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Identification of a heterogeneous orthotropic conductivity in a rectangular domain

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Abstract
This paper investigates the problem of identifying a heterogeneous transient orthotropic thermal conductivity in a two-dimensional rectangular domain using initial and Dirichlet boundary conditions and fluxes as overdetermination conditions. The measurement data represented by the heat fluxes are shown to ensure the unique solvability of the inverse problem solution. The finite-difference method is employed as the direct solver which is fed iteratively in a nonlinear minimization routine. Exact and noisy input data are inverted numerically. Numerical results indicate that accurate and stable solutions are obtained.

Keywords: Heat equation; Orthotropic material; Inverse problem

1 Introduction

The determination of coefficients in inverse heat conduction problems for the parabolic heat equation, [9], continues to receive significant attention in a variety of fields, such as heat transfer, oil recovery, groundwater flow, and finance. Many researchers investigated the case of simultaneous identification of coefficients in two-dimensional heat conduction problems, [3, 4, 13].

The identification of physical properties such as thermal conductivity using measured temperature or heat flux values at well sites is an important inverse problem. A common identification strategy is the indirect one where one can minimize the gap between a computed solution and the measured data (observations) via an iterative process, [12].

The main obstacle in this kind of problem is that there are usually so few observations that one finds hard to evaluate the spatial derivative of temperature by simple numerical differentiation. Therefore, heavier and more time-consuming optimization techniques are needed to obtain reliable results.

The estimation of thermal properties for the multi-dimensional inhomogeneous and anisotropic media is rather scarce in the literature [1, 7]. The aim of the present study is to consider a two-dimensional coefficient identification problem to estimate the space and time varying principal direction components of an orthotropic conductivity in a rectangular domain.

The structure of the paper is as follows. In Section 2 we give the mathematical formulation of the two-dimensional inverse problem. Sections 3 and 4 present the existence and uniqueness proofs, respectively. In Section 5 we briefly describe the explicit finite-difference method used to discretise the direct problem, whilst Section 6 introduces the constrained nonlinear minimization problem that has to be solved using the MATLAB routine *lsqnonlin*. In Section 7, numerical results are presented and discussed and finally conclusions of the paper are given in Section 8.

2 Statement of the inverse problem

Consider the nonlinear inverse coefficient identification problem which requires determining the principal direction components $a(y,t) > 0$ and $b(x,t) > 0$ of the two-dimensional heterogeneous orthotropic rectangular medium $D = (0,h) \times (0,\ell)$ together with the temperature $u(x,y,t)$ satisfying the heat equation

$$u_t = a(y,t)u_{xx} + b(x,t)u_{yy} + f(x,y,t), \quad (x,y,t) \in Q_T := D \times (0,T), \tag{1}$$

where f is a given heat source, subject to initial, boundary and overdetermination conditions

$$u(x,y,0) = \phi(x,y), \quad (x,y) \in \overline{D}, \tag{2}$$

$$u(0,y,t) = \mu_1(y,t), \quad u(h,y,t) = \mu_2(y,t), \quad (y,t) \in [0,l] \times [0,T], \tag{3}$$

$$u(x,0,t) = \mu_3(x,t), \quad u(x,l,t) = \mu_4(x,t), \quad (x,t) \in [0,h] \times [0,T], \tag{4}$$

$$a(y,t)u_x(0,y,t) = \mu_5(y,t), \quad (y,t) \in [0,l] \times [0,T], \tag{5}$$

$$b(x,t)u_y(x,0,t) = \mu_6(x,t), \quad (x,t) \in [0,h] \times [0,T]. \tag{6}$$

In the above setting one can see that Cauchy data are prescribed over the boundaries $x = 0$ and $y = 0$. Also by restricting the conductivity components $a(y,t)$ and $b(x,t)$ be independent of x and y, respectively, it then makes sense to study the injectivity/surjectivity of the mapping $(a,b) \mapsto (\mu_5, \mu_6)$. We finally mention that in the general case when $a(x,y,t)$ and $b(x,y,t)$ depend on all coordinates then the right hand side of (1) modifies as $(a(x,y,t)u_x)_x + (b(x,y,t)u_y)_y + f(x,y,t)$.

There is no theory available for this general orthotropic inverse coefficient identification, but at least in the isotropic case when $a = b$, all the knowledge of the temperature $u(x,y,t)$ for $(x,y,t) \in Q_T$ is necessary in order to render a unique solution, [5]. All this discussion warrants and justifies our assumption that $a(y,t)$ and $b(x,t)$ are independent on the variables x and y, respectively. Then, the measurements (5) and (6) are supplied as the correct traces of functionals, according to the illuminating discussion of Cannon et al. [2].

Suppose that the following assumptions hold:

(A1) $\phi \in C^{2+\gamma}(\overline{D}), \mu_i \in C^{2+\gamma,1+\gamma/2}([0,l] \times [0,T]), i \in \{1,2\}, \mu_k \in C^{2+\gamma,1+\gamma/2}([0,h] \times [0,T]), k \in \{3,4\}, \mu_5 \in C^{\gamma,\gamma/2}([0,l]\times[0,T]), \mu_6 \in C^{\gamma,\gamma/2}([0,h]\times[0,T]), f \in C^{\gamma,\gamma/2}(\overline{Q}_T)$ for some $\gamma \in (0,1)$;

(A2) $\phi_x(x,y) > 0, \phi_y(x,y) > 0, (x,y) \in \overline{D}, \mu_5(y,t) > 0, (y,t) \in [0,l] \times [0,T], \mu_6(x,t) > 0, (x,t) \in [0,h] \times [0,T]$;

(A3) consistency conditions of the zero and the first orders.

We remark that a formal elimination of $a(y,t)$ and $b(x,t)$ in (5) and (6), respectively, and substitution into (1) result in the nonlinear partial differential equation

$$u_t(x,y,t) = \frac{\mu_5(y,t)}{u_x(0,y,t)}u_{xx} + \frac{\mu_6(x,t)}{u_y(x,0,t)}u_{yy} + f(x,y,t), \quad (x,y,t) \in Q_T \tag{7}$$

to be solved for the temperature $u(x,y,t)$ subject to the initial and boundary conditions (2)–(4).

