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Nonaxisymmetric linear instability of cylindrical magnetohydrodynamic Taylor-Couette flow

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We consider the nonaxisymmetric modes of instability present in Taylor-Couette flow under the application of helical magnetic fields, mainly for magnetic Prandtl numbers close to the inductionless limit, and conduct a full examination of marginal stability in the resulting parameter space.

We allow for the azimuthal magnetic field to be generated by a combination of currents in the inner cylinder and fluid itself, and introduce a parameter governing the relation between the strength of these currents. A set of governing eigenvalue equations for the nonaxisymmetric modes of instability are derived and solved by spectral collocation with Chebyshev polynomials over the relevant parameter space, with the resulting instabilities examined in detail. We find that by altering the azimuthal magnetic field profiles the azimuthal magnetorotational instability, nonaxisymmetric helical magnetorotational instability, and Taylor instability yield interesting dynamics, such as different preferred mode types, and modes with azimuthal wave number $m > 1$. Finally, a comparison is given to the recent WKB analysis performed by Kirillov et al. [Kirillov, Stefani, and Fukumoto, J. Fluid Mech. 760, 591 (2014)] and its validity in the linear regime.

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INTRODUCTION

The dynamics of hydrodynamic Taylor-Couette flow, in which fluid is driven between two coaxially rotating cylinders, is well understood. It is known, by Rayleigh’s stability criterion, that fluid is stable as angular momentum increases radially outward. However, electrically conducting fluid can be destabilised through the addition of a weak axial magnetic field [1] and it is this magnetorotational instability (MRI) that is proposed to be the mechanism allowing for outward angular momentum transport in accretion disks [2]: a turbulent process that cannot be attributed to laminar viscous effects. Since its discovery the MRI has been the subject of numerous publications, for which we refer to the review by Julien and Knobloch [3].

Similarly, if one produces a purely azimuthal field by running a current through the fluid, even without rotation one may excite the current-driven Taylor instability (or Taylor-Vandakurov instability) [4, 5]. This is fundamentally different to the MRI in that instead of receiving energy from differential rotation, it is driven by the imposed current. The Taylor instability (TI) has numerous astrophysical applications, for example, in the stability of stars, as in Taylor’s original investigation, in radiative stellar cores for which there is near solid body rotation, and the Taylor-Spruit dynamo [6], for which it provides the $\alpha$-effect. Further industrial applications include liquid metal batteries [7], though it has recently been shown [8] that the Taylor instability is too weak to disrupt the electrolyte layer in the batteries.

It is reasonable to have a scenario in which there is differential rotation, as well as an imposed axial field and a current generated azimuthal field in the Taylor-Couette problem. The focus of this paper is then, given that MRI and Taylor instabilities can occur in similar parameter regimes, how would the interaction between them affect the resulting instability?

Experiments involving the MRI and Taylor instabilities are of particular interest, with the first involving the standard MRI (SMRI) suggested independently by Rüdiger and Zhang [9] and Goodman and Ji [10]. However, it was clear that any experiment would be particularly difficult to realise, due to the relevant parameters being $Rm = \Omega r_i^2/\eta$ and $S = B_0 r_i/\sqrt{\mu} \eta$, the magnetic Reynolds number and Lundquist numbers, respectively. As later reiterated by Hollerbach and Fournier [11], the difficulty arises due to the requirement that $Rm \sim O(10)$, which, given that laboratory fluids have $Pm = \eta/\nu \sim O(10^{-6})$ and $Rm = Re Pm$, would require $Re \sim O(10^5)$. At such large rotation rates, the Taylor-Proudman theorem states that any flow would be dominated by the conditions at the endplates. It is not surprising that the SMRI has not yet been achieved [12].

As noted by Hollerbach and Rüdiger [13], if an additional azimuthal magnetic field, $B_\phi = 1/r$, is applied to the fluid a new helical type of magnetorotational instability is found for which $Rm$ is no longer the relevant parameter, allowing instability at experimentally feasible values of $Re \sim O(10^3)$. This axisymmetric MRI variant [now known as the helical MRI (HMRI)] has been subsequently observed in the PROMISE experiments with good agreement with theoretical predictions [14, 15]. It has since been shown by Priede and Gerbeth [16] that this observed HMRI is self-sustaining, rather than a transient growth as was suggested previously.

A further nonaxisymmetric MRI variant is produced if one now considers the case in which only an azimuthal magnetic field $B_\phi = 1/r$ is present. This azimuthal MRI (AMRI), first discussed theoretically by Ogilvie and
Pringle [17], has also been shown to depend on the parameters Re and Ha instead of Rm and S [18] and is thus achievable in laboratory fluids. Indeed, recent work has shown experimental evidence for the AMRI [19].

