COMPUTATIONAL MODELING OF CRITICAL POINTS AND STRUCTURAL DISCONTINUITIES

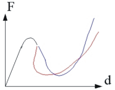
J L Curiel Sosa
Ingeniero Superior Industrial, Universidad de Sevilla
Master of Research, University of Wales Swansea

Civil and Computational Engineering Centre,
University of Wales Swansea, 2006

IN-TIME IMPLICIT/EXPLICIT ALGORITHM

Introduction

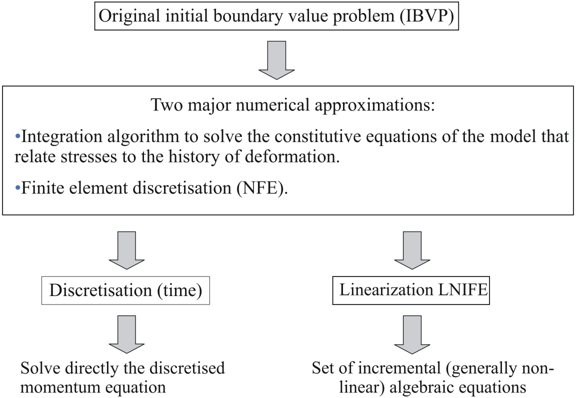
- Motivation – solution of convergence problems in FEM such as limit points
- Critical points. A structure is uncontrollable or partially uncontrollable at these points (snap-through, snap-back).



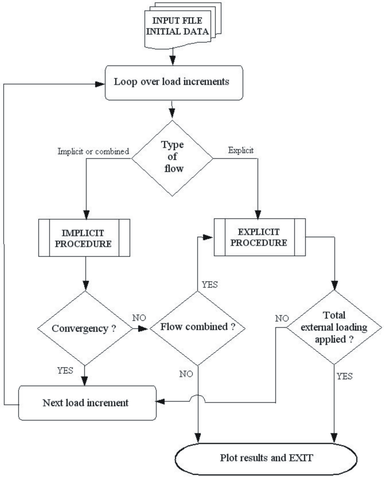
- Divergence in methods of solution
- An algorithmic tool to solve these problems in a combination of efficiency and CPU-time-saving is proposed – in-time implicit-explicit algorithm (I/E).
- Many software packages (such as ABAQUS, LS-DYNA, etc) use explicit methods from the beginning when modelling these critical singularities. It often expands the processing time (non-large scale problems).
- Implicit – iterative Newton-Raphson. Its quadratic rates of convergence $\longrightarrow$ faster than explicit methods in non-large scale problems (as the computer time varies with the square of the bandwidth of the mesh)
- It is remarked that there is not partitioning of the mesh.

LNIFE vs Direct Integration

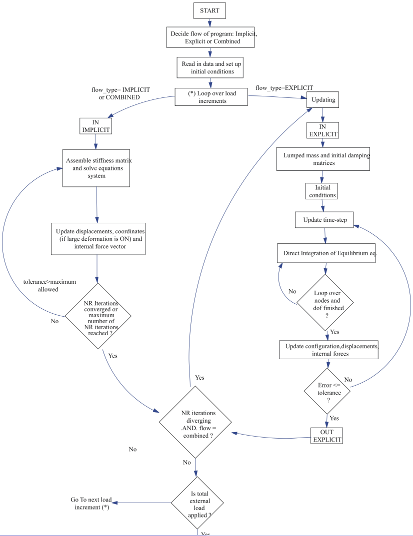
Numerical approximations of the schemes considered. Linearised Non-linear Incremental Finite Element equations (LNIFE) (implicit) and direct time integration by Central Difference Method (explicit).



