This is a repository copy of *Robust control of uncertain multi-inventory systems via Linear Matrix Inequality*.

White Rose Research Online URL for this paper:
http://eprints.whiterose.ac.uk/89762/

Version: Accepted Version

Article:
Bauso, D., Giarré, L. and Pesenti, R. (2010) Robust control of uncertain multi-inventory systems via Linear Matrix Inequality. International Journal of Control, 83 (8). pp. 1727-1740. ISSN 0020-7179

https://doi.org/10.1080/00207179.2010.491131

Reuse
Unless indicated otherwise, fulltext items are protected by copyright with all rights reserved. The copyright exception in section 29 of the Copyright, Designs and Patents Act 1988 allows the making of a single copy solely for the purpose of non-commercial research or private study within the limits of fair dealing. The publisher or other rights-holder may allow further reproduction and re-use of this version - refer to the White Rose Research Online record for this item. Where records identify the publisher as the copyright holder, users can verify any specific terms of use on the publisher's website.

Takedown
If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

eprints@whiterose.ac.uk
https://eprints.whiterose.ac.uk/

arXiv:0710.4765v1 [math.OC] 25 Oct 2007

Robust control of uncertain multi-inventory systems via Linear Matrix Inequality

D. Bauso, [a] L. Giarré [b] and R. Pesenti [c,1]

[a] *Dipartimento di Ingegneria Informatica*
Università di Palermo, Viale Delle Scienze, 90128 Palermo, Italy.
dario.bauso@unipa.it

[b] *Dipartimento di Ingegneria dell'Automazione e dei Sistemi*
Università di Palermo, Viale Delle Scienze, 90128 Palermo, Italy.
giarre@unipa.it

[c] *Dipartimento di Matematica Applicata - Università "Ca' Foscari" di Venezia,*
Dorsoduro 3825/e, 30123 Venezia, Italy.
pesenti@unive.it

Abstract

We consider a continuous time linear multi–inventory system with unknown demands bounded within ellipsoids and controls bounded within ellipsoids or polytopes. We address the problem of ε-stabilizing the inventory since this implies some reduction of the inventory costs. The main results are certain conditions under which ε-stabilizability is possible through a saturated linear state feedback control. All the results are based on a Linear Matrix Inequalities (LMIs) approach and on some recent techniques for the modeling and analysis of polytopic systems with saturations.

Key words: Impulse Control, Inventory Control, Hybrid Systems

1 Introduction

We consider a continuous time linear multi–inventory system with unknown demands bounded within ellipsoids and controls bounded within ellipsoids or polytopes. The system is modelled as a first order one integrating the discrepancy between controls and demands at different sites (buffers). Thus, the state represents the buffer levels. We wish to study conditions under

[1] Corresponding author R. Pesenti, Research supported by PRIN "Advanced control and identification techniques for innovative applications", and PRIN "Analysis, optimization, and coordination of logistic and production systems".

which the state can be driven within an a-priori chosen target set through a saturated linear state feedback control. Let ε be a maximal dimension of the target set, the above problem corresponds to ε-stabilizing the state.

Motivations for ε-stabilizing the state derive from the benefits associated to keeping the state and consequently also the inventory costs bounded. This work is in line with some recent literature on robust optimization [1,5] and control [2] of inventory systems. Here as well as in [2] we focus on saturated linear state feedback controls since such controls arise naturally in any system with bounded controls.

The main results of this work can be summarized as follows. Initially we introduce the necessary and sufficient conditions for the ε-stabilizability in the form of an inclusion between convex sets. In the case where both demands and controls are bounded within polytopes, it is well known that verifying such conditions is NP-hard [10]. Here, we prove that verification becomes easy when both demands and controls are bounded within ellipsoids (we will refer to it as the *ellipsoidal case*). This is possible by rewriting the inclusion between ellipsoids in terms of unconstrained quadratic maximization.

For the ellipsoidal case, we first characterize invariant sets through a fourth degree condition. As verifying such a condition is difficult, we then propose the best quadratic approximation of the same condition. We proceed by describing the region of linearity of the control and conclude by providing LMI conditions on the target set under which the saturated control ε-stabilizes the system. The case where demands are bounded within ellipsoids and controls are bounded within polytopes (we will refer to it as the *polytopic case*) is an open problem and we propose certain sufficient LMI conditions to solve it.

All the results are based on a Linear Matrix Inequalities (LMIs) approach in line with the recent work [6] on inventory/manufacturing systems. In particular, when addressing the politopic case, we use the same technique provided in [9] to rewrite the model with saturations in polytopic form. Once we do this, we can apply the LMI analysis covered in the book [7] for polytopic systems.

This paper is arranged as follows. In Section 2, we formulate the problem. In Section 3, we introduce necessary and sufficient conditions for the admissibility of the problem. In Sections 4 and 5 we study the problem with ellipsoidal and polytopic constraints respectively. Finally, in Section 6, we draw some conclusions.