Further analysis of the different types of MRI, and in particular the AMRI, was included in a WKB approximation performed by Kirillov et al. [20]. Previous research had shown that the HMRI was only able to exist for rotation profiles between two restrictive Liu limits [21], with the possibility of instability at Keplerian rotation profiles excluded [22] for all inductionless forms of the MRI. Kirillov and Stefani [23] showed that the Liu limits can be excluded [22] for all inductionless forms of the MRI. Kirillov and Stefani [23] showed that the Liu limits can be bypassed via variation of the azimuthal magnetic field’s radial profile, allowing inductionless MRI (Rm → 0) even under Keplerian rotation. Under further investigation [20], it was shown that this variation in azimuthal field profile leads to a connecting curve between the two Liu limits.

By allowing for these deviations from the previously defined azimuthal magnetic field profile through generating the field by a combination of currents running through the core and fluid, interesting dynamics may occur. It is then possible to have both MRI and current driven instabilities, which may occur in overlapping sections of parameter space. Rüdiger et al. [24] have examined this possibility for the case Pm = 1. More recently, [25] has fully explored this for the axisymmetric HMRI.

This paper aims to extend these works to include all possible azimuthal magnetic field profiles and the more experimentally appropriate parameter Pm = 10^-6 respectively for nonaxisymmetric instability. We derive a set of perturbation equations that incorporate the azimuthal magnetic field configuration in terms of the ratio of current run through the core and fluid, and discuss the effect this has on onset of instability, both in general and in conjunction with changes to the rotation profile. Furthermore, we comment on the applicability of the WKB theory results of Kirillov et al., such as the extension of the Liu limits in the linear regime. We show that the choice of magnetic field profile can determine the kind of mode of instability excited, with small changes to the ratio of currents fundamentally changing the flow structure. Finally, we examine the effects that axial field strength has on stability, and show evidence of the appearance of higher azimuthal mode-numbers for axially dominant magnetic fields as suggested by Kirillov et al. [20].

**MATHEMATICAL SETTING**

We consider cylindrical Taylor-Couette flow with radii r_i and r_o and angular velocity Ω_e and Ω_o for the inner and outer cylinders, respectively. In order to allow any results to be examined experimentally, we take the inner and outer radii to be 4 and 8 cm, respectively, to match the geometry of the PROMISE experiment [14], giving a relative gap width r_i/r_o = 0.5.

Now, let us include a helical basic state magnetic field with components in the azimuthal and axial directions such that the axial field is generated externally and the azimuthal field by running a combination of currents in the core and the fluid itself. We consider a number of possibilities for the current configuration, including running current in solely the core or fluid and running current in both the core and fluid in the same, or opposite, direction.

The system of equations we derive reduces to a onedimensional generalised eigenvalue problem, however the inherent difficulty arises from the number of control parameters that govern the setup. We have freedom in choosing the angular velocity of both the inner and outer boundaries; Ω_i and Ω_o, the relative strength of the axial magnetic field; B_z, and the relative strength of the components of the azimuthal field produced by the current in the core and the fluid; I_in and I_fl, respectively.

### Governing equations

We consider the basic state of the system, driven by the differential rotation of the two cylinders, as well as the background magnetic field. This gives a basic state velocity \( \textbf{U}_0 = (0, \Omega_i \phi_0, 0) \), with the standard Taylor-Couette angular velocity profile \( \Omega_i(r) \). Here the parameter \( \mu_\Omega \), the rotation ratio, prescribes the steepness of the rotation profile and is given by \( \mu_\Omega = \Omega_o/\Omega_i \).

The basic state magnetic field \( \textbf{B}_0 = (0, B_\phi, B_z) \), where \( B_z \) is the axial magnetic field and \( B_\phi \) is the azimuthal field produced by running currents in both the inner cylinder \( I_{in} \) and the fluid itself \( I_{fl} \), respectively.

\[
B_\phi = \frac{\mu_\Omega}{2\pi} \left[ \frac{1}{r} I_{in} + \frac{r^2 - r_i^2}{r^2 - r_o^2} I_{fl} \right].
\]

The nondimensionalizations of length, time and \( U \) are all straightforward: Lengths are scaled with \( r_i \), basic state velocity with \( \Omega_i r_i \), time on the viscous diffusive timescale \( r_i^2/\nu \), and perturbation velocity with \( \eta/r_i \). Due to the range of combinations of the azimuthal magnetic fields we wish to examine, scaling the magnetic field is nontrivial: Nondimensionalizations must be valid for zero current in either the fluid or core, as well as equal and opposite currents. The best choice is ultimately to scale both basic state and perturbation magnetic fields with \( B \), the rms mean of \( B_\phi \).

