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Cellular cohomology of posets with local coefficients☆

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Abstract

We describe a "cellular" approach to the computation of the cohomology of a poset with coefficients in a presheaf. A cellular cochain complex is constructed, described explicitly and shown to compute the cohomology under certain circumstances. The descriptions are refined further for certain classes of posets including the cell posets of regular CW-complexes and geometric lattices.

Introduction

The cohomology groups of a poset $\mathbf{P}$ with coefficients in a presheaf F are the derived functors (or higher limits) of the limit $\varprojlim_{\mathbf{P}} F$, and they can be computed using a canonical complex $S^*(\mathbf{P}; F)$. In [6] we showed that the Khovanov homology groups of a link diagram are the higher limits of a certain poset and presheaf. The definition of Khovanov homology involves a different complex however, which, while based on essentially the same poset and presheaf, appears ad hoc in its construction. The motivation for [6] was to place it within a more general framework. The relationship between the two constructions is analogous to that found in topology, where the cellular chain complex of a space is simple and well suited to explicit computation – like the definition of Khovanov homology – but has less flexibility than the singular chain complex.

So we are naturally led to the following questions: for what posets $\mathbf{P}$ and presheaves F can a "cellular" cohomology be defined? Under what circumstances will this compute the higher limits, that is to say, coincide with the usual cohomology? In this paper we propose a definition of cellular cohomology $HC^*(\mathbf{P}; F)$ applicable to a large class of posets, and show that for many naturally occurring examples, including the cell posets of regular CW-complexes and geometric lattices, this cellular cohomology computes the higher limits.

Specifically, we define a cochain complex $C^*(\mathbf{P}; F)$ for a (graded) poset $\mathbf{P}$ equipped with a presheaf F by mimicking the construction of the cellular chain complex in topology. We define first the relative cohomology of pairs and then apply this to adjacent degrees of a filtration of $\mathbf{P}$. The role of open cells is played by open intervals $\mathbf{P}_{>x}$. Our main result is that, as in topology, a vanishing condition on these relative cohomologies suffices for this cellular complex to compute the higher limits. If a poset has this condition we call it *cellular* and our main result (see Section 3) then reads:

Theorem 1. *Let* $\mathbf{P}$ *be graded, cellular, locally finite with a corank function and let* F *be a presheaf on* $\mathbf{P}$*. Then there is a natural isomorphism*

$$HS^*(\mathbf{P}; F) \cong HC^*(\mathbf{P}; F).$$

☆Dedicated to the memory of Colin Maclachlan

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The proof is in two steps: finite posets are handled via a spectral sequence and this is extended to locally finite posets using a projective resolution. On the other hand it is not hard to find examples for which the higher limits *cannot* be computed cellularly.

We also spend some time describing the cellular cochain complex explicitly. For example in Section 2 we describe the cochain groups as a product taken over the "cells":

Proposition 1. *Let* $\mathbf{P}$ *be graded with a corank function and* F *a presheaf on* $\mathbf{P}$*. Then there are natural isomorphisms*

$$C^n(\mathbf{P};F) \cong \prod_{|x|=n} \widetilde{HS}^{n-1}(\mathbf{P}_{>x};\Delta F(x))$$

Our most concrete result concerning the complex itself is Proposition 7 where we show that the form of the cellular cochain groups is shaped by abelian groups that reflect the structure of closed intervals $\mathbf{P}_{\geq x}$. This also makes it easy to see that the cochain groups need not necessarily be free, even if the presheaf takes values that are free.

Section 4 describes in some detail the cellular complex in a couple of important cases: when $\mathbf{P}$ is the cell poset of a regular CW-complex (this includes the Khovanov homology result mentioned above) and when $\mathbf{P}$ is a geometric lattice.

Colin Maclachlan, colleague, mentor and friend, died in November 2012. Colin maintained a healthy skepticism of "abstract mumbo jumbo", so we're not sure that he would have approved of this paper. Nevertheless, we dedicate it to him with much respect and affection.

1. Cohomology of posets with coefficients in a presheaf

We start by recalling definitions and elementary results concerning the cohomology of posets with coefficients in a presheaf. Everything is well-known (see [7, Appendix II.3]) but it is convenient to have it to hand and set down the notation we will be using. Many of the definitions and results carry straight through to the more general setting of small categories, but as we will be working with posets we prefer to state everything in these more concrete terms.

1.1. Definitions

Let $\mathbf{P} = (\mathbf{P}, \leq)$ be a poset. We will usually think of $\mathbf{P}$ as a category having objects the elements of the poset and with a unique morphism $x \to y$ if and only if $x \leq y$. The nerve $N^*\mathbf{P}$ of $\mathbf{P}$ is the simplicial set with n-simplicies $N^n\mathbf{P}$ the poset sequences $\sigma = \sigma_n \leq \cdots \leq \sigma_0$, where the $\sigma_i \in \mathbf{P}$, and with face maps $d_i : N^n\mathbf{P} \to N^{n-1}\mathbf{P}$ and degeneracy maps $s_i : N^n\mathbf{P} \to N^{n+1}\mathbf{P}$ given by

$$d_i\sigma = \sigma_n \leq \cdots \leq \widehat{\sigma_i} \leq \cdots \leq \sigma_0 \text{ and } s_i\sigma = \sigma_n \leq \cdots \leq \sigma_i \leq \sigma_i \leq \cdots \leq \sigma_0. \tag{1}$$

The geometrical realization of $N^*\mathbf{P}$ will be denoted by $|N^*\mathbf{P}|$. It is a CW-complex with a single n-cell for each non-degenerate n-simplex $\sigma = \sigma_n < \cdots < \sigma_0$.

A presheaf on $\mathbf{P}$ is a (covariant) functor $F : \mathbf{P}^{\mathrm{op}} \to \mathbf{Ab}$ to abelian groups (or, more generally, to R-modules; again, we specialize for concreteness). The category $\mathbf{PreSh}(\mathbf{P})$ has as objects the presheaves $F : \mathbf{P}^{\mathrm{op}} \to \mathbf{Ab}$ and as morphisms the natural transformations $\kappa : F \to G$. We write F_x^y for the homomorphism $F(y) \to F(x)$ induced by $x \leq y$ in $\mathbf{P}$, and κ_x for the map $F(x) \to G(x)$ that is the component at x of the natural transformation κ.

Example 1. For $A \in \mathbf{Ab}$ the *constant* presheaf ΔA is defined by $\Delta A(x) = A$ for every $x \in \mathbf{P}$ and $(\Delta A)_x^y = 1$ for every $x \leq y$ in $\mathbf{P}$.

Example 2. For $A \in \mathbf{Ab}$ and $x \in \mathbf{P}$ the *Yoneda* presheaf $\Upsilon_x A$ is defined by

$$\Upsilon_x A(y) = \begin{cases} A, & \text{if } y \le x \\ 0, & \text{otherwise,} \end{cases}$$

and with $(\Upsilon_x A)^z_y = 1$ when $y \le z \le x$; or $0 \to A$ when $y \le x$ and $z \not\le x$, and the zero map otherwise. Thus $\Upsilon_x A$ is the constant presheaf ΔA on the closed interval $\mathbf{P}_{\le x} = \{y \in \mathbf{P} \mid y \le x\}$ and the zero presheaf on the rest of $\mathbf{P}$. If $x \le y$ in $\mathbf{P}$ and $A \to B$ is a map of abelian groups, then there is an induced morphism of presheaves $\Upsilon_x A \to \Upsilon_y B$. Indeed, the most useful property of the Yoneda functor Υ_x is that is it left adjoint to the evaluation functor $\mathbf{PreSh(P)} \to \mathbf{Ab}$ taking F to $F(x)$. Explicitly, this adjunction is $\mathrm{Hom}_{\mathbf{PreSh(P)}}(\Upsilon_x A, F) \cong \mathrm{Hom}_{\mathbb{Z}}(A, F(x))$, via $\kappa \mapsto \kappa_x$, and if $\zeta : \Upsilon_x A \to \Upsilon_y B$ is the induced morphism above, then $\mathrm{Hom}_{\mathbf{PreSh(P)}}(-, F)$ applied to $\Upsilon_x A \to \Upsilon_y B$ is the map

$$\mathrm{Hom}_{\mathbb{Z}}(B, F(y)) \xrightarrow{\zeta^*} \mathrm{Hom}_{\mathbb{Z}}(A, F(x)) \text{ with } \kappa_y \mapsto F^y_x \kappa_y \zeta_x. \tag{2}$$

In particular $\Upsilon_x A$ is projective in $\mathbf{PreSh(P)}$ if and only if A is projective in $\mathbf{Ab}$ (i.e.: A is free).

For any presheaf F the limit $\varprojlim_{\mathbf{P}} F$ (or group of global sections) exists in $\mathbf{Ab}$ and is constructed by taking the subgroup of the product $\prod_{x \in \mathbf{P}} F(x)$ consisting of those $\mathbf{P}$-tuples $(a_x)_{x \in \mathbf{P}}$ with $a_x \in F(x)$, and such that for all $x \le y$ in $\mathbf{P}$ the induced map $F(y) \to F(x)$ sends a_y to a_x. Indeed we have a left exact functor $\varprojlim_{\mathbf{P}} : \mathbf{PreSh(P)} \to \mathbf{Ab}$ and the right derived functors

$$\varprojlim{}^i_{\mathbf{P}} := R^i \varprojlim_{\mathbf{P}}$$

are called the higher limits of F. By definition the *cohomology groups of* $\mathbf{P}$ *with coefficients in the presheaf* F are these higher limits evaluated at F.

The higher limits are computed as follows. There is a canonical projective resolution $A_* \to \Delta\mathbb{Z}$ in $\mathbf{PreSh(P)}$ with $A_n = \bigoplus_\sigma \Upsilon_{\sigma_n} \mathbb{Z}$, where the direct sum is over the simplicies $\sigma \in N^n\mathbf{P}$, and with the maps $A_n \to A_{n-1}$ induced by the simplicial structure of the nerve (see, for example, [12, Proposition II.6.1]). The associated cochain complex $S^*(\mathbf{P}; F) := \mathrm{Hom}_{\mathbf{PreSh(P)}}(A_*, F)$ thus computes the higher limits. It has nth cochain group

$$S^n(\mathbf{P}; F) = \prod_\sigma F(\sigma_n),$$

where the product is over the n-simplicies $\sigma \in N^n\mathbf{P}$ of the nerve. We adopt the convention that $S^*(\varnothing; F)$ is the zero complex. If $s \in S^n(\mathbf{P}; F)$ and $\sigma \in N^n\mathbf{P}$ we write $s \cdot \sigma$ for the component of s indexed by σ, so if $\sigma = \sigma_n \le \cdots \le \sigma_0$ then $s \cdot \sigma \in F(\sigma_n)$.