2 Problem Formulation

Consider the continuous time linear multi–inventory system

$$\dot{x}(t) = Bu(t) - w(t), \tag{1}$$

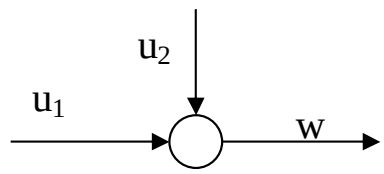

Fig. 1. Graph with one node and two arcs.

where $x(t) \in \mathbb{R}^n$ is a vector whose components are the buffer levels, $u(t) \in \mathbb{R}^m$ is the controlled flow vector, $B \in \mathbb{Q}^{n\times m}$, with $m \geq n$ and $rank(B) = n$ is the controlled process matrix and $w(t) \in \mathbb{R}^n$ is the unknown demand. To model backlog $x(t)$ may be less than zero. Demands are bounded within ellipsoids, i.e.,

$$w(t) \in \mathcal{W} = \{w \in \mathbb{R}^n : w^T R_w w \leq 1\}. \tag{2}$$

In a first case, in the following referred as *ellipsoidal case*, controls are bounded within ellipsoids,

$$u(t) \in \mathcal{U} = \{u \in \mathbb{R}^m : u^T R_u u \leq 1\}. \tag{3}$$

In a second case, in the following referred as *polytopic case*, controls are bounded within polytopes

$$u(t) \in \mathcal{U} = \{u \in \mathbb{R}^m : u^- \leq u \leq u^+\} \tag{4}$$

with assigned u^+, u^-.

For any positive definite matrix $P \in \mathbb{R}^{n\times n}$, define the function $V(x) = x^T P x$ and the ellipsoidal target set $\Pi = \{x \in \mathbb{R}^n : V(x) \leq 1\}$. In addition, for any matrix $K \in \mathbb{R}^{n\times n}$, define as saturated linear state feedback control any policy

$$u = -sat\{Kx\} = \begin{cases} -Kx & \text{if } Kx \in \mathcal{U} \\ u(x) \in \partial\mathcal{U} & \text{otherwise} \end{cases} \tag{5}$$

where hereafter ∂F indicates the frontier of a given set F.

Problem 1 *(ε-stabilizing) Consider a system (1) in the ellipsoidal or polytopic case. Find conditions on the positive definite matrix $P \in \mathbb{R}^{n\times n}$, under which there exists a saturated linear state feedback control $u = -sat\{Kx\}$ such that it is possible to drive the state $x(t)$ within the target set Π.*

Solving the above problem corresponds to ε-stabilizing the state x within Π.

Example 1 *Throughout this paper we consider, as illustrative example, the graph with one node and two arcs depicted in Fig. 1. The incidence matrix is $B = [1 \quad 1]$. The continuous time dynamics is*

$$\dot{x}(t) = \underbrace{[1 \quad 1]}_{B} \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{u} - w = u_1(t) + u_2(t) - w(t),$$

with demand bounded in the ellipsoid

$$w^2 \leq 1$$

and with the following either ellipsoidal or polytopic constraints on the control u

$$(u_1 + u_2)^2 \leq 1, \tag{6}$$

$$-2 \leq u_1 \leq 3, \; -2 \leq u_2 \leq 1. \tag{7}$$

Finally, the target set is the sphere of unitary radius $\Pi = \{x \in \mathbb{R} : x^2 \leq 1\}$.

3 Stability necessary and sufficient conditions

System (1) is ε-stabilizable if and only if for all $w \in \mathcal{W}$, there exists $u \in int\{\mathcal{U}\}$ such that $Bu = w$ (see, e.g., [3]). For the short of notation, the previous condition is usually expressed as

$$B\mathcal{U} \supset \mathcal{W}. \tag{8}$$

Deciding whether (8) holds is NP-hard, when $\mathcal{U}$ and $\mathcal{W}$ are polytopes. Here, we prove that verifying (8) becomes easy when both $\mathcal{U}$ and $\mathcal{W}$ are ellipsoids. Observe that we can rewrite $Bu = w$ as $u_{\mathcal{B}} = \mathcal{B}^{-1}w - \mathcal{B}^{-1}Nu_N$, where $B = [\mathcal{B}|N]$ being $\mathcal{B}$ a basis of B and N the remaining columns of B, correspondingly $u_{\mathcal{B}}$ are the n components of u associated to the basis $\mathcal{B}$ and u_N are the $m - n$ components of u associated to the columns in N.

As we observe that (8) is equivalent to

$$\max_{w \in \mathcal{W}} \min_{u \in \mathbb{R}^m : Bu = w} uR_u u < 1,$$

Condition (8) holds if and only if

$$\begin{aligned} &\max_{w \in \mathcal{W}} \min_{u_N \in \mathbb{R}^{m-n}} f(u_{\mathcal{B}}(w, u_N), u_N) = \\ &\qquad = \left[w^T \mathcal{B}^{-T} - u_N^T N^T \mathcal{B}^{-T} | u_N^T\right] R_u \begin{bmatrix} \mathcal{B}^{-1}w - \mathcal{B}^{-1}Nu_N \\ u_N \end{bmatrix} < 1 \end{aligned} \tag{9}$$

When we consider the illustrative example in Section 1, we have $\mathcal{B} = [1]$, $N = [1]$ then problem (9) becomes

$$\begin{aligned} &\max_{-1 \leq w \leq 1} \min_{u_2 \in \mathbb{R}} f(u_{\mathcal{B}}(w, u_2), u_2) = \\ &\qquad = [w - u_2 | u_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w - u_2 \\ u_2 \end{bmatrix} = (w - u_2)^2 + u_2^2 < 1 \end{aligned} \tag{10}$$

