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A ROUTE-SWAPPING DYNAMICAL SYSTEM AND LYAPUNOV FUNCTION FOR STOCHASTIC USER EQUILIBRIUM

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Abstract – An analysis of the continuous-time dynamics of a route-swap adjustment process is presented, which is a natural adaptation of that which was presented in Smith (1984) for deterministic choice problems, for a case in which drivers are assumed to make perceptual errors in their evaluations of travel cost, according to a Random Utility Model. We show that stationary points of this system are stochastic user equilibria. A Lyapunov function is developed for this system under the assumption of monotone, continuously differentiable and bounded cost-flow functions and a logit-based decision rule, establishing convergence and stability of trajectories of such a dynamical system with respect to a stochastic user equilibrium solution.

1. INTRODUCTION

There is now a growing body of literature examining the kinds of smooth, continuous-time trajectories which might approximate the day-to-day dynamic adjustment processes of car drivers in transport networks. These begin by postulating a continuously-varying state variable (e.g. representing network flows, costs or cost differences) together with some autonomous, continuous-time dynamical system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \quad (t \geq 0) \quad (1)$$

for some given, smooth, time-independent function $\mathbf{f} : \mathcal{X} \rightarrow \mathbb{R}^n$, where $\mathcal{X} \subseteq \mathbb{R}^n$, and the function $\mathbf{x} : [0, \infty) \rightarrow \mathcal{X}$ is differentiable at all times $t > 0$. Typically in such systems, we begin with specifying the form of \mathbf{f} , and then the following *initial value problem* is of interest: given some $\mathbf{x}_0 \in \mathcal{X}$, find a function $\mathbf{x}(\cdot)$ which is continuous at $t = 0$, differentiable at $t > 0$, differentiable on the right at $t = 0$, and solves the system of equations:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \quad (t \geq 0) \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X} .$$

With such dynamical systems, it is natural to explore the properties of *fixed/stationary point equilibria* of the system, namely those \mathbf{x} satisfying:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \quad (\mathbf{x} \in \mathcal{X})$$

and then to ask of such point equilibria: which ones are likely to emerge and persist as the convergent behaviour of system (1), i.e. which equilibria are *stable*?

Papers that have explored various dynamical route adjustment processes of the form (1) include those by Smith (1984), Friesz et al. (1994), Zhang and Nagurney (1996), Zhang et al. (2001), Yang and Liu (2007), Yang and Zhang (2009), He et al. (2010), Han and Du

(2012), Guo et al. (2013) and Zhang et al. (2015). All of these established the Wardrop Deterministic User Equilibrium (DUE) state as a fixed point of their dynamical system. A subset of these also established general conditions to ensure global asymptotic stability of the Wardrop equilibrium solution with respect to the particular dynamical system they specified; notably Smith (1984); Friesz et al. (1994); Zhang and Nagurney (1996); and Han and Du (2012). Yang and Zhang (2009) established that many of these rerouting dynamics are special cases of a general dynamic, which they termed a Rational Behavior Adjustment Process. However, as Zhang et al. (2015) recently noted, the process proposed originally by Smith (1984) is in some respects ‘the most natural and has the simplest formulation [and] ... has stimulated various extensional applications’. This process, which has subsequently been termed the Proportional-Switch Adjustment Process or simply the *Smith dynamic* (Sandholm, 2010), is an important reference case for the present paper.

Several authors have also considered dynamical route adjustment processes for which the fixed points coincide with the Stochastic User Equilibrium (SUE) model. Horowitz (1984) studied the convergence properties of a variety of discrete-time decision rules for two-route networks. Cantarella and Cascetta (1995) considered a very wide class of discrete-time dynamic processes, but also established specific results for a particular process in which (a) a fixed proportion α ($0 < \alpha \leq 1$) of travellers reconsider their previous day’s choice, and (b) forecasted path costs are based on a convex combination of the latest experience (with weight $0 < \beta \leq 1$) and the previous forecast. They showed that, under typically-assumed conditions, small enough values of α and β exist to ensure stability of SUE with respect to such a system. Watling (1999) considered a special case of such a process, with $\alpha = 1$, and set out sufficient conditions on β to ensure stability, which through a route re-labelling strategy were shown to be applicable to a quite wide class of such problems. In addition, relationships were explored between stability/instability properties in discrete and continuous time, which were further explored by Cantarella and Watling (2015). The continuous-time model explored in Watling (1999) is a second important reference-case for the present paper, being what Sandholm (2010) subsequently termed the *logit dynamic*. In addition, Watling presented methods for estimating domains of attraction for multiple equilibria, which were further refined and elaborated by Bie and Lo (2010). Yang and Liu (2007) established that various existing processes could be viewed as the mean dynamic of a stochastic process, mainly focusing on dynamical systems related to DUE, but also presented numerical experiments for the logit dynamic and its relation to SUE. Guo et al (2013, Appendix B), while mainly concerned with DUE, established convergence for a discrete-time form of the logit dynamic.

The purpose of the present paper is to formulate and analyse a new form of dynamical system for SUE, differing from the logit dynamic and developed from the logic of the Smith dynamic proposed for DUE. In doing so, we provide a kind of bridge between DUE- and SUE-based dynamical modelling, in the sense that the dynamic processes of the two are connected, not only the equilibrium states. We achieve this by developing a route-swapping dynamical system and a corresponding Lyapunov function, which can be seen to be the SUE analogue of the dynamical process and results presented in Smith (1984) for DUE.

2. BASIC STATIC NOTATION

We suppose that our network consists of k origin-destination (OD) movements with positive demands contained in the vector \mathbf{q} of length k . Consider the finite set of all routes that visit no link twice, across all OD movements, and suppose that there are n such routes in total. Let the matrix \mathbf{A} denote the (OD-movement)-route incidence matrix, of dimension $k \times n$, any element of which is 1 if the route serves the given OD movement and 0 otherwise. We may then define the convex set $D \subseteq \mathbb{R}^n$ of demand-feasible route flows as:

$$D = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ with } \mathbf{A}\mathbf{x} = \mathbf{q} \text{ and } \mathbf{x} \geq \mathbf{0}\}$$

where \mathbf{x} has the elements x_r denoting the steady flow along route r (for $r = 1, 2, \dots, n$). In addition, we define the set of n -dimensional vectors in Euclidean space with strictly positive elements as:

$$\mathcal{S} = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ with } x_r > 0 \text{ for } r = 1, 2, \dots, n\}.$$

Let $\mathbf{c} : D \rightarrow \mathbb{R}^n$ be the given route cost-flow function, such that $c_r(\mathbf{x})$ is the cost of travelling on route r when the route flow vector is \mathbf{x} (for $r = 1, 2, \dots, n$). We suppose that $\mathbf{c}(\mathbf{x})$ is a monotone, continuously differentiable function of \mathbf{x} defined throughout D .

