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Distributed Consensus in Noncooperative Inventory Games ⋆

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Abstract

This paper deals with repeated nonsymmetric congestion games in which the players cannot observe their payoffs at each stage. Examples of applications come from sharing facilities by multiple users. We show that these games present a unique Pareto optimal Nash equilibrium that dominates all other Nash equilibria and consequently it is also the social optimum among all equilibria, as it minimizes the sum of all the players' costs. We assume that the players adopt a best response strategy. At each stage, they construct their belief concerning others probable behavior, and then, simultaneously make a decision by optimizing their payoff based on their beliefs. Within this context, we provide a *consensus protocol* that allows the convergence of the players' strategies to the Pareto optimal Nash equilibrium. The protocol allows each player to construct its belief by exchanging only some aggregate but sufficient information with a restricted number of neighbor players. Such a networked information structure has the advantages of being scalable to systems with a large number of players and of reducing each player's data exposure to the competitors.

Key words: Game theory; Multi-agent systems, Inventory; Consensus protocols.

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1 Introduction

The main contribution of this paper is the design of a *consensus protocol* (see, e.g., [3,24]) that allows the convergence of strategies to a Pareto optimal Nash equilibrium [2,26] for repeated nonsymmetric congestion games under partial information.

The repeated games considered in this work are congestion games [25] as, at each stage or time period, the payoff (cost) that the ith player pays for playing a strategy is a monotonically nonincreasing function of the total number of players playing the same strategy. The games are also nonsymmetric as the payoffs are player-specific functions [22]. Finally, the games are under partial information as we assume that: each player cannot observe its payoff at each stage since the payments occur only on the long term; each player learns [30], i.e., constructs its belief concerning other players' probable behavior, by exchanging information with a restricted number of *neighbor players*.

A networked information structure has the advantages of being scalable to systems with a large number of players and of reducing each player's data exposure to the competitors. On the other hand, delay in the propagation of information through the network implies that players' strategies cannot converge immediately but only after some stages.

We prove that players can learn using a consensus protocol where the quality of the information exchanged does not force the players to reveal their past decisions (see the *minimal information paradigm* in [11]). In the last part of the work, we also prove that players can use linear predictors to increase the protocol speed of convergence.

Congestion games always admit at least one Nash equilibrium as established by Rosenthal in [25]. However, its efficient computation is a non trivial issue. Then, from a different perspective, we can review our results as a further attempt of providing an algorithm that finds Nash equilibria in polynomial time for special classes of congestion games [19,30,32]. Our results prove also that the Nash equilibrium that we find is the unique Pareto optimal one.

In addition, we prove that, in our problem, the Pareto optimal Nash equilibrium dominates all other Nash equilibria, i.e., it minimizes the cost of each player and therefore it is also social optimal, as it minimizes the social cost defined as the sum of players' costs. This is an important property as it implies that competition does not induce loss of efficiency in the system.

Examples of applications come from situations where multiple players share a service facility as airport facilities or telephone systems, drilling for oil, cooperative farming, and fishing (see also the literature on *cost-sharing games* [29], and on *externality games* [14]).

As motivating example, we consider a multi-retailer inventory application. The players, namely different competing retailers, share a common warehouse (or supplier) and cannot hold any private inventory from stage to stage, i.e., inventory left in excess at one stage is no longer utilizable in the future. The latter fact prevents the retailers from having large replenishment and stocks. Such a situation occurs when dealing with perishable goods as, for instance, the newspapers. Transportation is provided at the retailers' expense by the supplier at every stage, e.g., every day, but the players adjust their transportation payments with the supplier only every once in a while, e.g., once every two months.

Players aim at coordinating joint orders thus to share fixed transportation costs. As typical of repeated games, we assume that the retailers act myopically, that is, at each stage, they choose their best strategy on the basis of a payoff defined on single stage [23]. The reader is referred to [27] for a general introduction to multi-retailer inventory problems with particular emphasis on coordination among non cooperative retailers. Recent more specific examples are [4], [6], [7], [13] and [31]. The role of information is discussed, e.g., in [9] and [10]. The modelling of such problems as non cooperative games is in [1], [17], [20] and [28]. Our idea of selecting the best (Pareto optimal) among several Nash equilibria presents some similarities with [7], which however do not consider the possibility of sharing transportation costs. Alternative ways to achieve coordination proposed in the literature are either to centralize control at the supplier [8] and [18], or to allow side payments [16], and [21].

The rest of the paper is organized as follows. In Section 2, we develop the game theoretic model of the inventory system and formally state the problem. In Section 3, we prove the existence of and characterize the unique Pareto optimal Nash equilibrium. In Section 4 we prove some stability properties of the Pareto optimal Nash equilibrium. In Section 5, we design a distributed protocol that allows the convergence of the strategies to the Pareto optimal Nash equilibrium. In Section 6, we analyze the speed of convergence of the protocol. In Section 7, we introduce a numerical example. Finally, in Section 8, we draw some conclusions.

2 The Inventory Game

We consider a set of n players $\Gamma = \{1, \ldots, n\}$ where each player may exchange information only with a subset of neighbor players. Hereafter, we indicate with the same symbol i both the generic player and the associated index. More formally, we assume that the set Γ induces a single component graph

$G = (\Gamma, E)$ whose edgeset E includes all the non oriented couples (i, j) of players that exchange information with each other. In this context, we define the neighborhood of a player i the set $N_i = \{j : (i, j) \in E\} \cup \{i\}$.

Each player i faces a customer demand and must decide whether to fulfill it or to pay a penalty p_i (see it, for instance, as a missed revenue). Differently, we can review penalty p_i as the cost incurred by the player when, rather than participating in the game, it fulfills the demand by turning to a different supplier. We call *active* player the one who decides to meet the demand. At each stage of the game, the active players receive the items required by their customer from the common warehouse. Transportation occurs only once in a stage/day, and has a total cost equal to K. The transportation cost of each stage will be divided equally by all the players active on that stage, at the moment of adjusting their transportation payments with the supplier, e.g., once every two months. As the costs do not realize immediately, the players, before playing a strategy at a given stage, need to estimate the number of active players, and they do this by exchanging information.

Define the function $s_i(k) \in S_i = \{0, 1\}$ as the strategy of player i, for each player $i \in \Gamma$, at stage k. We indicate $s(k) = \{s_1(k), \ldots, s_n(k)\}$ as the vector of the players' strategies and $s_{-i} = \{s_1(k), \ldots, s_{i-1}(k), s_{i+1}(k), \ldots, s_n(k)\}$ as the vector of strategies of players $j \neq i$. At stage k, $s_i(k)$ is equal to 1 if player i meets the demand and equal to 0 otherwise. Then $s_i(k)$ has the following payoff defined on single stage

$$J_i(s_i(k), s_{-i}(k)) = \frac{K}{1 + \|s_{-i}(k)\|_1} s_i(k) + (1 - s_i(k))p_i, \tag{1}$$

where $\|s_{-i}(k)\|_1$ is trivially equal to the number of active players other than i.

Note that the above game is a nonsymmetric congestion game with only two strategies for each player [22], and admits the exact Rosenthal's potential function [25] defined as $\Phi(s) = \sum_{j=1}^{|A(s)|} \frac{K}{j} + \sum_{i \in N \setminus A(s)} p_i$ where $A(s) = \{i : s_i = 1\}$ is the set of active players.

For the above game, best response strategies are the only ones considered in this paper. In the case of *complete information*, each player i knows the other players' strategies $s_{-i}(k)$ and optimizes repeatedly over stages its payoff (1) choosing as *best response* (see, e.g., [26]) the following threshold strategy

$$s_i(k) = \begin{cases} 1 \text{ if } \|s_{-i}(k)\|_1 \geq l_i \\ 0 \text{ otherwise} \end{cases}, \tag{2}$$

where the threshold l_i is equal to $\frac{K}{p_i} - 1$.

Incomplete information means that player i may only estimate the number $\|s_{-i}(k)\|_1$ of all other active players. Note that, for the players, it is not possible to infer the number of active players from the cost as we assume that players realize their costs only in the long term and not immediately. In the rest of the paper, being $\hat{\chi}_i(k)$ the estimate of $\|s_{-i}(k)\|_1$, the best response strategy (2) slightly modifies as

$$s_i(k) = \begin{cases} 1 \text{ if } \hat{\chi}_i(k) \geq l_i \\ 0 \text{ otherwise} \end{cases}. \tag{3}$$

To compute $\hat{\chi}_i(k)$, at stage k, player i processes two types of public information: *pre-decision information*, $x_i(k)$, received from the neighbor players in N_i, and *post-decision information*, $z_i(k)$, transmitted to the neighbor players. The information evolves according to a *distributed protocol* $\Pi = \{\phi, h_i \; i \in \Gamma\}$ defined by the following dynamic equations:

$$x_i(k+1) = \phi_i(z_j(k) \text{ for all } j \in N_i) \tag{4}$$
$$z_i(k) = h_i(s_i(k), s_i(k-1), x_i(k)), \tag{5}$$

where the functions $\phi_i(\cdot)$ and $h_i(\cdot)$ are to be designed in Section 5.

The protocol must be such that $\hat{\chi}_i(k)$ can be inferred from the converging value of the pre-decision information $x_i(k)$. If this is true, then player i selects its strategy $s_i(k) = \mu_i(x_i(k))$ on the basis of the only pre-decision information.

In the rest of the paper we always refer to (3) as when we consider a best response strategy.