Now consider, function $f(u_{\mathcal{B}}(w, u_N), u_N)$. It is a differentiable convex function in u_N. Then, for any $w \in \mathcal{W}$ we can analytically determine the best response $u_N^*(d) = arg \min_{u_N \in \mathbb{R}^{m-n}} f(u_{\mathcal{B}}(w, u_N), u_N)$,

by imposing

$$\nabla_{u_N} f(u_{\mathcal{B}}(w,u_N),u_N) = 2\left[-N^T\mathcal{B}^{-T}|I\right] R_u \begin{bmatrix} \mathcal{B}^{-1}w - \mathcal{B}^{-1}Nu_N \\ u_N \end{bmatrix} = 0,$$

where I is the $(m-n)\times(m-n)$ identical matrix. We obtain

$$u_N^*(w) = -\underbrace{\left(\left[-N^T\mathcal{B}^{-T}|I\right] R_u \begin{bmatrix} -\mathcal{B}^{-1}N \\ I \end{bmatrix}\right)^{-1} \left[-N^T\mathcal{B}^{-T}|I\right] R_u \begin{bmatrix} \mathcal{B}^{-1} \\ 0 \end{bmatrix}}_{M} w = -Mw,$$

where 0 is the $(m-n)\times n$ null matrix. In the example under consideration, we have

$$u_2^*(w) = -\left([-1|1]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)^{-1} [-1|1]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} w = \frac{w}{2}.$$

For any $w \in \mathcal{W}$ the minimal value of $f(u_{\mathcal{B}}(w,u_N),u_N)$ is

$$f(u_{\mathcal{B}}(w,u_N^*(w)),u_N^*(w)) = w^{*T}\Phi w^*,$$

where

$$\Phi = \underbrace{[\mathcal{B}^{-T} + M^T N^T \mathcal{B}^{-T} | -M^T]}_{H^T} R_u \underbrace{\begin{bmatrix} \mathcal{B}^{-1} + \mathcal{B}^{-1}NM \\ -M \end{bmatrix}}_{H} = H^T R_u H \tag{11}$$

is a positive definite $n\times n$ matrix, as M is full rank. So far, we have shown that we can find the optimal value of problem (9) by solving problem

$$\max_{w\in\mathcal{W}} = w^T\Phi w, \tag{12}$$

and checking that the optimal value is less than one.

We are ready to observe that problem (12) is easy as it reduces to determining the eigenvectors of an $n\times n$ matrix.

Theorem 1 *System (1) is ε-stabilizable if and only if $w^{*T}\Phi w^* < 1$, for all w^* eigenvectors associated to the maximum eigenvalue of matrix $R_w^{-1}\Phi$ whose weighted quadratic norm $w^{*T}R_w w^*$ is equal to 1.*

Proof. As $w^T\Phi w$ is convex, its optimal value w^* lays on the frontier $\partial\mathcal{W}$ of the set $\mathcal{W}$, i.e., for $w^{*T}R_w w^* = 1$. Imposing the Karush Kuhn Tucker first order optimality condition, we obtain $2(\Phi - \lambda R_w)w^* = 0$. Then the optimal values of w^* are some of the matrix $R_w^{-1}\Phi$ eigenvectors whose weighted quadratic norm $w^{*T}R_w w^*$ is equal to 1. In particular, w^* are the eigenvectors associated to the maximal eigenvalues of $R_w^{-1}\Phi$.

□

In the example under consideration $\Phi = \left[\frac{1}{2}\right]$ and $w^* = \pm 1$ then $w^{*T}\Phi w^* = \frac{1}{2} < 1$, hence the associated system is ε-stabilizable.

In the following we discuss for which initial state the system is certainly ε-stabilizable through a (pure) linear state feedback control; hence we show that if we saturated the previous linear policy the system is ε-stabilizable for any initial state.

4 Ellipsoidal constraints

Let us start by considering only the constraints (2) on w and neglect the ellipsoidal constraints (3) on u. Among the saturated linear state feedback control (5) we prove that we can solve Problem 1 using controls of type $u = sat\{-kHx\}$, with $k \in \mathbb{R}$ and $H \in \mathbb{R}^n$ as defined in (11). Note that matrix H is a right inverse of B, that is $BH = I$. We motivate the choice of $u = -sat\{kHx\}$ with H as defined in (11) as such a control describes the best response of u under the worst w as proved in the previous section. Also, note that the scalar $k \in \mathbb{R}$ must be lower than a certain value, which means that we cannot use a bang-bang control. This is motivated by the following reason. If we use a control $u = sat\{-kHx\}$, then the necessary and sufficient condition (8) becomes

$$B\mathcal{U}_{lin} \supset \mathcal{W} \tag{13}$$

where

$$\mathcal{U}_{lin} = \{u \in \mathbb{R}^m : \, u = -kHx, \, k^2 x^T H^T R_u H x \leq 1\}.$$

Following the derivation of (12) in the previous Section, we have that (13) holds if and only if

$$k^2 w^{*T} \Phi w^* < 1.$$

For $k = 1$ the above condition holds true as it reduces to (12). Obviously, the value $\hat{k} = \sqrt{\frac{1}{w^{*T}\Phi w^*}}$ is an upper bound for k, namely, we must choose k such that $k < \hat{k}$ if we wish the necessary and sufficient condition (13) be satisfied.

With the above considerations in mind, we can conclude that the dimensions of the target Π where it is possible to drive the state are lower bounded.

Denote by $\lambda_{max}(Z)$ the maximum eigenvalue of a given matrix Z. In the following theorem we prove that $\dot{V}(x) < 0$ within a given set (*invariant* set). This result will allow exploiting $V(x)$ as a Lyapunov function to prove the convergence to the target set Π.