3. DEVELOPMENT OF A RUM-BASED, ROUTE-SWAPPING DYNAMIC

Since it was originally proposed in Smith (1984), the pairwise route-swapping dynamic has emerged as a standard reference model in the evolutionary game theory literature, where it is referred to as the *Smith dynamic* (Sandholm, 2010). The key behavioural aspects of this dynamic are: (a) pairwise path-swapping from more costly to less costly paths; and (b) for those pairs of paths in (a), an assumed rate of exchange of path flow proportional to the product of the path flow on the higher cost path and the cost surplus of the higher over the lower cost path. A limitation of this model, however, is that it does not allow for possible mis-perceptions of travellers in their evaluations of travel costs. Here, we aim to modify the Smith dynamic to incorporate mis-perception as represented by a Random Utility Model (RUM), while aiming to retain as much as possible of the originally-proposed behavioural process. In particular we assume pairwise path-swapping from less attractive to more attractive routes, but depart from the original model in how the ‘attractiveness’ is defined.

Consider a pair of paths (r, s) serving the same origin-destination (OD) movement at time t , with current flows $x_r(t)$ and $x_s(t)$, and current ‘measured’ travel costs $c_r(\mathbf{x}(t))$ and $c_s(\mathbf{x}(t))$. Suppose that traveller mis-perception is modelled by a multinomial logit model with dispersion parameter θ^{-1} . Suppose that we are an observer of this OD movement, then if we randomly selected a traveller making this movement, the current relative odds of that traveller being a path r rather than path s traveller would clearly be given by the ratio $x_r(t):x_s(t)$. If, on the other hand, travellers were able to immediately

readjust their route choice in response to the prevailing travel costs, then according to the multinomial logit model, these relative odds would be $\exp(-\theta^{-1}c_r(\mathbf{x}(t))):\exp(-\theta^{-1}c_s(\mathbf{x}(t)))$. The difference between these two cases (the current flows and the immediately readjusted flows) can be measured by the *odds ratio*:

$$O_{rs}(\mathbf{x}(t)) = \frac{x_r(t)/x_s(t)}{\frac{\exp(-\theta^{-1}c_r(\mathbf{x}(t)))}{\exp(-\theta^{-1}c_s(\mathbf{x}(t)))}} . \quad (2)$$

(Note that since we are supposing that $c_r(\mathbf{x})$ is a continuously differentiable function of \mathbf{x} throughout S , then so $O_{rs}(\mathbf{x})$ is also a continuously differentiable function of \mathbf{x} throughout S .) An odds ratio of $O_{rs}(\mathbf{x}(t)) = 1$ would denote that the split of traffic between the two paths was exactly in accordance with the prevailing travel costs, according to a multinomial logit model, whereas $O_{rs}(\mathbf{x}(t)) > 1$ indicates that path r has too much flow relative to path s , according to the prevailing travel costs. Therefore, *at the aggregate OD level*, the odds ratio can be used as the basis for suggesting an overall route-swapping dynamic for the population of travellers using that movement.

It should be noted that we do not suggest individual travellers perceive such a stimulus; rather we use an aggregate but probabilistically distributed model, with the random utility model applied to suggest the overall effect on the population of travellers for that movement. It is also noted that the Independence from Irrelevant Alternatives (IIA) property of the multinomial logit model means that it makes sense to make such pairwise comparisons as implied by (2), since under this RUM the flow-split between any pair of alternatives does not depend on the costs of other alternatives. In order to develop this dynamic in combination with something close to the original Smith dynamic constructed for deterministic choice models, it is natural to transform the odds ratio so that a stimulus level of zero suggests no flow-swap is needed, and a positive value suggests a flow swap is needed away from route r to route s .

In order to do so, a scalar function $h(y)$ ($y > 0$) is defined, such that:

- h is continuously differentiable and monotonically increasing on the open interval $(0, \infty)$; and
- $h(1) = 0$.

This scalar function is used to transform the odds ratio (2) for use in a route-swapping system, according to:

$$g_{rs}(\mathbf{x}) = h(O_{rs}(\mathbf{x})) \quad (3)$$

with (2) and (3) then used to define the system:

$$\dot{\mathbf{x}}(t) = \Phi(\mathbf{x}(t)) \quad (t \geq 0; \mathbf{x}(0) = \mathbf{x}_0 \in D \cap \mathcal{S}) \quad (4)$$

where $\Phi : D \cap \mathcal{S} \rightarrow \mathbb{R}^n$ is given by:

$$\Phi(\mathbf{x}) = \sum_{r=1}^n \sum_{s=1}^n x_r (g_{rs}(\mathbf{x}))_+ \Delta_{rs} \quad \text{for all } \mathbf{x} \in D \cap \mathcal{S} \quad (5)$$

where for any real number y

$$y_+ = \max(0, y)$$

and where Δ_{rs} is a *path-swap indicator vector* of dimension n . This latter vector is the zero vector if routes r and s serve different OD movements or if $r = s$; otherwise, it has -1 for the r^{th} element, $+1$ for the s^{th} element, and zeroes elsewhere. The sets D and \mathcal{S} were defined in section 2.

Equations (2)–(5) define a family of dynamical systems; a particular instance of this family corresponds to a particular choice of the function $h(\cdot)$ (satisfying the required conditions on h). We shall initially consider the two possibilities:

$$h(y) = \lambda (y - 1) \quad (\lambda > 0) \quad (6)$$

or

$$h(y) = \beta \ln y \quad (\beta > 0). \quad (7)$$

Both of (6) and (7) satisfy the required conditions on h , and both contain a free parameter allowing the rate of adjustment of the process to be ‘tuned’, independently of the equilibrium properties.

