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Mixed Integer Optimal Compensation: Decompositions and Mean-Field Approximations

Dario Bauso, Quanyan Zhu and Tamer Başar

***Abstract*— Mixed integer optimal compensation deals with optimizing integer- and real-valued control variables to compensate disturbances in dynamic systems. The mixed integer nature of controls might be a cause of intractability for instances of larger dimensions. To tackle this issue, we propose a decomposition method which turns the original n-dimensional problem into n independent scalar problems of lot sizing form. Each scalar problem is then reformulated as a shortest path one and solved through linear programming over a receding horizon. This last reformulation step mirrors a standard procedure in mixed integer programming. We apply the decomposition method to a mean-field coupled multi-agent system problem, where each agent seeks to compensate a combination of the exogenous signal and the local state average. We discuss a large population mean-field type of approximation as well as the application of predictive control methods.**

I. INTRODUCTION

Mixed integer optimal compensation arises when optimizing integer- and real-valued control variables in order to compensate for disturbances in dynamic systems. Mixed integer control in a receding horizon has been formulated in [3]. Mixed integer control is considered a specific subfield of optimal hybrid control [5]. Optimal integer control problems have been receiving growing attention and are often categorized under different names (e.g. alphabet control [7], [15]). Integer control requires more than standard convex optimization techniques. It is known that new structural properties of the problem play important roles in mixed integer control. As an example, see *multimodularity* presented as the counterpart of convexity in discrete action spaces [6]. We should note that there is vast literature on mixed integer programming [13], and it is in this context that we cast the problem addressed in this paper.

This paper is in the spirit of [14], which surveys solution methods for mixed integer lot sizing models. The paper has three main contributions. First, we formulate the mixed integer optimal compensation problem. Second, we provide a performance analysis of the decomposition method that reformulates the n-dimensional mixed integer problem as n independent uncertain lot sizing systems. Third, we view each mixed integer problem as a shortest path problem and solve the latter through linear programming. The conservativeness arising from the robust decomposition and approximation can be reduced if we operate in accordance with the predictive control technique: i) optimize controls for each independent system based on the prediction of other states, ii) apply the first control, iii) provide measurement updates of other states and repeat the procedure.

This paper differs from [3] as we focus on a smaller class of problems that can be solved exactly by simply relaxing the integer constraints. The lot sizing like model used here has much to do with the inventory example briefly mentioned in [5]. There, the authors simply include the example in a large list of hybrid optimal control problems but do not address the issue of how to fit general methods to this specific problem. On the contrary, in this work we emphasize the computational benefits that can be derived from the "nice structure" of the lot sizing constraints matrix. Binary variables, used to model impulses, match linear programming in [4]. There, the linear reformulation is a straightforward derivation of the *(inverse) dwell time* conditions that have first appeared in [10]. Analogies with [4] are, for instance, the use of total uni-modularity to prove the exactness of the linear programming reformulation. Differences are in the procedure itself upon which the linear program is built up. The shortest path model is an additional element which distinguishes the present approach from [4].

We also provide a discussion on a special case of interest where each agent seeks to compensate a combination of the exogenous signal and the local state average. In this case, our decomposition idea is similar to mean-field methods in large population consensus [9], [19], [18].

The theory of mean field games, as formulated by J. M. Lasry and P. L. Lions in [12] aims at studying situations with a large number of (indistinguishable) agents whose strategies are influenced by the mass of the other agents. This theory is very versatile and is attracting an ever increasing interest with several applications in economics, physics and biology (see [1], [8], [11]). From a mathematical point of view, the Mean Field approach leads to a study of a system of PDEs, where the classical Hamilton-Jacobi-Bellman equation is coupled with a Fokker-Planck equation for the density of the players, in an interesting forward-backward way. The decomposition method proposed here requires that each agent i computes in advance the time evolution of the local average (see, e.g., the Fokker-Planck-Kolmogorov equation in [2], [12], [16], [17]). However, since this is practically impossible, we use here the predictive control method to approximate the computation of the solution.

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D. Bauso is with Dipartimento di Ingegneria Chimica, Gestionale, Informatica, e Meccanica, Università di Palermo, V.le delle Scienze, 90128 Palermo, Italy dario.bauso@unipa.it

Q. Zhu and T. Başar are with Coordinated Science Laboratory and the Department of Electrical & Computer Engineering, University of Illinois at Urbana-Champaign, 1308 W. Main St., Urbana IL, USA, {zhu31, basar1}@illinois.edu

The paper is organized as follows. We present the problem statement in Section II. We then move to present the decomposition method in Section III. In Section IV, we turn to introducing the shortest path reformulation and the linear program. Finally, in Section V, we discuss the case where the local state average appears in the dynamics.