Theorem 2 *Consider system (1) subject to the only ellipsoidal constraints (2) on w, and controlled via linear state feedback $u = -kHx$, with H such that $BH = I$. Then condition $\dot{V} < 0$ holds if and only if*

$$k^2 (x^T P x)^2 - x^T P R_w^{-1} P x > 0. \tag{14}$$

Proof. For H such that $BH = I$, condition $\dot{V} < 0$ is equivalent to

$$2kx^T Px + 2w^T Px > 0. \tag{15}$$

We aim at proving that $\dot{V} < 0$ holds for any x external to an appropriate smooth closed surface. To do this, we look for an $x \in \mathbb{R}^n$ inducing a solution strictly greater than zero for the following problem

$$\min_{w \in \mathcal{W}} \zeta(x, w) = 2kx^T Px + 2w^T Px. \tag{16}$$

As $\zeta(x, w)$ is linear in w, the optimal w^* must lay on the boundary of set $\mathcal{W}$. The Karush Kuhn Tucker conditions impose that $Px = -\lambda R_w w^*$ for some $\lambda \geq 0$, that is $w^* = -\frac{1}{\lambda} R_w^{-1} Px$. Note that being P full rank, it necessarily holts that$\lambda \neq 0$ for all $x \neq 0$. Then, $\zeta(x, w^*) = 2kx^T Px - \frac{2}{\lambda} x^T P R_w^{-1} Px > 0$. As w^* lays on the boundary of $\mathcal{W}$, we have $w^{*T} R_w w^* = \frac{x^T P R_w^{-1} Px}{\lambda^2} = 1$ from which $\lambda = \sqrt{x^T P R_w^{-1} Px}$. Hence, $\zeta(x, w^*) > 0$, and therefore also (15) holds, if and only if (14) holds.

□

We now exploit $V(x) = x^T Px$ as a Lyapunov function to prove the convergence to the target set Π. We determine under which conditions on P and k we have that $\dot{V} < 0$ or, equivalently, inequality (14) hold for any $x \notin \Pi$.

When $P = \nu R_w$, (14) becomes $k^2 x^T Px > \nu$. Then, in this case, we can use $V(x)$ to prove the convergence of the system to Π for $k^2 \geq \nu$.

In the following, we consider the general case when $P \neq \nu R_w$.

Lemma 1 *Consider system (1) subject to the only ellipsoidal constraints (2) on w, and controlled via linear state feedback $u = -kHx$, with H such that $BH = I$. Then, $k^2(x^T Px)^2 - x^T P R_w^{-1} Px > 0$ holds for any $x \notin \Pi$ if and only if $k^2 - x^T P R_w^{-1} Px \geq 0$ holds for any $x \in \partial\Pi$.*

Proof. (*Necessity*). Assume that there exists $\hat{x} \in \partial\Pi$ such that $k^2 - x^T P R_w^{-1} Px < 0$. Then, there also exists a ball $Ball(\hat{x}, r)$ centered in $\hat{x}$ with a sufficiently small radius $r > 0$ such that for all $x \in Ball(\hat{x}, r)$ we have $k^2 - x^T P R_w^{-1} Px < 0$. This implies that there exist $x \notin \Pi$ for which condition (14) does not hold.

(*Sufficiency*). Assume that $k^2 - x^T P R_w^{-1} Px \geq 0$ holds for any $x \in \partial\Pi$. By contradiction, consider $\hat{x} \notin \Pi$, i.e., $\hat{x}^T P \hat{x} = \rho > 1$, such that $k^2(\hat{x}^T P \hat{x})^2 - \hat{x}^T P R_w^{-1} P \hat{x} < 0$, that is $k^2 \rho^2 - \hat{x}^T P R_w^{-1} P \hat{x} < 0$. Then, there exists $\tilde{x} = \frac{\hat{x}}{\sqrt{\rho}} \in \partial\Pi$ such that $k^2 \rho^2 - \rho \tilde{x}^T P R_w^{-1} P \tilde{x} < 0$, that is $k^2 \rho - \tilde{x}^T P R_w^{-1} P \tilde{x} < 0$. This latter result is contradictory as we cannot have $k^2 \rho < \tilde{x}^T P R_w^{-1} P \tilde{x} \leq k^2$, for $\rho > 1$.

□

Lemma 2 *Consider system (1) subject to the only ellipsoidal constraints (2) on w, and controlled via linear state feedback $u = -kHx$, with H such that $BH = I$. We can use $V(x)$ to prove the convergence of the system to Π for $k^2 \geq \lambda_{max}(R_w^{-1} P)$.*

Proof. Condition $k^2 - x^T P R_w^{-1} Px \geq 0$ holds for any $x \in \partial\Pi$ if and only if $\min_{x \in \partial\Pi}\{k^2 -$

$x^T P R_w^{-1} P x\} \geq 0$. Imposing the Karush Kuhn Tucker first order optimality condition, we obtain $2(PR_w^{-1}P - \lambda P)x^* = 0$. Then the optimal values of x^* are some of the matrix $R_w^{-1}P$ eigenvectors whose weighted quadratic norm $x^{*T}Px^*$ is equal to 1. In particular, x^* are the eigenvectors associated to the maximal eigenvalues of $R_w^{-1}P$. For vectors x^*, condition $k^2 - x^{*T}PR_w^{-1}Px^* \geq 0$ becomes $k^2 - \lambda_{max}(R_w^{-1}P)x^{*T}Px^* \geq 0$, that is $k^2 - \lambda_{max}(R_w^{-1}P) \geq 0$. □

Observe that the system converges to the target set $\Pi_R = \{x : k^2 x^T R_w x \leq 1\}$ as any feasible target set $\Pi = \{x : x^T P x \leq 1\}$, with $k^2 \geq \lambda_{max}(R_w^{-1}P)$ includes Π_R. Indeed, $\Pi \supseteq \Pi_R$ if $x^T P x - k^2 x^T R_w x = x^T(P - k^2 R_w)x \leq 0$ or equivalently if $P - k^2 R_w \preceq 0$. In turn, the latter condition is equivalent to $R_w^{-1}P - k^2 I \preceq 0$ that certainly holds as $k^2 \geq \lambda_{max}(R_w^{-1}P)$

In the next theorem we introduce the constraints on controls (3). To this end, we need to define the family of ellipsoid $\Sigma_0(\xi) = \{x \in \mathbb{R}^n : x^T P x \leq x(0)^T P x(0) := \xi\}$ parametrized in $\xi \geq 1$.