3 Existence of solution of the inverse problem

Theorem 1. *Suppose that the assumptions* **(A1)-(A3)** *hold. Then for some* $T_0 \in (0,T]$ *there exists a solution of the problem (1)-(6) such that* $(a,b,u) \in C^{\gamma,\gamma/2}([0,l]\times[0,T_0]) \times C^{\gamma,\gamma/2}([0,h]\times[0,T_0]) \times C^{2+\gamma,1+\gamma/2}(\overline{Q}_{T_0})$, $a(y,t)>0, (y,t)\in[0,l]\times[0,T_0], b(x,t)>0, (x,t)\in[0,h]\times[0,T_0]$.

Proof. In order to make the initial and boundary conditions (2)–(4) homogenous the following notation will be used:

$\psi(x,y,t) := \mu_1(y,t) - \mu_1(y,0) + \frac{x}{h}(\mu_2(y,t) - \mu_2(y,0) - \mu_1(y,t) + \mu_1(y,0)) + \mu_3(x,t) - \mu_3(x,0) - [\mu_1(0,t) - \mu_1(0,0) + \frac{x}{h}(\mu_2(0,t) - \mu_2(0,0) - \mu_1(0,t) + \mu_1(0,0))] + \frac{y}{l}[\mu_4(x,t) - \mu_4(x,0) - \mu_1(l,t) + \mu_1(l,0) - \frac{x}{h}(\mu_2(l,t) - \mu_2(l,0) - \mu_1(l,t) + \mu_1(l,0)) - \mu_3(x,t) + \mu_3(x,0) + \mu_1(0,t) - \mu_1(0,0) + \frac{x}{h}(\mu_2(0,t) - \mu_2(0,0) - \mu_1(0,t) + \mu_1(0,0))]$.

Then by the superposition

$$u(x,y,t) = v(x,y,t) + \phi(x,y) + \psi(x,y,t)$$

we reduce the equations (1)-(4) to the following ones:

$$v_t = a(y,t)v_{xx} + b(x,t)v_{yy} + F(x,y,t) + a(y,t)(\phi_{xx}(x,y) + \psi_{xx}(x,y,t)) + b(x,t)(\phi_{yy}(x,y) + \psi_{yy}(x,y,t)), \quad (x,y,t) \in Q_T, \tag{8}$$

$$v(x,y,0) = 0, \quad (x,y) \in \overline{D}, \tag{9}$$

$$v(0,y,t) = v(h,y,t) = 0, \quad (y,t) \in [0,l]\times[0,T], \tag{10}$$

$$v(x,0,t) = v(x,l,t) = 0, \quad (x,t) \in [0,h]\times[0,T], \tag{11}$$

where $F(x,y,t) := f(x,y,t) - \psi_t(x,y,t)$.

Supposing for the moment that the coefficients $a(y,t)$ and $b(x,t)$ are known, we find the solution v of the problem (8)-(11) as

$$v(x,y,t) = \int_0^t \iint_D G(x,y,t,\xi,\eta,\tau)\Big[F(\xi,\eta,\tau) + a(\eta,\tau)(\phi_{\xi\xi}(\xi,\eta) + \psi_{\xi\xi}(\xi,\eta,\tau)) + b(\xi,\tau)(\phi_{\eta\eta}(\xi,\eta) + \psi_{\eta\eta}(\xi,\eta,\tau)\Big]d\xi d\eta d\tau, \quad (x,y,t) \in \overline{Q}_T, \tag{12}$$

where $G(x,y,t,\xi,\eta,\tau)$ is the Green function of the problem (8)-(11). The assumptions **(A1)-(A3)** ensure the existence of such a Green function [8]. Then the solution u of the problem (1)-(4) is given by the formula

$$u(x,y,t) = \phi(x,y) + \psi(x,y,t) + \int_0^t \iint_D G(x,y,t,\xi,\eta,\tau)\Big[F(\xi,\eta,\tau) + a(\eta,\tau)(\phi_{\xi\xi}(\xi,\eta) + \psi_{\xi\xi}(\xi,\eta,\tau)) + b(\xi,\tau)(\phi_{\eta\eta}(\xi,\eta) + \psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau, \quad (x,y,t) \in \overline{Q}_T. \tag{13}$$

Substituting (13) into (5) and (6) we obtain

$$a(y,t) = \mu_5(y,t)\Bigg\{\phi_x(0,y) + \psi_x(0,y,t) + \int\limits_0^t \iint\limits_D G_x(0,y,t,\xi,\eta,\tau)\Big[F(\xi,\eta,\tau)$$
$$+ a(\eta,\tau)(\phi_{\xi\xi}(\xi,\eta) + \psi_{\xi\xi}(\xi,\eta,\tau))$$
$$+ b(\xi,\tau)(\phi_{\eta\eta}(\xi,\eta) + \psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg\}^{-1}, \quad (y,t) \in [0,l] \times [0,T], \tag{14}$$

$$b(x,t) = \mu_6(x,t)\Bigg\{\phi_y(x,0) + \psi_y(x,0,t) + \int\limits_0^t \iint\limits_D G_y(x,0,t,\xi,\eta,\tau)\Big[F(\xi,\eta,\tau)$$
$$+ a(\eta,\tau)(\phi_{\xi\xi}(\xi,\eta) + \psi_{\xi\xi}(\xi,\eta,\tau))$$
$$+ b(\xi,\tau)(\phi_{\eta\eta}(\xi,\eta) + \psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg\}^{-1}, \quad (x,t) \in [0,h] \times [0,T]. \tag{15}$$

So, the inverse problem (1)-(6) has been reduced to the system of integral equations (14) and (15).

