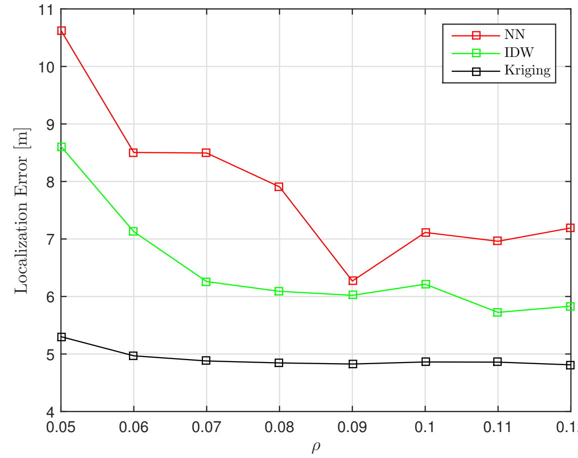

Fig. 9. Comparison of localization error of the NN, IDW and Kriging algorithms with respect to sparsity .

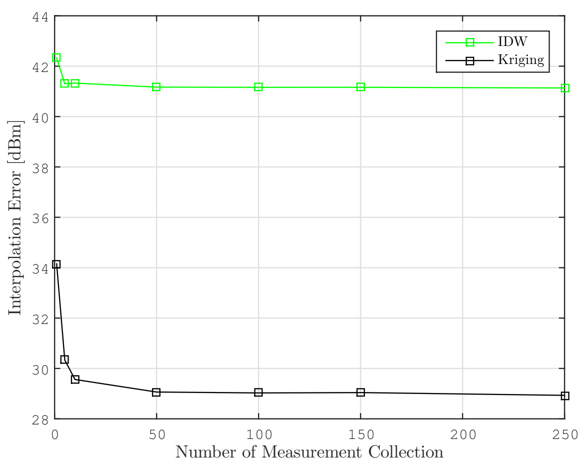

Fig. 8. Comparison of interpolation error of the NN, IDW and Kriging algorithms with respect to number of measurement collection.

locations. Later when ρ increases and the measurements are sufficient, the Kriging interpolation error is lower than the other two methods.

Fig. 8 shows the interpolation error against the number of offline measurements. The offline fingerprint values are averaged over the number of measurement collection. As shown, the interpolation error decrease more obviously when the number of collection increases from 1 to 50. When the number of measurement collection is more than 50 times, the interpolation error remains flat, which indicates that the measurement noise effect is almost eliminated.

Fig. 9 shows the mean error of locating static target at $[50, 50]$ utilizing the RSS fingerprint databases constructed by the NN, IDW based on 5 nearest neighbours and Kriging algorithms with different measurement sparsity. The target RSS are simulated using the log-distance path-loss model (11) with the same shadowing variance $\sigma^2 = 5$. As shown in Fig. 9, the error of locating the target based on the NN, IDW and Kriging constructed databases decreases with the increase of ρ. The localization error based on the Kriging constructed databases remained the lowest one even though the sparsity is extremely low between 0.05 and 0.06. Although it is seen in Fig. 7 that for $\rho < 0.07$ the Kriging interpolation error is worse than the IDW due to the inadequate variogram modelling. However, location performance in Fig. 9 shows that Kriging performs better than IDW at all ρ values. The reason is that even though the measurements are extremely sparse, it does not lead to significant interpolation error for all locations. The Kriging algorithm weights the neighbour data points based on the variogram models, while the IDW weights the neighbour points only according to the Euclidean distance.

V. Conclusions and Future Work

In this paper, the Kriging algorithm, which is a statistics-based spatial interpolation algorithm, is proposed to construct RSS fingerprint database with sparsely collected measurements. The simulation results demonstrate that the Kriging algorithm interpolates the RSS fingerprint at unknown location without the knowledge of the beacon locations. The performance is compared to the NN and IDW algorithms over different levels of sparsity. The results show that the Kriging algorithm generates more accurate RSS fingerprints than the NN and IDW algorithms. Employing the constructed databases, the accuracy of localization of a static target is studied with respect to data sparsity. The results illustrate that the Kriging algorithm performs better than the NN and IDW algorithms.