We consider the following problem.

Problem 1 *Given the n-player repeated inventory game with binary strategies $s_i(k) = \{0, 1\}$ and payoffs (1), determine a distributed protocol $\Pi = \{\phi, h_i \; i \in \Gamma\}$ as in (4)-(5) that allows the convergence of strategies (3) to a Pareto optimal Nash equilibrium s^*, if exists.*

Observe that all results presented in the rest of the paper require only that the strategies are binary and have a threshold structure. Therefore the structure of the payoff can be relaxed as long as the best responses maintain a threshold structure as defined in (2)-(3).

Let us show an example in which the game under consideration may arise.

Example 1 *(Newsboy) Consider a newsboy application where, at each stage, and for each newsboy i, the demand D_i is independent identically distributed,* *with probability distribution $d_i(x)$ and expected value constant over time, $E\{D_i\} = \nu_i$. If the newsboy orders a quantity S_i from the warehouse, then he will pay the cost*

$$E\left\{\frac{K}{n_i} + c_iS_i + h_i \max(0, S_i - D_i) + P_i \max(0, D_i - S_i) - \right.$$
$$\left. -B_i \min(D_i, S_i)\right\} = \frac{K}{n_i} + G(c_i, B_i, h_i, P_i, \nu_i) \tag{6}$$

where K is the set up cost, n_i the number of newsboys sharing the set up cost, c_i the purchase cost per unit stock, h_i and P_i respectively the penalty on storage and on shortage, and B_i the reward coefficient. In the right hand side of the above equation, the function $G(c_i, B_i, h_i, P_i, \nu_i)$ is obtained by averaging the cost with respect to the probability distribution $d_i(x)$.
On the other hand, if the newsboy does not order, then he will pay a penalty for not satisfying the demand

$$P_iE\{D_i\} = P_i\nu_i. \tag{7}$$

To choose whether to reorder or not the newsboy compares the costs or reordering (6) with the penalty for not satisfying the demand (7). As a consequence, he will reorder only if the number of newsboys sharing the set up cost n_i verifies $\frac{K}{n_i} \leq P_i\nu_i - G(c_i, B_i, h_i, P_i, \nu_i)$, which is equivalent to

$$n_i \geq \begin{cases} \frac{K}{P_i\nu_i - G(c_i,B_i,h_i,P_i,\nu_i)} & \text{if } P_i\nu_i \geq G(c_i, B_i, h_i, P_i, \nu_i) \\ \infty & \text{if } P_i\nu_i < G(c_i, B_i, h_i, P_i, \nu_i). \end{cases}$$

The right-hand term plays the role of threshold l_i in the inventory game under concern.

3 A Pareto optimal Nash equilibrium

The game under consideration always has a Nash equilibrium because it is a congestion game [25]. In this section we prove that there exists a unique Pareto optimal Nash equilibrium and we describe its characteristics. To this end, here and in the rest of the paper, we make, without loss of generality, the following assumptions:

Assumption 1 *The set Γ of players is ordered so that $l_1 \leq l_2 \leq \ldots \leq l_n$.*

Assumption 2 *There may exist other players $i = n+1, n+2, \ldots$ not included in Γ, all of them with thresholds $l_i = \infty$.*

Assumption 3 *The players in the empty subset of* Γ *have thresholds* $l_i = -\infty$.

The last assumption is obviously artificial, but simplifies the proofs of most results in the rest of the paper. Indeed, such an assumption allows us to prove the theorems without the necessity of introducing different arguments in the case when the set of active players is empty.

3.1 Characterization of Nash equilibria

In a Nash equilibrium $s^\star = \{s_1^\star, \ldots, s_n^\star\}$, each player i selects a strategy $s_i^\star$ such that

$$J_i(s_i^\star, s_{-i}^\star) \leq J_i(s_i, s_{-i}^\star) \quad \text{for all } s_i \in S_i,\, i \in \Gamma. \tag{8}$$

Hence, from (2), we obtain the following equilibrium conditions

$$s_i^\star = \begin{cases} 1 \text{ if } \|s_{-i}^\star\|_1 \geq l_i \\ 0 \text{ otherwise} \end{cases} \quad \text{for all}\, i \in \Gamma. \tag{9}$$

On the basis of (9), we can state the following property of any Nash Equilibrium.

Lemma 1 *If* $s^\star$ *is a Nash equilibrium then:*

i) if player i *is active, namely* $s_i^\star = 1$*, then all the preceding players* $1, \ldots, i-1$ *are also active, i.e.,* $s_1^\star = \ldots = s_{i-1}^\star = 1$*;*
ii) if player i *is not active, namely* $s_i^\star = 0$*, then neither all successive players* $i+1, \ldots, n$ *are active, i.e.,* $s_{i+1}^\star = \ldots = s_n^\star = 0$.

PROOF. To prove item i), we show that the assumption $s_i^\star = 1$ and $s_{i-1}^\star = 0$ are in contradiction. The equilibrium condition (9) and $s_i^\star = 1$ imply $\|s_{-i}\|_1 = \sum_{j\in\Gamma,\, j\neq i} s_j^\star \geq l_i$. Since $s_{i-1}^\star = 0$, the latter inequality is equivalent to

$$\sum_{j\in\Gamma,\, j\neq i,\, j\neq i-1} s_j^\star \geq l_i. \tag{10}$$

However, condition (9) and $s_{i-1}^\star = 0$ also imply $\|s_{-(i-1)}\|_1 = \sum_{j\in\Gamma,\, j\neq i-1} s_j^\star < l_{i-1}$. But this last inequality is in contradiction with (10) since

$$\sum_{j\in\Gamma,\, j\neq i,\, j\neq i-1} s_j^\star \leq \sum_{j\in\Gamma,\, j\neq i-1} s_j^\star.$$

A complementary argument can be used to prove item ii). $\square$

Let us now introduce two definitions.

Definition 1 *A set $C \subseteq \Gamma$ is* compatible *if $l_i \leq |C| - 1$ for all $i \in C$.*

In a compatible set C, each player finds convenient to meet the demand if all other players in C do the same.

Definition 2 *A set $C \subseteq \Gamma$ of cardinality $|C| = r$ is* complete *if it contains all the first r players, with $r \geq 0$, i.e., $C = \{1, \ldots, r\}$.*

Note that $C = \emptyset$ is both a complete and a compatible set.

Theorem 1 *The vector of strategies $s^\star$, defined as*

$$s_i^\star = \begin{cases} 1 \; \textit{if} \quad i \in C \\ 0 \; \textit{otherwise} \end{cases} \tag{11}$$

is a Nash equilibrium if and only if the set $C = \{1, \ldots, r\} \subseteq \Gamma$ is both complete and compatible and the following condition holds

$$l_{r+1} > r. \tag{12}$$

PROOF. *Sufficiency.* Assume that $s^\star$, defined as in (11), is a Nash equilibrium. Observe that if $C = \emptyset$ then it is complete and compatible by definition. Otherwise, C is complete by Lemma 1 and compatible by definition of a Nash equilibrium. Finally, note that if $C = \Gamma$, condition (12) holds since $l_{n+1} = \infty$. Otherwise, condition (12) holds since the player $r + 1 \notin C$ chooses a strategy $s_{r+1}^\star = 0$ that, together with (9), implies $l_{r+1} > \|s_{-(r+1)}^\star\|_1 = r$.
Necessity. Assume that C is complete and compatible and that condition (12) holds. Observe that $J_i(1, s_{-i}^\star) \leq J_i(0, s_{-i}^\star)$ and therefore $s_i^\star = 1$ holds, for all players $i \in C$, since C is compatible. Then note that, since C is complete, all $i \notin C$ are such that $i > r$. From condition (12) we also have $l_i > r$ for all $i > r$. Hence, $J_i(0, s_{-i}^\star) \leq J_i(1, s_{-i}^\star)$ holds and consequently $s_i^\star = 0$ for all players $i \notin C$. $\square$

From Theorem 1 we derive the following corollary.

Corollary 1 *There always exists a Nash equilibrium*

$$s_i^\star = \begin{cases} 1 \; \textit{if} \quad i \in \overline{C} \\ 0 \; \textit{otherwise} \end{cases} \tag{13}$$

where $\overline{C}$ is the maximal compatible set.

PROOF. First observe that the set $\overline{C}$ always exists since it may possibly be the empty set.
With Assumptions 1, 2, and 3 in mind, we show that if $\overline{C}$ is maximal then it is also complete. Assume by contradiction that $\overline{C}$ is not complete. Let player i be in $\overline{C}$ and player $i-1$ be not in $\overline{C}$. As $i \in \overline{C}$, then $l_i < |\overline{C}|$. Since $l_{i-1} \leq l_i$, then $l_{i-1} < |\overline{C}|$ which in turn implies that also $\overline{C} \cup \{i-1\}$ is a compatible set in contradiction with the maximality hypothesis on $\overline{C}$.
Now, assume that $\overline{C}$ is equal to $\{1, \ldots, r\}$. Since $\overline{C}$ is maximal, $\overline{C} \cup \{r+1\}$ is not compatible, so $l_{r+1} > r$, i.e., condition (12). Then, even for $\overline{C} \subset \Gamma$, the hypotheses of Theorem 1 hold true, and the vector of strategies $s^\star$, defined in (13) is a Nash equilibrium. Note that due to Assumptions 2 and 3 the above reasoning applies also to the cases $\overline{C} = \emptyset$ and $\overline{C} = \Gamma$. $\square$

Observe that, if $\overline{C}$ is the maximal compatible set, it trivially holds

$$r = |\overline{C}| = \max_{\lambda} \{\lambda \in \{1, \ldots, n\} : \, l_\lambda < \lambda\} \, . \tag{14}$$

The following example shows that two Nash equilibria may exist, only one associated to the maximal compatible set as defined in (13).