Restricting attention to \( r_i/r_o = 0.5 \), this is given by

\[
\bar{B} = \frac{\mu_\Omega}{2\pi \tau} I_{in} \left( \frac{2}{3} \right)^{1/2} \left[ \ln 2 \left( 1 - \frac{1}{3} \tau \right) + \tau + \frac{1}{12} \tau^2 \right]^{1/2},
\]

where \( \tau = I_{fl}/I_{in} \) is the ratio of currents flowing within the fluid and the inner core. The nondimensional equivalent of the field profile (1) then becomes
\[ B_\phi = \left( \frac{3}{2} \right)^{1/2} \left( \frac{1}{r} - \frac{1}{3} \frac{1}{r^2} + \frac{r}{2} \right), \]

Note that this profile is well defined for all choices of \( \tau \), including \( \tau = 0 \) (corresponding to currents only in the inner core) and \( \tau = \pm \infty \) (corresponding to currents only in the fluid). The new nondimensional axial field is simply given the name \( B_z = B_\delta \).

To summarize, the basic state whose stability is studied in (6)–(9) consists of the usual Couette flow profile \( \Omega(r) = c_1 + c_2/r^2 \), with \( c_1 \) and \( c_2 \) determined by \( \mu_1 \), together with magnetic fields determined by \( \tau \) and \( \delta \).

\[
\begin{align*}
\gamma(C_2 e + C_3 f) + C_4 e + C_5 f &= ReE_1 + ReF_1 + Ha^2G_1 + Ha^2H_1, \quad (6) \\
\gamma(C_2 e + C_4 f) + C_5 e + C_6 f &= ReE_2 + ReF_2 + Ha^2G_2 + Ha^2H_2, \quad (7) \\
Pm\gamma(C_1 g + C_2 h) + C_3 e + C_4 f &= E_3 + F_3 + RmG_3 + RmH_3, \quad (8) \\
Pm\gamma(C_2 g + C_3 h) + C_4 e + C_5 f &= E_4 + F_4 + RmG_4 + RmH_4. \quad (9)
\end{align*}
\]

where the coefficients are mostly the same as those given by Hollerbach et al. [18], with the differences being,

\[
\begin{align*}
E4 &= i\Delta \left( \frac{mB_\phi}{r} + k\delta \right) e, \quad (10) \\
F3 &= i\Delta \left( \frac{mB_\phi}{r} + k\delta \right) f, \quad (11) \\
F4 &= ik \left( \Delta B'_\phi - \frac{B_\phi(\Delta - \Delta)}{r} - \frac{2km\delta}{r^2} \right) f, \quad (12) \\
G1 &= i\Delta \left( \frac{mB_\phi}{r} + k\delta \right) g, \quad (13) \\
G2 &= -i\Delta \left( \frac{B_\phi}{r} + \frac{km\delta}{r^2} \right) g, \quad (14) \\
H1 &= -ik \left( \Delta B''_\phi + \frac{B_\phi(\Delta - \Delta) + 2km\delta}{r} \right) h, \quad (15) \\
H2 &= \frac{im\Delta}{r} \left( B''_\phi + \frac{B'_\phi}{r} - \frac{B_\phi}{r^2} \right) h + \frac{4imB_\phi k^2}{r^3} h + i \left( \frac{mB_\phi}{r} + k\delta \right) C_4 h. \quad (16)
\end{align*}
\]

Here, as in [18] we use the notation \( \Delta = m^2/r^2 + k^2 \) and \( \Delta = 4m^2/r^2 + 2k^2 \). The nondimensional numbers are the Reynolds number \( Re = \Omega r_i^2/\nu \), the Hartmann number \( Ha = Br_i/\sqrt{\mu_0 \rho_0 \nu} \), and the magnetic Prandtl number \( Pm = \nu/\eta \). Note that each of the new coefficients contains the nondimensional quantities \( B_\phi \) and \( \delta \), which describe the basic state magnetic field.

We linearise the governing equations about the basic state, expressing the perturbation flow and magnetic field, \( \mathbf{u} \) and \( \mathbf{b} \), as a toroidal-poloidal decomposition to satisfy the \( \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0 \) conditions,

\[
\begin{align*}
\mathbf{u} &= \nabla \times (e\mathbf{e}_r) + \nabla \times \nabla \times (f\mathbf{e}_r), \quad (4) \\
\mathbf{b} &= \nabla \times (g\mathbf{e}_r) + \nabla \times \nabla \times (h\mathbf{e}_r). \quad (5)
\end{align*}
\]