II. Mixed integer optimal compensation

In mixed integer optimal compensation problems, we have continuous states $x(k) \in \mathbb{R}^n$, continuous controls $u(k) \in \mathbb{R}^n$ and disturbances $w(k) \in \mathbb{R}^n$, and discrete controls $y(k) \in \{0,1\}^n$. Evolution of the state over a finite horizon of length N is described by a linear discrete-time (difference) equation in the general form (1), where A and E are matrices of compatible dimensions and $x(0) = \xi_0 \geq 0$ is the initial state. Continuous and discrete controls are linked together by general *capacity constraints* (2), where the (scalar) parameter c is an upper bound on control:

$$x(k+1) = Ax(k) + Ew(k) + u(k) \geq 0, \quad x(N) = 0, \qquad (1)$$
$$0 \leq u(k) \leq cy(k), \quad y(k) \in \{0,1\}^n. \qquad (2)$$

The above dynamics are characterized by one discrete and one continuous control variable per each state. Starting from nonnegative initial states, we force the states to remain confined to within the positive orthant, which may describe a safety region in engineering applications or the desire for preventing shortfalls in inventory applications. The final state, $x(N)$, has to be equal to zero, which corresponds to saying that the control $u(k)$ has to "compensate" the cumulative effects of the disturbances $Ew(k)$ and term $Ax(k)$ over the horizon.

The following assumption helps us to describe the common situation where the disturbance seeks to push the state out of the desired region.

Assumption 1 (Unstabilizing disturbance effects):

$$Ew(k) < 0, \qquad (3)$$

where the inequality is to be interpreted component-wise. Actually, the control actions push the state away from the boundaries into the positive orthant, thus counterbalancing the destabilizing effects of the disturbances. However, controlling the system has a cost and "over acting" on it is punished by introducing a cost/objective function. This function, to be minimized with respect to $y(k)$ and $u(k)$, is a linear one including proportional, holding and fixed cost terms expressed by parameters p^k, h^k, and f^k respectively:

$$\sum_{k=0}^{N-1} \left(\langle p^k, u(k) \rangle + \langle h^k, x(k) \rangle + \langle f^k, y(k) \rangle \right), \qquad (4)$$

where $\langle .,. \rangle$ denotes the Euclidean inner product. Conditions (1)-(4) describe the problem of interest. This can be turned into a mixed integer linear program by using the standard method discussed next.

A. Mixed integer linear program and exact solution.

Let us start by collecting states, continuous and discrete controls, proportional, holding and fixed costs all as appropriate vectors as shown below:

$$\begin{array}{ll} x = [x(0)^T \dots x(N)^T]^T, & u = [u(0)^T \dots u(N-1)^T]^T, \\ y = [y(0)^T \dots y(N-1)^T]^T, & p = [(p^0)^T \dots (p^{N-1})^T]^T, \\ h = [(h^0)^T \dots (h^{N-1})^T]^T, & f = [(f^0)^T \dots (f^{N-1})^T]^T. \end{array}$$

Furthermore, to put dynamics (1) into "constraints" form, let us define matrices $\mathbf{A}$, $\mathbf{B}$ and vector $\mathbf{b}$ as below, where I denotes the identity matrix:

$$\mathbf{A} = \begin{bmatrix} -I & 0 & \dots & 0 & 0 \\ A & -I & \dots & 0 & 0 \\ 0 & A & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A & -I \\ 0 & 0 & \dots & 0 & -I \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ I & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \\ 0 & 0 & \dots & 0 \end{bmatrix};$$

$$\mathbf{b} = \left[-\xi_0^T - (Ew(0))^T \dots - (Ew(N-1))^T \; 0 \right]^T$$

Finally, we are in a position to establish that problem (1)-(4) can be solved exactly through the following mixed integer linear program:

$$(MIPC) \quad \min_{u,y} \quad J(u,y) = \langle pu \rangle + \langle hx \rangle + \langle fy \rangle \qquad (5)$$
$$\mathbf{A}x + \mathbf{B}u = \mathbf{b}, \quad x \geq 0, \qquad (6)$$
$$0 \leq u \leq cy, \quad y \in \{0,1\}^{nN}. \qquad (7)$$

The mixed integer linear program (5)-(7) is the most natural mathematical programming representation of the problem of interest (1)-(4). For this reason, throughout this paper, unless otherwise stated, we will refer to (5)-(7) as the problem of interest instead of the original problem (1)-(4).

To overcome the intractability of the mixed integer linear program (5)-(7), we propose a new method whose underlying idea is to bring back dynamics (1) to the lot sizing model [14]. To do this, we introduce some additional assumptions on the structure of matrix A, which simplify the tractability and affect in no way the generality of the results.

B. Introducing some structure on A

With regard to (1), we can isolate the dependence of one component state on the other ones and rewrite (1) in a way that emphasizes the analogies with standard lot sizing models [14]:

$$x(k+1) = x(k) + \Delta x(k) + Ew(k) + u(k) \geq 0. \qquad (8)$$

Equation (8) is a straighforward representation of (1) once invoking (9). Let us denote by a_{ij} the dependence of state i on state j. So, matrix A can be decomposed as

$$A = I + \Delta, \qquad \Delta = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn-1} & a_{nn} \end{bmatrix}. \qquad (9)$$

To preserve the nature of the game which has stabilizing control actions playing against unstabilizing disturbances we suppose that the influence of other states on state i is relatively "weak".