Example 2 *Consider a set Γ of 8 players and assume $K = 60$. Penalties p_i and thresholds l_i are listed in Tab. 1.*

Tab. 1 about here

The maximal compatible set $\overline{C} = \{1, 2, 3, 4, 5, 6\}$ and the associated Nash equilibrium $s^1 = \{1, 1, 1, 1, 1, 1, 0, 0\}$. Pareto optimality is evident as any deviation from this equilibrium is disadvantageous for at least one player. Also we have another complete and compatible set, $C = \{1, 2, 3\}$, that verifies (12) and is therefore associated to a second Nash equilibrium $s^2 = \{1, 1, 1, 0, 0, 0, 0, 0\}$.

3.2 Characterization of the Pareto optimal Nash equilibrium

A vector of strategies $\hat{s} = \{\hat{s}_1, \ldots, \hat{s}_n\}$ is Pareto optimal if there is no other vector of strategies s such that

$$J_i(s_i, \, s_{-i}) \leq J_i(\hat{s}_i, \, \hat{s}_{-i}) \quad \text{for all } \; i \in \Gamma, \tag{15}$$

where the strict inequality is satisfied by at least one player.

We say that an equilibrium $\hat{s} = \{\hat{s}_1, \dots, \hat{s}_n\}$ dominates all the other equilibria if, for all equilibria s, it is such that

$$J_i(s_i,\, s_{-i}) \geq J_i(\hat{s}_i,\, \hat{s}_{-i}), \quad \text{for all } i \in \Gamma. \tag{16}$$

Trivially, if there exists an equilibrium that dominates all other equilibria then it is also *social optimal* as it verifies,

$$\sum_{i \in \Gamma} J_i(s_i,\, s_{-i}) \geq \sum_{i \in \Gamma} J_i(\hat{s}_i,\, \hat{s}_{-i}). \tag{17}$$

In other words, a social optimal Nash equilibrium is the one among all the Nash equilibria that minimizes the social cost defined as the sum of all the players' costs. However, note there may exist vectors of strategies different from equilibria that induce a smaller cost for the same social cost.

Theorem 2 *Let $s^\star$ be the Nash equilibrium associated to the maximal compatible set $\overline{C}$. If $p_i \neq \frac{K}{|\overline{C}|}$ for all $i \in \overline{C}$, then*

- Pareto optimality. *The Nash equilibrium $s^\star$ is Pareto optimal;*
- Uniqueness. *The Nash equilibrium $s^\star$ is the unique Pareto optimal Nash equilibrium.*
- Social optimality. *The Nash equilibrium $s^\star$ is social optimal.*

PROOF. *Pareto optimality.* We show that Nash equilibrium $s^\star$ is Pareto optimal since any other vector of strategies s induces a worse payoff for at least one player. In the Nash equilibrium $s^\star$, each $i \in \overline{C}$ gets a payoff $J_i(1, s^\star_{-i}) = \frac{K}{|\overline{C}|} < p_i$, each $i \notin \overline{C}$ gets a payoff $J_i(0, s^\star_{-i}) = p_i < \frac{K}{|\overline{C}|}$. Now, consider the vector of strategies s. Define $D = \{i \in \overline{C} : s_i = 0\}$ as the set of players with $l_i < |\overline{C}|$ that are not active in s and $E = \{i \notin \overline{C} : s_i = 1\}$ as the set of players with $l_i \geq |\overline{C}|$ that are active in s. Trivially, $D \cup E \neq \emptyset$ as $s \neq s^\star$. We deal with $E \neq \emptyset$ and $E = \emptyset$ separately.
If $E \neq \emptyset$ and $D = \emptyset$, each $i \in E$ gets a payoff $J_i(1, s_{-i}) = \frac{K}{|\overline{C} \cup E|}$ strictly greater than $J_i(0, s^\star_{-i}) = p_i$ as $\overline{C}$ is the maximal compatible sets. The latter condition trivially holds also when $D \neq \emptyset$ since, in this case, each player $i \in E$ incurs in a higher payoff $J_i(1, s_{-i}) = \frac{K}{|(\overline{C} \cup E) \setminus D|}$.
If $E = \emptyset$, then $D \neq \emptyset$, and each $i \in \overline{C} \setminus D$, if exists, gets a payoff $J_i(1, s_{-i}) = \frac{K}{|\overline{C} \setminus D|} > J_i(0, s^\star_{-i}) = \frac{K}{|\overline{C}|}$. At the same time, each $i \in D$ gets a payoff $J_i(0, s_{-i}) = p_i > J_i(1, s^\star_{-i}) = \frac{K}{|\overline{C}|}$. Finally, each $i \in \Gamma \setminus \overline{C}$ gets a payoff $J_i(0, s_{-i}) = p_i = J_i(0, s^\star_{-i})$.
Uniqueness and social optimality. We prove the uniqueness and the social optimality of the Pareto optimal Nash Equilibrium by showing that it dominates all the other equilibria. Consider a generic Nash equilibrium s associated to a complete and compatible set C different form $\overline{C}$. Since $\overline{C}$ is maximal then $C \subset$

$\overline{C}$. Then, each $i \in C$, if exists, gets a payoff $J_i(s_i, s_{-i}) = \frac{K}{|C|} > J_i(s_i^\star, s_{-i}^\star) = \frac{K}{|\overline{C}|}$; analogously, each $i \in \overline{C} \setminus C$ gets a payoff $J_i(s_i, s_{-i}) = p_i > J_i(s_i^\star, s_{-i}^\star) = \frac{K}{|\overline{C}|}$; finally, each player $i \in \Gamma \setminus \overline{C}$, gets a payoff $J_i(s_i, s_{-i}) = p_i = J_i(s_i^\star, s_{-i}^\star)$. Then, in any generic Nash equilibrium each player has a payoffs not better than the ones associated to $s^\star$. $\square$

Observe that if and only if $p_i = \frac{K}{|\overline{C}|}$ for all i, there exists two Pareto optimal Nash equilibria with equal payoff. They are associated respectively to the maximal compatible set $\overline{C}$ and to the empty set. In the rest of the paper, only the equilibrium $s^\star$ associated to the maximal compatible set $\overline{C}$ will be called the *Pareto optimal Nash equilibrium*.

4 Stability of Nash equilibria

In this section, we assume that at each stage k of the repeated game, each player i knows the number of active players at the previous stage, sets $\hat{\chi}_i(k) = \|s_{-i}(k-1)\|_1$ and applies the best response strategy (3). In this context, we prove that the Pareto optimal Nash equilibrium $s^\star$ is stable with respect to its neighborhood of strategies s such that $s \geq s^\star$ componentwise. On the basis of this result, in the next section, we will be able to study the convergence properties of the repeated inventory game. Under the above hypothesis on $\hat{\chi}_i(k)$, the best response strategy (3) yields the following dynamic model

$$s_i(k) = \begin{cases} 1 \text{ if } \|s_{-i}(k-1)\|_1 \geq l_i \\ 0 \text{ otherwise} \end{cases} \quad \text{for all } i \in \Gamma. \tag{18}$$

Given an equilibrium $s^\star$ and the associated complete and compatible set $C = \{1, \ldots, r\}$, we define a *positive (negative) perturbation* at stage 0, the vector $\Delta s(0) = s(0) - s^\star \geq 0$ ($\Delta s(0) \leq 0$). In other words, a positive (negative) perturbation is a change of strategies of a subset of players $P = \{i \in \Gamma \setminus C : \Delta s_i(0) = 1\}$ ($P = \{i \in C : \Delta s_i(0) = -1\}$), called *perturbed set*. The cardinality of the perturbed set $|P| = \|\Delta s(0)\|_1$ is the number of players that join the set C (leave the set C). In addition, a positive (negative) perturbation $\overline{\Delta s}(0)$ is *maximal* when $\|\overline{\Delta s}(0)\|_1 = |\Gamma \setminus C|$, ($\|\overline{\Delta s}(0)\|_1 = |C|$). In this last case, all the players in $\Gamma \setminus C$, (C) change strategy.

A Nash equilibrium $s^\star$ is called *stable with respect to positive perturbations* if there exist two integers $\delta > 0$ and $\bar{k} > 0$ such that if $\|\Delta s(0)\|_1 \leq \delta$, then $s(k) = s^\star$ for all $k \geq \bar{k}$, when the strategy (18) is applied in the repeated game by all the players. Analogously a Nash equilibrium $s^\star$ is called *maximally stable* *with respect to positive perturbations* if it is stable with respect to the maximal positive perturbation $\overline{\Delta s}(0)$.

In the following, we introduce some theorems concerning the stability of Nash equilibria with respect to positive perturbation. The motivation for analyzing positive perturbations stems from the fact that, as we will show later on, maximal stability with respect to positive perturbations is a property that distinguishes the Pareto optimal Nash equilibrium from all other Nash equilibria. We will exploit the above property in the consensus protocol to force the convergence of the strategies to the Pareto optimal Nash equilibrium.