We then expand in terms of normal modes, \( e = e(r)\exp(\gamma t + im\phi + ikz) \), etc., such that \( \gamma \) is the (complex) growth rate and \( m \) and \( k \) are the azimuthal and axial wave numbers, respectively. Taking the \( r \)-components of the curl and double curl of the Navier-Stokes equations, as well as the induction equation and its curl, we obtain the following set of eigenvalue equations:

\[
\begin{align*}
e &= f = \frac{\partial f}{\partial r} = 0, \quad (17) \\
and \text{insulating for the magnetic field, i.e. } J = \nabla \times B = 0 \text{ for } r < r_i \text{ and } r > r_o. \text{ This yields} \\
\Delta g - \frac{2km}{r_i} h = 0, \quad (18)
\end{align*}
\]

and

\[
\begin{align*}
m &= \frac{m}{r_i} g = 0, \quad (19) \\
k \left( h' + \frac{h}{r_i} \right) - \Delta \frac{I_m(kr_i)}{I'_m(kr_i)} h = 0. \quad (20)
\end{align*}
\]

at the inner boundary. For the outer boundary \( r_o \), replace \( r_i \) with \( r_o \) and \( I_m \) with \( K_m \) respectively. In Fig. 1 we also use perfectly conducting boundary conditions, such that \( B_r = \partial B_\phi/\partial r + B_\phi/r = 0 \), giving

\[
\begin{align*}
k g' + \frac{ikg}{r} + \frac{imh''}{r} - \frac{imh}{r^2} = 0. \quad (22)
\end{align*}
\]

We solve (6)–(9) by taking radial expansions of \( e, f, g \) and \( h \) in terms of Chebyshev polynomials, of degree up to \( N = 30 - 80 \) depending on the parameter regime, before collocating at \( N \) Chebyshev-Gauss-Lobatto nodes. After including the boundary conditions, this gives a
\((4N + 10) \times (4N + 10)\) matrix eigenvalue problem with eigenvalues \(\gamma\). We then optimise to obtain the most unstable axial wavenumber \(k\). The resulting code has been benchmarked against previous results from both [18, 24], as well as another independently written code.

**RESULTS**

\[
P_m = 1 - \text{A destabilisation of Chandrasekhar's equipartition solution}
\]

We first consider the case where \(P_m = 1\). In their paper, Rüdiger et al. [24] imposed perfectly conducting boundary conditions, and examined a number of different parameter ranges for the AMRI at this value of \(P_m\). One instance used a flat rotation law \(\mu_0 = 0.5\), as well as the nearly uniform azimuthal field corresponding to \(\tau = 1\). Rüdiger et al. claimed that for this value of \(P_m\) there is a stable separating region between the AMRI and Taylor instabilities where the differential rotation stabilises the Taylor instability. When run over the same \(R_e\) and \(H_a\) range, we recover the results of Rüdiger et al., however upon considering a larger range for \(R_e\) and \(H_a\) a relatively weak region of instability occurs between the two (Fig. 1). This feature is made more apparent when considering insulating boundary conditions, where the weak instability appears for values of \(R_e\) and \(H_a\) inside the range Rüdiger et al. examined. It has recently been established by Kirillov et al. [20] that this weak instability is due to the destabilisation of Chandrasekhar’s equipartition solution, and it is shown via the analytic WKB solution that the dissipation-induced instability develops into the AMRI as \(R_e \to 0\). This has since been confirmed numerically by Rüdiger et al. [26]. Chandrasekhar’s original investigation [27] showed that when a flow of velocity \(U\) is aligned with the magnetic field of Alfvén velocity \(U_{Alf}, U = U_{Alf}\), the flow is stable. This was shown to be destabilised [28] if \(U = \beta U_{Alf}\) with some constant \(\beta \neq 1\), and has since shown to be unstable even for \(\beta = 1\), albeit not as readily, by considering dissipation. \(\beta\), which determines the relative strength of the velocity and Alfvén velocities, can be worked out in our non-dimensionalization to be \(\beta = \text{Re}\sqrt{\frac{P_m}{H_a}}\). Here, taking the quasi-galactic rotation rate \(\mu_0 = 0.5\) we note that setting \(\tau = 1\) corresponds to the magnetic field steepness \(\mu_B = \frac{B_\phi(r_1)}{B_\phi(r_2)} = 1\) (in Rüdiger’s notation), resulting in the Chandrasekhar line \(2\mu_0 = \mu_B\). This has radially aligned profiles for \(U\) and \(U_{Alf}\), and thus for \(P_m = 1\) is the equipartition \(R_e = H_a\) as seen in Fig. 1. This destabilisation is marked in different in both its location in parameter space and its growth rate when comparing perfectly conducting and insulating boundaries and adds nicely to the cases of \(\mu_0 = 0.25\) and \(\mu_0 = 0.35\) examined by Rüdiger et al. [26], as it shows clearly the comparative weakness of the instability on the equipartition line.