Assumption 2 (Weakly coupling):

$$\Delta x(k)+Ew(k)<0, \tag{10}$$

where inequality is again component-wise. Essentially, the states' mutual dependence expressed by $\Delta x(k)$ only emphasizes or reduces "weakly" the destabilizing effects of the disturbances. In the next section, we present a decomposition approach that translates dynamics (8) into n scalar dynamics in "lot sizing" form [14].

III. ROBUST DECOMPOSITION

With the term "robust decomposition" we mean a mathematical manipulation through which dynamics (8) are replaced by n independent uncertain lot sizing models of the form (11) where $x_i(k)$ is the inventory, $d_i(k)$ the demand, $u_i(k)$ the reordered quantity and $\mathscr{D}_i^k \subset \mathbb{R}$ denotes the uncertainty set:

$$x_i(k+1)=x_i(k)-d_i(k)+u_i(k)\geq 0, \quad d_i(k)\in \mathscr{D}_i^k. \tag{11}$$

Replacing (8) with (11) is possible once we relate the demand $d_i(k)$ to the current values of all other state components and disturbances as expressed below:

$$\begin{array}{rcl} d_i(k) &=& -\left[\sum_{j=1}^n a_{ij}x_j(k)+\sum_{j=1}^n E_{ij}w_j(k)\right] \\ &=& -\left[\langle \Delta_{i\bullet}x(k)\rangle+\langle E_{i\bullet}w(k)\rangle\right], \end{array} \tag{12}$$

where we denote by $\Delta_{i\bullet}$ the ith row of matrix Δ. Same convention applies to $E_{i\bullet}$. To say it differently, we do assume that the influence that all other states have on state i enters into equation (11) through demand $d_i(k)$ defined in (12). Our next step is to make the n dynamics in the form (11) mutually independent.

To do this, we introduce X^k as the set of admissible state vectors $x(k)$ and observe that this set is always bounded for bounded $d_i(k)$. Then there always exists a scalar $\phi>0$ such that $\|x\|_\infty \leq \phi$ for all $x\in X^k$. In view of this, it is possible to decompose the system by replacing the current demand $d_i(k)$ by the maximal or minimal demand as computed below:

$$\begin{array}{rcl} d_i^+(k) &=& \max_{\xi\in X^k}\{-\langle\Delta_{i\bullet}\xi\rangle-\langle E_{i\bullet}w(k)\rangle\} \qquad (13)\\ &=& \sum_j [\Delta_{ij}]_-\phi-\langle E_{i\bullet}w(k)\rangle \\ d_i^-(k) &=& \min_{\xi\in X^k}\{-\langle\Delta_{i\bullet}\xi\rangle-\langle E_{i\bullet}w(k)\rangle\} \qquad (14)\\ &=& \sum_j [\Delta_{ij}]_+\phi-\langle E_{i\bullet}w(k)\rangle, \end{array}$$

where $[\Delta_{ij}]_+$ denotes the positive part of Δ_{ij}, i.e., $\max\{\Delta_{ij},0\}$ and $[\Delta_{ij}]_-$ the negative part. In the following we will write compactly $d_i^e(k)$, $e\in\{+,-,nil\}$ to generically address the maximal demand (13) when $e=+$, the minimal demand (14) when $e=-$, and the exact demand (12) when $e=nil$. From the above preamble we derive the uncertainty set as

$$\mathscr{D}_i^k=\{\eta\in\mathbb{R}: d_i^-(k)\leq\eta\leq d_i^+(k)\}.$$

The idea behind (13) is to take for estimated value the maximal demand, i.e., the demand that would push the state out of the positive orthant in the shortest time. Likewise, (14) describes the demand that would push the state out of the positive orthant in the longest time. To complete the decomposition, it remains to turn the objective function (4) into n independent components

$$J_i(u_i,y_i)=\sum_{k=0}^{N-1}\left(p_i^k u_i(k)+h_i^k x_i(k)+f_i^k y_i(k)\right).$$

Note that because of the linear structure of $J(u,y)$ in (5), we have $J(u,y)=\sum_{i=1}^n J_i(u_i,y_i)$. Then, we have translated our original problem into n independent mixed integer minimization problems of the form (15)-(17). In the spirit of predictive control, we solve, for $\tau=0,\ldots,N-1$, and $e(\tau)=nil$, $e(k)=e$, for $k>\tau$, $e\in\{nil,+,-\}$, and with ξ_i^τ being the measured state at time τ:

$$(MIPC_i)^e \quad \min_{u_i,y_i}\ \sum_{k=\tau}^{N-1}\left(p_i^k u_i(k)+h_i^k x_i(k)+f_i^k y_i(k)\right) \tag{15}$$

$$x_i(k+1)=x_i(k)-d_i^{e(k)}(k)+u_i(k)\geq 0, \tag{16}$$

$$x_i(\tau)=\xi_i^\tau,\ x_i(N)=0$$

$$0\leq u_i(k)\leq c y_i(k),\ y_i(k)\in\{0,1\}. \tag{17}$$

Denote by $(MIPC)^r$ the relaxation of $(MIPC)$ where $0\leq y\leq 1$.