While both (6) and (7) give rise to candidate dynamical systems that can be related to SUE, we propose that (7) has two key advantages:

Firstly, when (7) is combined with (2)/(3), it implies that the path-swapping is governed by the *log-odds ratio* $g_{rs}(\mathbf{x}) = \beta \ln O_{rs}(\mathbf{x})$, a commonly-used measure in statistics (e.g. Cramer, 2003; Hilbe, 2009). In that field, the logarithm is favoured over the direct use of the odds-ratio for a reason that also has relevance in our present context. Namely, directly using the odds-ratio as in (6) has a disadvantage that the scale of the implied stimulus is asymmetric in its implied sensitivity. For example, if $x_r(t) = 90$, $x_s(t) = 10$, $\exp(-\theta^{-1}c_r(\mathbf{x}(t))) = 0.02$, $\exp(-\theta^{-1}c_s(\mathbf{x}(t))) = 0.08$, then using the linear function (6) we find $g_{rs}(\mathbf{x}(t)) = \lambda(O_{rs}(\mathbf{x}(t)) - 1) = 35\lambda$; on the other hand, reversing the roles of routes r and s , the reverse stimulus is much smaller in magnitude, with $g_{sr}(\mathbf{x}(t)) = -\frac{35}{36}\lambda$. Adopting the logarithmic transform (7), on the other hand, the magnitude of the dynamic stimulus is insensitive to the route labelling, with $g_{rs}(\mathbf{x}(t)) = -g_{sr}(\mathbf{x}(t)) = \beta \ln 36$.

Secondly, the logarithmic transform (7) provides a direct link to the original Smith dynamic for DUE, as a limiting case. In order to see this, we note that (7) combined with (2)/(3) is readily simplified to:

$$g_{rs}(\mathbf{x}(t)) = \beta\theta^{-1} (c_r(\mathbf{x}(t)) - c_s(\mathbf{x}(t))) + \beta \ln \left(\frac{x_r(t)}{x_s(t)} \right)$$

and setting $\alpha = \beta\theta^{-1}$ ($\alpha > 0$), this may be re-written as:

$$g_{rs}(\mathbf{x}(t)) = \alpha \cdot \left(c_r(\mathbf{x}(t)) - c_s(\mathbf{x}(t)) + \theta \ln \left(\frac{x_r(t)}{x_s(t)} \right) \right). \quad (8)$$

In comparison, in the DUE case, we consider a slight generalisation of the Smith dynamic, in which the rate of change in route flows $\dot{\mathbf{x}}(t)$ is a constant $\alpha > 0$ multiplied by the original Smith dynamic (i.e. the original Smith dynamic for DUE corresponds to the choice $\alpha = 1$). This generalised Smith dynamic can then be expressed as the system (4) and (5) in combination with the choice:

$$g_{rs}(\mathbf{x}(t)) = \alpha \cdot (c_r(\mathbf{x}(t)) - c_s(\mathbf{x}(t))). \quad (9)$$

Although $\theta = 0$ is not permitted in our RUM-based dynamical system, it is clear from (8) and (9) that for small values of θ the dynamic implied by (8), will approximate that implied by (9). Although (9) may be interpreted either as an individual stimulus for travellers to change or as an aggregate OD-level stimulus, the dynamic implied by (8) only has an aggregate OD level interpretation, since the additional term is expressing the rate of adjustment of the population of travellers on that OD movement, given the assumptions about the distribution of perception errors across this population (as contained in θ).

As we shall exemplify with an example, the dynamical systems implied by (6) or (7) (in combination with (2)–(5)) are new ones, and in particular, they differ from the *logit dynamic* previously studied in the literature (as discussed in section 1). In order to illustrate this, consider a simple example of a single OD movement with a demand of $q_1 = 3$ served by two parallel routes, with route cost functions $c_1(\mathbf{x}) = 5 + x_1^2/2$ and

$c_2(\mathbf{x}) = 10 + x_2^2/4$. Suppose the logit parameter $\theta = 1$, that the parameters in (6) and

(7) are given by $\lambda = 1$ and $\beta = 1$, and suppose the initial conditions of the system are $\mathbf{x}(0) = (x_1(0), x_2(0)) = (2, 1)$. We refer to the system implied by equations (2)–(5) with (7) as the *logit-based Smith dynamic*, and the system implied by equations (2)–(6) as the *alternative logit-based Smith dynamic*. Figure 1 illustrates the trajectory of the flow on route 1 as a function of time (horizontal axis), for each of these dynamical systems. We can see that even for an example with only two routes, the three systems differ; i.e. they do not differ simply because of the pairwise way in which (5) is constructed, since in this small example this is not a relevant distinction. All three provide smooth trajectories that, at least for the initial condition and example network tested, converge to SUE. Although neither individual nor aggregate behaviour in real-life systems can be expected to be smooth in this way, the three models are all viable candidates as smooth approximations to the underlying real-life system, but with different rates of system

adjustment. Based on the nature of the observed flow adjustments over time, and in particular the manner in which they approach something akin to equilibrium, one such model could be chosen as a best approximation to the real-life system.

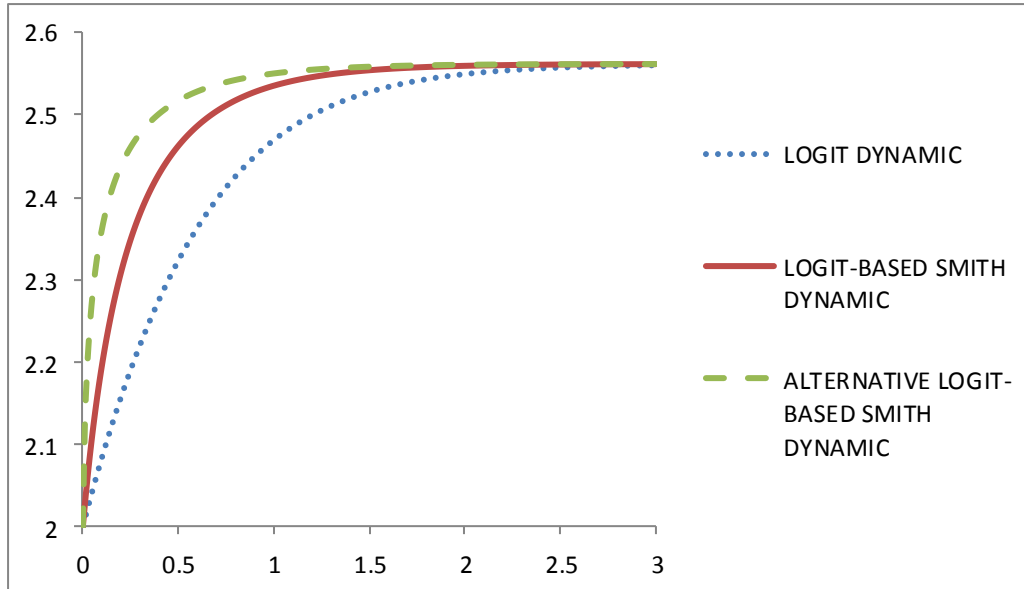


Figure 1: Route 1 flow as a function of time for three alternative logit-related dynamical systems