Theorem 3 *Consider a Nash equilibrium $s^\star$ associated to a set $C = \{1, \ldots, r\}$. The Nash equilibrium $s^\star$ is stable with respect to positive perturbations $\Delta s(0) : ||\Delta s(0)||_1 = j - r - 1$ if all players $i \notin C$, with $r < i \leq j$, have thresholds $l_i \geq i$. In addition, the Nash equilibrium $s^\star$ is not stable with respect to positive perturbations $\Delta s(0) : ||\Delta s(0)||_1 = \hat{j} - r$, if there exists a player $\hat{j} = arg \min\{i \in \Gamma \setminus C : l_i < i\}$*

PROOF. *Stability.* Keep in mind that each player observes the other players' strategies with a one-stage delay throughout this proof. Note that, since $s^\star$ is a Nash equilibrium, Theorem 1 and the definition of positive perturbation implies that the following two conditions hold: i) the threshold $l_{r+1} \geq r + 1$ and, ii) at stage $k = 0$, $s_i(0) = 1$, for all $i \in C \cup P$, whereas $s_i(0) = 0$, for all $i \in \Gamma \setminus C \cup P$. Note also that a positive perturbation $\Delta s(0)$ induces players $i \in \Gamma \setminus C$, with thresholds $l_i \leq |C \cup P|$ to change strategy from $s_i(0) = 0$ to $s_i(1) = 1$. Differently, all players $i \in C$, will not change strategy, since they have thresholds $l_i \leq |C| < |C \cup P|$.
Consider now a particular perturbation $\Delta s(0)$ with $||\Delta s(0)||_1 = j - r - 1$. Then, at stage $k = 1$, all players $i \geq j$ set $s_i(1) = 0$, since they observe $j - 1$ active players, and their threshold is $l_i \geq j$. Hence, at stage $k = 2$, player $j-1$ surely sets $s_{j-1}(2) = 0$, since it observes at most $j - 2$ active players and its threshold is $l_{j-1} \geq j - 1$. Following the same line of reasoning at the generic stage k with $1 < k < j-r$, player $j-k+1$ sets $s_{j-k+1}(k) = 0$, since it observes at most $j - k$ active players and $l_{j-k+1} \geq j - k + 1$. Hence, at most at stage $k = j - r$ the strategies converge to the desired Nash equilibrium $s^\star$.
Instability. It is enough to show that the Nash equilibrium $s^\star$ is not stable with respect to a perturbation $\Delta s(0) : ||\Delta s(0)||_1 = \hat{j} - r$ induced by the perturbed set $P = \{r + 1, \ldots, \hat{j}\}$. To see this fact, consider that, at stage $k = 1$, players $r + 1, \ldots, \hat{j}$ set $s_{r+1}(1) = \ldots = s_{\hat{j}}(1) = 1$ since their thresholds are lower than or equal to $l_{\hat{j}} = \hat{j} - 1$. The players $r + 1, \ldots, \hat{j}$ do not change their strategies in the following stages, then the desired equilibrium point $s^\star$ will never be reached. $\square$

In the previous theorem, it is immediate to observe that player $\hat{j}$ may exist only for $\hat{j} \geq r + 2$, since condition (12) of Theorem 1 imposes $l_{r+1} \geq r + 1$. In addition, $l_{\hat{j}} = \hat{j} - 1$ since for all i such that $r < i < \hat{j}$ there holds $l_i \geq i$ by minimality of $\hat{j}$.

Given a Nash equilibrium $s^\star$, Theorem 3 establishes that $s^\star$ is stable with respect to any positive perturbation $\Delta s(0)$ if a player $\hat{j} = arg \min\{i \in \Gamma \setminus C : l_i < i\}$ does not exist. On the other hand, $s^\star$ is stable for $||\Delta s(0)||_1 < \hat{j} - r - 1$ and is not stable for $||\Delta s(0)||_1 \geq \hat{j} - r$, if player $\hat{j}$ exists. Under this latter hypothesis, Theorem 3 does not hold for a perturbation $||\Delta s(0)||_1 = \hat{j} - r - 1$.

For the sake of clarity we compute $\hat{j}$ and simulate dynamics on strategies for the system of Example 2.

Example 3 *(Example 2 cont'd) Consider the complete and compatible set* $C = \{1, 2, 3\}$ *with associated the second Nash equilibrium* $s^2 = \{1, 1, 1, 0, 0, 0, 0, 0\}$. *Player*

$$\hat{j} = arg \min\{i \in \Gamma \setminus C : l_i < i\} = 6$$

then Theorem 3 states that s^* *is stable if* $\|\Delta s(0)\|_1 < 2$ *and instable if* $\|\Delta s(0)\|_1 \geq 3$.

Tab. 2 about here

In Table 2 we simulate the dynamics on strategies under a positive perturbations $\|\Delta s(0)\|_1 = 1$ *assuming that, for example, player 5 selects* $s_5(0) = 1$. *At stage* $k = 2$ *the dynamics converges again to the equilibrium* s^2. *Analogously in Table 3 we simulate the dynamics for* $\|\Delta s(0)\|_1 = 3$ *obtained when, for example, players 5, 7 and 8 select* $s_5(0) = s_7(0) = s_8(0) = 1$. *At stage* $k = 1$ *the dynamics converges to the first Nash equilibrium* $s^1 = \{1, 1, 1, 1, 1, 1, 0, 0\}$ *showing that* s^2 *is instable.*

Tab. 3 about here

Assuming that there exists player $\hat{j} = arg \min\{i \in \Gamma \setminus C : l_i < i\}$, the following theorem addresses the case $||\Delta s(0)||_1 = \hat{j} - r - 1$.

Theorem 4 *Consider a Nash equilibrium* $s^\star$ *associated to a set* $C = \{1, \ldots, r\}$. *Assume that there exists a player* $\hat{j} = arg \min\{i \in \Gamma \setminus C : l_i < i\}$ *and let* $\hat{i} = arg \min\{i \in \Gamma \setminus C : l_i = \hat{j} - 1\}$. *The vector of strategies* $s^\star$ *is not stable with respect to positive perturbations* $\Delta s(0) : ||\Delta s(0)||_1 = \hat{j} - r - 1$ *if and only if at least one of the following conditions holds:*
i) there exist players $\hat{j} + 1, \ldots, 2\hat{j} - \hat{i} - 1$ *with threshold equal to* $\hat{j} - 1$,
ii) there exist players $\hat{j} + 1, \ldots, 2\hat{j} - \hat{i}$.

PROOF. Let us initially observe that, due to the minimality of $\hat{j}$, $\hat{i}$ is less than or equal to $\hat{j} - 1$. If $\hat{i} = \hat{j} - 1$ then condition i) holds since it defines an empty set.
Sufficiency. We first prove condition i). For doing so, let the perturbed set P

be equal to $\{r+1,\ldots,\hat{j}-1\}$ then $s_{r+1}(0)=\ldots=s_{\hat{j}-1}(0)=1$, which implies, at stage $k=1$, $s_{r+1}(1)=\ldots=s_{\hat{i}-1}(1)=1$, $s_{\hat{j}}(1)=\ldots=s_{2\hat{j}-\hat{i}-1}(1)=1$. Actually, each player i such that $r+1\leq i\leq \hat{i}-1$ observes that at the previous stage, $k=0$, other $\hat{j}-2\geq l_i$ players are active and each player i such that $\hat{j}\leq i\leq 2\hat{j}-\hat{i}-1$ observes that other $\hat{j}-1=l_i$ players are active. Similarly, at stage $k=2$, it surely holds that $s_{r+1}(2)=\ldots=s_{\hat{j}-1}(2)=1$. Hence, from such a stage on, the players $r+1,\ldots,\hat{j}-1$ surely decide to meet the demand on at least the even stages, and therefore $s^\star$ is not stable.
It is left to prove condition ii). Let the perturbed set P be equal to $\{r+1,\ldots,\hat{i}-1,\hat{j}+1,\ldots,2\hat{j}-\hat{i}\}$ then $s_{r+1}(0)=\ldots=s_{\hat{i}-1}(0)=1$ and $s_{\hat{j}+1}(0)=\ldots=s_{2\hat{j}-\hat{i}}(0)=1$, which implies, at stage $k=1$, $s_{r+1}(1)=\ldots=s_{\hat{j}}(1)=1$. Actually, each player i such that $r+1\leq i\leq \hat{i}-1$ observes that at the previous stage, $k=0$, other $\hat{j}-2\geq l_i$ players are active and each player i such that $\hat{i}\leq i\leq \hat{j}$ observes that other $\hat{j}-1=l_i$ players are active. For an analogous reason, from stage $k=2$ on, the players $r+1,\ldots,\hat{j}$ surely decide to meet the demand at every stage and strategies converge to a new Nash equilibrium different from $s^\star$.
Necessity. Assume that condition i) and condition ii) do not hold. Then the set Γ includes at most $2\hat{j}-\hat{i}-1$ players and the threshold of the last player must satisfy the following condition $l_{2\hat{j}-\hat{i}-1}>\hat{j}-1$. Then, given a perturbation $\Delta s(0)$ with $||\Delta s(0)||_1=\hat{j}-r-1$, at stage $k=1$ it holds $s_i(1)=1$ for i such that either $i<\hat{i}$ or $l_i=\hat{j}-1$ but $i\notin P$, $s_i(1)=0$ otherwise. Assume without loss of generality that all i such that $l_i=\hat{j}-1$ but $i\notin P$ are smaller than the minimum $\tilde{i}$ such that $l_{\tilde{i}}=\hat{j}-1$ and $\tilde{i}\in P$, then the maximum number of active players at stage $k=1$ may be obtained for $P=\{r+1,\ldots,\hat{i}-1,\hat{j},\ldots,2\hat{j}-\hat{i}-1\}$. Indeed, by doing this, we preserve all players i with threshold $l_i=\hat{j}-1$ from being perturbed at $k=0$.
Having chosen such a P, the number of active players at stage $k=1$ is equal to $\hat{j}-1$. Indeed, all players $i=\hat{i},\ldots,\hat{j}-1$ have thresholds $l_i=\hat{j}-1$ and therefore $s_{\hat{i}}(1)=\ldots=s_{\hat{j}-1}(1)=1$. At the same time, all players $i=\hat{j},\ldots,2\hat{j}-\hat{i}-1$, whose thresholds are $l_i\geq\hat{j}-1$ observe only other $\hat{j}-2$ active players and therefore $s_{\hat{j}}(1)=\ldots=s_{2\hat{j}-\hat{i}-1}(1)=0$. Now, at stage $k=2$, we have $s_{\hat{i}}(2)=\ldots=s_{\hat{j}-1}(2)=0$ and $s_{2\hat{j}-\hat{i}-1}(2)=0$, since $l_{2\hat{j}-\hat{i}-1}>\hat{j}-1$. The situation at $k=2$ is equivalent to the one obtainable at $k=0$ in presence of a perturbation with $||\Delta s(0)||_1=\hat{j}-r-2$. Since for perturbations with $||\Delta s(0)||<\hat{j}-r-1$, see Theorem 3, the Nash equilibrium $s^\star$ is stable, we can affirm that even in this case the strategies will converge to the Nash equilibrium $s^\star$. $\square$