Lemma 1: The following relations hold true:

$$(MIPC_i)^-,\ (MIPC)^r\leq (MIPC_i)\leq (MIPC_i)^+.$$

Proof: The conditions $(MIPC_i)^-\leq(MIPC_i)\leq(MIPC_i)^+$ are true as $d_i^-(k)\leq d_i(k)\leq d_i^+(k)$ for all $k=0,\ldots,N-1$ and the cost (15) is increasing in the demand. The inequality $(MIPC)^r\leq(MIPC_i)$ derives from observing that in $(MIPC)^r$ we relax the integer constraints on y and therefore the cost cannot be higher than in $(MIPC)$. ■

IV. SHORTEST PATH AND LINEAR PROGRAMMING

What we will establish is that, for the problem at hand, relaxing and massaging the problem in a certain manner, leads to a shortest path reformulation of the original problem. Shortest path formulations are based on the notion of regeneration interval as discussed next.

Let us borrow from [14] the concept of regeneration interval and adapt it to the generic minimization problem i expressed by (15)-(17).

Definition 1 (Pochet and Wolsey 1993): A pair of periods $[\alpha,\beta]$ forms a *regeneration interval* for (x_i,u_i,y_i) if $x_i(\alpha-1)=x_i(\beta)=0$ and $x_i(k)>0$ for $k=\alpha,\alpha+1,\ldots,\beta-1$.

Given a regeneration interval $[\alpha,\beta]$, we can define the accumulated demand over the interval $d_i^{\alpha\beta}$, and the residual demand $r_i^{\alpha\beta}$ as

$$d_i^{\alpha\beta}=\sum_{k=\alpha}^{\beta}d_i^{e(k)}(k),\quad r_i^{\alpha\beta}=d_i^{\alpha\beta}-\left\lfloor\frac{d_i^{\alpha\beta}}{C}\right\rfloor C. \tag{18}$$

Our idea is now to translate problem (15)-(17) into new variables. More formally, let us consider variables $y_i^{\alpha\beta}(k)$ and $\varepsilon_i^{\alpha\beta}(k)$ defined below with the following interpretation. Variable $y_i^{\alpha\beta}(k)$ is equal to one in the presence of a saturated control at time k, and zero otherwise. Similarly, variable $\varepsilon_i^{\alpha\beta}(k)$ is equal to one in the presence of a non-saturated control at time k, and zero otherwise:

$$y_i^{\alpha\beta}(k) = \begin{cases} 1 & u_i(k) = c \\ 0 & \text{otherwise.} \end{cases} \quad \varepsilon_i^{\alpha\beta}(k) = \begin{cases} 1 & 0 < u_i(k) < c \\ 0 & \text{otherwise.} \end{cases}$$

Variables $y_i^{\alpha\beta}(k)$ and $\varepsilon_i^{\alpha\beta}(k)$ tell us on which period full or partial batches are ordered. Then, we can rely on well known results from the lot sizing literature which convert the original mixed integer problem (15)-(17) into a number of linear programs $\left(LP_i^{\alpha\beta}\right)$, each one associated to a specific regeneration interval $[\alpha,\beta]$. Denoting by $e_i^k = p_i^k + \sum_{j=k+1}^{N-1} h_i^j$ and after some standard manipulation, the linear program $\left(LP_i^{\alpha\beta}\right)$ for fixed regeneration interval $[\alpha,\beta]$ appears as:

$$\min_{y_i^{\alpha,\beta},u_i^{\alpha,\beta}} \sum_{k=\alpha}^{\beta} \left(ce_i^k + f_i^k\right) y_i^{\alpha\beta}(k) + \sum_{k=\alpha}^{\beta} \left(r^{\alpha\beta} e_i^k + f_i^k\right) \varepsilon_i^{\alpha\beta}(k)$$

$$\sum_{k=\alpha}^{\beta} y_i^{\alpha\beta}(k) + \sum_{k=\alpha}^{\beta} \varepsilon_i^{\alpha\beta}(k) = \left\lceil \frac{d_i^{\alpha\beta}}{c} \right\rceil$$

$$\sum_{k=\alpha}^{t} y_i^{\alpha\beta}(k) + \sum_{k=\alpha}^{t} \varepsilon_i^{\alpha\beta}(k) \geq \left\lceil \frac{d_i^{\alpha t}}{c} \right\rceil, \quad t = \alpha, \ldots, \beta - 1$$

$$\sum_{k=\alpha}^{\beta} y_i^{\alpha\beta}(k) = \left\lfloor \frac{d_i^{\alpha\beta} - r_i^{\alpha\beta}}{c} \right\rfloor$$

$$\sum_{k=\alpha}^{t} y_i^{\alpha\beta}(k) \geq \left\lfloor \frac{d_i^{\alpha t} - r_i^{\alpha t}}{c} \right\rfloor, \quad t = \alpha, \ldots, \beta - 1$$

$$y_i^{\alpha\beta}(k), \varepsilon_i^{\alpha\beta}(k) \geq 0, \quad k = \alpha, \ldots, \beta.$$