In the present paper, for the reasons already explained above, we shall henceforth focus on the logit-based Smith dynamic, which as we noted above can be expressed by the system (4)/(5)/(8). In doing so, we provide evidence on the theoretical properties of a new candidate model, which can be considered alongside existing results for the logit dynamic. As noted above the model then has the attractive feature that it provides a bridge to the seminal work with DUE on the Smith dynamic, which the logit-based Smith dynamic approaches as the assumed variance in travellers' perceptual errors (as controlled by θ) approaches zero. Figure 2 illustrates this for the two-route example network considered earlier (for $\alpha = 1$). That is to say, it is not simply that as $\theta \rightarrow 0$, there is a limit point that approaches DUE, but that also the *trajectory* of the dynamical adjustment process towards equilibrium for the logit case approaches the deterministic one (labelled as $\theta = 0$).

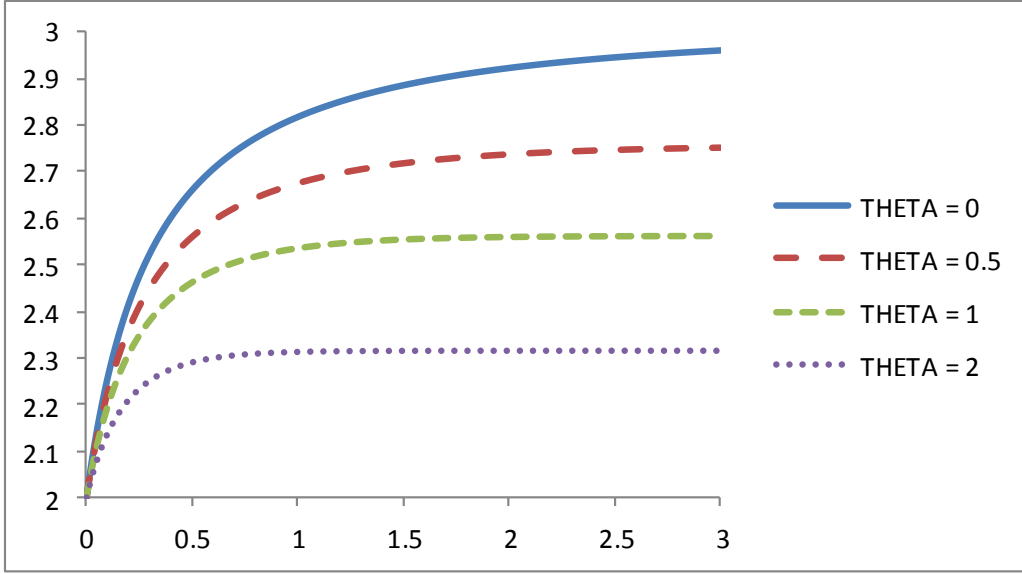


Figure 2: Route 1 flow as a function of time for three cases of the Logit-based Smith Dynamic, and (for 'THETA= 0') the original Smith Dynamic

4. THEORETICAL ANALYSIS OF THE LOGIT-BASED SMITH DYNAMIC

In the present section we establish a series of theoretical results concerning the *Logit-based Smith Dynamic* given by (2)–(5) with (7), which we showed in the previous section to be expressible in an equivalent form (4)/(5)/(8) which is re-stated below to avoid any ambiguity. Since the case $\alpha = 1$ provides the direct generalisation of the original Smith dynamic for DUE (and since the results are anyway trivially extended for any $\alpha > 0$), we shall restrict attention to the case $\alpha = 1$. The system is then:

$$\dot{\mathbf{x}}(t) = \Phi(\mathbf{x}(t)) \quad (t \geq 0; \mathbf{x}(0) = \mathbf{x}_0 \in D \cap \mathcal{S}) \quad (10)$$

$$\Phi(\mathbf{x}) = \sum_{r=1}^n \sum_{s=1}^n x_r (g_{rs}(\mathbf{x}))_+ \Delta_{rs} \quad \text{for all } \mathbf{x} \in D \cap \mathcal{S} \quad (11)$$

$$g_{rs}(\mathbf{x}(t)) = c_r(\mathbf{x}(t)) - c_s(\mathbf{x}(t)) + \theta \ln \left(\frac{x_r(t)}{x_s(t)} \right) \quad \text{for all } \mathbf{x}(t) \in D \cap \mathcal{S}, \quad (12)$$

where all other relevant notation was defined in section 2. We note that under the assumptions stated in section 2, each g_{rs} given in (12) is a continuously differentiable function of \mathbf{x} in S since we are assuming that each c_r is a continuously differentiable function of \mathbf{x} in S . It further follows that $\Phi(\mathbf{x})$ given by (11), when combined with (12), is a Lipschitz continuous function of \mathbf{x} on any compact subset of S . (Continuous differentiability is lost due to the “+” suffix in the term $(g_{rs}(\mathbf{x}))_+$; but Lipschitz continuity remains.)

We begin, in Lemma 1, by formally establishing the relationship of this dynamical system with the Stochastic User Equilibrium (SUE) model (Sheffi, 1985).

Lemma 1

If $\mathbf{x} \in D \cap \mathcal{S}$ is an SUE, then it is a point equilibrium of system (10)–(12). Further if $\mathbf{x} \in D \cap \mathcal{S}$ is an equilibrium point of (10)–(12), then it is an SUE.