\[
P_m = 10^{-6} - \text{Purely azimuthal magnetic fields i.e.} \delta = 0
\]

**Changing \(\tau\)**

We now focus solely on the near-inductionless limit, \(P_m = 10^{-6}\), a value typical of liquid metals. We first consider the effect of changing the parameter \(\tau\), corresponding to the ratio of currents running though the fluid and core, when there is no axial field \(\delta\) applied. We recall that all values of \(\tau\) are valid, with \(\tau = 0\) corresponding to current free in the fluid, and \(\tau = \pm \infty\) corresponding to current solely in the fluid, and focus on the range \(-3 < \tau < 3\). Whereas both Rüdiger et al. [24] and Kirillov et al. [20] have considered varying magnetic field profiles in this manner, a range such as this has not been examined. In particular, the negative values of \(\tau\) such as \(\tau < -1\) have not been included when expressing the magnetic field profile in terms of \(\mu_B\) as in [24], or the magnetic Rossby number as in [20].

We find that, for \(\tau > 0\) at least, increasing \(\tau\) makes the AMRI more unstable to both \(R_e\) and \(H_a\) (Fig. 2). The \(H_a\) threshold of instability of the Taylor instability is also affected, with more positive values of \(\tau\) allowing for the Taylor instability to be excited at lower values of \(H_a\), before reaching a limit as \(\tau \to \infty\). This is not surprising, as it is well known that the Taylor instability is at its most unstable when the azimuthal field is produced solely by current in the fluid [5, 29]. The onset of the Taylor instability can be seen to rapidly approach its fully current-driven limit for values of \(\tau\) as low as \(\tau = 5\), with a value \(\tau = 100\) giving an excellent approximation for solely current-driven instability.

Further interesting behaviour is seen when we allow \(\tau\) to be negative, i.e. running current through the core and fluid in different directions (Fig. 3). Keeping in mind the radial magnetic field profile of \(B_\theta\), we may expect only subtle changes in the AMRI, much like for positive \(\tau\) values. However, it is clear from taking negative values of \(\tau\) that this is not the case at all: Between the values of \(\tau = -1\) and \(0\) the AMRI is in fact stabilised, an effect that can be seen more greatly at \(\mu_0\) further above the Rayleigh limit.

By looking at the \(m = 1\) eigenmodes for \(\tau = -1\) and \(\tau = -0.2\) (Fig. 4), we can see that the two modes are quite different in physical structure. In fact, it can be shown that there exists two separate branches of AMRI modes that are more unstable for different values of \(\tau\). The apparent stabilisation of the AMRI around \(\tau = -0.6\) can be seen to be an exchange between these modes as to which is most unstable. This phenomenon can be seen more clearly at lower values of \(R_e\) when considering helical magnetic fields.

It should be noted that for more negative values of \(\tau\) we see the reappearance of the Taylor instability, and as the field becomes predominantly current-driven again
FIG. 1. (Color online) Weak instability of Chandrasekhar’s equipartition between AMRI and TI for Pm=1. Note that in each plot the upper left region of instability is the AMRI and the lower right region is the TI. The contours are taken as the growth rate $\gamma$. Positive growth rates are plotted, with the thickest line indicating the case of marginal stability. Here $\mu_\Omega = 0.5, \tau = 1, m = 1$ and $\delta = 0$, giving a purely azimuthal field. Left: insulating boundary conditions. Right: perfectly conducting boundary conditions. This compares directly with Fig. 7 of [24], where only the range $Ha \leq 400$, and $Re \leq 500$ is considered.

FIG. 2. (Color online) Instability curves for purely azimuthal magnetic fields at $Pm = 10^{-6}$, where $\mu_\Omega = 0.26, \delta = 0, m = 1$, with contours taken as $\log_{10}\gamma$, showing the effect of changing positive $\tau$: a) $\tau = 0$, b) $\tau = 0.4$, and c) $\tau = 100$. We can see that increasing $\tau$ causes the AMRI to be more unstable and means that the Taylor instability occurs at lower values of $Ha$.

we see the line of marginal stability approaching its fully current-driven limit. For moderately negative $\tau$ the Taylor instability is much more difficult to excite in the Hartmann parameter than the corresponding positive $\tau$ value, likely due to unfavourable alignment of field and flow.

Changing $\mu_\Omega$

As seen in [24], the rotation profile $\mu_\Omega$ can have a large effect on the stability of the AMRI and Taylor instabilities. Rüdiger et al. examined cases with the steep rotation law $\mu_\Omega = 0$, and flat rotation laws such as $\mu_\Omega = 0.5$. Hollerbach et al. [18] focused attention on the Rayleigh limit up to $\mu_\Omega = 0.35$, showing the way in which $\mu_\Omega$ increases the critical Reynolds number to unachievable experimental levels, but only for the case where $\tau = 0$. We expand upon this to include $\mu_\Omega$ upwards of the Rayleigh limit for the non-zero $\tau$. We recall that it is well known that increasing $\mu_\Omega$ makes the AMRI more stable, however as noted by Kirillov et al. [20], carefully changing the field profile $\tau$ can allow for instability even at large values of $\mu_\Omega$, though one might expect the Reynolds number required for its onset to be experimentally unachievable.