Now, we specialize the previous theorems to the Pareto optimal Nash equilibrium.

Corollary 2 *The unique Pareto optimal Nash equilibrium is maximally stable with respect to positive perturbations.*

PROOF. From definition of maximal stability, we must show that $s^\star$ is stable with respect to the maximal positive perturbation $\overline{\Delta s}(0)$, with $\|\overline{\Delta s}(0)\|_1 = |\Gamma - \overline{C}|$. From maximality of $\overline{C}$ it must hold $l_i \geq i$ for all i, such that $r < i < n$. As a consequence, see Theorem 3, $s^\star$ is stable with respect to $\overline{\Delta s}(0)$ and therefore it is also maximally stable. $\square$

Let us conclude this section remarking that the Pareto optimal Nash equilibrium may not be globally stable with respect to negative perturbations. It is straightforward to prove this fact when, e.g., several Nash equilibria exist. Consider, for instance, the Nash equilibrium $s^1 = \{1, 1, 1, 1, 1, 1, 0, 0\}$ of Example 2: any negative perturbation (any of the players 1-6 selects 0 instead of 1) makes the strategies converge to the second equilibrium s^2.

5 A Consensus Protocol

In this section, we exploit the stability properties introduced in the previous section to design a protocol $\widehat{\Pi} = \{\phi_i,\, h_i,\;i \in \Gamma\}$ that allows the distributed convergence of the best response strategies (3) to the Pareto optimal Nash equilibrium.

Consider the graph G induced by the set of players Γ as defined in Section 2. Let L be the Laplacian matrix of G and use L_{ij} and $L_{i\bullet}$ to denote respectively the i, j entry and the i-th row of L.
Let us consider the *almost-linear* protocol $\widehat{\Pi}$ defined by the following dynamics:

$$x_i(k+1) = z_i(k) + \alpha \sum_{j \in N_i} L_{ij} z_j(k) + \delta_T(k) \qquad (19)$$

$$z_i(k) = x_i(k) + s_i(k) - s_i(k-1) \quad \text{for all } k \geq 1 \qquad (20)$$

$$z_i(0) = x_i(0) = s_i(0) \qquad (21)$$

where α is a negative scalar such that the eigenvalues of the matrix $(I + \alpha L)$ are inside the unit circle, except for the largest one that is equal to one. We will show that the pre-decision information $x_i(k)$ in (19) is a local estimate of the percentage of the active players at each stage $k-1$. The post-decision information $z_i(k)$ in (20) updates the estimate in the light of the strategy $s_i(k)$.

Almost linearity is due to the non linear correcting term $\delta_T(k)$ acting any T stages in (19). This term describes the use of linear predictors, which will be discussed in Section 6.1. There, we will show that, when using linear predictors, the presence of a non null δ_T increases the speed of convergence of the protocol. We will also emphasize this last argument in the numerical example of Section 7. Throughout this section, we disregard this term by assuming $\delta_T(k)$ constantly equal to 0.

In the following we introduce two lemmas. The first one states that, at each stage k, the average value $Avg(x(k)) = \frac{\sum_i x_i(k)}{n}$ is the percentage of active players at the previous stage $k-1$. The second lemma states that if no player changes its strategy for a sufficient number of stages the pre-decision information $x_i(k)$ converges to the $Avg(x(k))$. For this last reason, protocol $\widehat{\Pi}$ may also be referred to as an *average consensus protocol* (see, e.g., [24]).

Now, let us initially rewrite the dynamic of the pre-decision information (19) for $k \geq 1$ as

$$\begin{aligned} x(k+1) = (I + \alpha L)(x(k) + s(k) - s(k-1)) = \\ s(k) + \textstyle\sum_{r=0}^{k}(I + \alpha L)^{k-r}\alpha L s(r). \end{aligned} \tag{22}$$

To obtain the second term of (22) we substitute in (19) the value of $z_i(k)$ in (20). Then we observe that from (19) it holds $x(1) = s(0) + \alpha L s(0)$ hence, by induction, if we assume $x(k) = s(k-1) + \alpha \sum_{r=0}^{k-1}(I + \alpha L)^{(k-1)-r} L s(r)$, we obtain the last term of (22).

Lemma 2 *Given the dynamic of the pre-decision and the post-decision information vectors as described in (19), (20) and (21) at each stage k, the following condition holds*

$$Avg(x(k)) = \frac{\|x(k)\|_1}{n} = \frac{\|s(k-1)\|_1}{n}. \tag{23}$$

PROOF. Consider the pre-decision information vector $x(k)$ as expressed in (22). Then, observe that $1'L = 0$. Hence, $1'x(k) = 1's(k-1) + \alpha 1' \sum_{r=0}^{k-1}(I + \alpha L)^{(k-1)-r} L^r s = 1's(k-1) + \alpha \sum_{r=0}^{k-1} 1' I L^r s = 1's(k-1) = \|s(k-1)\|_1$, which in turn implies $Avg(x(k)) = \frac{\|x(k)\|_1}{n} = \frac{\|s(k-1)\|_1}{n}$. □

Lemma 3 *Consider the dynamic of the pre-decision and the post-decision information vectors as described in (19), (20) and (21) and assume that no player changes strategy from stage r on, then there exists a finite integer $\hat{r} \geq 1$ such that, for player i, it holds $x_i(r + \hat{r}) = \frac{\|s(r+\hat{r}-1)\|_1}{n} = \frac{\|s(r)\|_1}{n}$, i.e., $x_i(r + \hat{r})$ is equal to the percentage of active players at stage r.*

PROOF. We extend to the discrete-time system (22) the results established for continuous-time systems in [24]. In particular, when no players change strategy for $k > r$, we have $s_i(k) - s_i(k-1) = 0$ and the system (22) is equivalent to

$$x(k+1) = (I + \alpha L)x(k), \quad \text{for } k \geq r. \tag{24}$$

Given the discrete-time system above, there exists $\hat{r} \geq 0$ such that, for each player i, $x_i(\hat{r}) = Avg(x(r+\hat{r})) = \frac{\|s(r+\hat{r}-1)\|_1}{n} = Avg(x(r+1))$ (see, e.g., Corollary 1 in [24]). $\square$

In the assumption that no player changes strategy from a generic stage r on, the above arguments guarantee that each player i may estimate the percentage of the active players in a finite number of stages T. Lemma 3 shows that T is finite and precisely $T \leq \hat{r} - r$. It will be shown in Section 6, that $T \leq 2n$ in presence of linear predictors.
At stage $r + T$ player i estimates the number of all other active players as

$$\begin{aligned} \hat{\chi}_i(r+T) &= \|s_{-i}(r+T-1)\| = \|s_{-i}(r)\| \\ &= nx_i(r+T) - s_i(r). \end{aligned} \tag{25}$$

Now, assume that players can change strategy only at stages $\hat{k} = qT$, $q = 0, 1, 2, \ldots$. At stages $\hat{k} \geq 1$, we can generalize (25) as $\hat{\chi}_i(\hat{k}) = nx_i(\hat{k}) - s_i(\hat{k}-T)$. At stage $\hat{k} = 0$ let the players estimate all the other players active, i.e., $\hat{\chi}_i(0) = n - 1$.