The above model has been extensively used in the lot sizing context. The first and third equality constraints tell us that the ordered quantity over the interval has to be equal to the accumulated demand over the same interval. This makes sense as the initial and final states of a regeneration interval are null by definition. The second and fourth inequality constraints impose that the accumulated demand in any subinterval may not exceed the ordered quantity over the same subinterval. Again, this is due to the condition that the states are nonnegative in any period of a regeneration interval. Finally, the objective function is simply a rearrangement of (15) induced by the variable transformation seen above and specialized to the regeneration interval $[\alpha,\beta]$ rather than on the entire horizon $[0,N]$.

The solutions of $(LP_i^{\alpha\beta})$ that are binary are called "feasible". Then, we are ready to recall the following "nice property" of $(LP_i^{\alpha\beta})$ presented first by Pochet and Wolsey in [14].

Theorem 1 (Total Uni-modularity): The optimal solution of $(LP_i^{\alpha\beta})$ is feasible.

Proof: Observe that the constraint matrix of $(LP_i^{\alpha\beta})$ is a $0-1$ matrix. We can reorder the constraints in a certain manner, so that matrix has the consecutive 1's property on each column and turns out to be totally unimodular. It follows that $y_i^{\alpha,\beta}$ and $\varepsilon_i^{\alpha,\beta}$ are $0-1$ in any extreme solution. ■

A. Shortest path

We now resort to well-known results on lot sizing to come up with a shortest path model which links together the linear programming problems of all possible regeneration intervals.

So, let us define variables $z_i^{\alpha\beta} \in \{0,1\}$, which yield 1 when a regeneration interval $[\alpha,\beta]$ appears in the solution of (15) -(17), and 0 otherwise. The linear programming problem (LP_i) solving (15) -(17) takes on the form below. For $\tau = 0,\ldots,N-1$, solve

$$\min_{y_i^{\alpha\beta},u_i^{\alpha\beta},z_i^{\alpha\beta}} \sum_{\alpha=\tau+1}^{N-1} \sum_{\beta=\alpha}^{N-1} \sum_{k=\alpha}^{\beta} \left[\left(ce_i^k + f_i^k\right) y_i^{\alpha\beta}(k) + \sum_{k=\alpha}^{\beta} \left(r^{\alpha\beta} e_i^k + f_i^k\right) \varepsilon_i^{\alpha\beta}(k) \right]$$

$$\sum_{\beta=\tau+1}^{N} z_i^{\tau+1,\beta} = 1$$

$$\sum_{\alpha=\tau+1}^{t-1} z_i^{\alpha,t-1} - \sum_{\beta=t}^{N} z_i^{t\beta} = 0 \quad t = \tau+2,\ldots,N, \quad \tau+1 \leq \alpha \leq \beta \leq N$$

$$\sum_{k=\alpha}^{\beta} y_i^{\alpha\beta}(k) + \sum_{k=\alpha}^{\beta} \varepsilon_i^{\alpha\beta}(k) = \left\lceil \frac{d_i^{\alpha\beta}}{c} \right\rceil z_i^{\alpha\beta}, \quad \tau+1 \leq \alpha \leq \beta \leq N$$

$$\sum_{k=\alpha}^{t} y_i^{\alpha\beta}(k) + \sum_{k=\alpha}^{t} \varepsilon_i^{\alpha\beta}(k) \geq \left\lceil \frac{d_i^{\alpha t}}{c} \right\rceil z_i^{\alpha\beta}, \quad t = \alpha,\ldots,\beta-1, \quad \tau+1 \leq \alpha \leq \beta \leq N$$

$$\sum_{k=\alpha}^{\beta} y_i^{\alpha\beta}(k) = \left\lfloor \frac{d_i^{\alpha\beta} - r_i^{\alpha\beta}}{c} \right\rfloor z_i^{\alpha\beta} \quad \tau+1 \leq \alpha \leq \beta \leq N$$

$$\sum_{k=\alpha}^{t} y_i^{\alpha\beta}(k) \geq \left\lfloor \frac{d_i^{\alpha t} - r_i^{\alpha t}}{c} \right\rfloor z_i^{\alpha\beta}, \quad t = \alpha,\ldots,\beta-1, \quad \tau+1 \leq \alpha \leq \beta \leq N$$

$$y_i^{\alpha\beta}(k), \varepsilon_i^{\alpha\beta}(k), z_i^{\alpha\beta} \geq 0, \quad k = \alpha,\ldots,\beta.$$