The differential $d\colon S^{n-1}(\mathbf{P}; F) \to S^n(\mathbf{P}; F)$ is defined for $s \in S^{n-1}(\mathbf{P}; F)$ and $\sigma \in N^n\mathbf{P}$ by

$$ds \cdot \sigma = \sum_{i=0}^{n-1} (-1)^i s \cdot d_i\sigma + (-1)^n F^{\sigma_{n-1}}_{\sigma_n}(s \cdot d_n\sigma) \tag{3}$$

with the d_i the face maps (1) of the nerve. By defining

$$HS^*(\mathbf{P}; F) := H(S^*(\mathbf{P}; F), d)$$

we have $\varprojlim{}^*_{\mathbf{P}} F \cong HS^*(\mathbf{P}; F)$.

With a constant presheaf we recover the topology:

$$HS^*(\mathbf{P}; \Delta A) \cong H^*(|N^*\mathbf{P}|, A) \tag{4}$$

where the right hand side is the ordinary singular cohomology of the geometrical realisation $|N^*\mathbf{P}|$ (see for example [1, Theorem 2.1]).

The complex $S^*(\mathbf{P}; F)$ has a factor for each simplex in the nerve, degenerate or not. There is a version which uses only non-degenerate simplices: let $T^n(\mathbf{P}; F) = \prod_\sigma F(\sigma_n)$, taking the product over $N^n_\circ\mathbf{P}$, the non-degenerate simplicies $\sigma = \sigma_n < \cdots < \sigma_n$, and the differential also given by (3). Then $T^*(\mathbf{P}; F)$ is a quotient of $S^*(\mathbf{P}; F)$ by the subcomplex consisting of the degenerate simplicies, itself homotopy equivalent to the zero complex, hence $T^*(\mathbf{P}; F)$ and $S^*(\mathbf{P}; F)$ are homotopy equivalent. Explicitly, define $f : S^*(\mathbf{P}; F) \to T^*(\mathbf{P}; F)$ by $fs \cdot \sigma = s \cdot \sigma$ for $\sigma \in N^n_\circ\mathbf{P}$, and $g : T^*(\mathbf{P}; F) \to S^*(\mathbf{P}; F)$ by

$$gt \cdot \sigma = \begin{cases} t \cdot \sigma, & \text{if } \sigma \in N^n_\circ\mathbf{P}, \\ 0, & \text{otherwise.} \end{cases}$$

Then fg is the identity on $T^*(\mathbf{P}; F)$. Let $h : S^n(\mathbf{P}; F) \to S^{n-1}(\mathbf{P}; F)$ be given by

$$hs \cdot \sigma = \begin{cases} (-1)^p s \cdot s_p\sigma, & \text{if } \sigma = \sigma_{n-1} \cdots \sigma_{p+\ell}\sigma_p \cdots \sigma_p\sigma_{p-1} \cdots \sigma_0, \\ 0, & \text{otherwise,} \end{cases}$$

where the $\sigma_p, \ldots, \sigma_0$ are distinct, there are $\ell \geq 2$ with ℓ *even* repeats of $\sigma_p \neq \sigma_{p+\ell}$, and $s_p \colon N^{n-1}\mathbf{P} \to N^n\mathbf{P}$ is a degeneracy map from (1). Then h is a chain homotopy between gf and the identity on $S^*(\mathbf{P}; F)$. Much of what we say about $S^*(\mathbf{P}; F)$ holds analogously for $T^*(\mathbf{P}; F)$. We will content ourselves with pointing this out where appropriate and leaving the details to the reader.

1.2. Induced maps

If $f \colon \mathbf{Q} \to \mathbf{P}$ is a map of posets then there are a number of induced maps and functors.

- There is an induced map of simplicial sets $N^*\mathbf{Q} \to N^*\mathbf{P}$ sending $\sigma = \sigma_n \leq \cdots \leq \sigma_0 \in N^n\mathbf{Q}$ to $f\sigma = f\sigma_n \leq \cdots \leq f\sigma_0 \in N^n\mathbf{P}$.
- There is an induced functor $\mathbf{PreSh}(\mathbf{P}) \to \mathbf{PreSh}(\mathbf{Q})$ sending $F \in \mathbf{PreSh}(\mathbf{P})$ to $f^*F := F \circ f$ and $\kappa : F \to G$ to $f^*\kappa : f^*F \to f^*G$ with $f^*\kappa_x = \kappa_{fx}$. If f is an inclusion $\mathbf{Q} \hookrightarrow \mathbf{P}$ then we will just write F for f^*F.
- There is an induced map of groups $f^* \colon S^*(\mathbf{P}; F) \to S^*(\mathbf{Q}; f^*F)$, the *pull-back*, defined for $s \in S^n(\mathbf{P}; F)$ and $\sigma \in N^n\mathbf{Q}$ by

$$f^*s \cdot \sigma = s \cdot f\sigma. \tag{5}$$

Lemma 1. *The pull-back f^* is a chain map. If $g \colon \mathbf{R} \to \mathbf{Q}$ is another poset map then $(fg)^* = g^*f^* : S^*(\mathbf{P}, F) \to S^*(\mathbf{R}, (fg)^*F)$.*

If $f : \mathbf{Q} \to \mathbf{P}$ is injective then (5) gives a pull-back $T^*(\mathbf{P}; F) \to T^*(\mathbf{Q}; f^*F)$ and the analogue of Lemma 1 holds.

- If $f \colon \mathbf{Q} \to \mathbf{P}$ is finite-to-one, i.e. for each $x \in \mathbf{P}$ the pre-image $f^{-1}x$ is a finite set, then there is an induced map of groups $f_* \colon S^*(\mathbf{Q}; f^*F) \to S^*(\mathbf{P}; F)$, the *push-forward*, defined for $s \in S^n(\mathbf{Q}; f^*F)$ and $\sigma \in N^n\mathbf{P}$ by

$$f_*s \cdot \sigma = \sum_{\tau \in f^{-1}\sigma} s \cdot \tau \tag{6}$$

  If $f^{-1}\sigma$ is empty the right-hand side is taken to be zero. By definition of the presheaf f^*F each element of $f^{-1}\sigma$ has associated to it the same abelian group – namely $F(\sigma_n)$ – and the sum in (6) takes place in this group. The push-forward is not in general a chain map: for example if $f : \mathbf{Q} \to \mathbf{P}$ is injective but not surjective.

Remark 1. Our notation differs from that found in [7, Appendix 2] where f_* is used for the induced functor $\mathbf{PreSh}(\mathbf{P}) \to \mathbf{PreSh}(\mathbf{Q})$ and f^* denotes a left adjoint to f_*.

Lemma 2. *If f is injective then $f^* f_* = id$, so that f^* is surjective and f_* is injective.*

– Morphisms of presheaves also induce maps of complexes. Let F and G be presheaves on $\mathbf{P}$ and $\kappa\colon F \to G$ a natural transformation. Then there is an induced map $\kappa_*\colon S^*(\mathbf{P}; F) \to S^*(\mathbf{P}; G)$ defined for $s \in S^n(\mathbf{P}; F)$ and $\sigma = \sigma_n \le \cdots \le \sigma_0 \in N^n\mathbf{P}$ by

$$\kappa_* s \cdot \sigma = \kappa_{\sigma_n}(s \cdot \sigma).$$

Lemma 3. *The induced map κ_* is a chain map and if $f\colon \mathbf{Q} \to \mathbf{P}$ is a poset map then the following diagram commutes:*

$$\begin{tikzcd}
S^*(\mathbf{P}; F) \arrow[r, "\kappa_*"] \arrow[d, "f^*"'] & S^*(\mathbf{P}; G) \arrow[d, "f^*"] \\
S^*(\mathbf{Q}; f^*F) \arrow[r, "f^*\kappa_*"'] & S^*(\mathbf{Q}; f^*G)
\end{tikzcd}$$

1.3. Reduced cohomology

For $A \in \mathbf{Ab}$ we can augment $S^*(\mathbf{P}; \Delta A)$ in degree -1 by defining $d^{-1}\colon A \to S^0(\mathbf{P}; \Delta A)$ to be $d^{-1}a \cdot \sigma = a$ for $\sigma \in N^0\mathbf{P}$ (i.e. d^{-1} injects A diagonally). Then $\widetilde{S}^*(\mathbf{P}; \Delta A) := A \to S^*(\mathbf{P}; \Delta A)$ is a cochain complex, and the *reduced* cohomology is defined by

$$\widetilde{HS}^*(\mathbf{P}; \Delta A) := H(\widetilde{S}^*(\mathbf{P}; \Delta A), d).$$

This is isomorphic to $\widetilde{H}^*(|N^*\mathbf{P}|; A)$, the ordinary reduced cohomology of the realization of the nerve. Sometimes it will be convenient to set $N^{-1}\mathbf{P} = N_\circ^{-1}\mathbf{P} = \star$, the one element set, and define $a \cdot \star = a$ for $a \in A$.

An alternative construction is as follows: let $\star$ be the one-element poset and let $F \in \mathbf{PreSh}(\star)$ be the presheaf with $F(\star) = A$. The collapse map $f\colon \mathbf{P} \to \star$ induces the constant presheaf ΔA on $\mathbf{P}$ via $f^*F = \Delta A$. By considering the map $f^*\colon HS^*(\star; F) \to HS^*(\mathbf{P}; \Delta A)$ induced by the pullback, we have

$$\widetilde{HS}^*(\mathbf{P}; \Delta A) \cong \operatorname{coker} f^*.$$

Analogously we can define $\widetilde{T}^*(\mathbf{P}; \Delta A)$ and $\widetilde{HT}^*(\mathbf{P}; \Delta A)$ – although only the first of the two approaches above now works – and we have a homotopy equivalence $\widetilde{T}^*(\mathbf{P}; \Delta A) \simeq \widetilde{S}^*(\mathbf{P}; \Delta A)$.