To begin with, we establish the existence of a positive solution $(a(y,t), b(x,t))$ of the system of integral equations (14) and (15) in the space $C([0,l]\times[0,T])\times C([0,h]\times[0,T])$ by applying the Schauder fixed-point theorem. For this, we need to find the estimates for the solution. It follows from assumption **(A2)** that $\phi_x(0,y) \geq M_1 > 0, y \in [0,l], \phi_y(x,0) \geq M_2 > 0, x \in [0,h]$. As the rest of terms in the denominators of (14) and (15) are equal to zero for $t = 0$, there exists $T_0 \in (0,T]$ such that the following inequalities hold:

$$\Big|\psi_x(0,y,t) + \int\limits_0^t \iint\limits_D G_x(0,y,t,\xi,\eta,\tau)\Big[F(\xi,\eta,\tau) + a(\eta,\tau)(\phi_{\xi\xi}(\xi,\eta)$$
$$+ \psi_{\xi\xi}(\xi,\eta,\tau)) + b(\xi,\tau)(\phi_{\eta\eta}(\xi,\eta) + \psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Big| \leq \frac{M_1}{2},$$

$$\Big|\psi_y(x,0,t) + \int\limits_0^t \iint\limits_D G_y(x,0,t,\xi,\eta,\tau)\Big[F(\xi,\eta,\tau) + a(\eta,\tau)(\phi_{\xi\xi}(\xi,\eta)$$
$$+ \psi_{\xi\xi}(\xi,\eta,\tau)) + b(\xi,\tau)(\phi_{\eta\eta}(\xi,\eta) + \psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Big| \leq \frac{M_2}{2}, \quad (x,y,t) \in \overline{Q}_{T_0}. \tag{16}$$

Then we find from (14) and (15) that

$$a(y,t) \le A_1 := \frac{\max\limits_{[0,l]\times[0,T]} \mu_5(y,t)}{M_1/2}, \quad (y,t) \in [0,l]\times[0,T],$$

$$b(x,t) \le B_1 := \frac{\max\limits_{[0,h]\times[0,T]} \mu_6(x,t)}{M_2/2}, \quad (x,t) \in [0,h]\times[0,T], \tag{17}$$

$$a(y,t) \ge A_0 := \frac{\min\limits_{[0,l]\times[0,T]} \mu_5(y,t)}{\max\limits_{[0,l]} \phi_x(0,y) + M_1/2}, \quad (y,t) \in [0,l]\times[0,T],$$

$$b(x,t) \ge B_0 := \frac{\min\limits_{[0,h]\times[0,T]} \mu_6(x,t)}{\max\limits_{[0,h]} \phi_y(x,0) + M_2/2}, \quad (x,t) \in [0,h]\times[0,T]. \tag{18}$$

Denote $\omega := (a(y,t), (b(x,t))$ and rewrite the system (14) and (15) as an operator equation

$$\omega = P\omega. \tag{19}$$

Introduce the set $\mathcal{N} := \{\omega \in C([0,l]\times[0,T_0])\times C([0,h]\times[0,T_0]) : A_0 \le a(y,t) \le A_1, B_0 \le b(x,t) \le B_1\}$. It is easy to see that the operator P maps $\mathcal{N}$ onto $\mathcal{N}$. The compactness of the operator P may be easily established by the same procedure as in [9].

It follows that the Schauder theorem may be applied to the equation (19) and, hence, there exists a continuous solution of the system of integral equations (14) and (15). Taking into account the assumption **(A1)**, we conclude that $a \in C^{\gamma,\gamma/2}([0,l]\times[0,T_0]), b \in C^{\gamma,\gamma/2}([0,h]\times[0,T_0])$. Then, it also follows [8] that $u \in C^{2+\gamma,1+\gamma/2}(\overline{Q}_{T_0})$. The proof is complete.

Remark. Having the estimates (17) and (18), one can easily estimate from (16) the value of T_0.

4 Uniqueness of solution of the inverse problem

Theorem 2. *Suppose that* $\mu_5(y,t) \ne 0, (y,t) \in [0,l]\times[0,T], \mu_6(x,t) \ne 0, (x,t) \in [0,h]\times[0,T]$. *Then a solution* $(a(y,t), b(x,t), u(x,y,t))$ *of the problem (1)-(6) is unique in the space* $C^{\gamma,\gamma/2}([0,l]\times[0,T])\times C^{\gamma,\gamma/2}([0,h]\times[0,T])\times C^{2+\gamma,1+\gamma/2}(\overline{Q}_T), a(y,t) > 0, (y,t) \in [0,l]\times[0,T], b(x,t) > 0, (x,t) \in [0,h]\times[0,T]$.

Proof. Suppose that there are two solutions $(a_i(y,t), b_i(x,t), u_i(x,y,t)), i \in \{1,2\}$ of the problem (1)-(6) in the indicated class. Denote $a := a_1 - a_2, b := b_1 - b_2, u := u_1 - u_2$. Then (a,b,u) is a solution of the following problem:

$$u_t = a_1(y,t)u_{xx} + b_1(x,t)u_{yy} + a(y,t)u_{2xx}(x,y,t) + b(x,t)u_{2yy}(x,y,t), \quad (x,y,t) \in Q_T, \tag{20}$$

$$u(x,y,0) = 0, \quad (x,y) \in \overline{D}, \tag{21}$$

$$u(0,y,t) = u(h,y,t) = 0, \quad (y,t) \in [0,l]\times[0,T], \tag{22}$$

$$u(x,0,t) = u(x,l,t) = 0, \quad (x,t) \in [0,h]\times[0,T], \tag{23}$$

$$a_1(y,t)u_x(0,y,t) = -a(y,t)u_{2x}(0,y,t), \quad (y,t) \in [0,l]\times[0,T], \tag{24}$$

$$b_1(x,t)u_y(x,0,t) = -b(x,t)u_{2y}(x,0,t), \quad (x,t) \in [0,h]\times[0,T]. \tag{25}$$