The above constraints have already appeared in $\left(LP_i^{\alpha\beta}\right)$. The only difference here is that, now, because of the presence of $z_i^{\alpha\beta}$ in the right hand term, the constraints referring to a given regeneration interval come into play only if that interval is chosen as part of the solution, that is, whenever $z_i^{\alpha\beta}$ is set equal to one. Furthermore, a new class of constraints appear in the first line of constraints. These constraints are typical of shortest path problems and in this specific case helps us to force the variables $z_i^{\alpha\beta}(k)$ to describe a path from 0 to N. Finally, note that for $\tau = 0$, the linear program (LP_i) coincides with the linear program presented by Pochet and Wolsey in [14].

At this point, we are in a position to recall the important result established by Pochet and Wolsey in [14] and adapt it to $(MIPC_i)$ within the assumption of null final state (high values of h_i^N).

Theorem 2: The linear program (LP_i) solves $(MIPC_i)$ with null final state.

Proof: It turns out that the linear program (LP_i) is a shortest path problem on variables $z_i^{\alpha,\beta}$. Arcs are all associated to a different regeneration interval $[\alpha,\beta]$ and the respective costs are the optimal values of the objective functions of the corresponding linear programs $(LP_i^{\alpha,\beta})$ (cf. [14]). ■

B. Receding horizon implementation of (LP_i)

The main difference between the lot sizing model [14] and the (LP_i) of the present paper is that in the (LP_i) the initial state is non null. Actually, consecutive linear programs (LP_i) are linked together by the initial state condition expressed in (16), and which we rewrite below

$$x_i(\tau) = \xi_i^\tau.$$

To counter this little issue, we need to elaborate more on how to compute the accumulated demand in (18). Actually, take for $[\tau,t]$ any interval with $x(\tau) = \xi_i^\tau > 0$. Then, condition (18) needs to be revised as

$$d_i^{\tau t} = \max\left\{\sum_{k=\tau}^{t} d_i^{e(k)}(k) - \xi_i^\tau, 0\right\}. \tag{19}$$

Actually, the effective demand over an interval is the accumulated demand reduced by the inventory stored and initially available at the warehouse. From a computational standpoint, the revised formula (19) has a different effect depending on the cases where the accumulated demand exceeds the initial state or not, as discussed next.

1) $\sum_{k=\alpha}^{\beta} d_i^{e(k)}(k) \geq \xi_i^\tau$: the mixed linear program $(MIPC_i)$ with initial state $x(\tau) = \xi_i^\tau > 0$ and accumulated demand $\sum_{k=\alpha}^{\beta} d_i^{e(k)}(k)$ is turned into an (LP_i) characterized by null initial state $x(\alpha-1) = 0$ and effective demand $d_i^{\alpha\beta} = \sum_{k=\alpha}^{\beta} d_i^{e(k)}(k) - \xi_i^\tau$ as in the example below:

$$(MIPC_i) \quad \sum_{k=\alpha}^{\beta} d_i^{e(k)}(k) = 12, \quad x(\tau) = \xi_i^\tau = 10$$
$$\Longrightarrow (LP_i) \quad x(\alpha-1) = 0, \quad d_i^{\alpha\beta} = 2;$$

2) $\sum_{k=\alpha}^{\beta} d_i^{e(k)}(k) < \xi_i^\tau$: the mixed linear program $(MIPC_i)$ with initial state $x(\tau) = \xi_i^\tau > 0$ and accumulated demand $\sum_{k=\alpha}^{\beta} d_i^{e(k)}(k)$ is infeasible. The solution obtained at the previous period $\tau-1$ applies. A second example is shown next:

$$(MIPC_i) \quad \sum_{k=\alpha}^{\beta} d_i^{e(k)}(k) = 7, \quad x(\tau) = \xi_i^\tau = 10$$
$$\Longrightarrow (LP_i) \text{ unfeasible.}$$

V. Mean field coupling

In this section, we provide a discussion on a special case of interest where each agent seeks to compensate a combination of the exogenous signal and the local state average. In this case, our decomposition idea is similar to mean-field methods in large population consensus [9], [19], [18]. We discuss the mean-field approximations as well as the application of predictive control methods to approximate the computation.