1.4. Relative cohomology

Let $f\colon \mathbf{Q} \to \mathbf{P}$ be a poset map, F a presheaf on $\mathbf{P}$ and $f^*\colon S^*(\mathbf{P}; F) \to S^*(\mathbf{Q}; f^*F)$ the pull-back. Define

$$S^*(\mathbf{P}, \mathbf{Q}; F) := \ker f^*.$$

We will mostly consider the case where f is an inclusion, which is why we omit it from the notation. Observe that for $s \in S^*(\mathbf{P}; F)$ we have $s \in S^*(\mathbf{P}, \mathbf{Q}; F)$ if and only if $s \cdot \sigma = 0$ for all $\sigma \in f(N^*\mathbf{Q}) \subset N^*\mathbf{P}$. The differential on $S^*(\mathbf{P}; F)$ restricts to a differential on $S^*(\mathbf{P}, \mathbf{Q}; F)$ and we define

$$HS^*(\mathbf{P}, \mathbf{Q}; F) := H(S^*(\mathbf{P}, \mathbf{Q}; F), d),$$

the (*relative*) cohomology of the pair $(\mathbf{P}, \mathbf{Q})$ with coefficients in the presheaf F. We adopt the convention that $S^*(\mathbf{P}, \varnothing; F) = S^*(\mathbf{P}; F)$. If f is an injection we can analogously define $T^*(\mathbf{P}, \mathbf{Q}; F)$. The maps given at the end of §1.1 then restrict to the various relative complexes to give a homotopy equivalence $T^*(\mathbf{P}, \mathbf{Q}; F) \simeq S^*(\mathbf{P}, \mathbf{Q}; F)$.

If f is injective then Lemma 2 gives a short exact sequence

$$0 \longrightarrow S^*(\mathbf{P},\mathbf{Q};F) \longrightarrow S^*(\mathbf{P};F) \xrightarrow{f^*} S^*(\mathbf{Q};f^*F) \longrightarrow 0$$

and hence a long exact sequence

$$\cdots \xrightarrow{\beta} HS^n(\mathbf{P},\mathbf{Q};F) \longrightarrow HS^n(\mathbf{P};F) \longrightarrow HS^n(\mathbf{Q};f^*F) \xrightarrow{\beta} HS^{n+1}(\mathbf{P},\mathbf{Q};F) \longrightarrow \cdots \quad (7)$$

Lemma 4. *Let $\mathbf{P},\mathbf{Q},\mathbf{R}$ be posets with $j\colon \mathbf{R} \hookrightarrow \mathbf{Q}$ and $i\colon \mathbf{Q} \hookrightarrow \mathbf{P}$ inclusions and let F be a presheaf on $\mathbf{P}$. Then there is a short exact sequence*

$$0 \longrightarrow S^*(\mathbf{P},\mathbf{Q};F) \longrightarrow S^*(\mathbf{P},\mathbf{R};F) \xrightarrow{i^*} S^*(\mathbf{Q},\mathbf{R};F) \longrightarrow 0$$

and hence a long exact sequence

$$\cdots \xrightarrow{\delta} HS^n(\mathbf{P},\mathbf{Q};F) \longrightarrow HS^n(\mathbf{P},\mathbf{R};F) \longrightarrow HS^n(\mathbf{Q},\mathbf{R};F) \xrightarrow{\delta} HS^{n+1}(\mathbf{P},\mathbf{Q};F) \longrightarrow \cdots$$

Lemma 5. *Let $(\mathbf{P},\mathbf{Q},\mathbf{R})$ be the triple of Lemma 4 with $\iota\colon HS^n(\mathbf{Q},\mathbf{R};F) \to HS^n(\mathbf{Q};F)$ induced by the inclusion $S^n(\mathbf{Q},\mathbf{R};F) \to S^n(\mathbf{Q};F)$ and $\beta : HS^n(\mathbf{Q};F) \to HS^{n+1}(\mathbf{P},\mathbf{Q};F)$ the connecting homomorphism of the pair $(\mathbf{P},\mathbf{Q})$. If δ is the connecting homomorphism of Lemma 4 then $\delta = \beta\iota$.*

2. Cellular cohomology of posets with coefficients in a presheaf

Singular cohomology can be defined for any space X. If X has a cellular structure – for example is a CW-complex – then cellular cohomology can also be defined and singular cohomology can be computed using it. In this section we define, for a large class of posets, a cellular cohomology with coefficients in a presheaf. A primordial version, along the lines of Proposition 5 below, appears in [17, Section 4]. As in topology we define a cochain complex using the relative cohomology of pairs – which is not hard – and then expend some effort describing it more explicitly. In the next section we will describe a situation in which the $\varprojlim^*_{\mathbf{P}} F$ can be computed via this cellular cohomology.

2.1. The definition of cellular cohomology

We begin by recalling some poset terminology. If $x \leq y \in \mathbf{P}$ and for any $x \leq z \leq y$ we have either $z = x$ or $z = y$, then y is said to *cover* x, written $x \prec y$. $\mathbf{P}$ is *graded* if there exists a *rank function* $rk : \mathbf{P} \to \mathbb{Z}$, i.e. a function such that (i) $x < y$ implies $rk(x) < rk(y)$, and (ii) $x \prec y$ implies $rk(y) = rk(x) + 1$.

Fix a rank function rk on graded $\mathbf{P}$ and assume further that rk is bounded above with $r = \max_{x\in\mathbf{P}}\{rk(x)\}$. Define the *corank* function $|\cdot| : \mathbf{P} \to \mathbb{Z}^{\geq 0}$ by $|x| = r - rk(x)$. Filter $\mathbf{P}$ by corank by letting

$$\mathbf{P}^k = \{x \in \mathbf{P} \mid |x| \leq k\}. \quad (8)$$

There is thus a sequence of inclusions $\mathbf{P}^0 \subset \mathbf{P}^1 \subset \mathbf{P}^2 \subset \cdots$ If F is a presheaf on $\mathbf{P}$ then we get induced presheaves F on each $\mathbf{P}^k$ via the inclusions $\mathbf{P}^k \hookrightarrow \mathbf{P}$.

From now on all posets will be graded with a bounded rank function and with a corank function $|\cdot| : \mathbf{P} \to \mathbb{Z}^{\geq 0}$; we will abbreviate this set-up to "graded with a corank function".

Definition 2.1. *Let $\mathbf{P}$ be graded with a corank function, $\{\mathbf{P}^k\}_{k\in\mathbb{Z}}$ the associated filtration in (8) and F a presheaf on $\mathbf{P}$. The cellular cochain complex $C^*(\mathbf{P};F)$ has chain groups*

$$C^n(\mathbf{P};F) := HS^n(\mathbf{P}^n,\mathbf{P}^{n-1};F)$$

and differential $C^{n-1}(\mathbf{P};F) \xrightarrow{\delta} C^n(\mathbf{P};F)$ by taking $\mathbf{R} = \mathbf{P}^{n-2}$, $\mathbf{Q} = \mathbf{P}^{n-1}$ and $\mathbf{P} = \mathbf{P}^n$ in Lemma 4 and defining δ to be the coboundary map in the associated the long exact sequence

$$C^{n-1}(\mathbf{P};F) = HS^{n-1}(\mathbf{P}^{n-1},\mathbf{P}^{n-2};F) \xrightarrow{\delta} HS^n(\mathbf{P}^n,\mathbf{P}^{n-1};F) = C^n(\mathbf{P};F).$$

The cellular cohomology of $\mathbf{P}$ with coefficients in the presheaf F is defined to be the homology of this complex:

$$HC^*(\mathbf{P};F) := H(C^*(\mathbf{P};F),\delta).$$

That $\delta^2 = 0$ is a standard argument following from Lemma 5. Clearly $C^n(\mathbf{P};F) = 0$ for $n < 0$ and in degree zero we have

$$C^0(\mathbf{P};F) = HS^0(\mathbf{P}^0,\varnothing;F) = HS^0(\mathbf{P}^0;F) = \varprojlim\nolimits_{\mathbf{P}^0} F = \prod_{|x|=0} F(x).$$

We can write an explicit formula for δ: if $s \in S^{n-1}(\mathbf{P}^{n-1},\mathbf{P}^{n-2};F)$ is a cocycle with homology class $[s]$ then $\delta[s] = [t]$ where for $\sigma \in N^n\mathbf{P}^n$ we have

$$t\cdot\sigma = \begin{cases} (-1)^n F^{\sigma_{n-1}}_{\sigma_n}(s\cdot d_n\sigma), & \text{if } |\sigma_n| = n \text{ and } |\sigma_{n-1}| = n-1,\\ 0, & \text{otherwise.}\end{cases} \tag{9}$$

This comes about by writing a formula for the connecting homomorphism of the pair $(\mathbf{P}^n,\mathbf{P}^{n-1})$ and then using Lemma 5.

2.2. Describing the cellular cochain complex

Continuing the analogy with the cellular cohomology of a CW-complex, the role of cells is played by the closed intervals $\mathbf{P}_{\geq x}$ and the role of boundary spheres of cells by the open intervals $\mathbf{P}_{>x}$:

$$\mathbf{P}_{\geq x} = \{y\in\mathbf{P}\,|\,y\geq x\} \text{ and } \mathbf{P}_{>x} = \{y\in\mathbf{P}\,|\,y > x\}.$$

$\mathbf{P}$ is *locally finite* if for any $x\in\mathbf{P}$ there are only finitely many y with $x \prec y$. If $\mathbf{P}$ is graded with a corank function and locally finite, then the interval $\mathbf{P}_{\geq x}$ is a finite poset for each x.

We now describe the cellular cochain complex $C^*(\mathbf{P};F)$ in terms of these intervals. The exposition is complicated slightly by the fact that there may be infinitely many elements of a given corank. In any case, let x be a fixed element of corank n. Then the diagram of inclusions below left induces, by Lemmas 1 and 2, the commuting diagram on the right:

$$\begin{tikzcd} \mathbf{P}_{\geq x} \arrow[r, tail, "\varepsilon_x"] & \mathbf{P}^n \\ \mathbf{P}_{>x} \arrow[u, tail, "j"] \arrow[r, tail] & \mathbf{P}^{n-1} \arrow[u, tail, "i"'] \end{tikzcd} \qquad \begin{tikzcd} S^*(\mathbf{P}_{\geq x};F) \arrow[d, two heads, "j^*"'] & S^*(\mathbf{P}^n;F) \arrow[l, two heads, "\varepsilon_x^*"'] \arrow[d, two heads, "i^*"] \\ S^*(\mathbf{P}_{>x};F) & S^*(\mathbf{P}^{n-1};F) \arrow[l, two heads] \end{tikzcd}$$

As the diagram commutes, ε_x^* restricts to a chain map $\ker i^* \to \ker j^*$, i.e. a chain map

$$\varepsilon_x^* : S^*(\mathbf{P}^n,\mathbf{P}^{n-1};F) \to S^*(\mathbf{P}_{\geq x},\mathbf{P}_{>x};F)$$

which for $s\in S^*(\mathbf{P}^n,\mathbf{P}^{n-1};F)$ and $\sigma\in N^*\mathbf{P}_{\geq x}$ is given by $\varepsilon_x^* s\cdot\sigma = s\cdot\varepsilon_x\sigma$.