Proof

By the Independence from Irrelevant Alternatives property of the multinomial logit model, a necessary and sufficient condition for logit SUE is that $\mathbf{x} \in D \cap \mathcal{S}$ and for any pair of routes (r, s) serving the same OD movement:

$$\frac{x_r}{x_s} = \frac{\exp(-\theta^{-1}c_r(\mathbf{x}))}{\exp(-\theta^{-1}c_s(\mathbf{x}))}.$$

This condition holds (according to (12)) if and only if $x_r (g_{rs}(\mathbf{x}))_+ = 0$ for all r, s such that $r \sim s$. Hence:

$$\Phi(\mathbf{x}) = \sum_{r=1}^n \sum_{s=1}^n x_r (g_{rs}(\mathbf{x}))_+ \Delta_{rs} = \mathbf{0}$$

thus establishing the “if” part of the Lemma.

Conversely, suppose $\mathbf{x} \in D \cap \mathcal{S}$ and that it is an equilibrium point of (10)–(12). Then,

$$\Phi(\mathbf{x}) = \sum_{r=1}^n \sum_{s=1}^n x_r (g_{rs}(\mathbf{x}))_+ \Delta_{rs} = \mathbf{0}.$$

Now put

$$u_r(\mathbf{x}) = -[c_r(\mathbf{x}) + \theta \ln(c_r(\mathbf{x}))];$$

so that

$$\Phi(\mathbf{x}) = \sum_{r=1}^n \sum_{s=1}^n x_r (\mathbf{u}(\mathbf{x}) \cdot \Delta_{rs})_+ \Delta_{rs} = \mathbf{0}.$$

Projecting this zero vector onto $\mathbf{u}(\mathbf{x})$,

$$\mathbf{u}(\mathbf{x}) \cdot \Phi(\mathbf{x}) = \mathbf{u}(\mathbf{x}) \cdot \sum_{r=1}^n \sum_{s=1}^n x_r (\mathbf{u}(\mathbf{x}) \cdot \Delta_{rs})_+ \Delta_{rs} = \mathbf{u}(\mathbf{x}) \cdot \mathbf{0} = 0.$$

It follows that

$$\begin{aligned} \sum_{r=1}^n \sum_{s=1}^n x_r (\mathbf{u}(\mathbf{x}) \cdot \Delta_{rs})_+ (\mathbf{u}(\mathbf{x}) \cdot \Delta_{rs})_+ &= \sum_{r=1}^n \sum_{s=1}^n x_r (\mathbf{u}(\mathbf{x}) \cdot \Delta_{rs})_+ (\mathbf{u}(\mathbf{x}) \cdot \Delta_{rs}) \\ &= \mathbf{u}(\mathbf{x}) \cdot \sum_{r=1}^n \sum_{s=1}^n x_r (\mathbf{u}(\mathbf{x}) \cdot \Delta_{rs})_+ \Delta_{rs} = \mathbf{u}(\mathbf{x}) \cdot \mathbf{0} = 0. \end{aligned}$$

Each term $x_r (\mathbf{u}(\mathbf{x}) \cdot \Delta_{rs})_+ (\mathbf{u}(\mathbf{x}) \cdot \Delta_{rs})_+$ is non-negative and so there is no cancellation. It follows (since the sum of these terms is zero) that each term is zero and hence

$$x_r (g_{rs}(\mathbf{x}))_+ = x_r (\mathbf{u}(\mathbf{x}) \cdot \Delta_{rs})_+ = 0$$

for all r, s such $r \sim s$. Consider any (r, s) on the *same* OD movement, written as $r \sim s$. Then since $\mathbf{x} \in S$ by hypothesis, all components of this vector $x_r > 0$, and so it follows from the above that for all such route pairs $r \sim s$:

$$(\mathbf{u}(\mathbf{x}) \cdot \Delta_{rs})_+ \Delta_{rs} = (g_{rs}(\mathbf{x}))_+ = \left(c_r(\mathbf{x}(t)) - c_s(\mathbf{x}(t)) + \theta \ln \left(\frac{x_r(t)}{x_s(t)} \right) \right)_+ = 0.$$

It follows that \mathbf{x} is an SUE (noting the necessary and sufficient condition for SUE stated at the start of the “if” part of this proof), and so we have proven the “only if” part of the result. \square .

We note that it is known that if the cost function \mathbf{c} is continuous and monotone on D then there exists a unique SUE solution \mathbf{e} in $D \cap \mathcal{S}$ (Cantarella and Cascetta, 1995). These properties of existence and uniqueness will be exploited in our subsequent results.

We now explore the dynamics of system (10)–(12) through a series of results. These results in turn show that (for monotone, continuously differentiable $\mathbf{c}(\cdot)$):

- (i) a (smooth) locally unique solution trajectory exists to differential equation (10)–(12) (Lemma 2);
- (ii) a Lyapunov function V may be constructed on $D \cap \mathcal{S}$ (Lemma 3);
- (iii) any solution trajectory stays away from the boundary of \mathcal{S} (Lemma 4);
- (iv) solution trajectories of (10)–(12), in staying away from the boundary of S , may be defined for all $t \geq 0$ — thus, for example, no solution trajectory is “prematurely” terminated by hitting the boundary of S (Lemma 5 and Corollary 1); and
- (v) a convergence/stability result on system (10)–(12), in relation to SUE, may then finally be established (Theorem 1).

Lemma 2

Let \mathbf{c} be continuously differentiable¹ on $D \cap \mathcal{S}$ and let $\mathbf{x}_0 \in D \cap \mathcal{S}$. Then there exists $\delta(\mathbf{x}_0) > 0$ and a unique solution to (10)–(12) for $t \in [-\delta(\mathbf{x}_0), \delta(\mathbf{x}_0)]$.

Proof

Let $\mathbf{x}_0 \in D \cap \mathcal{S}$. Now \mathbf{c} is a continuously differentiable function of \mathbf{x} throughout $D \cap \mathcal{S}$. So there is $r(\mathbf{x}_0) > 0$ such that Φ is defined and Lipschitz continuous throughout the closed neighbourhood $\text{cl}B(\mathbf{x}_0, r(\mathbf{x}_0)) \cap D$ of \mathbf{x}_0 . It follows from Picard’s theorem (see appendix A) that there is $\delta(\mathbf{x}_0) > 0$ and exactly one solution of (10)–(12) defined for all $t \in [-\delta(\mathbf{x}_0), \delta(\mathbf{x}_0)]$. Moreover:

¹ Continuous differentiability is unnecessarily strong but since it is assumed in a later result, we shall suppose it here.