Consider a graph $G = (V,E)$ with a set of vertices $V = \{1,\ldots,n\}$ and a set of edges $E \subseteq V \times V$. Denote by N_i the neighborhood of agent i, i.e., $N_i = \{j \in V : (i,j) \in E\}$. We can associate with the graph G the normalized graph Laplacian matrix $L \in \mathbb{R}^{n\times n}$ whose ij-th entry is

$$l_{ij} = \begin{cases} \frac{-1}{|N_i|} & j \in N_i \\ 1 & j = i \end{cases}.$$

Now, a special case of interest is when $\Delta = -\varepsilon L$ for any small enough scalar $\varepsilon > 0$. In this case dynamics (8) turns into:

$$x(k+1) = x(k) - \varepsilon L x(k) + E w(k) + u(k) \geq 0.$$

Essentially, the above dynamics together with the constraint $x_N = 0$ is paradigmatic of all those situations where each agent $i = 1,\ldots,n$ tries to compensate a combination of the exogenous signal $w(k)$ and the local state average

$$\bar{m}_i(k) = \frac{1}{|N_i|}\sum_{j\in N_i}(x_j(k) - x_i(k)).$$

Elaborating along the line of the robust decomposition (11), we then can compute the disturbance taking into account the influence of the local average on the exogenous signal as follows:

$$d_i(k) = -\left[\varepsilon \bar{m}_i(k) + \langle E_{i\bullet}, w(k)\rangle\right].$$

Note that Assumption (2) means that the exogenous signal is dominant if compared to the weak influence from neighbors.

In principle, for the decomposition method to be exact, each agent i should know in advance the time evolution of the local average $\bar{m}_i(k)$ for $k = 0,\ldots,N$. However, this may not be tractable. One way to approximate the mean $\bar{m}_i(k)$ is through mean-field methods. Under the further assumption that the number of agents is large and the agent dynamics are symmetric, the mean can be characterized through the finite-difference approximation of the continuity or advection equation that describes the transport of a conserved quantity [19]. Another way to deal with the problem is to use the predictive control method to approximate the computation. More specifically, when we solve the problem over the horizon from $\tilde{k} \geq 0$ to N, we assume that neighbor agents communicate their state and so at least the first sample $\bar{m}_i(\tilde{k})$ is exact. In the later stages of the horizon each agent approximates the local average by specializing (13)-(14) to our case. Actually, observe that maximal and minimal demand can be obtained by assuming that all agents $j \neq i$ are in 0 or ϕ respectively, and so we have for agent i:

$$\begin{aligned} d_i^+(k) &= \varepsilon x_i - \langle E_{i\bullet}, w(k) \rangle \\ d_i^-(k) &= -\left[\varepsilon(\phi - x_i) + \langle E_{i\bullet}, w(k) \rangle\right]. \end{aligned}$$

Alternatively, this also corresponds to assuming for the uncertain set $\mathscr{D}_i^k$ the following expression:

$$\mathscr{D}_i^k = \{\eta \in \mathbb{R} : -\varepsilon(\phi - x_i) - \langle E_{i\bullet}, w(k)\rangle \leq \eta \leq \varepsilon x_i - \langle E_{i\bullet}, w(k)\rangle\}.$$

The above set up introduces the following numerical example.

Consider a complete network of $n = 10$ agents. The local state average is the same for all i and also equal to the global average, i.e., for all i it holds $\bar{m}_i(k) = \frac{1}{n}\sum_{j \in V, j \neq i}(x_j(k) - x_i(k))$. The horizon length is $N = 15$, the scalar $\varepsilon = 0.1$, the initial state is $x(0) = [4 \dots 13]$, and the disturbance is $E_{i\bullet} w(k) = 1$ if k is odd and $E_{i\bullet} w(k) = 2$ otherwise for all agents i. The bound on input is $C = 3$ and the objective function is displayed below where $\mathbf{1}^n$ indicates the n-dimensional row vector on 1's:

$$J(u,y) = \sum_{k=0}^{N-1} \left(\langle \mathbf{1}^n, u(k)\rangle + \langle \mathbf{1}^n, x(k)\rangle + 100\langle \mathbf{1}^n, y(k)\rangle\right). \quad (20)$$

We also take $\phi = 13$. We plot in Fig. 1 the time evolution of the state $x(k)$. As expected, the state is non-negative for all

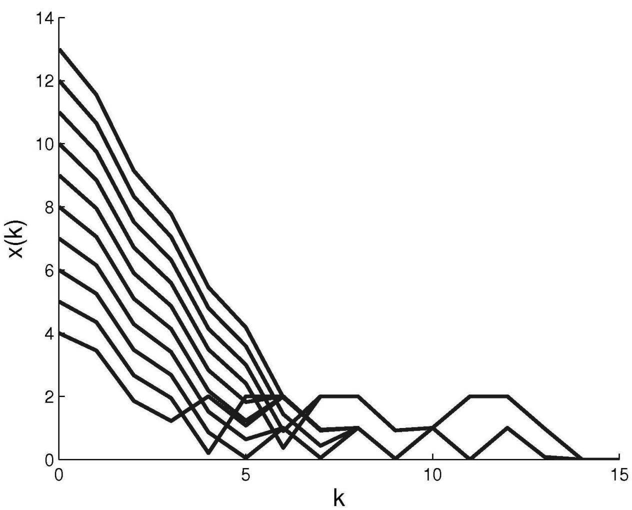

Fig. 1. Time evolution of state $x(k)$.

k. Also, the state $x(k)$ converges to a neighborhood of zero of size $c - \min_k\{d_i^-(k)\} = 2$.