Much like the weak instability of Chandrasekhar’s equipartition solution found when benchmarking at $Pm = 1$, it is likely that a similarly weak instability occurs at the inductionless limit. Here in conjunction with $\tau, \mu_\Omega$ determines the existence of the stable equipartition region between the two instabilities, with lower values of $\mu_\Omega$ giving a smooth transition between the AMRI and Taylor instability. In Fig. 5, which is near a point in parameter space on the Chandrasekhar line that intersects
FIG. 3. (Color online) Instability curves for purely azimuthal magnetic fields at $Pm = 10^{-6}$, with contours taken for $\log_{10} \gamma$, showing the effect of changing negative $\tau$, where $\mu_\Omega = 0.26$, $\delta = 0$, $m = 1$. (a) $\tau = -1.4$, (b) $\tau = -1$, (c) $\tau = -0.6$, and (d) $\tau = -0.2$. Here we can see stabilisation of the AMRI around $\tau = -0.6$, whereas further decreasing $\tau$ again makes the AMRI more unstable in Re and Ha. As $|\tau|$ becomes larger the Taylor instability becomes more unstable in Ha, but at values different from that of positive $\tau$.

with the Liu limit connecting curve given in [20], we vary $\mu_\Omega$ while keeping $\tau$ constant. We see evidence of the way in which Chandrasekhar’s weak equipartition instability and corresponding stable separating region change by altering the value $\beta$, with the onset of instability becoming less attainable as $\beta$ decreases. Figure 5 also alludes to the validity of Kirillov et al.’s connecting curve in the linear regime through the behaviour of the AMRI, where the slight increases in $\mu_\Omega$ does indeed bring the regime closer to the stable region Kirillov et al. identified. It can be seen that the AMRI does become stable to linear instability as this region is entered. Note that the values $R_o$ and $R_b$ used by Kirillov et al. may be expressed in terms of $\mu_\Omega$ and $\tau$ using $\mu_\Omega = (r_i/r_o)^{2\mu_\Omega}$ and $\tau = (r_i/r_o)^{2\mu_\Omega-2} - 1$. Thus, the Keplerian value of $R_o = -0.75$ corresponds to $\mu_\Omega \approx 0.354$.

Through further computations at various values of $\mu_\Omega$ and $\tau$ we may confirm, for at least a small section, the validity of Kirillov et al.’s [20] connecting curve between the upper and lower Liu limits. It can be seen that, much like in fig. 5, for different values of $\mu_\Omega > 0.25$ that adjusting $\tau$ heavily affects the stability of the AMRI, with an earlier onset of stability, (in Re and Ha), as $\tau$ increases away from the stable region. For $\tau$ close to the connecting curve, as well as higher values of $\mu_\Omega$, the Re instability threshold increases rapidly, meaning that even though by increasing $\tau$ it is theoretically possible to attain the AMRI for very flat rotation profiles, the Reynolds number required eliminates the possibility of reproducing this experimentally. Finally we note that our linear stability analysis is only able to confirm a small segment of Kirillov et al.’s curve, between $R_o = -1$ and $R_o \approx 1$. We do however show strong agreement with their WKB analysis for this region.

Helical magnetic fields – $\delta \neq 0$

Having fully examined the sensible parameter ranges for purely azimuthal fields, we now add an axial field to the system with relative strength given by $\delta$. Much of the previous discussion involving the AMRI in the presence of purely azimuthal magnetic fields still holds: $\tau$ and $\mu_\Omega$
understandably have the same effects on stability, and
the results regarding Kirillov et al.’s connecting curve
remain largely the same. The main additions here stem
from separation of $\pm m$ due to the innate handedness of
the helical field and the stabilising or destabilising effect
of the relative strength of the poloidal field. This axial
field also allows for a resonance-like appearance of higher
$\delta$ modes, as suggested by Kirillov et al. [20]. Note that
though we may take $\delta$ to be negative, it is unnecessary as
this is equivalent to positive $\delta$ with an oppositely signed
azimuthal wave number $m$.