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \Phi(\mathbf{x}(s)) ds \quad (-\delta(\mathbf{x}_0) \leq t \leq \delta(\mathbf{x}_0)) \quad (14)$$

thus establishing the result. \square

Discussion

Suppose in this discussion that the conditions stated in Lemma 2 hold. Then Lemma 2 ensures that for each $\mathbf{x}_0 \in D \cap \mathcal{S}$ there is a positive $\delta(\mathbf{x}_0)$ such that there exists a unique solution $\mathbf{x}(\cdot)$ to (10)–(12) defined on the time interval $[-\delta(\mathbf{x}_0), \delta(\mathbf{x}_0)]$ given by (14) (and therefore on the interval $[0, \delta(\mathbf{x}_0)]$). Now, we wish to show that this unique solution $\mathbf{x}(\cdot)$ to (10)–(12) can be extended from the time interval $[-\delta(\mathbf{x}_0), \delta(\mathbf{x}_0)]$ to the time interval $[-\delta(\mathbf{x}_0), +\infty)$ (or from $[0, \delta(\mathbf{x}_0)]$ to $[0, +\infty)$); so that the solution is uniquely defined for all future time.

Suppose now that we apply Lemma 2 for various \mathbf{x}_0 , and make the further key assumption that the $\delta(\mathbf{x}_0)$ which arise may be chosen (for all relevant \mathbf{x}_0 ; those which arise) to be independent of \mathbf{x}_0 ; let this be denoted $\delta > 0$. Under this key assumption, we may now use Picard's theorem successively at our particular initial \mathbf{x}_0 , then at $\mathbf{x}(\delta)$, then at $\mathbf{x}(2\delta)$, then at $\mathbf{x}(3\delta)$, etc. There must then be (by these successive applications of Picard's theorem) unique continuously differentiable solutions of (10)–(12) over each of the equal-length time intervals:

$$[-\delta, \delta], [0, 2\delta], [\delta, 3\delta], [2\delta, 4\delta], [3\delta, 5\delta], \dots$$

In this case the above successively generated solutions clearly fit together to yield a unique solution defined over the time interval:

$$[-\delta, \delta] \cup [0, 2\delta] \cup [\delta, 3\delta] \cup [2\delta, 4\delta] \cup [3\delta, 5\delta], \dots$$

which contains $[0, +\infty)$. *Thus, based on our assumptions, we have proved that there is a unique solution with start point \mathbf{x}_0 and defined for all future time.*

But can such a δ be chosen to justify our key supposition above? We need δ to be independent of these relevant (successively generated) initial points (namely the points $\mathbf{x}_0, \mathbf{x}(\delta), \mathbf{x}(2\delta), \mathbf{x}(3\delta), \dots$). To show that this choice is possible we need to utilise a Lyapunov argument for the system (10)–(12), as follows.

Lemma 3 (A Lyapunov result when \mathbf{c} is monotone)

Consider the (scalar) objective function $V: \mathcal{S} \rightarrow \mathbb{R}$ where

$$V(\mathbf{x}) = \sum_{r \sim s} x_r \left((c_r(\mathbf{x}) + \theta \ln x_r) - (c_s(\mathbf{x}) + \theta \ln x_s) \right)_+^2 \quad (\mathbf{x} \in \mathcal{S}; \theta > 0). \quad (15)$$

Suppose the route cost-flow function $\mathbf{c}(\cdot)$ is non-negative, continuously differentiable (and so bounded) on D , and monotone on D . (Thus any unboundedness in (15) must arise from the “ $\ln x_r$ ” terms.) Let $\mathbf{x}_0 \in D \cap \mathcal{S}$. Let $\mathbf{x}(\cdot)$ be the unique solution of the dynamical system (10)–(12) starting at $\mathbf{x}(0) = \mathbf{x}_0$ and defined for all $t \in [-\delta(\mathbf{x}_0), \delta(\mathbf{x}_0)]$ where $\delta(\mathbf{x}_0) > 0$. Then for all $t \in [0, \delta(\mathbf{x}_0)]$:

$$V(\mathbf{x}(t)) \leq V(\mathbf{x}_0).$$

Proof

We begin by noting that it is possible to re-write (12) as:

$$g_{rs}(\mathbf{x}) = (c_r(\mathbf{x}(t)) + \theta \ln x_r(t)) - (c_s(\mathbf{x}(t)) + \theta \ln x_s(t))$$

and so we may imagine that drivers are motivated by a deterministic Smith dynamic (Smith, 1984) but with the original route cost function $\mathbf{c}(\mathbf{x})$ replaced with $\mathbf{c}(\mathbf{x}) + \theta \ln \mathbf{x}$ (where $\ln \mathbf{x}$ denotes $(\ln x_1, \ln x_2, \dots, \ln x_n)$). Then, the result is established by applying the proof of descent in Smith (1984, Appendix), but showing in this case that $\Phi(\mathbf{x}(t))$ is a descent direction for V whenever $\mathbf{x}(t)$ belongs to $D \cap \mathcal{S}$ rather than D , and for the route cost function $\mathbf{c}(\mathbf{x}) + \theta \ln \mathbf{x}$ rather than $\mathbf{c}(\mathbf{x})$. This modified result however must hold, since if $\mathbf{c}(\mathbf{x})$ is monotone on D then $\mathbf{c}(\mathbf{x}) + \theta \ln \mathbf{x}$ is monotone on $D \cap \mathcal{S}$. \square

Lemma 4

Suppose that the conditions stated in Lemma 3 hold and that V is defined by (15). Let

$$M_0 > 0 \text{ and } F = \{\mathbf{x} \in D \cap \mathcal{S}; V(\mathbf{x}) \leq M_0\}.$$

Then there is a constant $R > 0$ such that

$$\text{dist}(\mathbf{x}, \text{boundary}(\mathcal{S})) \geq R \text{ for all } \mathbf{x} \in F.$$

Proof

Let $M_0 > 0$ and $F = \{\mathbf{x} \in D \cap \mathcal{S}; V(\mathbf{x}) \leq M_0\}$. Let \mathbf{x} belong to F . Then:

$$x_r \left(\left((c_r(\mathbf{x}) + \theta \ln x_r) - (c_s(\mathbf{x}) + \theta \ln x_s) \right)_+ \right)^2 \leq M_0$$

for all (r, s) such that $r \sim s$.