Proposition 2. *The map of abelian groups*

$$\varepsilon : S^*(\mathbf{P}^n,\mathbf{P}^{n-1};F) \to \prod_{|x|=n} S^*(\mathbf{P}_{\geq x},\mathbf{P}_{>x};F) \tag{10}$$

given by $\varepsilon s\cdot x = \varepsilon_x^* s$ is a chain isomorphism.

Proof. The differential on the product is $\prod d_x$ with d_x the differential on $S^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F)$, and ε is a chain map as each ε_x^* is. Let $s, s' \in S^i(\mathbf{P}^n, \mathbf{P}^{n-1}; F)$ be such that $\varepsilon s = \varepsilon s'$. If $\sigma \in N^i\mathbf{P}^n \setminus N^i\mathbf{P}^{n-1}$ then $|\sigma_i| = n$ and so $\sigma = \varepsilon_{\sigma_i}(\sigma)$ for $\sigma \in N^i\mathbf{P}_{\geq\sigma_i}$. In particular $s \cdot \sigma = s \cdot \varepsilon_{\sigma_i}(\sigma) = s' \cdot \varepsilon_{\sigma_i}(\sigma) = s' \cdot \sigma$ and so ε is injective. On the otherhand let $s \in \prod_{|x|=n} S^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F)$ be of degree i. If $\tau \in N^i\mathbf{P}^n \setminus N^i\mathbf{P}^{n-1}$ then $\tau = \varepsilon_{\tau_i}(\tau)$ with $\tau \in N^i\mathbf{P}_{\geq\tau_i}$ but is not in the image of any other ε_x. Let $t \in S^i(\mathbf{P}^n; F)$ be such that

$$t \cdot \tau = \begin{cases} (s \cdot x) \cdot \sigma, & \text{if } \tau \in N^i\mathbf{P}^n \setminus N^i\mathbf{P}^{n-1} \text{ where } \tau = \varepsilon_x(\tau) \\ 0, & \tau \in N^i\mathbf{P}^{n-1}. \end{cases}$$

Then $t \in S^i(\mathbf{P}^n, \mathbf{P}^{n-1}; F)$ with $\varepsilon t = s$ and so ε is surjective. $\square$

Writing ε as well for the composition

$$HS^*(\mathbf{P}^n, \mathbf{P}^{n-1}; F) \xrightarrow{\cong} H\Big(\prod_{|x|=n} S^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F)\Big) \xrightarrow{\cong} \prod_{|x|=n} HS^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F)$$

the differential of the cellular cochain complex can be described in terms of the isomorphism (10) as the map making the following diagram commute:

$$\begin{tikzcd} C^{n-1} = HS^{n-1}(\mathbf{P}^{n-1}, \mathbf{P}^{n-2}; F) \arrow[r, "\delta"] \arrow[d, "\varepsilon"'] & HS^n(\mathbf{P}^n, \mathbf{P}^{n-1}; F) = C^n \arrow[d, "\varepsilon"] \\ \prod_{|y|=n-1} HS^{n-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; F) \arrow[r, dotted] & \prod_{|x|=n} HS^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F) \end{tikzcd}$$

We will call this map δ as well. An explicit formula for ε^{-1} can be extracted from the surjectivity part of the proof of Proposition 2 and by combining this with (9) and (10) we have proved the following alternative description of the cellular cochain complex:

Proposition 3. *Let* $\mathbf{P}$ *be graded with a corank function and* F *a presheaf on* $\mathbf{P}$*. Then there are isomorphisms*

$$C^n(\mathbf{P}; F) \cong \prod_{|x|=n} HS^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F)$$

with respect to which the differential in the cellular cochain complex $C^{n-1}(\mathbf{P}; F) \xrightarrow{\delta} C^n(\mathbf{P}; F)$ *has the following effect on an element* $s \in \prod_{|y|=n-1} HS^{n-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; F)$*. Suppose that* $s \cdot y = [s_y]$ *for* s_y *a cocycle in* $S^{n-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; F)$*. Then* $\delta s \cdot x = [t_x]$ *where* $t_x \in S^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F)$ *is given by*

$$t_x \cdot \sigma = \begin{cases} (-1)^n F_x^y(s_y \cdot d_n\sigma), & \textit{if } \sigma_n = x \prec y = \sigma_{n-1}, \\ 0, & \textit{otherwise,} \end{cases} \tag{11}$$

where $\sigma \in N^n\mathbf{P}_{\geq x}$*.*

In particular for a fixed element x of corank n, the diagram

$$\begin{tikzcd} C^{n-1} \cong \prod_{|y|=n-1} HS^{n-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; F) \arrow[r, "\delta"] \arrow[d, "\text{proj.}"'] & \prod_{|x|=n} HS^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F) \cong C^n \arrow[d, "\text{proj.}"] \\ \prod_{x \prec y} HS^{n-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; F) \arrow[r] & HS^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F) \end{tikzcd} \tag{12}$$

commutes, where the bottom horizontal map is the restriction of δ to the y covering x followed by projection onto the x-coordinate. In words, if $s \in \prod_{|y|=n-1} HS^{n-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; F)$ then the component of δs indexed by x depends only on the components of s indexed by the y covering x.

If y is a fixed element of corank $n-1$ and x a fixed element of corank n, then the map δ^y_x making the diagram

$$\begin{tikzcd} C^{n-1} \cong \prod_{|y|=n-1} HS^{n-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; F) \arrow[r, "\delta"] \arrow[d, "\text{proj.}"'] & \prod_{|x|=n} HS^{n}(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F) \cong C^n \arrow[d, "\text{proj.}"] \\ HS^{n-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; F) \arrow[r, dotted, "\delta^y_x"] & HS^{n}(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F) \end{tikzcd}$$

commute is called the *matrix element* corresponding to the pair (x, y). Explicitly, if s is a cocycle in $HS^{n-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; F)$ then $\delta^y_x[s] = [t]$ where for $\sigma \in N^n\mathbf{P}_{\geq x}$, the coordinate $t \cdot \sigma$ is given by (11) with s_y replaced by s. In general δ is not determined by its matrix elements. If $\mathbf{P}$ is locally finite however then the bottom left term in (12) is a direct sum, and for $s \in C^{n-1}$ we have

$$\delta s \cdot x = \sum_{x<y} \delta^y_x(s \cdot y).$$

We can further refine the chain groups of the cellular complex:

Proposition 4. *Let $x \in \mathbf{P}$. Then*

$$HS^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F) \cong HS^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x)) \cong \widetilde{HS}^{*-1}(\mathbf{P}_{>x}; \Delta F(x))$$

where $\Delta F(x)$ is the constant presheaf with value $F(x) \in \mathbf{Ab}$.

Proof. Consider F and $\Delta F(x)$ as presheaves on the closed interval $\mathbf{P}_{\geq x}$. In $\mathbf{PreSh}(\mathbf{P}_{\geq x})$ there is a natural transformation $\kappa\colon F \to \Delta F(x)$ defined by $\kappa_y = F^y_x$ which induces a chain map $\kappa_*\colon S^*(\mathbf{P}_{\geq x}; F) \to S^*(\mathbf{P}_{\geq x}; \Delta F(x))$. The inclusion $\mathbf{P}_{>x} \hookrightarrow \mathbf{P}_{\geq x}$ and Lemma 3 mean that κ_* restricts to a chain map $\kappa_*\colon S^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F) \to S^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x))$ which turns out to be an inclusion. Surjectivity is also not hard to show, and we have an induced isomorphism in cohomology:

$$HS^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F) \xrightarrow{\cong} HS^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x)).$$

We observed in §1.1 that $HS^*(\mathbf{P}_{\geq x}; \Delta F(x)) \cong H^*(|N^*\mathbf{P}_{\geq x}|, F(x))$, the ordinary singular cohomology of the classifying space $|N^*\mathbf{P}_{\geq x}|$. As $\mathbf{P}_{\geq x}$ has a unique minimal element the space $|N^*\mathbf{P}_{\geq x}|$ is contractible, giving

$$HS^i(\mathbf{P}_{\geq x}; \Delta F(x)) \cong \begin{cases} F(x), & i = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Thus for $i > 1$ the coboundary map in the long exact sequence (7) of the pair $(\mathbf{P}_{\geq x}, \mathbf{P}_{>x})$ is an isomorphism

$$\widetilde{HS}^{i-1}(\mathbf{P}_{>x}; \Delta F(x)) \cong HS^{i-1}(\mathbf{P}_{>x}; \Delta F(x)) \xrightarrow{\cong} HS^i(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x)).$$

$HS^0(\mathbf{P}_{\geq x}; \Delta F(x))$ is the diagonal copy of $F(x)$ in $S^0(\mathbf{P}_{\geq x}; \Delta F(x))$, so $HS^0(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x))$ is trivial when $|x| > 0$ and isomorphic to $F(x)$ when $|x| = 0$ (and thus $\cong \widetilde{HS}^{-1}(\mathbf{P}_{>x}; \Delta F(x))$ in either case). The long exact sequence of the pair $(\mathbf{P}_{\geq x}, \mathbf{P}_{>x})$ collapses to the short exact sequence:

$$0 \to HS^0(\mathbf{P}_{\geq x}; \Delta F(x)) \to HS^0(\mathbf{P}_{>x}; \Delta F(x)) \to HS^1(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x)) \to 0.$$

When $|x| > 0$ the first map can be identified with $HS^0(\star; \Delta F(x)) \to HS^0(\mathbf{P}_{>x}; \Delta F(x))$, giving $HS^1(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x)) \cong \widetilde{HS}^0(\mathbf{P}_{>x}; \Delta F(x))$. For $|x| = 0$ these two are both trivial. □