Changing $\delta$

The relative strength of the azimuthal field in the sys-

tem has a large effect on the instabilities that exist. With
a very weak axial component the magnetic field is still
azimuthally dominated, producing the Taylor instability
and nonaxisymmetric helical MRI with a particularly
similar curve of marginal stability to that of a purely
azimuthal field. Progressively stronger axial fields however
work to either stabilise or destabilise (depending on the
handedness of the flow) the Taylor instability and MRI
to nonaxisymmetric instabilities. Axial fields that far
outweigh the azimuthal fields simply return the SMRI,
which is far more stable to differential rotation [13].

For $m = 1$ modes, increasing $\delta$ from zero shows this
transition from weak to strongly poloidal field clearly,
with obvious stabilisation of the Taylor instability at even
the relatively weak $\delta = 0.05$ (Fig. 6). Even when ac-
counting for the inherent destabilisation gained by gener-
ating the azimuthal field through current in the fluid,
increasing $\delta$ by small values has a marked effect, making
the nonaxisymmetric helical MRI more stable to both
increasing rotation speed and total field strength. By
$\delta = 0.5$, the instability is almost completely stabilised up
to $Re = 10^5$, which is much greater than that attainable
experimentally.

If we now consider the more interesting $m = -1$ modes,
an analog to the stabilisation of the AMRI in purely
azimuthal fields can be observed by examining the values $\tau$
between $-1$ and 0 (Fig. 7). It appears as though
the instability switches between two of its most unstable
modes, which is seen more clearly at greater $|\delta|$. Due
to the continuous nature of the transition between the
AMRI and nonaxisymmetric helical MRI, we may assume
that this is the same effect as seen in Fig. 3.

We can see that these modes are physically very differ-
ent (see fig. 8), with the left-hand mode being localised
to the inner boundary and of much smaller axial wave-
length, suggesting some kind of boundary mode, whereas
the right-hand mode spreads over the full gap width, indi-
cating a global mode.

Now, let us consider the $m = -1$ equivalent of Fig.
6. By increasing $\delta$ in the range $[0,0.5]$ we see that,
whereas the $m = 1$ modes would have been stabilised,
the $m = -1$ modes are in fact more unstable, most likely due
to favourable alignment of the flow and field directions,
(Fig. 9). The onset of instability can be seen at the low-
est values of $Ha$ and $Re$ at $\delta \approx 0.3$, after which increasing
the magnitude of $\delta$ again acts to stabilise the system, as
one would expect. We can even see the Taylor instability,
usually only excited by predominantly toroidal magnetic
fields, to be present at such values of $\delta$. The profound
symmetry-breaking between the $m = \pm 1$ modes as $\delta$ is
increased can be seen in Fig. 10.

This destabilisation by the axial field can be seen more
clearly when considering specific $Re$ (or $Ha$) and plotting
the marginal stability in $Ha$ (or $Re$) vs $\delta$. Here, moti-
vated by the assertion by Kirillov et al. that higher modes
appear predominantly in regimes with a much stronger
axial than azimuthal field, we examine the stability of
$m = -1,-2$ and $-3$ modes in both $Re$ and $Ha$ as the ax-
ial field strength is increased. We look for evidence of a
banded structure, as given by Kirillov et al., though due

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FIG. 5. (Color online) The $Pm = 10^{-6}$ analog of the weak instability seen at $Pm = 1$, and its dependence on $\mu_3$, where $\tau = 0.5$, $m = 1$, $\delta = 0$, and a) $\mu_3 = 0.3975$, b) $\mu_3 = 0.4$, c) $\mu_3 = 0.4025$. The AMRI can clearly be seen to be present in this parameter regime, however as $\mu_3$ is increased, greater Reynolds numbers are needed for its onset. Increasing $\mu_3$ further such that the parameters approach the connecting curve that Kirillov identified would cause the AMRI to become completely stable.
there is an obvious stabilisation of the Taylor instability as well as the nonaxisymmetric helical MRI. 

there is little difference between the helical and toroidal magnetic fields, however as the field becomes more axially dominated 

\[ \mu \text{ stabilising effect of the poloidal field at } \tau = 0 \]

FIG. 7. (Color online) Instability curves at \( Pm = 10^{-6} \), with contours taken for \( \log_{10} \gamma \), showing the stabilising effect of the poloidal field at \( \mu_0 = 0.26, \tau = 0.8, m = 1 \). a) \( \delta = 0.01 \) b) \( \delta = 0.05 \) and c) \( \delta = 0.5 \). With a small \( \delta \) there is little difference between the helical and toroidal magnetic fields, however as the field becomes more axially dominated there is an obvious stabilisation of the Taylor instability as well as the nonaxisymmetric helical MRI.