Now, at each \mathbf{x} in F , choose a route s (joining any OD movement) so as to minimise $\ln x_s$ and denote this chosen s by $s_{\min}(\mathbf{x})$. That is to say:

$$s_{\min}(\mathbf{x}) \in \{1, 2, \dots, n\} \text{ and } \ln x_{s_{\min}(\mathbf{x})} \leq \ln x_r \quad \forall r = 1, 2, \dots, n.$$

If there is a tie for the minimum, then arbitrarily choose one of these routes and call this $s_{\min}(\mathbf{x})$.

Then for each \mathbf{x} in F , having chosen $s_{\min}(\mathbf{x})$, choose a route r on the same OD movement so as to maximise x_r , and denote this by $r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))$. That is to say:

$$r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x})) \sim s_{\min}(\mathbf{x}) \text{ and } \ln x_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))} \geq \ln x_r \quad \forall r \sim s_{\min}(\mathbf{x}).$$

If there is a tie for the maximum on that OD movement, arbitrarily choose one of the routes and call this $r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))$.

Now, this maximum route flow is certainly bounded below since:

$$x_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))} \geq \frac{\min(q_1, q_2, \dots, q_k)}{n} = M_1 > 0$$

where we have defined M_1 for the first time above. To see this result, suppose that we chose an OD movement with smallest demand flow (i.e. one with OD demand

$\min(q_1, q_2, \dots, q_k)$) and then spread traffic evenly across the feasible routes for that movement. Certainly the number of feasible routes for that movement is less than or equal to the total number of routes n , and so certainly this fractional spread must be greater than or equal to M_1 , as defined above. Note that $M_1 > 0$ since we suppose that every OD movement has a strictly positive flow.

Now from our earlier remark, with $s = s_{\min}(\mathbf{x})$ and $r = r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))$:

$$x_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))}(\mathbf{x}) \left(\left(c_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))}(\mathbf{x}) + \theta \ln x_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))}(\mathbf{x}) \right) - \left(c_{s_{\min}(\mathbf{x})}(\mathbf{x}) + \theta \ln x_{s_{\min}(\mathbf{x})}(\mathbf{x}) \right) \right)_+^2 \leq M_0$$

and with our bound above on the route flows, it then follows that:

$$\left(\left(c_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))}(\mathbf{x}) + \theta \ln x_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))}(\mathbf{x}) \right) - \left(c_{s_{\min}(\mathbf{x})}(\mathbf{x}) + \theta \ln x_{s_{\min}(\mathbf{x})}(\mathbf{x}) \right) \right)_+^2 \leq \frac{M_0}{M_1}$$

whence:

$$\left(c_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))}(\mathbf{x}) + \theta \ln x_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))}(\mathbf{x}) \right) - \left(c_{s_{\min}(\mathbf{x})}(\mathbf{x}) + \theta \ln x_{s_{\min}(\mathbf{x})}(\mathbf{x}) \right)_+ \leq \left(\frac{M_0}{M_1} \right)^{\frac{1}{2}}$$

and so:

$$\left(c_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))}(\mathbf{x}) + \theta \ln x_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))}(\mathbf{x}) \right) - \left(c_{s_{\min}(\mathbf{x})}(\mathbf{x}) + \theta \ln x_{s_{\min}(\mathbf{x})}(\mathbf{x}) \right) \leq \left(\frac{M_0}{M_1} \right)^{\frac{1}{2}}.$$

Since by hypothesis, $c_r(\cdot)$ is non-negative and bounded above on D (by B , say), it follows that $c_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))}(\mathbf{x}) - c_{s_{\min}(\mathbf{x})}(\mathbf{x}) \geq -B$, and hence:

$$-B + \theta \ln x_{r_{\max}(\mathbf{x} | s_{\min}(\mathbf{x}))}(\mathbf{x}) - \theta \ln x_{s_{\min}(\mathbf{x})}(\mathbf{x}) \leq \left(\frac{M_0}{M_1} \right)^{\frac{1}{2}}.$$

Using once more the bound on the maximum route flow, together with the fact that the \ln function is increasing

$$-B + \theta \ln M_1 - \theta \ln x_{s_{\min}(\mathbf{x})}(\mathbf{x}) \leq \left(\frac{M_0}{M_1} \right)^{\frac{1}{2}}.$$

Rearranging:

$$-\ln x_{s_{\min}(\mathbf{x})}(\mathbf{x}) \leq \frac{1}{\theta} \left(\left(\frac{M_0}{M_1} \right)^{\frac{1}{2}} + B - \theta \ln M_1 \right)$$

and hence:

$$x_{s_{min}(\mathbf{x})}(\mathbf{x}) \geq \exp\left(-\frac{1}{\theta}\left(\left(\frac{M_0}{M_1}\right)^{\frac{1}{2}} + B - \theta \ln M_1\right)\right) = R > 0 \text{ (say).}$$

Recalling that, at any fixed \mathbf{x} in $D \cap \mathcal{S}$, the component $x_{s_{min}(\mathbf{x})}(\mathbf{x})$ of \mathbf{x} is chosen to be the smallest component of \mathbf{x} , it follows that the distance between *any* component of \mathbf{x} and the boundary of \mathcal{S} must exceed this positive constant R . \square

Corollary 1

Under the conditions of Lemma 4, the local solutions defined for each possible start point $\mathbf{x}_0 \in D \cap \mathcal{S}$ fit together to create a solution $\mathbf{x}(\cdot)$ starting at any \mathbf{x}_0 in $D \cap \mathcal{S}$ and defined for all time t in $[0, +\infty)$.

Proof

This result follows from the previous discussion and Lemma 4. We need to show that we can choose a $\delta(\mathbf{x})$ that is independent of the relevant \mathbf{x} . So let \mathbf{x}_0 belong to $D \cap \mathcal{S}$, let $M_0 = V(\mathbf{x}_0)$ and let $F = \{\mathbf{x} \in D \cap \mathcal{S}; V(\mathbf{x}) \leq M_0\}$. Although \mathcal{S} is not closed the set F is closed since V is continuous. Also F is bounded and so (being both closed and bounded) is compact. Relevant \mathbf{x} here are those \mathbf{x} in F .