This leads to our third description of the cellular chain complex in terms of the reduced cohomology of open intervals. Writing α for the isomorphism

$$\prod_{|x|=n} \widetilde{HS}^{n-1}(\mathbf{P}_{>x}; \Delta F(x)) \xrightarrow{\cong} \prod_{|x|=n} HS^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F),$$

induced by Proposition 4, then if s is an element of the left hand side with $s \cdot x = [s_x]$ for s_x a cocycle in $\widetilde{S}^{n-1}(\mathbf{P}_{>x}; \Delta F(x))$, we have $\alpha s = t$ with $t \cdot x = [t_x]$, where $t_x \in S^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x))$ is given by

$$t_x \cdot \sigma = \begin{cases} (-1)^n s_x \cdot d_n\sigma, & \text{if } \sigma_n = x < \sigma_{n-1}, \\ 0 & \text{otherwise,} \end{cases}$$

and $\sigma \in N^n\mathbf{P}_{\geq x}$. Computing $\alpha^{-1}\delta\alpha$ with the δ of (11) gives:

Proposition 5. *Let* $\mathbf{P}$ *be graded with a corank function and* F *a presheaf on* $\mathbf{P}$*. Then there are isomorphisms*

$$C^n \cong \prod_{|x|=n} \widetilde{HS}^{n-1}(\mathbf{P}_{>x}; \Delta F(x))$$

and differential $C^{n-1} \xrightarrow{\delta} C^n$ *where if* $s \in \prod_{|y|=n-1} \widetilde{HS}^{n-2}(\mathbf{P}_{>y}; \Delta F(y))$ *with* $s \cdot y = [s_y]$ *for* s_y *a cocycle in* $\widetilde{S}^{n-2}(\mathbf{P}_{>y}; \Delta F(y))$*, then* $\delta s = t$ *with* $t \cdot x = [t_x]$ *where* $t_x \in \widetilde{S}^{n-1}(\mathbf{P}_{>x}; \Delta F(x))$ *is given by*

$$t_x \cdot \sigma = \begin{cases} (-1)^{n-1} F_x^y(s_y \cdot d_{n-1}\sigma), & \text{if } x < y = \sigma_{n-1} < \sigma_{n-2}, \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

and $\sigma \in N^{n-1}\mathbf{P}_{>x}$.

Similar comments pertain to this description of the cellular complex as those following Proposition 3. In particular if y is fixed of corank $n-1$ and x fixed of corank n then the matrix element δ_x^y sends $[s]$ to $[t]$ with $s \in \widetilde{HS}^{n-2}(\mathbf{P}_{>y}, \Delta F(y))$ a cocycle and $t \cdot \sigma$ given by (13) with s_y replaced by s.

2.3. Locally finite posets

When $\mathbf{P}$ is locally finite the middle of the three terms in Proposition 4 can be further reduced. For $\mathbf{P}_{\geq x}$ is then a finite poset, so that $S^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta\mathbb{Z})$ has free cochain groups and hence $S^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x)) \cong S^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta\mathbb{Z}) \otimes F(x)$. In any case – locally finite or not – $HS^i(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F) \cong HT^i(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; F) = 0$ for $i > |x|$ and so

$$HS^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x)) \cong HT^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta\mathbb{Z}) \otimes F(x) \tag{14}$$

when $|x| = n$, giving

$$C^n \cong \prod_{|x|=n} HT^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta\mathbb{Z}) \otimes F(x).$$

The differential is determined by the matrix elements δ_x^y in the locally finite case, and the image of an $s \in HT^{n-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; \Delta\mathbb{Z}) \otimes F(y)$ is determined by the images of elements of the form $[s_\sigma] \otimes a$, where $a \in F(y)$, and for $\sigma \in N_\circ^{n-1}\mathbf{P}_{\geq y}$ (with necessarily $\sigma_{n-1} = y$) the tuple $s_\sigma \in T^{n-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; \Delta\mathbb{Z})$ has 1 in the σ-coordinate and 0's elsewhere. Then

$$\delta_x^y : [s_\sigma] \otimes a \mapsto [s_{x\sigma}] \otimes (-1)^n F_x^y(a) \tag{15}$$

where $x\sigma$ is the result of pre-appending x to σ.

Similarly we have

$$\widetilde{HS}^{n-1}(\mathbf{P}_{>x}; \Delta F(x)) \cong \widetilde{HT}^{n-1}(\mathbf{P}_{>x}; \Delta\mathbb{Z}) \otimes F(x)$$

and so

$$C^n \cong \prod_{|x|=n} \widetilde{HT}^{n-1}(\mathbf{P}_{>x};\Delta\mathbb{Z}) \otimes F(x) \tag{16}$$

with

$$\delta_x^y : [s_\sigma] \otimes a \mapsto [s_{y\sigma}] \otimes (-1)^{n-1} F_x^y(a), \tag{17}$$

where $\sigma \in N_\circ^{n-2}\mathbf{P}_{>y}$ and s_σ, a are analogous to (15).

We now turn to a description of $C^*(\mathbf{P}; F)$ in terms of generators and relations. If X is a CW-complex then the cellular cochains in degree n are free on the n-cells – one chooses free generators by fixing orientations for the cells in the usual way. Looking forward to §4.1, if $\mathbf{P}$ is the cell poset of a regular CW-complex then C^n will turn out – not surprisingly – to be free on the elements of corank n, and free generators can easily be found. If $\mathbf{P}$ is a geometric lattice (see §4.4) then C^n is also free, but with each x of corank n now contributing $(-1)^n\mu(x, \mathbf{1})$ free factors, rather than just one, where μ is the Möbius function of the lattice[1]. In general the situation is more complex. First of all C^n may not be free, and even when it is we prefer not to privelege a particular choice of basis.

We thus give a non-free presentation for C^n when $\mathbf{P}$ is locally finite (see also [16, §1.6]). Let $x \in \mathbf{P}$ be fixed with $|x| = n$ and recall that $N_\circ^n\mathbf{P}_{\geq x}$ are the non-degenerate n-simplicies in the nerve of $\mathbf{P}_{\geq x}$. The elements of $N_\circ^n\mathbf{P}_{\geq x}$ thus have the form $\sigma = x\sigma_{n-1}\cdots\sigma_0$ with $|\sigma_i| = i$.

Definition 2.2. *Let $\tau = x\tau_{n-1}\cdots\tau_{j+1}\tau_{j-1}\cdots\tau_0$ be a fixed $(n-1)$-simplex in $N_\circ^{n-1}\mathbf{P}_{\geq x}$ with $0 \leq j < n$ and $|\tau_i| = i$. We call the set B_τ of all n-simplicies in $N_\circ^n\mathbf{P}_{\geq x}$ of the form*

$$x\tau_{n-1}\cdots\tau_{j+1}y\tau_{j-1}\cdots\tau_0$$

(where necessarily $|y| = j$) the compatible family given by τ. Let $s \in \bigoplus_\sigma F(x)$, the direct sum over the $\sigma \in N_\circ^n\mathbf{P}_{\geq x}$, and let B_τ be some compatible family. We say that s is B_τ-constant if there is a fixed $a \in F(x)$ such that

$$s \cdot \sigma = \begin{cases} a, & \sigma \in B_\tau, \\ 0, & \textit{otherwise.} \end{cases}$$

Now let $K_x \subset \bigoplus_\sigma F(x)$ be the subgroup generated by the s that are B_τ-constant for some τ, where τ ranges over the $(n-1)$-simplicies of $N_\circ^{n-1}\mathbf{P}_{>x}$.

Proposition 6. *Let $\mathbf{P}$ be locally finite and let $x \in \mathbf{P}$ with $|x| = n$. Then there are isomorphisms*

$$HS^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x)) \cong \Big(\bigoplus_\sigma F(x)\Big)/K_x \cong A_x \otimes F(x)$$

where the direct sum is over the $\sigma \in N_\circ^n\mathbf{P}_{\geq x}$, and A_x is the abelian group having presentation with generators the $\sigma \in N_\circ^n\mathbf{P}_{\geq x}$ and relations the $\sum_{\sigma\in B_\tau}\sigma = 0$, for each compatible family B_τ.

Proof. We have $T^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x))$ is the direct sum $\bigoplus_\sigma F(x)$ over the $\sigma \in N_\circ^n\mathbf{P}_{\geq x}$. Moreover $T^{n+1} = 0$ so that $HS^n \cong HT^n = (\bigoplus_\sigma F(x))/\operatorname{im} d^{n-1}$, and it remains to show that $\operatorname{im} d^{n-1} = K_x$.

Firstly, let $t \in T^{n-1}(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x))$ be an element that is non-zero only in the coordinate indexed by $\tau = x\tau_{n-1}\cdots\tau_{j+1}\tau_{j-1}\cdots\tau_0$ where $|\tau_i| = i$. We have for $\sigma \in N_\circ^n\mathbf{P}$ that $dt\cdot\sigma \neq 0$ if and only if σ is of the form $x\tau_{n-1}\cdots\tau_{j+1}y\tau_{j-1}\cdots\tau_0$, in which case $dt\cdot\sigma = t\cdot\tau$. Thus, dt is B_τ-constant. As every element of $T^{n-1}(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x))$ is a sum of such t, we have $\operatorname{im} d^{n-1} \subset K_x$.

Conversely, let s be B_τ-constant with value a and with τ as in the previous paragraph. Define $t \in T^{n-1}$ to have value $(-1)^j a$ in the coordinate indexed by τ and 0 elsewhere. Then $dt = s$ and so $K_x \subset d^{n-1}$. The second isomorphism follows from the first and (14). □

[1] The Möbius function is the inverse of the zeta function in the incidence algebra of $\mathbf{P}$. Explicitly, for k a field it is the k-valued function on the intervals given by $\mu(x, x) = 1$ and $\mu(x, y) = -\sum_{x\leq z<y}\mu(x, z)$ when $x < y$. See [15, Section 3.7]

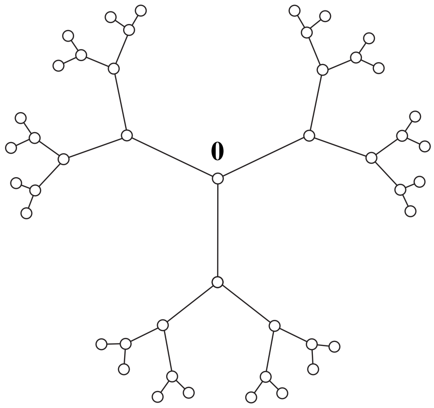

Figure 1: 3-valent tree $\mathbf{P}$ for which $HS^*(\mathbf{P}; F)$ cannot be computed cellularly.