Theorem 5 *The average consensus protocol $\widehat{\Pi}$ defined in (19), (20) and (21) allows the best response strategy (3) to converge in $(n-1)T$ stages to the unique Pareto optimal Nash equilibrium.*

PROOF. Initially observe that no player changes its strategy at stages $k \neq qT$. Then note that the best response strategy, sampled at stages $\hat{k}$, evolves as

$$s_i(\hat{k}) = \begin{cases} 1 \text{ if } \|s_{-i}(\hat{k}-T)\|_1 \geq l_i \\ 0 \text{ otherwise} \end{cases}.$$

Such a dynamic is exactly as in (18). The hypothesis $\hat{\chi}_i(0) = n - 1$ implies that, at stage $\hat{k} = 0$, the initial strategy $s(0) \geq s^\star$ with $\|\Delta s(0)\|_1 \leq |\Gamma \setminus \overline{C}|$. Since (18) is maximally stable with respect to positive perturbations even the system of the sampled strategies will converge to the Pareto optimal Nash equilibrium. The system of sampled strategies converges in at most $n-1$ stages. Actually, assume that the Pareto optimal Nash equilibrium is associated to $\overline{C} = \emptyset$. Then, in the worst case, at stage $\hat{k} = 0$, $n - 1$ players decide to meet the demand and at each successive stage $\hat{k} = qT$ only a single player changes its strategy and decide not to meet the demand any more. $\square$

Note that the convergence properties of the protocol established in the previous theorem still hold for any initial estimate $z_i(0)$ in (21) that is an upper bound of the $|\overline{C}|$.

Let us finally observe that the value of T depends on the information available to the players. If, at each stage k, each player can infer the number of active players either because it is connected with all the other players or because it can observe the costs, then $T = 1$ and the system converge to the Pareto optimal Nash equilibrium in $n-1$ stages. We discuss the value of T for more generic situations in the following section.

6 A-priori information and speed of convergence of the protocol

In this section, we determine the values of both α and T as functions of the players' computation capabilities and their knowledge about the structure of graph G. We show that T grows linearly with n when players can use linear predictors and discuss the non linear correcting term $\delta_T(k)$ in (19). Differently, in absence of linear predictors ($\delta_T(k) = 0$ for all k) the players must wait that the pre-decision information converges to the desired percentage of currently active players. In this latter case the number of stages T may become proportional to $n^2 log(n)$ or even to $n^3 log(n)$ depending on the knowledge that players have on the eigenvalues of the Laplacian matrix L.
Throughout this section we recall the hypotheses of Lemma 3, i.e., players are interested in determining the value of $Avg(s(r)) = Avg(x(r+1))$ and do not change strategy from stage r on.

6.1 Linear Predictors

With focus on (19) the non-linear correcting term must i) compensate the linear dynamics $-z_i(k) - \alpha \sum_{j \in N_i} L_{ij} z_j(k)$ and ii) correct the estimate of the percentage of active players. For doing so, the non linear correction may take the form

$$\begin{aligned} \delta_T(k) = & -z_i(k) - \alpha \textstyle\sum_{j \in N_i} L_{ij} z_j(k) \\ & + \rho(x_i(k), x_i(k-1), \ldots, x_i(k-T)). \end{aligned} \tag{26}$$

Now, we show that it is possible to design ρ linearly as follows

$$\rho(x_i(r+T), x_i(r+T-1), \ldots, x_i(r)) = \sum_{k=0}^{n-1} \gamma_k x_i(r+k), \tag{27}$$

where γ_k are the coefficients of the characteristic polynomial of the matrix $I + \alpha L$ and therefore depend on the structure of graph G [12].
The next theorem shows that each player i may determine the value of $Avg(x(r+1))$ in $n-1$ stages.

Theorem 6 *Given the protocol $\widehat{\Pi}$ as in (19), (20) and (21), if the players know the characteristic polynomial of the matrix $I + \alpha L$, then the number of stages necessary to estimate the percentage of active players is at most $n-1$, i.e., $x_i(r+T) = \frac{\|s(r)\|_1}{n}$ with $T \leq n-1$ for all players $i \in \Gamma$ and for any generic stage $r \geq 0$.*

PROOF. Consider the generic player i. We know from Lemma 3 that there exists a finite $\hat{r}$ such that $x_i(r+\hat{r}) = Avg(x(r+1)) = Avg(s(r))$ independently of the value of the vector $s(r)$. If $\hat{r} \leq n-1$ set $T = \hat{r}$ and the theorem is proven. If $\hat{r} > n-1$ first observe that, if $s(r+k) = s(r)$ for $k \geq 0$, equation (22) implies $x(r+k) = (I+\alpha L)^k (x(r) + s(r) - s(r-1))$ and in particular $x(r+\hat{r}) = (I+\alpha L)^{\hat{r}}(x(r)+s(r)-s(r-1))$, which in turn becomes $x(r+\hat{r}) = \sum_{k=0}^{n-1} \gamma_k (I+\alpha L)^k (x(r)+s(r)-s(r-1))$, where γ_k are appropriate coefficients that can be determined if the characteristic polynomial of $I+\alpha L$ is known making use of the Cayley–Hamilton theorem. Hence, $Avg(x(r+1)) = \sum_{k=0}^{n-1} \gamma_k x_i(r+k)$. $\square$

An immediate consequence of the above theorem is that, in the worst case, no other distributed protocol may determine the number of active players faster than $\widehat{\Pi}$, provided that players know the characteristic polynomial of the matrix $I + \alpha L$. If G is a path graph, the value of T can never be less than n, since information takes $n-1$ stages to propagate end to end all over the path. Now, consider the case in which the players have no knowledge on the structure of the graph G, then the values of the parameters γ_k cannot be a priori fixed. The next theorem proves that $2n$ stages are sufficient for the generic player to estimate $Avg(x(r+1))$.

Theorem 7 *Given the protocol $\widehat{\Pi}$ as in (19), (20) and (21), the number of stages necessary to estimate the percentage of active players is at most $2n$, i.e., $x_i(r+T) = \frac{\|s(r)\|_1}{n}$ with $T \leq 2n$ for all players $i \in \Gamma$ and for any generic stage $r \geq 0$.*

PROOF. Consider the generic player i and follow the same line of reasoning of the proof of Theorem 7. From Theorem 7 we know that player i can express the value of $Avg(x(r+1))$ as a linear combination of the value of the pre-decision information available at stages $r, \ldots, r+n-1$, that is $Avg(x(r+1)) = \sum_{k=0}^{n-1} \gamma_k x_i(r+k)$. However, in this case, since the structure of the graph G is not known, player i cannot a-priori fix the values of the coefficients γ_k. Nevertheless, under the hypothesis that no player changes its strategy form stage r on, we know that $Avg(x(r+1)) = Avg(x(r+2)) = \ldots = Avg(x(r+n))$

by Lemma 2. Then the following set of $n+1$ conditions must hold

$$Avg(x(r+1)) = \sum_{k=0}^{n-1} \gamma_k x_i(r+k) \tag{28}$$

$$Avg(x(r+1)) = \sum_{k=0}^{n-1} \gamma_k x_i(r+k+1) \tag{29}$$

$$\cdots\cdots\cdots$$

$$Avg(x(r+1)) = \sum_{k=0}^{n-1} \gamma_k x_i(r+k+n-1) \tag{30}$$

$$1 = \sum_{k=0}^{n-1} \gamma_k. \tag{31}$$

Condition (31) derives straightforward from

$$Avg(x(r+1)) = \sum_{k=0}^{n-1} \gamma_k x_i(r+\hat{r}+k) = \sum_{k=0}^{n-1} \gamma_k Avg(x(r+1)).$$

Set the unknown value of $Avg(x(r+1))$ equal to $-\gamma_n$, collect sample data of the pre-decision information in the coefficient matrix X and rewrite (31) as

$$\begin{bmatrix} X & 1 \\ 1' & 0 \end{bmatrix} \begin{bmatrix} \gamma \\ \gamma_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{32}$$

When the coefficient matrix in (32) is not singular, player i can easily determine both the desired value $Avg(x(r+1))$ and the coefficients γ_k. On the contrary, singularity of X implies that system (32) has multiple solutions. Even in this case we show that the players may determine $Avg(x(r+1))$ once proved that all feasible solutions have the same γ_n component. The last statement holds true if all the eigenvectors associated to a possible null eigenvalue 0 of the coefficient matrix have the nth component equal to 0. In other words, it must happen that the projection of the kernel of the coefficient matrix on the variable γ_n is always 0.
The proof is by contradiction. Indeed, assume, without loss of generality, that a null eigenvalue exists with associated eigenvector $\begin{bmatrix} q \\ 1 \end{bmatrix}$ having a non null nth component. Then, $[X \quad 1] \begin{bmatrix} q \\ 1 \end{bmatrix} = 0$. Since X is symmetric it also holds $[q \quad 1] \begin{bmatrix} X \\ 1' \end{bmatrix} = 0$. However, the last condition would imply that row $[1' \quad 0]$ can

be expressed as a linear combination of rows $[X \quad 1]$, but this contradicts the fact that system (32) surely has a feasible solution. $\square$

6.2 No Predictors

We now compare the previous results with the ones obtainable when no predictors are used.