Simplified Implicit/Explicit Flowchart



Implicit/Explicit Flowchart



Implicit Sub-algorithm I

1. Initiate $(k=0)$

$$\boldsymbol{u}_{n+1}^{(0)} = \boldsymbol{u}_n \qquad \boldsymbol{R} = \boldsymbol{f}^{int}(\boldsymbol{u}_n) - \boldsymbol{f}^{ext} \tag{1}$$

2. For all elements, calculate consistent tangent stiffness matrix.

$$\hat{\mathbf{D}} = \frac{\partial \hat{\boldsymbol{\sigma}}}{\partial \boldsymbol{\varepsilon}_{\mathbf{n+1}}} \tag{2}$$

3. Assembly of stiffness matrices

$$\mathbf{k}_T^{(e)} = \sum_{j=1}^{n_{gaus}} \xi_j \, \mathbf{B}_j^T \, \hat{\mathbf{D}}_j \, \mathbf{B}_j \tag{3}$$

4. Increment iteration counter $(k=k+1)$, assembly, solve the linearized equilibrium equation

$$\mathbf{K}_T \delta \boldsymbol{u}^{(k)} = -\boldsymbol{R}^{(k-1)}(\boldsymbol{u}_{n+1}) \tag{4}$$

and update stresses and internal variables:

$$\boldsymbol{u}_{n+1}^{(k)} = \boldsymbol{u}_{n+1}^{(k-1)} + \delta \boldsymbol{u}^{(k)} \tag{5}$$

$$\tag{6}$$

Implicit Sub-algorithm II

4. (continued)

$$\boldsymbol{\varepsilon}_{n+1}^{(k)} = \mathbf{B}\, \boldsymbol{u}_{n+1}^{(k)} \qquad \boldsymbol{\sigma}_{n+1}^{(k)} = \hat{\boldsymbol{\sigma}}(\boldsymbol{\alpha}_n, \boldsymbol{\varepsilon}_{n+1}^{(k)}) \qquad \boldsymbol{\alpha}_{n+1}^{(k)} = \hat{\boldsymbol{\alpha}}(\boldsymbol{\alpha}_n, \boldsymbol{\varepsilon}_{n+1}^{(k)}) \tag{7}$$

5. New internal forces at each element

$$\boldsymbol{f}_{(e)}^{int} = \sum_{j=1}^{n_{gaus}} \xi_j \, J_j \mathbf{B}_j^T \, \boldsymbol{\sigma}_{n+1,\,j}^{(k)} \tag{8}$$

6. Gathering of element internal forces vector and updating residual.
7. If iterations diverge then go to the EXP scheme
   else:
   - 7.a If $\frac{\|\boldsymbol{f}^{ext} - \boldsymbol{f}^{int}\|}{\|\boldsymbol{f}^{ext}\|} \leq \epsilon$ then the solution for current external load is reached and values for this load are from the last iteration $(\bullet)_{n+1} = (\bullet)_{n+1}^{(k)}$
   - 7.b else go to (II).
8. If the total load is not completely applied, increment the external load and go to 2, else exit.

Explicit Sub-algorithm I

The following central differences are utilized for the approximation of the corresponding derivatives,

$$\dot{\mathbf{u}}_{n-\frac{1}{2}} = \frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{\Delta t_{n-\frac{1}{2}}} \tag{9}$$

$$\ddot{\mathbf{u}}_n = \frac{\dot{\mathbf{u}}_{n+\frac{1}{2}} - \dot{\mathbf{u}}_{n-\frac{1}{2}}}{\Delta t_n} \tag{10}$$

$$\dot{\mathbf{u}}_n = \frac{\dot{\mathbf{u}}_{n+\frac{1}{2}} + \dot{\mathbf{u}}_{n-\frac{1}{2}}}{2} \tag{11}$$

where Δt_n has been set in a general form as an adaptive time each increment. Substituting Eqs (10, 11) in the momentum equations results in the following expressions for the velocity,

$$\dot{\mathbf{u}}_{n+\frac{1}{2}} = \frac{2\mathbf{M} - \Delta t_n \mathbf{C}}{2\mathbf{M} + \Delta t_n \mathbf{C}} \dot{\mathbf{u}}_{n-\frac{1}{2}} + \frac{\mathbf{f}^{ext}(t_n) - \mathbf{f}^{int}(t_n)}{2\mathbf{M} + \Delta t_n \mathbf{C}} \tag{12}$$