Using the Green function $\tilde{G}(x,y,t,\xi,\eta,\tau)$ of the problem (20)-(23), we find its solution as

$$u(x,y,t)=\int_0^t\iint_D \tilde{G}(x,y,t,\xi,\eta,\tau)(a(\eta,\tau)u_{2\xi\xi}(\xi,\eta,\tau)$$
$$+b(\xi,\tau)u_{2\eta\eta}(\xi,\eta,\tau))d\xi d\eta d\tau, \quad (x,y,t)\in\overline{Q}_T. \tag{26}$$

We obtain from (24)-(26) the following integral equations:

$$a(y,t)=-\frac{a_1(y,t)}{u_{2x}(0,y,t)}\int_0^t\iint_D \tilde{G}_x(0,y,t,\xi,\eta,\tau)(a(\eta,\tau)u_{2\xi\xi}(\xi,\eta,\tau)$$
$$+b(\xi,\tau)u_{2\eta\eta}(\xi,\eta,\tau))d\xi d\eta d\tau, \quad (y,t)\in[0,l]\times[0,T], \tag{27}$$
$$b(x,t)=-\frac{b_1(x,t)}{u_{2y}(x,0,t)}\int_0^t\iint_D \tilde{G}_y(x,0,t,\xi,\eta,\tau)(a(\eta,\tau)u_{2\xi\xi}(\xi,\eta,\tau)$$
$$+b(\xi,\tau)u_{2\eta\eta}(\xi,\eta,\tau))d\xi d\eta d\tau, \quad (y,t)\in[0,l]\times[0,T]. \tag{28}$$

Note that $u_{2x}(0,y,t)\neq 0, u_{2y}(x,0,t)\neq 0$ as (a_2,b_2,u_2) is a solution of (1)-(6) and, hence, it verifies the conditions (5), (6) with functions μ_5 and μ_6 which do not vanish. Consequently, (27) and (28) is a homogeneous system of Volterra integral equations of the second kind and has only the trivial solution $a(y,t)\equiv 0, b(x,t)\equiv 0$. From this and (20)–(23) we obtain that $u(x,y,t)\equiv 0,(x,y,t)\in\overline{Q}_T$. The proof is complete.

5 Solution of direct problem

In this section, we consider the direct initial boundary value problem (1)–(4) where $a(y,t)$, $b(x,t)$, $f(x,y,t)$, $\phi(x,y)$, and μ_i, $i=1,2,3,4$, are known and the solution $u(x,y,t)$ is to be determined. To achieve this, we use the Forward-Time-Central-Space (FTCS) finite-difference scheme which is conditionally stable.

We subdivide the solution domain Q_T into M_x, M_y and N subintervals of equal step lengths Δx and Δy, and uniform time step Δt, where $\Delta x=h/M_x$, $\Delta y=\ell/M_y$ and $\Delta t=T/N$, for space and time, respectively. At the node (i,j,k) we denote $u_{i,j}^k:=u(x_i,y_j,t_k)$, where $x_i=i\Delta x$, $y_j=j\Delta y$, $t_k=k\Delta t$, $a_j^k:=a(y_j,t_k)$, $b_i^k:=b(x_i,t_k)$ and $f_{i,j}^k:=f(x_i,y_j,t_k)$ for $i=\overline{0,M_x}$, $j=\overline{0,M_y}$ and $k=\overline{0,N}$.

The simplest explicit difference scheme for equation (1) is given by

$$\frac{u_{i,j}^{k+1}-u_{i,j}^k}{\Delta t}=a_j^k\frac{u_{i+1,j}^k-2u_{i,j}^k+u_{i-1,j}^k}{(\Delta x)^2}+b_i^k\frac{u_{i,j+1}^k-2u_{i,j}^k+u_{i,j-1}^k}{(\Delta y)^2}+f_{i,j}^k \tag{29}$$

for $i=\overline{1,M_x-1}$, $j=\overline{1,M_y-1}$ and $k=\overline{0,N}$. The initial and boundary conditions (2)–(4) give

$$u_{i,j}^0=\phi_{i,j}, \quad i=\overline{0,M_x}, \quad j=\overline{0,M_y}, \tag{30}$$
$$u_{0,j}^k=\mu_1(y_j,t_k), \quad u_{M_x,j}^k=\mu_2(y_j,t_k), \quad j=\overline{0,M_y}, \quad k=\overline{1,N}, \tag{31}$$
$$u_{i,0}^k=\mu_3(x_i,t_k), \quad u_{i,M_y}^k=\mu_4(x_i,t_k), \quad i=\overline{0,M_x}, \quad k=\overline{1,N}. \tag{32}$$

Let $\tilde{a}$ and $\tilde{b}$ be the maximum values of $a(y,t)$ and $b(x,t)$, respectively, then, the stability condition for the explicit FDM scheme (29) will be [10].

$$\frac{\tilde{a}\Delta t}{(\Delta x)^2} + \frac{\tilde{b}\Delta t}{(\Delta y)^2} \leq \frac{1}{2}. \tag{33}$$

The heat fluxes (5) and (6) can be calculated using the second-order FDM approximations:

$$\mu_5(y_j,t_k) = a_j^k \frac{4u_{1,j}^k - u_{2,j}^k - 3u_{0,j}^k}{2\Delta x}, \quad j=\overline{1,M_y-1}, \quad k=\overline{1,N}, \tag{34}$$

$$\mu_6(x_i,t_k) = b_i^k \frac{4u_{i,1}^k - u_{i,2}^k - 3u_{i,0}^k}{2\Delta y}, \quad i=\overline{1,M_x-1}, \quad k=\overline{1,N}. \tag{35}$$

6 Numerical solution of inverse problem

In this section we aim to obtain stable reconstructions for the principal direction components $a(y,t)>0$ and $b(x,t)>0$ of the two-dimensional heterogeneous orthotropic rectangular medium together with the temperature $u(x,y,t)$ satisfying the equations (1)–(6). One can remark that at initial time $t=0$ the values of the principal direction components are known and they can easily be obtained form the overdetermination conditions (5) and (6) as

$$a(y,0) = \frac{\mu_5(y,0)}{\phi_x(0,y)}, \quad b(x,0) = \frac{\mu_6(x,0)}{\phi_y(x,0)}, \quad y\in[0,\ell], \quad x\in[0,h]. \tag{36}$$