Lemma 3 states that, in any case, the pre-decision information converges to the desired average value $Avg(s(r)) = Avg(x(r+1))$. We are then interested in deriving after how many stages a player can determine $Avg(x(r+1))$ by rounding the pre-decision information currently available. To this end let us consider the following autonomous discrete time system of order n

$$x(k+1) = (I + \alpha L)x(k). \tag{33}$$

System (33) describes the evolution of the pre-decision information when players do not change their strategies from stage r on. Actually, equation (33) is trivially equivalent to (22) when the players' strategies are disregarded.
Starting from any initial state $x(r+1)$ the system (33) converges to $Avg(x(r+1))$. Then, observe that $Avg(x(r+1))$ must be equal to a multiple of $\frac{1}{n}$ due to its physical meaning. As a consequence, we could choose T as equal to the minimal k such that $|x_i(k+r+1) - Avg(x(r+1))| < \frac{1}{2n}$ for each player i and let the players determining $Avg(x(r+1))$ by simply rounding $x_i(k+r+1)$ to its closest multiple of $\frac{1}{n}$.
To determine the value of T, consider first the modal decomposition of the undriven response of system (33) given by

$$x(k+r+1) = (I + \alpha L)^k x(r+1) = \sum_{i=1}^{n} \beta_i \bar{\lambda}_i^k v_i,$$

where, for $i = 1, \ldots, n$, $\bar{\lambda}_i$ is an eigenvalue of $I + \alpha L$, v_i is the associate eigenvector, and β_i depends on the initial state according to

$$x(r+1) = \sum_{i=1}^{n} \beta_i v_i.$$

Note that since the smallest eigenvalue of L is always $\lambda_1 = 0$, then $\bar{\lambda}_1 = 1$ and hence $\beta_1 v_1 = Avg(x(r+1))$. Note also that $I + \alpha L$ is symmetric then, due to the spectral theorem for Hermitian matrices, all its eigenvectors are orthonormal. Hence, $|\beta_i| = \|v_i' x(r+1)\|_\infty \leq \|v_i\|_\infty \|x(r+1)\|_\infty \leq 1$ since the initial state $x(r+1)$ satisfies $\|x(r+1)\|_\infty \leq 1$.
We can now state that (subscript ∞ is dropped)

$$
\begin{aligned}
&\|x(k+r+1) - Avg(x(r+1))\| = \|x(k+r+1) - \beta_1 v_1\| = \\
&\quad = \|\sum_{i=2}^{n} \bar{\lambda}_i^k \beta_i v_i\| \le \sum_{i=2}^{n} \|\bar{\lambda}_i^k \beta_i v_i\| \le \sum_{i=2}^{n} |\bar{\lambda}_i^k| |\beta_i| \|v_i\| \le \\
&\qquad \le |\hat{\lambda}|^k \sum_{i=2}^{n} \|v_i\|^2 \|x(r+1)\| \le |\hat{\lambda}|^k (n-1),
\end{aligned}
$$

where $\hat{\lambda}$ is the eigenvalue of $I + \alpha L$ with the second greatest absolute value. Indeed, the eigenvalue of $I + \alpha L$ with the greatest absolute value is $\bar{\lambda}_1$.
Given the above arguments a conservative condition on T is to impose $|\hat{\lambda}|^T (n-1) < \frac{1}{2n}$, from which we obtain

$$
T \ge \frac{-log(2(n-1)n)}{log(|\hat{\lambda}|)} + 1. \tag{34}
$$

In condition (34) T depends indirectly on the value of α through the eigenvalue $\hat{\lambda}$. In the following we discuss how to choose α in order to minimize T and, at the same time, to guarantee the stability of system (33). In (34), T is minimized if $|\hat{\lambda}|$ is minimum, since $|\hat{\lambda}| < 1$ for system (33) to be stable. Note that $|\hat{\lambda}|$ is equal

$$
|\hat{\lambda}| = \max \left\{ |1 + \alpha \lambda_n|, |1 + \alpha \lambda_2| \right\}. \tag{35}
$$

The optimal $\alpha^\star$ is then the solution of the following equation

$$
\alpha^\star = \arg\min_{\alpha} |\hat{\lambda}| = \arg\min_{\alpha} \max \left\{ |1 + \alpha \lambda_n|, |1 + \alpha \lambda_2| \right\}. \tag{36}
$$

It is easy to show that the solutions of the above equation are

$$
\alpha^\star = -\frac{2}{\lambda_2 + \lambda_n} \tag{37}
$$

$$
\hat{\lambda}^\star = 1 - \frac{2\lambda_2}{\lambda_2 + \lambda_n}. \tag{38}
$$

Consider now the stability of system (33). System (33) is stable if $|\bar{\lambda}_i| < 1$, $i = 2, \ldots, n$, which in turns implies that $|1+\alpha\lambda_i| < 1$. Since $\alpha < 0$ and $\lambda_i > 0$, the latter condition is certainly satisfied if and only if $1 + \alpha\lambda_n > -1$. From this last inequality, system (33) is stable if and only if $-\frac{2}{\lambda_n} < \alpha < 0$. In this context, note that $-\frac{2}{\lambda_n} < \alpha^\star < 0$.
Let us now introduce the following lemma that collects well-known properties on the eigenvalues λ_2 and λ_n that turn useful in the rest of the section. The interested reader is referred to [5], [12], and [15] for the proofs of the lemma.

Lemma 4 *Let $G_1 = (\Gamma, E_1)$ and $G_2 = (\Gamma, E_2)$ be two connected graphs on the same set of vertices Γ, and let $\lambda_2(G_1)$ and $\lambda_2(G_2)$ the second smallest eigenvalues of the Laplacian matrices associated to G_1 and G_2, respectively. Analogously, let $\lambda_n(G_1)$ and $\lambda_n(G_2)$ the greatest eigenvalues of the Laplacian matrices associated to G_1 and G_2, respectively. Then, the following properties hold*

(1) $\lambda_2(G_1) \leq \lambda_2(G_2)$, *if* $E_1 \subseteq E_2$;
(2) $\lambda_n(G_1) \leq \lambda_n(G_2)$, *if* $E_1 \subseteq E_2$;
(3) $\lambda_n(G_1) = \lambda_n(G_2) = n$, *if G_1 is complete graph;*
(4) $\lambda_2(G_1) = 2(1 - cos(\frac{\pi}{n}))$, *if G_1 is a path graph;*
(5) $\lambda_n(G_1) = 2(1 + cos(\frac{\pi}{n}))$, *if G_1 is a path graph.*

An immediate consequence of the previous lemma is that, if players know λ_2 and λ_n and the graph G is complete, then $\alpha^\star = -\frac{1}{n}$, $\hat{\lambda}^\star = 0$, and from (34) we have $T = 1$, whereas if G is a path graph, $\alpha^\star = -\frac{1}{2}$, $\hat{\lambda}^\star = cos(\frac{\pi}{n})$, and hence $T \to \frac{2n^2 log(2(n-1)n)}{\pi^2} + 1$ as n increases. Differently, if players know neither the structure of the graph G nor the eigenvalues λ_2 and λ_n. To guarantee the stability of system (33), condition $-\frac{2}{\lambda_n} < \alpha < 0$ must hold for any possible value of λ_n. By Lemma 4, the largest λ_n occurs when G is a complete graph, where $\lambda_n = n$. Then, α must be chosen within the interval $-\frac{2}{n} < \alpha < 0$. Now, consider a path graph. The fastest convergence occurs for the greatest $|\alpha|$, and when $\alpha \to -\frac{2}{n}$ we obtain $T \to \frac{n^3 log(2(n-1)n)}{2\pi^2} + 1$ as n increases.

7 Simulation Results

In this section we provide a numerical example and some simulation results for a set Γ of 8 players implementing the designed protocol with and without predictors. We will see that in both cases the strategies converge to the Pareto optimal Nash equilibrium though with different speed of convergence. Fig. 1 reports the induced graph G, whereas Tab. 4 lists the players' thresholds l_i and the initial strategies $s_i(0)$. Note that at $k = 0$ the strategies are not in the Pareto optimal Nash equilibrium $s^* = \{1, 1, 1, 0, 0, 0, 0, 0\}$.

Fig. 1 about here

Tab. 4 about here

Fig. 2 displays the evolution of the pre-decision information according to the protocol $\hat{\Pi}$ defined in (19)-(21) when the players use the linear predictors as in (26)-(27). Fig. 3 shows the evolution of the pre-decision information when the linear predictors are not present.
Both Fig. 2 and Fig. 3 show that at $k = 0$ players $1 - 2 - 3 - 4 - 5$ are active.

Differently, players $6-7-8$ are not active, as they can immediately estimate that the number of active players is below their threshold values. At stage $k = T$ all the players estimate the number of active players as equal to 5. In particular, player 5 observes other four active players and since its threshold is $l_1 = 5$ it changes strategy from $s_1(T-1) = 1$ to $s_1(T) = 0$ (circles in Fig. 2-3). At $k = 2T$, the players' new estimate is 4 and player 4 changes strategy, too. Finally, at stage $k = 3T$, the players strategies converge to the Pareto optimal Nash equilibrium with $\|s^\star\|_1 = 3$.
The difference between the two figures is that, in Fig. 2 the value of T is 15 whereas in Fig. 3 the value of T is 80.

Fig. 2 and 3 about here

8 Conclusion

In this paper, we have introduced a consensus protocol to achieve distributed convergence to a Pareto optimal Nash equilibrium, for a class of repeated nonsymmetric congestion games under partial information. We have specialized the game to a multi-retailer application, where transportation or set up costs are shared among all retailers, reordering from a common warehouse. The main results concern: i) the existence and the stability of Pareto optimal Nash equilibria, ii) the structure of the consensus protocol and its convergence properties.