With this expression the displacement at step $n+1$ is simply given by,

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t_n \, \dot{\mathbf{u}}_{n+\frac{1}{2}} \tag{13}$$

Explicit Sub-algorithm II

The mid-time increments are calculated as follows,

$$\Delta t_{n-\frac{1}{2}} = \frac{\Delta t_n + \Delta t_{n-1}}{2} \tag{14}$$

To preserve an explicit iterative scheme the mass and damping matrices must be diagonal which results in an uncoupled system of equations,

$$\dot{\mathbf{u}}_{n+\frac{1}{2}} = \frac{2-\Delta t_n p}{2+\Delta t_n p}\dot{\mathbf{u}}_{n-\frac{1}{2}} + 2\Delta t_n \mathbf{M}^{-1}\frac{\mathbf{f}^{ext}(t_n) - \mathbf{f}^{int}(t_n)}{2+\Delta t_n p} \tag{15}$$

As any other explicit method, it is conditionally stable.

$$\dot{u}_{i,\,n+\frac{1}{2}} = \frac{2-\Delta t_n p}{2+\Delta t_n p}\dot{u}_{i,\,n-\frac{1}{2}} + 2\Delta t_n \frac{f_i^{ext}(t_n) - f_i^{int}(t_n)}{(2+\Delta t_n p)m_{ii}} \tag{16}$$

The step is completed with the update of displacements by Eq (13) which is written in notation index in Eq (17).

$$u_{i,\,n+1} = u_{i,\,n} + \Delta t_n\, \dot{u}_{i,\,n+\frac{1}{2}} \tag{17}$$

Explicit Sub-algorithm III

Initiation of CDM The Eqs (15,16) need the not-known velocity at $t_{-\frac{1}{2}}$. We need other initial condition to start the iterations. Thus, using Eq(11) and $\dot{\mathbf{u}}_0 = 0$ is obtained that,

$$\dot{\mathbf{u}}_{-\frac{1}{2}} = -\dot{\mathbf{u}}_{\frac{1}{2}} \tag{18}$$

For the first iteration, the Eqs (15,16) become,

$$\dot{\mathbf{u}}_{\frac{1}{2}} = \Delta t_n \mathbf{M}^{-1} \frac{\mathbf{f}^{ext}(t_0) - \mathbf{f}^{int}(t_0)}{2} \tag{19}$$

$$\dot{u}_{i,\frac{1}{2}} = \Delta t_n \frac{f_i^{ext}(t_0) - f_i^{int}(t_0)}{2m_{ii}} \tag{20}$$

Dynamic Relaxation The problem is reduced to optimize mass matrix, damping matrix and time step to get the maximum velocity of convergence to obtain the displacements such that internal and external forces are in equilibrium.

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t_{n+\frac{1}{2}} \dot{\mathbf{u}}_{n+\frac{1}{2}} \tag{21}$$

$$\mathbf{C}_\beta = \sqrt{\frac{\mathbf{u}_n^T \mathbf{K}_n \mathbf{u}_n}{\mathbf{u}_n^T \mathbf{M} \mathbf{u}_n}} \qquad K_{ii} = \frac{f_i(\mathbf{u}_n) - f_i(\mathbf{u}_{n-1})}{\Delta t_{n-\frac{1}{2}} \dot{u}_{n-\frac{1}{2}}} \tag{22}$$

Adaptive Timestep

Step time adaptivity in function of the natural frequencies (faster performance).