The inverse problem is solved based on the nonlinear minimization of the least-squares objective function

$$F(a,b) := \left\|a(y,t)u_x(0,y,t) - \mu_5(y,t)\right\|^2 + \left\|b(x,t)u_y(x,0,t) - \mu_6(x,t)\right\|^2, \tag{37}$$

or, in discretised form

$$F(\underline{a},\underline{b}) = \sum_{k=1}^{N}\sum_{j=0}^{M_y}\Big[a_{j,k}u_x(0,y_j,t_k) - \mu_5(y_j,t_k)\Big]^2 + \sum_{k=1}^{N}\sum_{i=0}^{M_x}\Big[b_{i,k}u_y(x_i,0,t_k) - \mu_6(x_i,t_k)\Big]^2. \tag{38}$$

The minimization of the objective functional (38), subjected to the physical simple bound constraints $\underline{a} > \underline{0}$ and $\underline{b} > \underline{0}$ is accomplished using the MATLAB optimization toolbox routine *lsqnonlin*, which does not require supplying (by the user) the gradient of the objective function, [11]. Furthermore, within *lsqnonlin* we use the Trust-Region algorithm which is based on the interior-reflective Newton method. Each iteration involves a large linear system of equations whose solution, based on a preconditioned conjugate gradient method, allows a regular and sufficiently smooth decrease of the objective functional (38). Since the MATLAB routine *lsqnonlin* accepts only a vector of unknowns we make the matrices a and b be a long vector by renumbering their components accordingly.

Upper and lower bounds on the thermal conductivities a and b can be specified according to *a priori* information on these physical parameters.

In the numerical computation, we take the parameters of the routine *lsqnonlin* as follows:

- Maximum number of iterations = $10^5\times$ (number of variables).
- Maximum number of objective function evaluations = $10^6\times$ (number of variables).
- Solution and objective function tolerances = 10^{-10}.

The inverse problem (1)–(6) is solved subject to both exact and noisy measurements (5) and (6). The noisy data is numerically simulated as

$$\mu_5^{\epsilon 1}(y_j,t_k)=\mu_5(y_j,t_k)+\epsilon 1_{j,k},\quad j=\overline{0,M_y},\quad k=\overline{1,N}, \tag{39}$$
$$\mu_6^{\epsilon 2}(x_i,t_k)=\mu_6(x_i,t_k)+\epsilon 2_{i,k},\quad i=\overline{0,M_x},\quad k=\overline{1,N}, \tag{40}$$

where $\epsilon 1_{j,k}$ and $\epsilon 2_{i,k}$ are random variables generated from a Gaussian normal distribution with mean zero and standard deviation $\sigma 1$ and $\sigma 2$ given by

$$\sigma 1=p\times\max_{(y,t)\in[0,\ell]\times[0,T]}|\mu_5(y,t)|,\quad \sigma 2=p\times\max_{(x,t)\in[0,h]\times[0,T]}|\mu_6(x,t)|, \tag{41}$$

where p represents the percentage of noise. We use the MATLAB function *normrnd* to generate the random variables $\underline{\epsilon 1}=(\epsilon 1_{j,k})_{j=\overline{0,M_y},k=\overline{1,N}}$ and $\underline{\epsilon 2}=(\epsilon 2_{i,k})_{i=\overline{0,M_x},k=\overline{1,N}}$ as follows:

$$\underline{\epsilon 1}=normrnd(0,\sigma 1,M_y+1,N),\quad \underline{\epsilon 2}=normrnd(0,\sigma 2,M_x+1,N). \tag{42}$$

In the case of noisy data (39) and (40), we replace $\mu_5(y_j,t_k)$ and $\mu_6(x_i,t_k)$ by $\mu_5^{\epsilon 1}(y_j,t_k)$ and $\mu_6^{\epsilon 2}(x_i,t_k)$, respectively, in (38).

7 Numerical results and discussion

In this section, we present numerical results for the orthotropic thermal conductivity components $a(y,t)$, $b(x,t)$ and the temperature $u(x,y,t)$, in the case of exact and noisy data (39) and (40). To assess the accuracy of the numerical solution we employ the root mean square errors ($rmse$) defined by:

$$rmse(a)=\left[\frac{1}{N(M_y+1)}\sum_{k=1}^{N}\sum_{j=0}^{M_y}\left(a_{numerical}(y_j,t_k)-a_{exact}(y_j,t_k)\right)^2\right]^{1/2}, \tag{43}$$
$$rmse(b)=\left[\frac{1}{N(M_x+1)}\sum_{k=1}^{N}\sum_{i=0}^{M_x}\left(b_{numerical}(x_i,t_k)-b_{exact}(x_i,t_k)\right)^2\right]^{1/2}. \tag{44}$$

For simplicity, we take $h=\ell=T=1$.

7.1 Example 1

Consider the inverse problem (1)–(6) with unknown coefficients $a(y,t)$ and $b(y,t)$, with the input data ϕ and μ_i, $i=\overline{1,6}$, as follows:

$$\phi(x,y)=u(x,y,0)=-(-2+x)^2-(-2+y)^2, \quad f(x,y,t)=\frac{101.5+3t+x+y}{50},$$
$$\mu_1(y,t)=u(0,y,t)=-4+2t-(-2+y)^2, \quad \mu_2(y,t)=u(h,y,t)=-1+2t-(-2+y)^2,$$
$$\mu_3(x,t)=u(x,0,t)=-4+2t-(-2+x)^2, \quad \mu_4(x,t)=u(x,\ell,t)=-1+2t-(-2+x)^2,$$
$$\mu_5(y,t)=a(y,t)u_x(0,y,t)=\frac{y+t+1}{25}, \quad \mu_6(x,t)=b(x,t)u_y(x,0,t)=\frac{x+2t+0.5}{25}.$$