Theorem 3 *Given system (1) in the ellipsoidal case, we can drive the state $x(t)$ from any initial value $x(0) \in \Sigma_0(\xi)$ to the target set Π via linear state feedback $u = -kHx$ if the following conditions hold*

$$k^2 \geq \lambda_{max}(R_w^{-1}P) \tag{17}$$

$$k^2 \xi \lambda_{max}(P^{-1}\Phi) \leq 1. \tag{18}$$

Proof. By Lemma 2, under condition (17) it holds $\dot{V}(t) < 0$ for all $x(t) \notin \Pi$ and then $V(x)$ can be considered as a Lyapunov function for the convergence of the state to the set Π when the linear control $u = -kHx$ is implemented. Condition $\dot{V}(t) < 0$ also implies that $\Sigma_0(\xi)$ is invariant with respect to the same linear feedback as $\xi \geq 1$ which means $\Sigma_0(\xi) \supseteq \Pi$. Then

$$\max_{t \geq 0} u^T(t) R_u u(t) \leq \max_{x \in \Sigma_0(\xi)} k^2 x^T H^T R_u H x = \max_{x \in \Sigma_0(\xi)} k^2 x^T \Phi x = k^2 \xi \lambda_{max}(P^{-1}\Phi).$$

Therefore the constraint $u = -kHx(t) \in \mathcal{U}$ for all $t \geq 0$ is enforced if (18) holds true. □

The following theorem provides a solution to Problem 1. Let us denote by X the set of states x where we can define a linear control $u(x) = -kHx$, i.e., $X = \{x : -kHx \in \mathcal{U}\}$. Consider the saturated linear state feedback control of type

$$u(x) = \begin{cases} -kHx & \text{if } x \in X \\ -\frac{Hx}{\sqrt{x^T H^T R_u H x}} & \text{if } x \notin X \end{cases}. \tag{19}$$

Theorem 4 *Consider a system (1) in the ellipsoidal case. For any positive definite matrix $P \in \mathbb{R}^{n \times n}$ satisfying condition (17), the saturated linear state feedback control (19) drives the state $x(t)$ within the target set Π for any initial state $x(0)$.*

Proof. By construction, $u(x)$ is a continuous function with $\mathcal{U}$ as codmain. When we use such a control, we know that $\dot{V}(x) < 0$ also holds for any $x \notin \Pi$, if $\Pi \subset X$ and $k^2 \geq \lambda_{max}(R^{-1}P)$

(see Lemma 2).

First observe that, for all $x \in \partial X$, we have $x^T P x > k^2 x^T H^T R_u H x = 1$, where the latter inequality holds as $\Pi \subset X$. Then, for any $x \notin X$, that is for $k^2 x^T H^T R_u H x > 1$, we have $\frac{x^T P x}{x^T H^T R_u H x} > k^2 \geq \lambda_{max}(R^{-1}P)$ since both $x^T P x$ and $x^T H^T R_u H x$ are positive definite quadratic forms.

In Lemma 2, we have proved that $\dot{V}(x) < 0$ for $x \in X \setminus \Pi$. Now, we consider $x \notin X$. We have $\dot{V}(x) < 0$ if and only if $-x^T P B u(x) + x^T P w > 0$, for all $w \in \mathcal{W}$, that is

$$\min_{w \in \mathcal{W}} \left\{ \frac{x^T P x}{\sqrt{x^T H^T R_u H x}} + x^T P w \right\} > 0 \tag{20}$$

must hold. Applying the Karush-Kuhn-Tuker conditions, we transform (20) in $\frac{x^T P x}{\sqrt{x^T H^T R_u H x}} - \sqrt{x^T P^T R_w^{-1} P x} > 0$. In turn, the latter inequality holds if $\frac{x^T P x}{x^T H^T R_u H x} - \lambda_{max}(R^{-1}P) > 0$, as $x^T P^T R_w^{-1} P x \leq \lambda_{max}(R^{-1}P) x^T P x$. We then conclude that $\dot{V}(x) < 0$ since $\frac{x^T P x}{x^T H^T R_u H x} > k^2 \geq \lambda_{max}(R^{-1}P)$.

□

Observe that the saturated linear state feedback control (19) is not decentralized in the sense that the generic ith control u_i in general depends on the demand at different nodes and on the other controls u_j, $j \neq i$. This is due to either the structure of matrix H or the ellipsoidal constraints (3).

Remark 1 *Consider the two equivalent matrix inequalities on P and $Q = P^{-1}$,*

$$(2k-1)P - P R_w^{-1} P \geq 0, \quad (2k-1)Q - R_w^{-1} \leq 0. \tag{21}$$

Trivially, any P satisfying condition (17) also satisfies the two above matrix inequalities.

Matrix inequalities of the above form will be used in the following sections.