Then by Lemma 4 there is a constant $R > 0$ such that

$$\text{dist}(\mathbf{x}, \text{boundary}(\mathcal{S})) \geq R \text{ for all } \mathbf{x} \in F.$$

Now \mathbf{c} is continuously differentiable on $D \cap \mathcal{S}$ and so on

$$F_\theta = (D \cap \mathcal{S}) \cap \{\mathbf{x}; \text{dist}(\mathbf{x}, \text{boundary}(\mathcal{S})) \geq R\}.$$

Also F_θ is (like F) a closed and bounded set in Euclidean space and so is compact. Hence the derivative \mathbf{c}' of \mathbf{c} , being continuous on F_θ , is also bounded on F_θ (which contains F). It follows (as remarked above) that \mathbf{c} and hence Φ is Lipschitz continuous on F_θ and so also on the subset F . So there exists $K > 0$ such that

$$\|\Phi(\mathbf{x}) - \Phi(\mathbf{y})\| \leq K \|\mathbf{x} - \mathbf{y}\| \text{ for all } \mathbf{x}, \mathbf{y} \in F.$$

By choice of R above, *for all* \mathbf{x} in F : the closed ball $B = \text{cl}[B_{R/2}(\mathbf{x})]$ is a subset of \mathcal{S} . It now follows from Picard's theorem (see Hunter (1996) and appendix A) that if we put

$$\delta = R/(2M_0)$$

then for each $\mathbf{x}_0 \in F$ there is a unique solution of (10)–(12) defined on $[-\delta, \delta]$, *where δ is not dependent on \mathbf{x}_0 so long as $\mathbf{x}_0 \in F$* . Now we know that each trajectory starting in F stays in F , since V decreases along a trajectory by Lemma 3. Thus, as indicated in the discussion, using Picard's theorem successively at our particular initial \mathbf{x}_0 , then at $\mathbf{x}(\delta)$, then at $\mathbf{x}(2\delta)$, then at $\mathbf{x}(3\delta)$,, there must be unique solutions of (10)–(12) over each of the equal-length time intervals:

$$[-\delta, \delta], [0, 2\delta], [\delta, 3\delta], [2\delta, 4\delta], [3\delta, 5\delta], \dots$$

(Of course, by the Lyapunov result in Lemma 3, $\mathbf{x}_0 \in F$ implies that $\mathbf{x}(\delta) \in F$, which in turn implies that $\mathbf{x}(2\delta) \in F$, which in turn implies that $\mathbf{x}(3\delta) \in F$ ) The proof is completed. \square

Lemma 5

Under the conditions given in Lemma 3, any solution $\mathbf{x}(\cdot)$ of (10)–(12) starting at $\mathbf{x}_0 \in D \cap \mathcal{S}$ (and so defined for all $t \geq 0$, by Lemma 4) satisfies:

$$V(\mathbf{x}(t)) \rightarrow 0 \text{ as } t \rightarrow +\infty.$$

Proof

We have shown that no solution trajectory starting at $\mathbf{x}_0 \in D \cap \mathcal{S}$ ever leaves $F = \{\mathbf{x} \in D \cap \mathcal{S}; V(\mathbf{x}) \leq V(\mathbf{x}_0)\}$. So by Lemma 4 such trajectories run for all time $t \geq 0$. Then the proof of the Lyapunov result in Smith (1984) may be applied to show that $V(\mathbf{x}(t)) \rightarrow 0$ as $t \rightarrow +\infty$. \square

Theorem 1

Let \mathbf{e} be the unique SUE². Under the conditions of Lemma 3, given any start point \mathbf{x}_0 in $D \cap \mathcal{S}$, any solution $\mathbf{x}(\cdot)$ of (10)–(12) starting at \mathbf{x}_0 must satisfy $\text{dist}(\mathbf{x}(t), \mathbf{e}) \rightarrow 0$ as $t \rightarrow \infty$.

Proof

By Lemma 5, $V(\mathbf{x}(t)) \rightarrow 0$ as $t \rightarrow \infty$. V is continuous and $V(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{e}$. Also $\mathbf{x}(t)$ belongs to $F = \{\mathbf{x} \in D \cap \mathcal{S}; V(\mathbf{x}) \leq V(\mathbf{x}_0)\}$ for all $t > 0$; and F is compact. Therefore

$$\text{dist}(\mathbf{x}(t), \mathbf{e}) \rightarrow 0 \text{ as } t \rightarrow \infty. \quad \square$$

5. CONCLUDING REMARKS

In this paper we have presented a path-swapping, continuous-time dynamical system that is a RUM-based analogue of the system proposed by Smith (1984) for deterministic choice systems. We have established that equilibria of this system are SUE solutions, and have gone on to establish a corresponding Lyapunov-style result for such a system.

This work opens up several opportunities for further research, including: i) the possibility to extend existing stability results for SUE in *discrete* time, which currently require a case-by-case analysis of network properties; ii) possibilities to devise stabilising control and pricing measures that exploit such properties; iii) the connection of the results presented with classes of dynamic process that have been identified for DUE-related systems (such as ‘rational behaviour adjustment processes’); and iv) the extension of the results to other choice models that may naturally be formulated as pairwise swaps, such as weibit (Castillo et al, 2008) and path-size logit/weibit (Kitthamkesorn and Chen, 2013), as well as more general choice models adopting the kinds of swapping dynamics suggested in Watling (1998).

² As we remarked earlier, existence and uniqueness of SUE follows from our hypotheses on the cost functions made in section 2; see, for example, Cantarella & Cascetta (1995).

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Appendix

A Picard Existence Theorem using our notation (see Hunter (section 2.3, 1996)).

In our setting, suppose that $R > 0$ is such that for all \mathbf{x} in F

- (a) the closed ball $B = \text{cl}[B_{R/2}(\mathbf{x})]$ is a subset of S and
- (b) $\Phi(\cdot)$ is defined and Lipschitz continuous with Lipschitz constant $K = K(\mathbf{x})$, on the set $D \cap \text{cl}[B_{R/2}(\mathbf{x})]$.

Let

$M = \sup\{\|\Phi(\mathbf{y})\|; \mathbf{y} \text{ belongs to the union of the closed sets } F \cap D \cap \text{cl}[B_{R/2}(\mathbf{x})] \text{ as } \mathbf{x} \text{ varies over } F.\}$ Also let $h = R/2M$. (N.B. h does not depend on \mathbf{x}_0 in F .)

Then, for each \mathbf{x}_0 in F , the system defined by (10)–(12) has a unique continuously differentiable solution $\mathbf{x}(\cdot)$ defined for all times t such that

$$-2h < -h \leq t \leq h < 2h.$$

Proof. This follows Hunter (1996).