Remark 2. The group A_x of Proposition 6 need not be free: let X be a finite $(n-1)$-dimensional regular CW-complex with homology $H_{n-2}(X;\mathbb{Z})$ containing the torsion subgroup $T_{n-2} \neq 0$ (for example X is the result of repeatedly suspending $\mathbb{R}P^2$). Let $\mathbf{Q}$ be the cell poset of X (see §4.1) and let $\mathbf{P}$ be $\mathbf{Q}$ with a unique minimal element $\mathbf{0}$ formally attached. Then,

$$A_{\mathbf{0}} \cong H^{n-1}(|N^*\mathbf{P}_{>\mathbf{0}}|, \mathbb{Z}) \cong H^{n-1}(|N^*\mathbf{Q}|, \mathbb{Z}) \cong H^{n-1}(X, \mathbb{Z}) \cong T_{n-2} \oplus \text{free part of } H_{n-1}(X, \mathbb{Z})$$

where we have used the fact (see §4.1) that X and $|N^*\mathbf{Q}|$ are homeomorphic.

Here is our final version of the cellular complex:

Proposition 7. *Let $\mathbf{P}$ be graded locally finite with a corank function and F a presheaf on $\mathbf{P}$. Then there are isomorphisms*

$$C^n \cong \prod_{|x|=n} A_x \otimes F(x)$$

where A_x is the abelian group having presentation with generators the $\sigma \in N^n_\circ \mathbf{P}_{\geq x}$ and relations the $\sum_{\sigma \in B_\tau} \sigma = 0$ for each compatible family B_τ in $N^n_\circ \mathbf{P}_{\geq x}$. If $|y| = n-1$ and $x \prec y$ then the matrix element $\delta^y_x : A_y \otimes F(y) \to A_x \otimes F(x)$ of the differential $\delta : C^{n-1} \to C^n$ is given by

$$\delta^y_x : \sigma \otimes a \mapsto x\sigma \otimes (-1)^n F^y_x(a) \tag{18}$$

where $\sigma \in N^{n-1}_\circ \mathbf{P}_{\geq y}$ is a generator of A_y with $a \in F(y)$ and $x\sigma$ is the result of pre-appending x to σ.

We finish with an example of a $\mathbf{P}$ for which $HS^*(\mathbf{P}; F) \ncong HC^*(\mathbf{P}; F)$.

Example 3. Let $\mathbf{P}$ be a finite 3-valent tree with a distinguished vertex $\mathbf{0}$ – as for example in Figure 1 – with vertices ordered by $x \leq y$ when the unique path without backtracking from $\mathbf{0}$ to y passes through x. Then $\mathbf{P}$ is graded with the rank of a vertex the number of edges between it and $\mathbf{0}$. The maximal elements are called *leaves*. Assume for simplicity that the leaves are all equidistant from $\mathbf{0}$ (or have the same rank). By Proposition 7, and if the maximum rank r is > 1, we have $C^0(\mathbf{P}; \Delta\mathbb{Z})$ is free abelian on the leaves; $C^1(\mathbf{P}; \Delta\mathbb{Z})$ is free abelian on the corank 1 elements, and $C^i(\mathbf{P}; \Delta\mathbb{Z}) = 0$ for $i > 1$. Thus $HC^i(\mathbf{P}; \Delta\mathbb{Z}) = 0$ for $i > 0$ and $HC^0(\mathbf{P}; \Delta\mathbb{Z})$ is free abelian on the pairs of leaves. On the other hand $\mathbf{P}$ has a unique minimum $\mathbf{0}$ so that $|N^*\mathbf{P}|$ is a cone on the space $|N^*\mathbf{P}_{>\mathbf{0}}|$, hence contractible. In particular $HS^0(\mathbf{P}; \Delta\mathbb{Z}) \cong \mathbb{Z}$ and $HS^i(\mathbf{P}; \Delta\mathbb{Z}) = 0$ for $i > 0$.

3. Computing cohomology cellularly

In general the cohomology groups $HS^*(\mathbf{P};F)$ and $HC^*(\mathbf{P};F)$ are not isomorphic as Example 3 shows. For a large class of posets however we have $HS^*(\mathbf{P};F) \cong HC^*(\mathbf{P};F)$ and so the higher limits can be computed cellularly. The situation is analogous to topology: if X is a filtered space then one can construct the cellular cochain complex of X, although in general the resulting cellular cohomology is not isomorphic to the singular cohomology. If a vanishing condition on the relative (singular) cohomologies of successive pairs of the filtration is satisfied then the two are isomorphic.

Definition 3.1. *Let* $\mathbf{P}$ *be graded with a corank function. Then* $\mathbf{P}$ *is cellular if and only if for every presheaf F we have*

$$HS^i(\mathbf{P}^n, \mathbf{P}^{n-1}; F) = 0 \text{ for } i \neq n. \tag{19}$$

Thus the cohomology of the pair $(\mathbf{P}^n, \mathbf{P}^{n-1})$ vanishes in every degree except the one that carries the cochains of the cellular complex. Using the results of the previous section we have $\mathbf{P}$ cellular when

$$HS^i(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta F(x)) \cong \widetilde{HS}^{i-1}(\mathbf{P}_{>x}; \Delta F(x)) = 0 \quad (i \neq |x|) \tag{20}$$

for every $x \in \mathbf{P}$ and every presheaf F. Moreover a locally finite $\mathbf{P}$ is cellular when

$$HS^i(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta\mathbb{Z}) \cong \widetilde{HS}^{i-1}(\mathbf{P}_{>x}; \Delta\mathbb{Z}) \cong \widetilde{H}^{i-1}(|N^*\mathbf{P}_{>x}|, \mathbb{Z}) = 0 \quad (i \neq |x|) \tag{21}$$

for every $x \in \mathbf{P}$ and with the last term the ordinary reduced cohomology of the space $|N^*\mathbf{P}_{>x}|$. Cellularity for locally finite $\mathbf{P}$ thus has nothing to do with the presheaf.

Remark 3. It is easy to find non-cellular posets, arguing topologically as in Remark 2 of §2.3. If X is a regular CW-complex with non-vanishing cohomology in some non-zero degree $< \dim X$, with cell poset $\mathbf{Q}$ (see §4.1) and $\mathbf{P}$ the result of formally adjoining a unique minimal element $\mathbf{0}$ to $\mathbf{Q}$, then (21) fails for $\mathbf{P}$ at $x = \mathbf{0}$.

We devote §4 to examples of posets that *are* cellular.

Remark 4. Even if $\mathbf{P}$ is cellular the cochains of the cellular complex need not be free: let X be a finite n-dimensional regular CW-complex with homology $H_i(X;\mathbb{Z})$ the finite group $T \neq 0$ in degree $\dim X - 1$ and vanishing in degrees $0 < i < \dim X - 1$. Again, suspending $\mathbb{R}P^2$ some number of times provides an example. If $\mathbf{P}$ is the result of formally adjoining a unique minimal element to the cell poset of X, then $\mathbf{P}$ is graded, cellular and with corank function, and $C^{n+1} \cong T$.

Remark 5. On the other hand if $\mathbf{P}$ is locally finite cellular and the cochains of the cellular complex *are* free then we have

$$C^n \cong \prod_{|x|=n} \mathbb{Z}^{\mu_x} \otimes F(x)$$

where $\mu_x = (-1)^{|x|-1}\mu(x, \mathbf{1})$ with μ the Möbius function of the poset obtained by adjoining a unique maximum $\mathbf{1}$ to $\mathbf{P}$. This follows by [15, Proposition 3.8.6] which interprets μ_x in terms of the reduced Euler characteristic of the space $|N^*\mathbf{P}_{>x}|$, and this characteristic has only one non-zero term by cellularity.

Theorem 2. *Let* $\mathbf{P}$ *be graded, cellular, locally finite with a corank function and let F be a presheaf on* $\mathbf{P}$*. Then there is an isomorphism*

$$HS^*(\mathbf{P};F) \cong HC^*(\mathbf{P};F).$$

Proof. (i). Assume in addition to the conditions stated in the theorem that $\mathbf{P}$ is also finite. We use a spectral sequence by filtering $S^* = S^*(\mathbf{P}; F)$ with $F^p S^* = S^*(\mathbf{P}, \mathbf{P}^p; F)$. By Lemma 4 we have a short exact sequence

$$0 \longrightarrow S^*(\mathbf{P}, \mathbf{P}^{p+1}; F) \longrightarrow S^*(\mathbf{P}, \mathbf{P}^p; F) \longrightarrow S^*(\mathbf{P}^{p+1}, \mathbf{P}^p; F) \longrightarrow 0 \tag{22}$$

hence $F^{p+1}S^*$ is a subcomplex of $F^p S^*$ and we have a bounded filtration

$$0 \subset F^c S^* \subset \cdots \subset F^{p+1}S^* \subset F^p S^* \subset \cdots \subset F^{-1}S^* = S^*$$

where c is the maximum corank. The E_0 page of the associated spectral sequence has

$$E_0^{p,q} = \frac{F^p S^{p+q}}{F^{p+1}S^{p+q}}$$

which is just $S^{p+q}(\mathbf{P}^{p+1}, \mathbf{P}^p; F)$ by (22). Since the differential on the E_0 page is induced by that on S^* we get an E_1 page with

$$E_1^{p,q} = HS^{p+q}(\mathbf{P}^{p+1}, \mathbf{P}^p; F)$$

When $q = 1$ this is simply C^p, and by the cellular assumption all other entries on the E_1 page are zero. One may check that on this one line the spectral sequence differential d^1 agrees with the differential in C^*. So the spectral sequence collapses at E_2 with the cellular cohomology on the line $q = 1$ and hence the result.