Example 2 *Consider the graph depicted in Fig. 1, with one node and two arcs and incidence matrix $B = [1 \quad 1]$. Controls are subject to ellipsoidal constraints (6). Then we have, $R_w = 1$, $R_u = I$ and $\Phi = \frac{1}{2}$. We can stabilize the system within $\Pi = \{x \in \mathbb{R} : x^2 \leq 1\}$ for any initial state $x(0) \leq \sqrt{2}$ via a pure linear state feedback $u = -kHx$. To see this take $Q = I$, and observe that the matrix inequality on Q (21) is satisfied for any $k \geq 1$. Furthermore, if we assume $k = 1$, then from (18) we must have $k^2 = 1 \leq \frac{2}{\xi^2} = \frac{2}{x(0)^2}$.*

5 Polytopic constraints

Controls u are subject to the polytopic constraints (4). Again, we study under which conditions we can solve Problem 1 using controls of type $u = -sat\{kHx\}$, with $k \in \mathbb{R}$ and $H \in \mathbb{R}^n$

such that $BH = I$. In this case, we interpret the $sat\{.\}$ operator as componentwise. More specifically, we choose the control

$$u_i = sat_{[u_i^-, u_i^+]}\{-kH_{i\bullet}x\}, \tag{22}$$

with H such that $BH = I$, $H_{i\bullet}$ denoting the ith row of H and where, for any given scalar a and b

$$sat_{[a,b]}\{\zeta\} = \begin{cases} b, & \text{if } \zeta > b, \\ \zeta, & \text{if } a \leq \zeta \leq b, \\ a, & \text{if } \zeta < a. \end{cases}$$

Henceforth we omit the indices of the sat function.

Under the control $u = sat\{-kHx\}$, the closed loop dynamics becomes

$$\dot{x} = Bsat\{-kHx\} - w. \tag{23}$$

Our idea is to rewrite the above dynamics in the following polytopic form

$$\dot{x} = A(t)x(t) - w(t), \quad w(t)^T R_w w(t) \leq 1, \tag{24}$$

where the time varying matrices $A(t)$ are expressed as convex combinations of 2^m matrices A_j, $j = 1, \ldots, 2^m$. More precisely the expressions for $A(t)$ are

$$A(t) = \sum_{j=1}^{2^m} \sigma_j(t) A_j, \quad \sum_{j=1}^{2^m} \sigma_j(t) = 1. \tag{25}$$

The procedure to compute matrices A_j's is borrowed from [9] and recalled below. Let us rewrite the control policy as

$$u_i = sat\{-kH_{i\bullet}x\} = \theta_i(x)(-kH_{i\bullet}x),$$

where $\theta_i(x)$ are the "degree of saturation" of the control components defined as follows

$$\theta_i(x) = \begin{cases} \frac{u_i^-}{-kH_{i\bullet}x} & \text{if } -kH_{i\bullet}x < u_i^- \\ 1 & \text{if } u_i^- \leq -kH_{i\bullet}x \leq u_i^+ \\ \frac{u_i^+}{-kH_{i\bullet}x} & \text{if } -kH_{i\bullet}x > u^+ \end{cases}. \tag{26}$$

Let $\underline{\theta} = [\underline{\theta}_1, \ldots, \underline{\theta}_m]$ be a vector whose components $\underline{\theta}_i$ are such that $0 \leq \underline{\theta}_i \leq 1$ and represent lower bounds of $\theta_i(x(t))$, for $t \geq 0$. Lower bounds depend on $x(0)$ and can be computed as $\underline{\theta}_i = \min_{x \in \Sigma_0(\xi)} \theta_i(x)$ where we remind the definition of $\Sigma_0(\xi) = \{x \in \mathbb{R}^n : x^T P x \leq x(0)^T P x(0) := \xi\}$. Also define the vector $\psi^\theta = [\psi_1^\theta, \ldots, \psi_m^\theta]$ with $\psi_i^\theta = \frac{1}{\underline{\theta}_i}$ and the associated portion of the state space

$$S(\psi^\theta) = \{x \in \mathbb{R}^n : -\psi^\theta \leq -kHx \leq \psi^\theta\}.$$

According to the above definition of the θ_is we derive that $S(\psi^\theta) \supseteq \Sigma_0(\xi)$. Note that we can affirm that $\underline{\theta}_i$ are lower bounds because the state trajectory never exits $S(\psi^\theta)$ as we will show in the proof of Theorem 5.

Consider now the 2^m vectors $\gamma_j \in \{1, \underline{\theta}_1\} \times \ldots \times \{1, \underline{\theta}_m\}$, with $j = 1, \ldots, 2^m$. In other words, γ_j is an m component vector with ith component γ_{ji} taking value 1 or $\underline{\theta}_i$. Then, each matrix A_j can be expressed as $A_j = -Bkdiag(\gamma_j)H$. Roughly speaking each vector γ_j stores the minimum and or maximum degree of saturation of all control components. Also, note that matrices A_js induce a partition of $S(\psi^\theta)$ into regions X_j, with $j = 1 \ldots, 2^m$. Each region is defined as the set of state values such that the control components are saturated with degree of saturation equal to γ_{ji}, namely

$$X_j = \{x \in \mathbb{R}^n : \theta_i(x) = \gamma_{ji},\ i = 1, \ldots, m\}.$$

We remind here that γ_{ji} is the ith component of γ_j.

To complete the derivation of the polytopic form (24) it is left to be noted that given any $x(t) \in S(\psi^\theta)$ we can compute the associated degree of saturation from (26) and derive the weights $\sigma_j(t)$ of the convex combination (25). All the results in the rest of this section try to give an answer to Problem 1 with respect to the polytopic system (24). For each A_j, let us define a matrix

$$M_j = QA_j^T + A_jQ + \alpha Q + \frac{1}{\alpha}R_w^{-1}$$

for a given positive and arbitrarily chosen scalar α and let (λ_j^r, v_j^r) with $r \in \{1, \ldots, n\}$ be the negative eigenvalues and corresponding eigenvectors of M_j.