$$\Delta t\left(t_{n+1}\right) = \frac{2}{max_i\{\omega_i(t_n)\}} \quad (23)$$

The natural frequencies are determined substituting $\mathbf{u}(t) = \widetilde{\mathbf{u}}e^{-j\omega t}$ and $\ddot{\mathbf{u}}(t) = -\omega^2\widetilde{\mathbf{u}}\exp(-j\omega t)$ into the homogeneous problem $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = 0$. It renders,

$$-\omega^2\mathbf{M}(cos(\omega t) - jsin(\omega t)) + \mathbf{K}(cos(\omega t) - jsin(\omega t)) = 0$$

which leads to the classical eigenvalue problem

$$| - \omega^2\mathbf{M} + \mathbf{K}| = 0$$

where ω^2 are the eigenvalues of the system that provide the natural frequencies for each node and degree of freedom, i.e. each variable.
Approximating stiffness matrix as,

$$K_{ii}(t_n) \simeq \frac{f_i^{int}(u_n) - f_i^{int}(u_{n-1})}{\dot{u}_i^{n-1/2}\Delta t_n} \quad (24)$$

$$\omega_i(t_n) = \sqrt{\frac{K_i(t_n)}{M_i}} \quad (25)$$

Transition implicit/explicit

The last converged iteration at IMP (displacement $\tilde{\mathbf{u}}$) is transferred to EXP as initial conditions. Thus, the internal forces and displacements, from IMP, are used to determine initial accelerations and velocities for EXP.

$$\mathbf{f}^{int}(\tilde{\mathbf{u}}) \quad \rightarrow \quad \mathbf{f}^{int}(0) \tag{26}$$

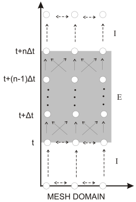
$$\tilde{\mathbf{u}} \quad \rightarrow \quad \mathbf{u}(0) \tag{27}$$

Initial velocities for EXP are approximated as follows:

$$\ddot{u}_i(0) = \frac{f_i^{ext} - f_i^{int}(0)}{M_{ii}} \tag{28}$$

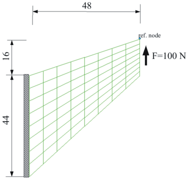
$$\dot{u}_i(0) = \ddot{u}_i(0)\,\Delta t(0) + \dot{u}_i^-(0) \tag{29}$$

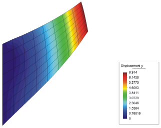
$$\dot{u}_i^-(0) = 0.0 \tag{30}$$



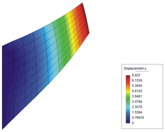
Once the solution is reached the flow is returned back to the implicit sub-algorithm if the external load is not still totally applied (otherwise, the execution is ended). Obviously, the converged solution values of the explicit sub-algorithm are passed on to the implicit one.

Cook's Membrane

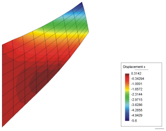




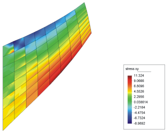
Vertical displacement $u_y\ (mm)$ by IMP



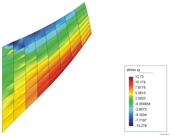
Vertical displacement $u_y\ (mm)$ by I/E



Displacement $u_x\ (mm)$ by I/E



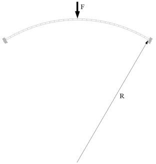
Shear stress $\sigma_{xy}\ (N/mm^2)$ by IMP (division of total external loads by increments).

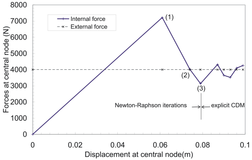


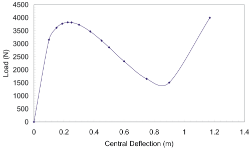
Shear stress $\sigma_{xy}\ (N/mm^2)$ by I/E