First considering \( Re = 10^4 \), we note the existence of more unstable modes for \( m = -2 \) and \( m = -3 \) (Fig. 11), and that these are highly dependent on the value of \( \tau \), much like the banded structure changed depending on \( Rb \). For negative values of \( \tau \) it is clear that the \( m = -2 \) and \( m = -3 \) modes are more unstable (to \( Ha \)) than the \( m = -1 \) modes at moderate values of \( \delta \), on the upper and lower branches between \( \delta = 0.1 \) and \( \delta = 1 \). For \( \tau = 0 \), we may note that the upper branch of the \( m = -2 \) mode is the most unstable (to \( Ha \)) at similar values of \( \delta \). It is at larger values of \( \tau \), (positive due to the field alignment), that we see the higher \( m \) modes occurring at much larger values of \( \delta \), as predicted. The reliance of the existence of higher \( m \) modes on \( \tau \) can be justified by referring to Kirillov et al.’s proposed banded structure for the \( m \) modes, with the value of \( Rb \) (or \( \tau \)) changing the ‘width’ in \( \tau \) of each band. Indeed, for the negative values of \( \tau \) given here the banded nature of the mode number \( m \) can be seen in the \( m = -1, 0, -2, -1, \ldots \) pattern given by Kirillov et al. (if one removes the \( m = 0 \) HMRI mode).

The same overall dependences can be seen for the Taylor instability when taking \( Re = 1 \).

When considering the Reynolds stability as \( \delta \) is varied, we note that, aside from a small range around \( \tau = -1 \), it is the \( m = -1 \) mode that is most unstable for all values \( \delta \). This is interesting in that it suggests that, while more unstable \( m = -2, -3 \) modes do appear at higher \( \delta \) in a resonant fashion, they are more unstable to increases in the magnetic field strength, not rotation rate. This suggests that increasing the magnetic field strength drives the higher modes comparatively more than \( m = -1 \), whereas the rotation favours no mode in particular.
FIG. 8. (Color online) Plot of the two different eigenmodes from fig. 7, showing the azimuthal velocity in a meridional cross-section with Re = 10^4 and (left) Ha = 10^3, (right) Ha = 10^2.3.

CONCLUSION

In the present work we have fully explored the relevant parameter space for the nonaxisymmetric MRI variants, highlighting a number of instability results as well as confirming others in the linear regime. We have shown clearly the large effect that the steepness of the magnetic field profile has on the onset of stability for both the MRI and Tayler instability, with fields generated by running current in the fluid allowing for earlier onset of marginal instability, and the effect this has on the flow structure of the non-axisymmetric helical MRI modes, showing in fig. 8 the two different types of modes that may be excited. We have added to previous work concerning Chandrasekhar’s equipartition, showing clearly the comparative weakness in the growth of this instability as well as giving an alternative example. Further to this, we have shown evidence of the location of sections of Kirillov et al.’s connecting curve between the Liu limits in the linear regime, and have shown the existence of the higher m modes previously predicted for axially dominated fields, offering confirmation to the validity of their WKB analysis in the linear regime, as well as the ‘banded’ nature of the modes.

We identify a number of possible avenues for further work. In particular, for the two different modes structures present in Fig. 8, the question remains as to why each of the two modes is dominant for either side of $\tau \approx -0.6$, and what significance, if any, the value $\tau \approx -0.6$, (at which both modes are stabilised in a purely toroidal field), holds. A full asymptotic analysis of our perturbation equations around this parameter range may prove to be fruitful in exploring this question. The regime $\mu_\Omega > 1$ also seems to yield some interesting results [30, 31], and may warrant further investigation.

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FIG. 10. (Color online) Contour plots at $Pm = 10^{-6}$ showing the symmetry breaking for $m = \pm 1$ modes as $\delta$ is increased ($\mu_\Omega = 0.26, \tau = 1$). a) $\delta = 0$, b) $\delta = 0.3, m = -1$ and c) $\delta = 0.3, m = 1$. One can see that whereas the $m = 1$ mode stabilises, the $m = -1$ mode becomes more unstable when a weak axial field is added.

FIG. 11. (Color online) Marginal stability at $Pm = 10^{-6}$ and $Re = 10^4$ for various $\tau$. a) $\tau = -100$, b) $\tau = -3$, c) $\tau = -0.75$, d) $\tau = -0.5$, e) $\tau = 0$ and f) $\tau = 100$. Plain curves correspond to $m = -1$, dashed to $m = -2$ and dot-dashed to $m = -3$. Here we can see the effect on the higher $m$ modes that $\tau$ has. We notice that for $-3 < \tau < 0$ there exists a more unstable $m = -2$ mode at moderate values of $\delta$, as well as a more unstable $m = -3$ upper branch. For negative values of $\tau$, getting closer to $\tau = 0$ allows for the higher $m$ modes are lesser values of $\delta$, and then for positive $\tau$ the higher $m$ modes are stable for all but relatively high values of $\delta$ for large $\tau$.

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