One can remark that conditions of Theorems 1 and 2 are satisfied and therefore, the local solvability of the solution is guaranteed. In fact, it can easily be checked by direct substitution that the analytical solution is given by

$$a(y,t)=\frac{y+t+1}{100}, \quad (y,t)\in[0,1]\times[0,1], \tag{45}$$
$$b(x,t)=\frac{x+2t+0.5}{100}, \quad (x,t)\in[0,1]\times[0,1], \tag{46}$$
$$u(x,y,t)=-(x-2)^2-(y-2)^2+2t, \quad (x,y,t)\in\overline{Q}_T. \tag{47}$$

We take a coarse mesh size with $N=M_x=M_y=5$, i.e. $\Delta x=\Delta y=\Delta t=1/5=0.2$. Then we need to choose an upper bound UB for a and b such that the stability condition (33) is satisfied. This yields $UB=1/20=0.05$. Also since a and b represent positive physical quantities we take a lower bound for a and b be given by $LB=0.01$. Keeping the sought parameters inside the lower and upper prescribed bounds through all the minimization process increases the performance of identification, [6].

We start our investigation for simultaneously determining the principal direction components $a(y,t)$ and $b(x,t)$ in a heterogeneous orthotropic with the case of exact input data, i.e. $p=0$ in (41). To test the robustness of the iterative method with respect to the independence on the initial guess, we take three different initial guesses namely:

initial A: $a^0=a_{exact}+3\times10^{-4}\text{randn(size(a))}$, $b^0=b_{exact}+3\times10^{-4}\text{randn(size(b))}$,
initial B: $a^0=a_{exact}+3\times10^{-3}\text{randn(size(a))}$, $b^0=b_{exact}+3\times10^{-3}\text{randn(size(b))}$,
initial C: $a^0=\text{ones(size(a))}$, $\quad b^0=\text{ones(size(b))}$.

where randn(:) is a MATLAB function.

Figure 1 shows the convergence of the objective function (38) with exact input data (5) and (6) for the various initial guesses A, B and C. Table 1 gives more details of these computations including the computational time and the *rmse* values (43) and (44). From Figure 1 and Table 1, it can be seen that, as expected, the farther the initial guess is, e.g. initial C, the more iterations and longer computational time are required to achieve convergence. However, for all initial guesses, the objective function (38) converges to the same very small minimum value of $O(10^{-20})$. This shows robustness with respect to the independence on the initial guess. Furthermore, one can notice that a rapid convergence is achieved for each initial guesses in more than eight iterations within no more than 64 seconds. Moreover, from Table 1 it can be seen that there is an excellent agreement between exact and numerically obtained solutions for all initial guesses with *rmse* values being very low of $O(10^{-12})$ to $O(10^{-11})$ for $a(y,t)$ and $b(x,t)$.

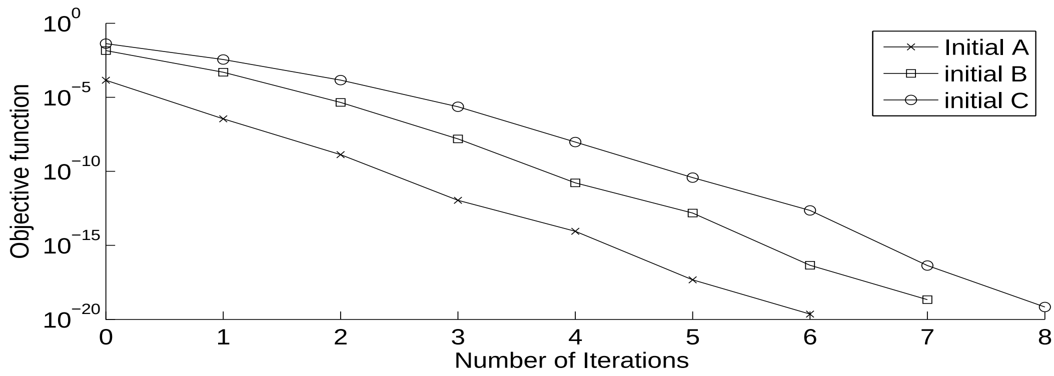
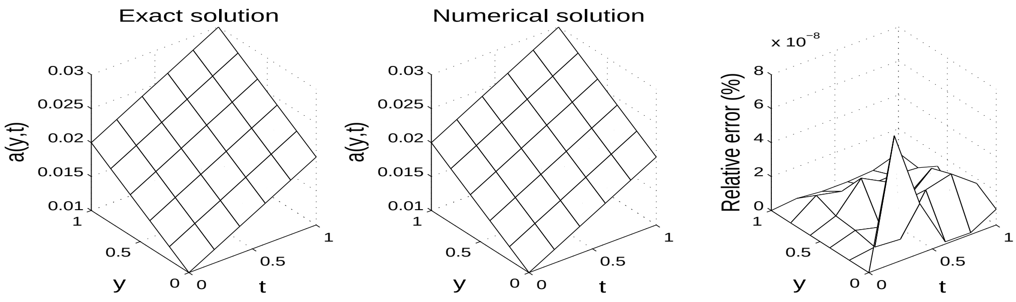


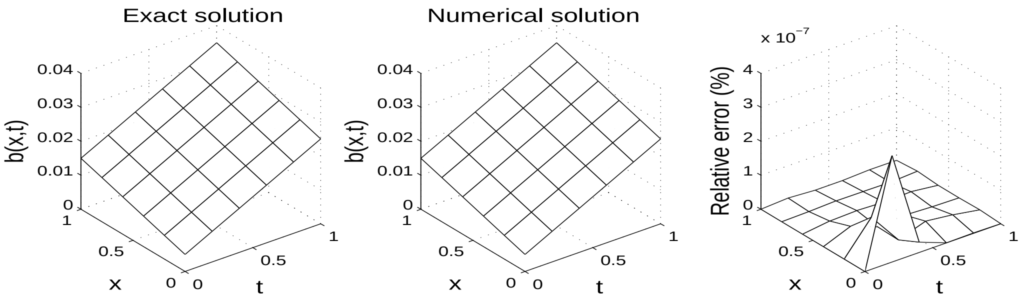
Figure 1: The objective function (38) with no noise, for various initial guesses, for Example 1.