VI. Conclusions and Future Developments

In a nutshell, we have proposed a robust decomposition method which reframes an n-dimensional problem into n independent tractable scalar problems of lot sizing form. Through an example, we have illustrated the mean-field coupling in a multi-agent system problem, where each agent seeks to compensate a combination of the exogenous signal and the local state average. We have discussed a large population mean-field type of approximation as well as the application of predictive control methods.

There are at least three possibilities for future developments. First, we will analyze connections between regeneration intervals and reverse dwell time conditions developed in hybrid/impulsive control. Second, we would like to zoom in on the exploitation of cutting planes methods to increase the efficiency of linear relaxation approximations. Third, we need to investigate the mean-field large population approximations that arise from the decomposition of the mixed-integer optimal compensation problem.

References

[1] Y. Achdou, F. Camilli, I. Capuzzo Dolcetta, "Mean field games: numerical methods for the planning problem", *SIAM J. of Control Opt.* vol. 50, 2012, pp. 77–109.

[2] Y. Achdou and I. Capuzzo Dolcetta, "Mean field games: numerical methods", *Siam J. Numer. Anal.*, vol. 48, 2010, pp. 1136–1162.

[3] D. Axehill, L. Vandenberghe, and A. Hansson, "Convex relaxations for mixed integer predictive control", *Automatica*, vol. 46, no. 9, 2010, pp. 1540–1545.

[4] D. Bauso, "Boolean-controlled systems via receding horizon and linear programing", *Mathematics of Control, Signals, and Systems (MCSS)*, vol. 21, no. 1, 2009, pp. 69-91.

[5] M. S. Branicky, V. S. Borkar, and S. K. Mitter, "A unified framework for hybrid control: model and optimal control theory", *IEEE Trans. on Automatic Control*, vol. 43, no. 1, 1998, pp. 31–45.

[6] P. R. De Waal and J. H. Van Schuppen, "A class of team problems with discrete action spaces: optimality conditions based on multimodularity," *SIAM Journal on Control and Optimization*, vol. 38, 2000, pp. 875–892.

[7] G.C. Goodwin and D.E. Quevedo, "Finite alphabet control and estimation," *International Journal of Control, Automation, and Systems*, vol. 1, no. 4, 2003, pp. 412–430.

[8] O. Gueant, J. M. Lasry, and P. L. Lions, "Mean field games and applications", *Paris-Princeton Lectures*, Springer, 2010, pp. 1–66.

[9] M.Y. Huang, P.E. Caines, and R.P. Malhamé, "Large population cost-coupled LQG problems with non-uniform agents: individual-mass behaviour and decentralized ε-Nash equilibria," *IEEE Trans. on Automatic Control*, vol. 52. no. 9, September, 2007, pp. 1560–1571.

[10] J. Hespanha, D. Liberzon, and A. Teel, "Lyapunov characterizations of input-to-state stability for impulsive systems", *Automatica*, vol. 44, no. 11, 2008, pp. 2735–2744.

[11] A. Lachapelle, J. Salomon, and G. Turinici, "Computation of Mean Field Equilibria in Economics", *Math. Models Meth. Appl. Sci.*, vol. 20, 2010, pp. 1–22.

[12] J.-M. Lasry and P.-L. Lions, "Mean field games", *Jpn. J. Math.*, vol. 2, 2007, pp. 229-260.

[13] G. L. Nemhauser and L. A. Wolsey, *Integer and Combinatorial Optimization*, John Wiley & Sons Ltd, New York, 1988.

[14] Y. Pochet and L. A. Wolsey, "Lot sizing with constant batches: Formulations and valid inequalities", *Mathematics of Operations Research*, vol. 18, no. 4, 1993, pp. 767–785.

[15] D. C. Tarraf, A. Megretski, and M. A. Dahleh, "A framework for robust stability of systems over finite alphabets", *IEEE Transactions on Automatic Control*, vol. 53, no. 5, 2008, pp. 1133–1146.

[16] H. Tembine, Q. Zhu, and T. Başar, "Risk-sensitive mean-field stochastic differential games," in Proc. 2011 IFAC World Congress, Milan, Italy, August 29 - September 2, 2011.

[17] Q. Zhu, H. Tembine, and T. Başar, "Hybrid risk-sensitive mean-field stochastic differential games with application to molecular biology," in Proc. of Conference on Decision and Control, Orlando, FL, Dec. 14 - Dec. 16, 2011.

[18] Q. Zhu and T. Başar, "A multi-resolution large population game framework for smart grid demand response management," in Proc. of Intl. Conference on Network Games, Control and Optimization (NETGCOOP 2011), Paris, France. Oct. 12 - 14, 2011.

[19] Q. Zhu, D. Bauso and T. Başar, "Large population consensus in an adversarial environment," in Proc. of Eighth International ISDG Workshop, July 21-23, 2011, Padova, Italy.