Theorem 5 *Consider system (1) in the polytopic case. The saturated linear state feedback control (22) drives the state $x(t)$ within the target set Π if*

$$X_j \subseteq Span\{v_j^r\}, \quad \textit{for all } j = 1, \ldots, 2^n. \tag{27}$$

Proof. First of all, note that if (27) holds true then $\Sigma_0(\xi)$ is invariant. Consequently, as $\Sigma_0(\xi) \subseteq S(\psi^\theta)$ and by definition $x(0) \in \Sigma_0(\xi)$, we also have that the state trajectory $x(t)$ will never exit $S(\psi^\theta)$. Now, we must show that $\dot{V}(x) < 0$ for all x and w such that $x \notin \Pi$, $u \in \mathcal{U}$ and $w \in \mathcal{W}$. In formulas, we must have

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T Px + x^T P\dot{x} = [A(t)x - w]^T Px + x^T P[A(t)x - w] \ = \\ &= x^T A(t)^T Px + x^T PA(t)x - w^T Px - x^T Pw < 0 \end{aligned} \tag{28}$$

for all x and w satisfying

$$1 - x^T Px \leq 0 \tag{29}$$

$$w^T R_w w - 1 \leq 0. \tag{30}$$

Using the $\mathcal{S}$-procedure, we can say that condition (28) is implied by conditions (29)-(30) if

there exist $\alpha, \beta \geq 0$, such that for all x and w

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A(t)^T P + PA(t)^T + \alpha P & -P \\ -P & -\beta R_w \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} - \alpha + \beta \leq 0. \tag{31}$$

Trivially it must hold $\beta \leq \alpha$. Assume without loss of generality $\beta = \alpha$. Remind that α and β can be chosen arbitrarily. After pre and post-multiplying by $Q = P^{-1}$, the above condition becomes

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} QA(t)^T + A(t)^T Q + \alpha Q & -I \\ -I & -\alpha R_w \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0. \tag{32}$$

Now, as the state never leaves the region $S(\psi^\theta)$, i.e., $x(t) \in S(\psi^\theta)$, we can always express $A(t)$ as convex combination of the A_js as in (25).

By convexity, the above condition is true if it holds, for all $j = 1, \ldots, 2^n$,

$$\begin{bmatrix} x_{(j)} \\ w_{(s)} \end{bmatrix}^T \begin{bmatrix} QA_j^T + A_j^T Q + \alpha Q & -I \\ -I & -\alpha R_w \end{bmatrix} \begin{bmatrix} x_{(j)} \\ w_{(s)} \end{bmatrix} \leq 0. \tag{33}$$

Using the Shur complement the condition (33) is implied by (27).

□

Stronger conditions are established in the following theorem which also highlights the dependence of M_j on the scalar α.

Theorem 6 *Consider system (1) in the polytopic case. The saturated linear state feedback control (22) drives the state $x(t)$ within the target set Π if there exists a scalar $\alpha \geq 0$ such that*

$$M_j < 0, \quad \textit{for all } j = 1, \ldots, 2^n. \tag{34}$$

Proof. Trivially, if we observe that (34) implies (27).

□

Both (27) and (34) are sufficient, but not necessary, conditions. When they hold, we are sure that the system state converge to a state strictly included in the target set Π. We discuss more on this topic in the next section.

5.1 Approximation error

We wish to estimate the difference in terms of volumes between the target set Π and the target set obtained from conditions (34) and we will call such a difference as *approximation error*.

On this purpose, denote by Q_j the matrix of the smallest (in volume) ellipsoid satisfying $M_j < 0$, which is given by

$$Q_j = \arg\inf_Q \min_\alpha \{det(Q),\ M_j = QA_j^T + A_jQ + \alpha Q + \frac{1}{\alpha}R_w^{-1} < 0\}. \tag{35}$$

To do this, let matrix $\underline{A}$ be the matrix A_j with $j = 1, \ldots, 2^m$ obtained when no controls are saturated and note that the dynamics associated to this single matrix is the same as if we assumed the controls unbounded. To be more precise, $\underline{A} = -BkH$ as all components of γ_j are equal to one. Remind that γ_j stores the degree of saturation of each control component. Also let us define $\underline{Q}$ the solution of (35) for $A_j = \underline{A}$. We do this, as the target set Π within which we can stabilize the state, must inscribe the ellipsoid defined by $\underline{Q}$, i.e.,

$$\Pi \supset \{x \in \mathbb{R}^n :\ x^T\underline{Q}^{-1}x \leq 1\}.$$

Similarly, let matrix $\overline{A}$ be the matrix A_j with $j = 1, \ldots, 2^m$ obtained when all controls are saturated at their lowest degree of saturation. To be more precise, $\overline{A} = -Bkdiag([\underline{\theta}_1, \ldots, \underline{\theta}_m])H$ as all components of γ_j are equal to $\underline{\theta}_i$ for $i = 1, \ldots, m$. If we also define $\overline{Q}$ the solution of (35) for $A_j = \overline{A}$, the target set Π must be inscribed in the ellipsoid defined by $\overline{Q}$, namely,

$$\Pi \subset \{x \in \mathbb{R}^n :\ x^T\overline{Q}^{-1}x \leq 1\}.$$

The approximation error can be measured by the ratio

$$e = \frac{det(\overline{Q}^{-1}) - det(\underline{Q}^{-1})}{det(\underline{Q}^{-1})}.$$