Table 1: Number of iterations, number of function evaluations, value of the objective function (38) at final iteration, the *rmse* values and the computational time, with no regularization and no noise for Example 1 for various initial guesses.

	initial A	initial B	initial C
No. of iterations	6	7	8
No. of function evaluations	511	584	657
Value of objective function (38) at final iteration	2.3E-20	2.1E-19	6.9E-20
$rmse(a)$	3.6E-12	1.1E-11	2.9E-12
$rmse(b)$	5.8E-12	1.7E-11	1.1E-11
Computational time	49 sec	57 sec	64 sec



(a)



(b)

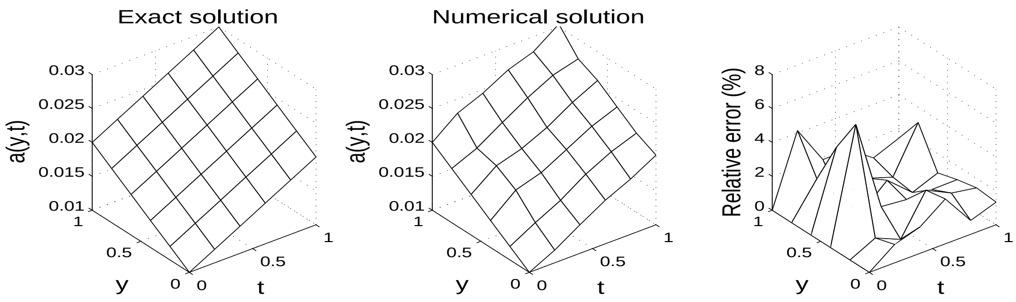
Figure 2: The exact solution (left), numerical solution (middle), error between them (right), with initial guess C, for: (a) $a(y,t)$ and (b) $b(x,t)$, with no noise, for Example 1.

In what follows, we take the initial guess for the unknown coefficients equal to the constant matrix of ones, i.e. we choose the initial guess C. The numerically obtained results for a and b are illustrated in Figure 2 and an excellent agreement can be observed.

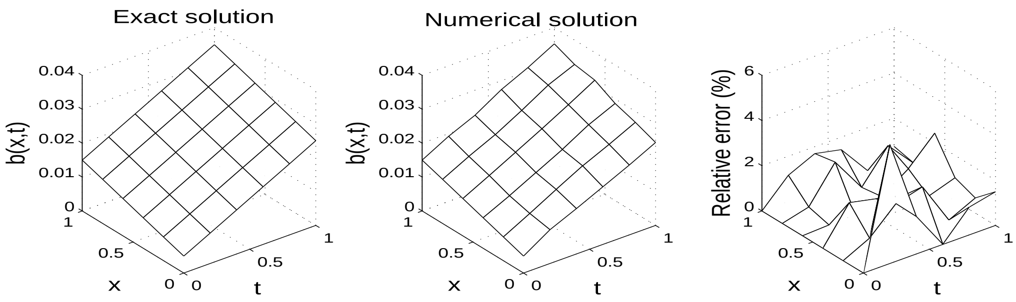
Next we consider the case of noisy data (39) and (40) with $p \in \{1, 5, 10\}\%$. The numerically obtained results are illustrated in Figures 3–5 for $p = 1\%$, 5% and 10%, respectively, and summarised in Table 2. From these figures and table it can be seen that as the percentage of noise p decreases from 10% to 5% and then to 1% the numerically obtained solution becomes more stable and accurate.

Table 2: Number of iterations, number of function evaluations, value of the objective function (38) at final iteration, the $rmse$ values and the computational time, with $p \in \{1, 5, 10\}\%$ noise, for Example 1.

	$p = 1\%$	$p = 5\%$	$p = 10\%$
No. of iterations	8	8	8
No. of function evaluations	657	657	657
Value of objective function (38) at final iteration	2.4E-20	3E-20	7.4E-20
$rmse(a)$	4.2E-4	0.0021	0.0043
$rmse(b)$	3.3E-4	0.0017	0.0034
Computational time	61 sec	61 sec	63 sec

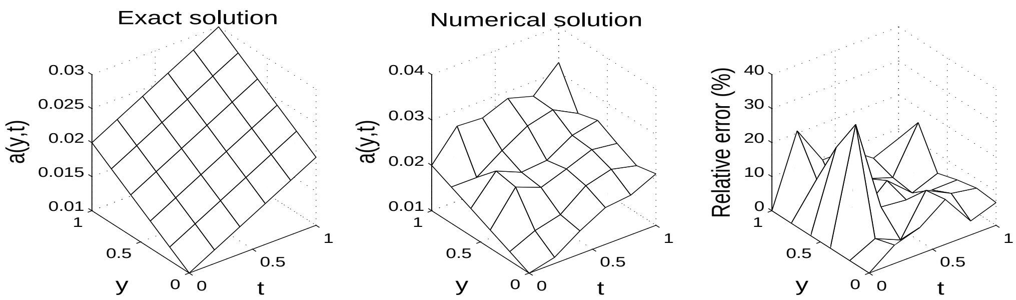


(a)

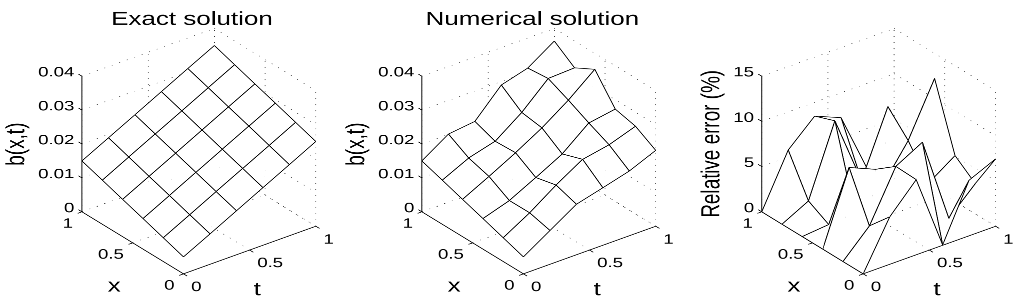


(b)

Figure 3: The exact solution (left), numerical solution (middle), error between them (right), for: (a) $a(y,t)$ and (b) $b(x,t)$, with $p = 1\%$ noisy data, for Example 1.