(ii). Returning to the general case of a locally finite $\mathbf{P}$, we proceed differently so as to avoid questions about the convergence of the spectral sequence used in (i). We define a projective resolution $B_* \to \Delta\mathbb{Z}$ in $\mathbf{PreSh(P)}$ such that $\mathrm{Hom}_{\mathbf{PreSh(P)}}(B_*, F) = C^*(\mathbf{P}, F)$. Let $B_n = \bigoplus_x \Upsilon_x$, the direct sum over the x of corank n, where $\Upsilon_x := \Upsilon_x A_x$ are Yoneda presheaves (§1.1) with

$$A_x = \mathrm{Hom}_{\mathbb{Z}}(HS^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta\mathbb{Z}), \mathbb{Z}).$$

A_x is free and so Υ_x, and hence B_n, is projective. To define $\zeta : B_n \to B_{n-1}$ let x have corank n and $x < y$. Define $\zeta_x^y : S^{*-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; \Delta\mathbb{Z}) \to S^*(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta\mathbb{Z})$ by

$$\zeta_x^y s \cdot \sigma = \begin{cases} (-1)^n s \cdot d_k \sigma, & \text{if } \sigma = xy\sigma_{k-2}\ldots\sigma_0, \\ 0, & \text{otherwise} \end{cases}$$

where $s \in S^{k-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; \Delta\mathbb{Z})$ and $\sigma \in N^k\mathbf{P}_{\geq x}$. Then ζ_x^y is a chain map, inducing a map, which we will also call ζ_x^y:

$$A_x = \mathrm{Hom}_{\mathbb{Z}}(HS^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta\mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(HS^{n-1}(\mathbf{P}_{\geq y}, \mathbf{P}_{>y}; \Delta\mathbb{Z}), \mathbb{Z}) = A_y,$$

and hence a presheaf morphism $\zeta_x^y : \Upsilon_x \to \Upsilon_y$. Let $\zeta = \sum_{x<y} \zeta_x^y : B_n \to B_{n-1}$. The sequence $\cdots \to B_n \to B_{n-1} \to \cdots$ is exact at B_n precisely when it is exact pointwise, i.e. for every x the sequence $\cdots \to B_n(x) \to B_{n-1}(x) \to \cdots$ is exact at $B_n(x)$. But this sequence is nothing other than the result of applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to the cellular cochain complex $C^*(\mathbf{P}_{\geq x}; \Delta\mathbb{Z})$. By local finiteness $\mathbf{P}_{\geq x}$ is a finite poset, and is cellular, and so by part (i) we have $HC^*(\mathbf{P}_{\geq x}; \Delta\mathbb{Z}) \cong HS^*(\mathbf{P}_{\geq x}; \Delta\mathbb{Z})$, which in turn is $\cong H^*(|N^*\mathbf{P}_{\geq x}|; \mathbb{Z})$, and this vanishes outside degree 0 as $|N^*\mathbf{P}_{\geq x}|$ is contractible. Thus $H_n B_*(x)$ vanishes when $n > 0$ and is $\cong \mathbb{Z}$ when $n = 0$. To augment B_* consider

$$B_1 \xrightarrow{\zeta} B_0 \to \operatorname{coker}\zeta \to 0$$

with the second map the quotient. Then for all x we have $\operatorname{coker}\zeta(x) \cong H_0 B_*(x) \cong \mathbb{Z}$ and for $x \leq y$ the map $\operatorname{coker}\zeta(y) \to \zeta(x)$ can be identified with the identity $\mathbb{Z} \to \mathbb{Z}$. Thus $\operatorname{coker}\zeta \cong \Delta\mathbb{Z}$ and we have our augmentation.

Finally, if $F \in \mathbf{PreSh(P)}$ then

$$\begin{aligned}\mathrm{Hom}_{\mathbf{PreSh(P)}}(B_n, F) = \mathrm{Hom}_{\mathbf{PreSh(P)}}\Big(\bigoplus_x \Upsilon_x, F\Big) &\cong \prod_x \mathrm{Hom}_{\mathbf{PreSh(P)}}(\Upsilon_x, F)\\ &\cong \prod_x \mathrm{Hom}_{\mathbb{Z}}(A_x, F(x)) \cong \prod_x \mathrm{Hom}_{\mathbb{Z}}(A_x, \mathbb{Z}) \otimes F(x)\\ &\cong \prod_x HS^n(\mathbf{P}_{\geq x}, \mathbf{P}_{>x}; \Delta\mathbb{Z}) \otimes F(x) \cong C^n(\mathbf{P}, F),\end{aligned}$$

and by (2) in §1.1 we have $\mathrm{Hom}_{\mathbf{PreSh(P)}}(B_n \xrightarrow{\zeta} B_{n-1}, F) \cong C^{n-1}(\mathbf{P}; F) \xrightarrow{\delta} C^n(\mathbf{P}; F)$. □

The spectral sequence in the proof of Theorem 2 has its origins in the work of Godement [8] (see also [1]). A special case of the projective resolution appears in [6, §1.3].

4. Examples

In this section we identify some important classes of posets as coming under the auspices of Theorem 2 and describe the resulting cellular chain complexes, in increasing order of complexity.

4.1. Cell posets

A CW-complex X is *regular* if the attaching map of every cell is a homeomorphism. In this case the *cell poset* $\mathbf{P}_X$ has elements the cells of X with cells $x \leq y$ iff $\overline{x} \supset \overline{y}$ (note: reverse inclusion). Since the closure of a cell meets only finitely many other cells this poset is locally finite. It is also graded and if $\dim X < \infty$ then the rank function is bounded with corank function given by $|x| = \dim x$. We have a topology to poset to topology progression given by $X \to \mathbf{P}_X \to |N^*\mathbf{P}_X|$ where X and $|N^*\mathbf{P}_X|$ are homeomorphic (see the proof of Theorem III.1.7 in [10]).

If x is an n-cell then $\mathbf{P}_{>x}$ is the cell poset of the induced CW-decomposition of the boundary ∂x, which is itself an $(n-1)$-sphere. Thus

$$\widetilde{HS}^{i-1}(\mathbf{P}_{>x}; \Delta\mathbb{Z}) \cong \widetilde{H}^{i-1}(|N^*\mathbf{P}_{>x}|, \mathbb{Z})$$

vanishes outside degree $i = |x|$ and $\widetilde{HS}^{|x|-1}(\mathbf{P}_{>x}; \Delta\mathbb{Z}) \cong \mathbb{Z}$. Cell posets are thus cellular with

$$C^n(\mathbf{P}; F) \cong \prod_{|x|=n} A_x \otimes F(x)$$

where $A_x \cong \mathbb{Z}$.

Cell posets also enjoy the $\diamond$-property: if u is an $(i+1)$-cell and v an $(i-1)$-cell ($0 \leq i \leq \dim X$) with $u < v$, then there are exactly two i-cells z_1, z_2 with $u < z_i < v$ (if $i = 0$ then there is an u but no v, and if $i = \dim X$ then there is a v but no u).

A compatible family B_τ in $\mathbf{P}_{\geq x}$ thus has one of the two forms illustrated in Figure 2. The group A_x of Proposition 7 thus has presentation with generators the $\sigma \in N^n_\circ \mathbf{P}_{\geq x}$ and relations of the form $\sigma + \sigma' = 0$, where σ, σ' are the two poset sequences running around each side of a $\diamond$.

We have the description (18) of the matrix element $\delta^y_x : A_y \otimes F(y) \to A_x \otimes F(x)$ of the differential, but we can also explicitly describe it as a map $\mathbb{Z} \otimes F(y) \to \mathbb{Z} \otimes F(x)$ as follows. If $\sigma, \sigma' \in N^n_\circ \mathbf{P}_{\geq x}$ then σ can be turned into σ' by successively moving poset sequences across $\diamond$'s – i.e.: replacing one sequence in Figure 2 by the other (sketch of proof: $\mathbf{P}_{\geq x}$ is the cell poset of an n-ball with the induced decomposition of the bounding $(n-1)$-sphere; the $\sigma \in N^n_\circ \mathbf{P}_{\geq x}$ correspond to simplices in the barycentric subdivision and moving them across $\diamond$'s corresponds to exchanging simplices sharing a common face of

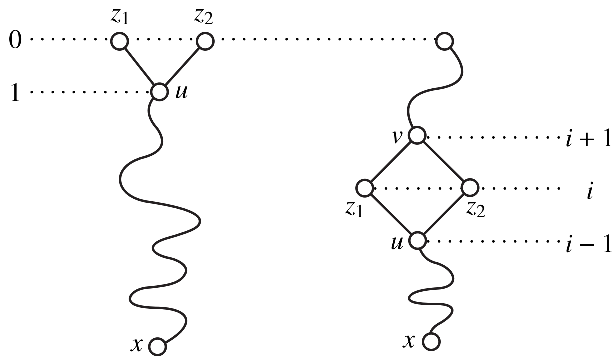

Figure 2: Compatible families in $\mathbf{P}_{\geq x}$ when $\mathbf{P}$ is a cell poset.

dimension $n-2$). In particular $\sigma' = \pm\sigma$, with the sign determined by the parity number of such maneuvers, and $A_x \cong \mathbb{Z}$ is freely generated by any of the $\sigma \in N^n_\circ \mathbf{P}_{\geq x}$.

For each x, fix a free generator σ_x of A_x and for each $x \prec y$ let $[x, y] = \pm 1$ be determined by

$$x\sigma_y = [x, y]\sigma_x, \tag{23}$$

where $x\sigma_y$ is the result of pre-appending x onto σ_y. If x, y, y', z form a $\diamond$-configuration then one can check that

$$[x, y][y, z] = -[x, y'][y', z]. \tag{24}$$

The matrix element $\delta^y_x : A_y \otimes F(y) \to A_x \otimes F(x)$ is then given by

$$\delta^y_x : \sigma_y \otimes a \mapsto x\sigma_y \otimes (-1)^n F^y_x(a) = [x, y]\sigma_x \otimes (-1)^n F^y_x(a).$$

By [11, Chapter IX, Theorem 7.2] orientations can be chosen for the cells of X in such a way that the $[x, y]$ – which are defined above in a purely combinatorial way – are the incidence numbers of the cells.

4.2. *Posets with unique extrema and Khovanov homology*

If $\mathbf{P}$ is a poset with a unique extremal – that is, maximal or minimal – element then the classifying space $|N^*\mathbf{P}|$ is contractible; indeed if x is the extremal element then $|N^*\mathbf{P}|$ is a cone on $|N^*(\mathbf{P} \setminus x)|$. If we have a constant presheaf $F = \Delta A$ on $\mathbf{P}$ then $HS^i(\mathbf{P}; \Delta A) \cong H^i(|N^*\mathbf{P}|, A)$ vanishes for $i > 0$ and is $\cong A$ in degree 0.

More generally if $\mathbf{P}$ has a unique maximum x then the limit functor

$$F \mapsto \varprojlim_{\mathbf{P}} F$$

is naturally isomorphic to the evaluation functor $F \mapsto F(x)$, which is exact. Hence the higher limits $HS^i(\mathbf{P}; F)$ vanish for $i > 0$ here also.