Example 3 *Consider the graph depicted in Fig. 1, with one node and two arcs, incidence matrix* $B = [1 \quad 1]$, *and target set* $\Pi = \{x \in \mathbb{R} : x^2 \leq 1\}$. *Controls are subject to polytopic constraints (7). Take* $H = [\frac{1}{2} \quad \frac{1}{2}]^T$ *and* $k = 1$. *Then according to (26) we have (here* x *is a scalar)*

$$\theta_1(x) = \begin{cases} \frac{2}{x/2} & \text{if } x/2 > 2 \\ 1 & \text{if } -3 \leq x/2 \leq 2 \\ -\frac{3}{x/2} & \text{if } x/2 < -3 \end{cases} \qquad \theta_2(x) = \begin{cases} \frac{2}{x/2} & \text{if } x/2 > 2 \\ 1 & \text{if } -1 \leq x/2 \leq 2 \\ -\frac{1}{x/2} & \text{if } x/2 < -1 \end{cases}.$$

If we consider initial states $x(0)$ *satisfying* $-10 \leq x(0) \leq 10$, *possible lower bounds for the* θ*'s are* $\underline{\theta}_1 = \frac{2}{5}$ *and* $\underline{\theta}_2 = \frac{1}{5}$. *Note that* $S(\psi^\theta) = \{x \in \mathbb{R}^n : -10 \leq x \leq 10\}$. *Vectors* γ*'s and matrices* A*'s turn out to be*

$$\begin{array}{llll} \gamma_1 = [1 \quad 1]^T & \gamma_2 = [0.4 \quad 1]^T & \gamma_3 = [1 \quad 0.2]^T & \gamma_4 = [0.4 \quad 0.2]^T \\ A_1 = -2 & A_2 = -1.4 & A_3 = -1.2 & A_4 = -0.6 \end{array}. \tag{36}$$

Dynamics (24) is then

$$\dot{x} = [-\sigma_1(t)2 - \sigma_2(t)1.4 - \sigma_3(t)1.2 - \sigma_4(t)0.6]x + w, \tag{37}$$

with $\sum_{j=1}^{4} \sigma_j(t) = 1$. *Furthermore, we have*

$$\begin{array}{ll} M_1 = [-4+\alpha]Q + \frac{1}{\alpha} & M_2 = [-2.8+\alpha]Q + \frac{1}{\alpha} \\ M_3 = [-2.4+\alpha]Q + \frac{1}{\alpha} & M_4 = [-1.2+\alpha]Q + \frac{1}{\alpha} \end{array}.$$

To apply Theorem 5 and 6, note that $\underline{A} = A_4$ *and that* $M_4 < 0$ *implies consequently* $M_j < 0$ *for all* j. *The solution of (35), for* $j = 4$ *is* $Q_4 = \underline{Q} = \frac{1}{0.36}$ *and* $\alpha = 0.6$, *then the approximation error is* $e = \frac{1-0.36}{0.36} = 1.78$.

6 Conclusions and future works

We have addressed the problem of ε-stabilizing the inventory of a continuous time linear multi–inventory system with unknown demands bounded within ellipsoids and controls bounded within ellipsoids or polytopes. Motivations are due to the cost reduction associated with a bounded inventory. As main results we have provided certain LMIs conditions under which ε-stabilizability is possible through a saturated linear state feedback control. We have also exploited some recent techniques for the modeling and analysis of polytopic systems with saturations.

This work is a continuation of [2] and is in line with some recent applications of LMI techniques to inventory/manufacturing systems [6]. In a future work, we will study the validity in probability of the LMI conditions derived in this paper. This is in accordance with some recent literature on *chance LMI constraints* developed in the area of robust optimization [4,8].

References

[1] E. Adida, and G. Perakis, "A Robust Optimization Approach to Dynamic Pricing and Inventory Control with no Backorders", *Mathematical Programming*, Ser. B vol. 107, 2006, pp. 97–129.

[2] D. Bauso, F. Blanchini, R. Pesenti, "Robust control policies for multi-inventory systems with average flow constraints", *Automatica*, Special Issue on Optimal Control Applications to Management Sciences, vol. 42, no. 8, pp. 1255-1266, Aug. 2006.

[3] F. Blanchini, F. Rinaldi and W. Ukovich, "A network design problem for a distribution system with uncertain demands", *SIAM Journal on Optimization*, 7 (1997), pp. 560–578.

[4] A. Ben-Tal, A. Nemirovsky, "On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty", *SIAM Journal on Optimization*, vol. 12, pp. 811–833, 2002.

[5] D. Bertsimas, A. Thiele, "A Robust Optimization Approach to Inventory Theory ", *Operations Research*, vol. 54, no. 1, Jan-Feb 2006, pp. 150–168.

[6] E. K. Boukas, "Manufacturing Systems: LMI Approach", *IEEE Transactions on Automatic Control*, vol. 51, no. 6, pp. 1014-1018, June 2006.

[7] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[8] G. Calafiore, M. C. Campi, "Uncertain Convex Programs: Randomized Solutions and Confidence Levels", *Mathematical Programming*, 102, pp. 25–46, 2005.

[9] J. M. Gomes da Silva, Jr. and S. Tarbouriech "Local Stabilization of Discrete-Time Linear Systems with Saturating Controls: An LMI-based Approach", *IEEE Trans. on Automatic Control*, vol. 46, no. 1, pp. 119-124, Jan. 2001.

[10] S.T. McCormick, "Submodular containment is hard, even for networks", *Operations Research Letters*, 19 (1996), pp. 95–99.

[11] J. A. Primbs, M. Giannelli, "Kuhn-Tucker-Based Stability Conditions for Systems With Saturation", *IEEE Transactions on Automatic Control*, vol. 46, no. 10, pp. 1643-1647, Oct. 2001.