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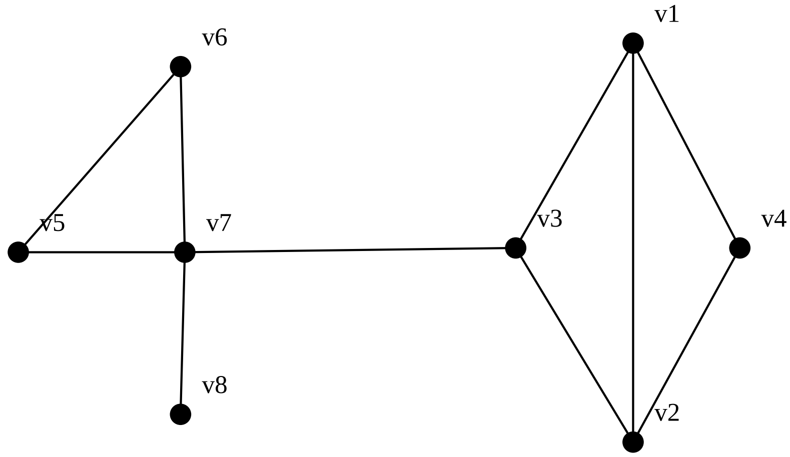

Fig. 1. An example of graph G for a set Γ of 8 players

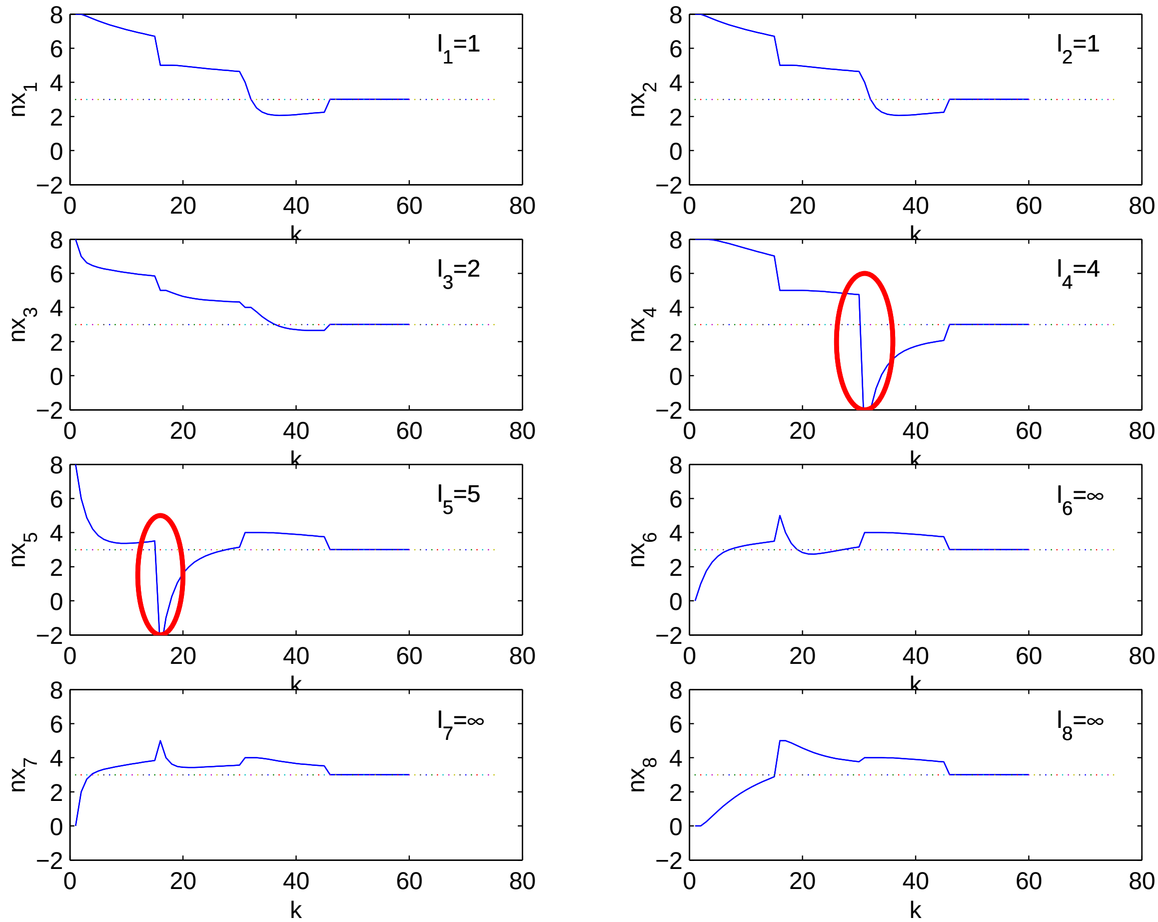

Fig. 2. Evolution of $nx_i(k)$ in presence of linear predictors as in (26)-(27). The circles indicate when a player changes strategy.

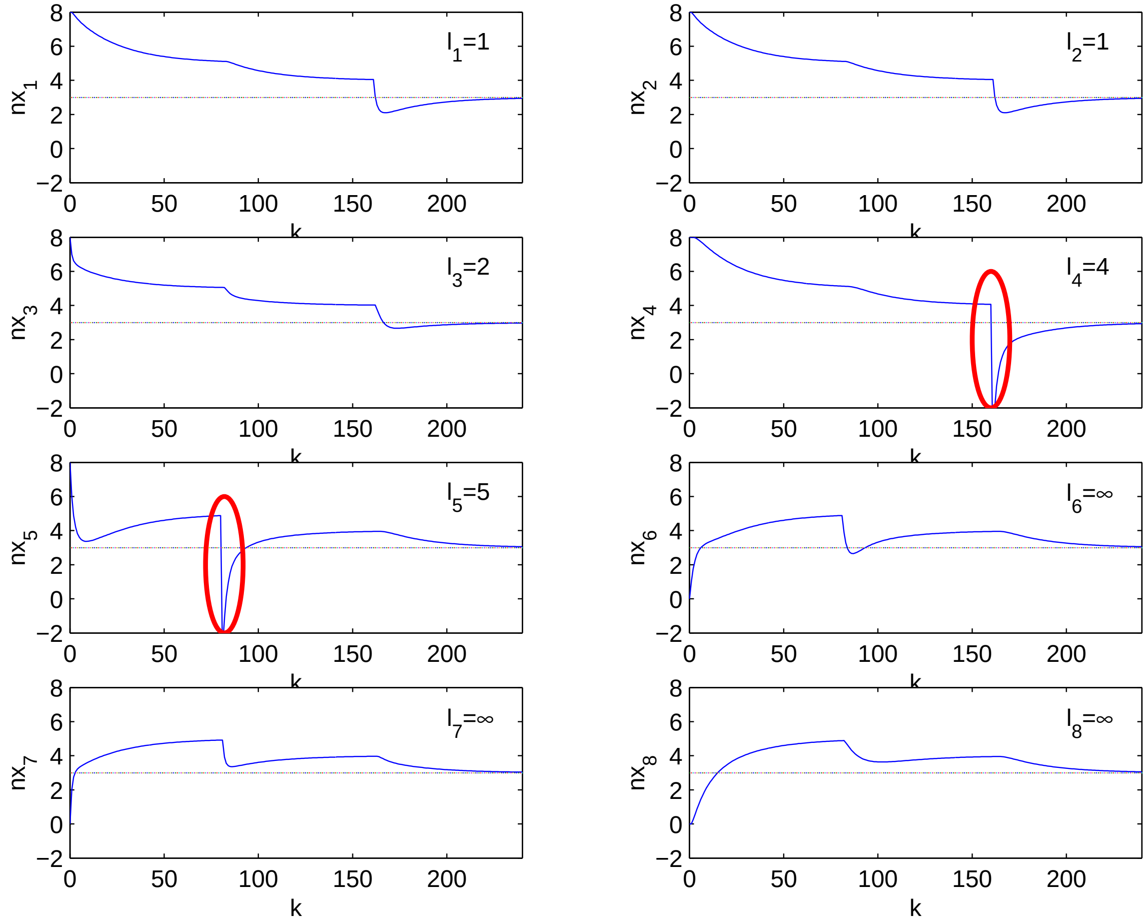

Fig. 3. Evolution of $nx_i(k)$ in absence of linear predictors. The circles indicate when a player changes strategy.

players	1	2	3	4	5	6	7	8
p_i	30	30	20	12	10	10	0	0
$l_i = \frac{K}{p_1} - 1$	1	1	2	4	5	5	∞	∞

Table 1
Players' thresholds and initial strategies

players	1	2	3	4	5	6	7	8
l_i	1	1	2	4	5	5	∞	∞
$s_i(0)$	1	1	1	0	1	0	0	0
$\|s_{-i}(0)\|_1$	3	3	3	4	3	4	4	4
$s_i(1)$	1	1	1	1	0	0	0	0
$\|s_{-i}(1)\|_1$	3	3	3	3	4	4	4	4
$s_i(k \geq 2)$	1	1	1	0	0	0	0	0
$\|s_{-i}(k \geq 2)\|_1$	2	2	2	3	3	3	3	3

Table 2
Stability of s^2 for $\|\Delta s(0)\|_1 = 1$.

players	1	2	3	4	5	6	7	8
l_i	1	1	2	4	5	5	∞	∞
$s_i(0)$	1	1	1	0	1	0	1	1
$\|s_{-i}(0)\|_1$	5	5	5	6	5	6	5	5
$s_i(k \geq 1)$	1	1	1	1	1	1	0	0
$\|s_{-i}(k \geq 1)\|_1$	5	5	5	5	5	5	6	6

Table 3
Instability of s^2 for $\|\Delta s(0)\|_1 = 3$.

players	1	2	3	4	5	6	7	8
l_i	1	1	2	4	5	∞	∞	∞
$s_i(0)$	1	1	1	1	1	0	0	0

Table 4
Players' thresholds and initial strategies