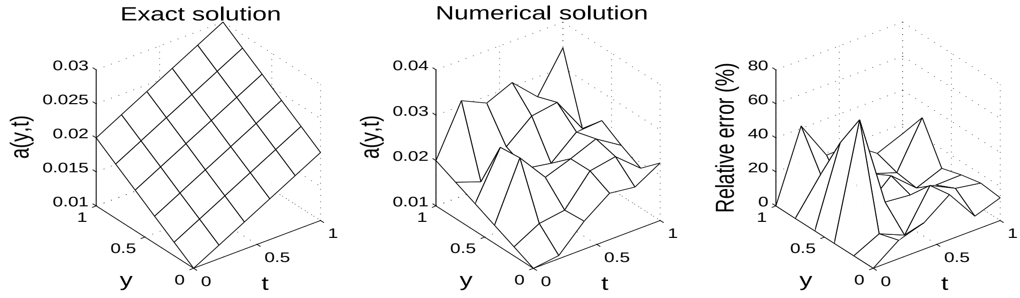


(a)

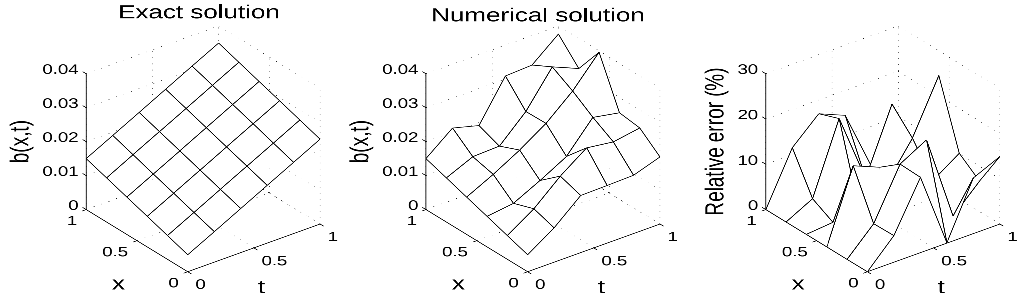


(b)

Figure 4: The exact solution (left), numerical solution (middle), error between them (right), for: (a) $a(y,t)$ and (b) $b(x,t)$, with $p = 5\%$ noisy data, for Example 1.



(a)



(b)

Figure 5: The exact solution (left), numerical solution (middle), error between them (right), for: (a) $a(y,t)$ and (b) $b(x,t)$, with $p = 10\%$ noisy data, for Example 1.

8 Conclusions

The inverse problem concerning the identification of the principal direction thermal conductivity components $a(y,t)$ and $b(x,t)$ of an orthotropic material and the temperature $u(x,y,t)$ in the two-dimensional heat equation in a rectangular domain has been investigated. The additional conditions which ensure a unique solvability of solution are given by the heat fluxes μ_5 and μ_6. The direct solver based on an explicit finite difference scheme has been developed. The inverse solver based on a nonlinear least-squares minimization has been solved using the MATLAB toolbox routine *lsqnonlin*. For both exact and noisy data, the numerical results obtained are accurate and stable.

Acknowledgments

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References

[1] Cannon, J.R. and Jones Jr., B. F. (1963) Determination of the diffusivity of an anisotropic medium, *International Journal of Engineering Science*, **1**, 457–460.

[2] Cannon, J.R., DuChateau, P. and Steube, K. (1990) Unknown ingredient inverse problems and trace-type functional differential equations, *Inverse Problems in Differential Equations*, (eds. D.L. Colton, R.E. Ewing and W. Rundell), pp. 187–202.

[3] Coles, C. and Murio, D.A. (2000) Identification of parameters in the 2-D IHCP, *Computers and Mathematics with Applications*, **40**, 939–956.

[4] Coles, C. and Murio, D.A. (2001) Simultaneous space diffusivity and source term reconstruction in 2D IHCP, *Computers and Mathematics with Applications*, **42**, 1549–1564.

[5] Huang, C.-H. and Chin, S.-C. (2000) A two-dimensional inverse problem in imaging the thermal conductivity of a non-homogeneous medium, *International Journal of Heat and Mass Transfer*, **43**, 4061–4071.

[6] Huang, L., Sun, X., Liu, Y. and Cen, Z. (2004) Parameter identification for two-dimensional orthotropic material bodies by the boundary element method, *Engineering Analysis with Boundary Elements*, **28**, 109–121.

[7] Knowles, I. and Yan, A. (2002) The recovery of an anisotropic conductivity in groundwater modelling, *Applicable Analysis*, **81**, 1347–1365.

[8] Ladyzenskaja, O.A., Solonnikov, V.A. and Uralceva, N.N. (1968) *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs 23, Providence, R.I., American Mathematical Society, 1968.

[9] Ivanchov, M.I. (2003) *Inverse Problems for Equations of Parabolic Type*, VNTL Publications, Lviv, Ukraine.

[10] Morton, K.W. and Mayers, D.F. (2005) *Numerical Solution of Partial Differential Equations: An Introduction*, Cambridge University Press.

[11] Mathworks R2012 Documentation Optimization Toolbox-Least Squares (Model Fitting) Algorithms, available from www.mathworks.com/help/toolbox/optim/ug /brnoybu.html.

[12] Tervola, P. (1989) A method to determine the thermal conductivity from measured temperature profiles, *International Journal of Heat and Mass Transfer*, **32**, 1425–1430.

[13] Yi, Z. and Murio, D.A. (2004) Identification of source terms in 2-D IHCP, *Computers and Mathematics with Applications*, **47**, 1517–1533.