If instead $\mathbf{P}$ has a unique minimum, but no unique maximum, then given an interesting enough presheaf the higher limits can be very rich. A good source of examples comes from the Khovanov homology [9] mentioned in the Introduction: we describe in [6, Theorem 1] how the Khovanov homology of a link diagram with n crossings arises as the cohomology of the cell poset of the suspension X of an $(n-1)$-simplex equipped with the Khovanov presheaf F_{Kh} – see Figure 3 (and also [5]). The cellular cochain complex $C^*(\mathbf{P}_X; F_{Kh})$ is then the standard cube complex found in Khovanov homology, and the $[x, y]$ of (23) are the signs "sprinkled" on the cube to make its faces anti-commute.

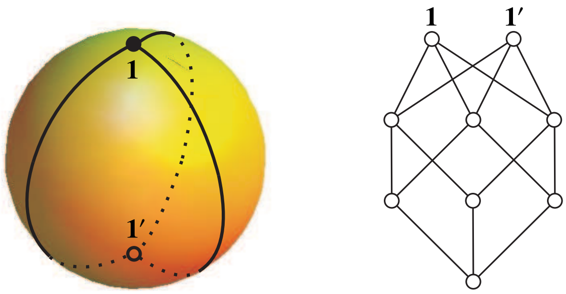

Figure 3: The X *(left)* and $\mathbf{P}_X$ *(right)* for the Khovanov homology of a link diagram with 3 crossings.

4.3. *The Bruhat order and the symmetric group*

This is another example of a cell poset, arising from a partial order on a finite Coxeter group. We illustrate with a particular example.

Let S_n be the symmetric group and write an $x \in S_n$ as a string $x = x(1)\cdots x(n)$. Then $S_n \setminus \text{id}$ can be given the structure of a cell poset in the following way. If $x, y \in S_n \setminus \text{id}$ then write $x \to y$ if

$$x = x(1)\cdots i \cdots j \cdots x(n) \text{ where } i > j \text{ and } y = x(1)\cdots j \cdots i \cdots x(n).$$

Define $x \leq y$ when there are x_i with

$$x = x_0 \to x_1 \to \cdots \to x_k = y.$$

The resulting $\leq$ is called the *Bruhat order* on S_n (actually, our Bruhat order is the opposite of that normally found in the literature, but the Bruhat order is isomorphic to its opposite anyway). For basic facts concerning the Bruhat order, including some of the constructions below, see [4, Chapter 2]. The corank function is $|x| = \ell(x) - 1$, where $\ell(x)$ is the number of inversions in x: pairs $i > j$ with $x = x(1)\cdots i \cdots j \cdots x(n)$. The poset $\mathbf{P} = S_n \setminus \text{id}$ has maxima the $n-1$ transpositions $s_i = 1\cdots i+1, i \cdots n$ with $\ell(s_i) = 1$ and unique minimum the permutation $x_0 = n\cdots 21$ with $\ell(x_0) = \binom{n}{2}$. There is then a regular CW decomposition X of the $(\ell(x_0) - 1)$-ball with $\mathbf{P}_X = \mathbf{P}$.

To describe the cellular complex $C^*(\mathbf{P}; F)$ for a presheaf F on $\mathbf{P}$ we need only give a free generator σ_x for the group A_x as in §4.1 and determine the signs $[x, y]$ of (23) for all $x \prec y$ in the Bruhat order. Let $x = x(1)\cdots i \cdots j \cdots x(n)$ where $i > j$ and call (i, j) a *swap pair* if for each k of the string appearing between i and j we have either $k < j$ or $k > i$. Then if $y = x(1)\cdots j \cdots i \cdots x(n)$ we have $x \prec y$, and all the y covering x arise by interchanging swap pairs in this way. Totally order pairs by $(n, n-1) > \cdots > (n, 2) > \cdots > (3, 2) > (n, 1) > \cdots > (3, 1) > (2, 1)$, and restrict this ordering to the swap pairs. Let $\sigma_x = x \prec \sigma_{n-1} \prec \cdots \prec \sigma_0$ where σ_{i-1} is the result of interchanging the minimal swap pair of σ_i. For example if $x = 4321 \in S_4$ then

$$\sigma_x = 43\underline{21} \prec 4\underline{31}2 \prec \underline{41}32 \prec 14\underline{32} \prec 1\underline{42}3 \prec 1243$$

with the minimal swap pairs underlined (and $12\underline{43} \prec 1234 = \text{id}$). Now to the signs. If $x \prec y$ with y the result of interchanging the minimal swap pair in x, then clearly $[x, y] = 1$. If now x has corank 1 and y is the result of interchanging a non-minimal swap pair in x then $[x, y] = -1$ (as $x\sigma_y + \sigma_x = 0$ via a relation of the form given on the left of Figure 2). For a general covering $x \prec y$ it is possible to find a $\diamond$-configuration x, y, y', z, so that $[x, y] = -[x, y'][y, z][y', z]$ by (24), where $[x, y']$, $[y, z]$ and $[y', z]$ are already known, the last two by induction on the corank. We leave the details to the reader. Figure 4 illustrates the case $n = 4$.

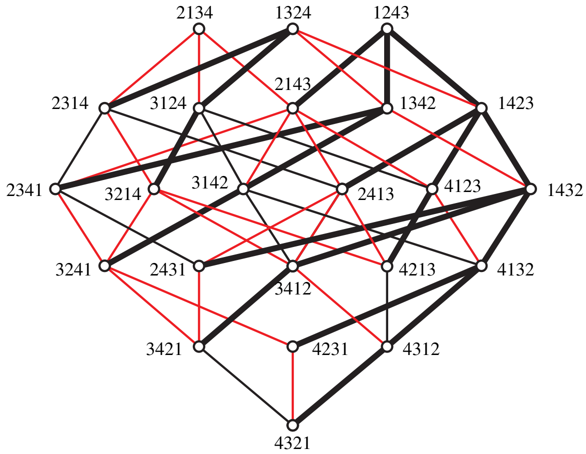

Figure 4: $S_4 \setminus \mathrm{id}$ equipped with the Bruhat order: the thick edges give the generators σ_x; the black edges (both thick and thin) are the $x \prec y$ with $[x, y] = 1$ and the red edges are the $x \prec y$ with $[x, y] = -1$.

4.4. Geometric lattices

A *lattice* is a poset $\mathbf{P}$ such that any two elements x and y have a supremum (or join) $x \vee y$ and an infimum (or meet) $x \wedge y$. $\mathbf{P}$ has *finite length* is there is an absolute bound on the number of elements in any poset chain $x_0 \leq \cdots \leq x_n$. If $\mathbf{P}$ has finite length and a unique minimum $\mathbf{0}$, then define a grading by taking $rk(x)$ to be the supremum of the lengths of all poset chains from $\mathbf{0}$ to x. $\mathbf{P}$ is a *geometric* lattice if every element can be expressed as a join of elements of rank 1 (called *atoms*) and for any x, y we have

$$rk(x \vee y) + rk(x \wedge y) \leq rk(x) + rk(y) \tag{25}$$

The motivating example is the linear subspaces of a vector space V over some field k, ordered by reverse inclusion. See [2, Chapter IV] or [15, Chapter 3] for general facts about geometric lattices.

Let $\mathbf{P}$ be a locally finite geometric lattice – in particular $\mathbf{P}$ is finite and hence also has a unique maximum $\mathbf{1}$, the join of the elements of $\mathbf{P}$. In the light of Section 4.2, let $\mathbf{Q} = \mathbf{P} \setminus \mathbf{1}$. For every $x \in \mathbf{P}$ the interval $\mathbf{P}_{\geq x}$ is also a geometric lattice. Let $\mu_x := (-1)^{|x|-1}\mu(x, \mathbf{1})$ where μ is the Möbius function of $\mathbf{P}$ and $|\cdot|$ is the corank function of $\mathbf{Q}$. For any x the space $|N^*\mathbf{Q}_{>x}|$ has the homotopy type of a bouquet of μ_x spheres of dimension $|x| - 1$ ([14], see also [13, Theorem 4.109] and [3]), hence

$$\widetilde{HS}^{|x|-1}(\mathbf{Q}_{>x}; \Delta\mathbb{Z}) \cong \mathbb{Z}^{\mu_x}$$

and the homology vanishes in all other degrees. Thus geometric lattices (minus their maximal elements) are cellular, and for any presheaf F on $\mathbf{Q}$ we have

$$C^n(\mathbf{Q}; F) \cong \bigoplus_{|x|=n} A_x \otimes F(x)$$

where $A_x \cong \mathbb{Z}^{\mu_x}$. One can find explicit free generators for A_x using R-labelings [15, Theorem 3.13.2] and hence an explicit description of the differential from Proposition 7.

References

[1] Kenneth Bacławski. Whitney numbers of geometric lattices. *Advances in Math.*, 16:125–138, 1975.

[2] Garrett Birkhoff. *Lattice theory*, volume 25 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, R.I., third edition, 1979.

[3] Anders Björner. On the homology of geometric lattices. *Algebra Universalis*, 14(1):107–128, 1982.

[4] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.

[5] Brent Everitt and Paul Turner. Homology of coloured posets: a generalisation of Khovanov's cube construction. *J. Algebra*, 322(2):429–448, 2009.

[6] Brent Everitt and Paul Turner. The homotopy theory of Khovanov homology. *Algebr. Geom. Topol.*, 14(5):2747–2781, 2014.

[7] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[8] Roger Godement. *Topologie algébrique et théorie des faisceaux*. Hermann, Paris, 1973. Troisième édition revue et corrigée, Publications de l'Institut de Mathématique de l'Université de Strasbourg, XIII, Actualités Scientifiques et Industrielles, No. 1252.

[9] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[10] Albert T Lundell. *The Topology of CW Complexes*. Van Nostrand Reinhold, New York, 1969.

[11] William S. Massey. *A basic course in algebraic topology*, volume 127 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[12] I. Moerdijk. *Classifying spaces and classifying topoi*, volume 1616 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.

[13] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[14] Daniel Quillen. Homotopy properties of the poset of nontrivial p-subgroups of a group. *Adv. in Math.*, 28(2):101–128, 1978.

[15] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[16] Michelle L. Wachs. Poset topology: tools and applications. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 497–615. Amer. Math. Soc., Providence, RI, 2007.

[17] Sergey Yuzvinsky. Cohen-Macaulay rings of sections. *Adv. in Math.*, 63(2):172–195, 1987.