It has been validated that the Kriging algorithm is a powerful tool for spatial analysis. As a future work, the proposed method will be applied to real data and the performance shall be investigated. The Kriging algorithm can be also applied in other wireless applications, such as pollution mapping and precision agriculture in wireless sensor networks (WSNs).

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Appendix

Under the intrinsic stationarity assumption, a sufficient condition for unbiasedness can be given as,

$$\sum_{i=1}^{k} \lambda_i - 1 = 0. \tag{14}$$

Taking advantage of unbiasedness, the kriging estimator error variance can be derived as

$$\begin{aligned}
&\mathrm{Var}(\bar{z}_u^* - \bar{z}_u)\\
=&\mathbb{E}[\bar{z}_u^* - \bar{z}_u]^2\\
=&\mathbb{E}\Big[\sum_{i=1}^{k} \lambda_i\big(\bar{z}_i - \bar{z}_u\big)\Big]^2\\
=&\sum_{i=1}^{k}\sum_{j=1}^{k} \lambda_i\lambda_j \cdot \frac{1}{2}\mathbb{E}\big[\bar{z}_i - \bar{z}_u\big]^2 + \sum_{i=1}^{k}\sum_{j=1}^{k} \lambda_i\lambda_j \cdot \frac{1}{2}\mathbb{E}\big[\bar{z}_j - \bar{z}_u\big]^2\\
&-\sum_{i=1}^{k}\sum_{j=1}^{k} \lambda_i\lambda_j \cdot \frac{1}{2}\mathbb{E}\big[\bar{z}_i - \bar{z}_j\big]^2\\
=&2\sum_{i=1}^{k} \lambda_i\gamma_{i,u} - \sum_{i=1}^{k}\sum_{j=1}^{k} \lambda_i\lambda_j\gamma_{i,j}.
\end{aligned} \tag{15}$$

To minimize the expression (14) under the unbiasedness (13) by introducing a *Lagrange multiplier* L, the aim now is,

$$\min_{\lambda_k \in R}\; 2\sum_{i=1}^{k} \lambda_i\gamma_{i,u} - \sum_{i=1}^{k}\sum_{j=1}^{k} \lambda_i\lambda_j\gamma_{i,j} + 2L\cdot\Big(1 - \sum_{i=1}^{k}\lambda_i\Big). \tag{16}$$

Differentiate expression (15) with respect to a weight λ_i and let it equal to 0, that is,

$$\sum_{j=1}^{k} \lambda_j\gamma_{i,j} + L = \gamma_{i,u}. \tag{17}$$

So the Kriging weights that minimize the error variance can be obtained by solving the equation set:

$$\begin{cases} \displaystyle\sum_{j=1}^{k} \lambda_j\gamma_{i,j} + L = \gamma_{i,u}, \quad i = 1...k \\ \displaystyle\sum_{i=1}^{k} \lambda_i = 1. \end{cases} \tag{18}$$

Expressing the equation set (17) as matrices, that is,

$$\begin{pmatrix} \gamma_{1,1} & \dots & \gamma_{1,k} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \gamma_{k,1} & \dots & \gamma_{k,k} & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \\ L \end{pmatrix} = \begin{pmatrix} \gamma_{1,u} \\ \vdots \\ \gamma_{k,u} \\ 1 \end{pmatrix}. \tag{19}$$

Thus, the Kriging weights can be calculated as,

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_K \\ L \end{pmatrix} = \begin{pmatrix} \gamma_{1,1} & \dots & \gamma_{1,k} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \gamma_{k,1} & \dots & \gamma_{k,k} & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_{1,u} \\ \vdots \\ \gamma_{k,u} \\ 1 \end{pmatrix}. \tag{20}$$