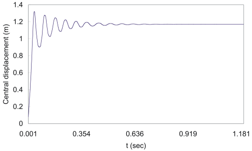
Trahair's Arch

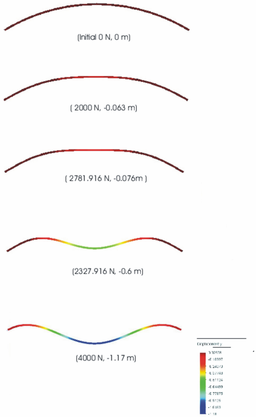
The central load is increased up to $4000\ N$. NRM began to diverge at deflection $|\delta_{crit}| = 0.076\ m\ (2781.917N)$. Explicit sub-alg. is initiated (with the last converged solution of NRM – point (3)) and executed. Excellent agreement with the equilibrium path obtained by YL Pin and Trahair (1998), Calhoum and DaDeppo (1983), Handbook of Structural Stability, Japan (1971)











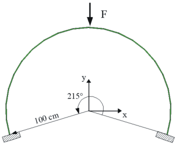
Relative Residual Norm

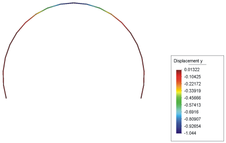
Relative residual norm (%) (error(%) in the table) and maximum residual in arch. Vertical internal force (N) and displacement (m) at central node of the arch(k_{cn}) . In the second column, the iteration number(i) is shown for the IMP sub-algorithm and the time (t_n) for the EXP sub-algorithm

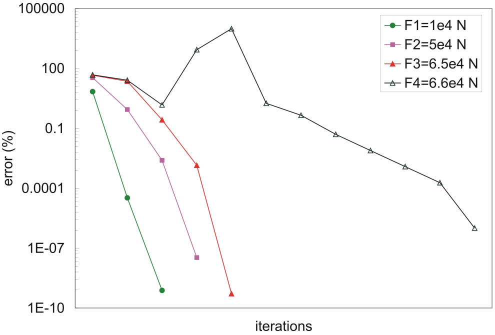
Flow	i/t_n	Error(%)	Max. Resid.	$\mathbf{f}_y^{int}(k_{cn})$	$\mathbf{u}_y(k_{cn})$	Stage
IMP	1	3,118.820	197,857.00	-	-	-
IMP	2	59.346	3,767.75	-7,220.6	-0.061	diverging
IMP	3	3,116.590	202,416.00	-4,015.2	-0.074	swap
EXP	0.0001	73.404	1,616.89	-3,477.9	-0.074	oscillat.
EXP	0.001	64.204	785.12	-4,389.4	-0.075	oscillat.
EXP	0.1	2.386	336.19	-3,971.2	-0.466	oscillat.
EXP	0.5	0.092	29.74	-3,998.5	-1.193	converging
EXP	0.641	0.033	9.98	-3,999.3	-1.17	converging
EXP	1.0	0.007	2.016	-4,001.27	-1.17	converging
EXP	1.5	10^{-3}	0.293	-4,000.16	-1.17	converging
EXP	2.0	$4.4 \cdot 10^{-5}$	0.017	-4,000.01	-1.17	solution

Circular Arch

- Point loading at the centre of the arch. No rotation is permitted at the supports.
- 20 quadrilateral elements with 4 nodes and 4 gauss points
- Thickness $10\,cm$, Young Modulus $210 \cdot 10^5\ \frac{N}{cm^2}$ and Poisson ratio 0.3
- Von-Mises perfectly plastic, yield stress $24500\,N/cm^2$





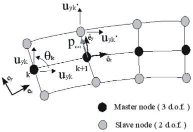


CONT.-BASED BEAM ELEMENT (CBE)

Continuum-based beam elements (CBE)

- To be used in reinforcement in materials such as RC
- Structural elements require a smaller timestep ⇒ CPU-time expanded.
- The CBE used only need to pose momentum equations at the master nodes ⇒ half of the nodes needed

$$\dot{\mathbf{u}}^T = \{\ \dot{u}_{x1} \quad \dot{u}_{y1} \quad \dot{\theta}_1 \quad \dots \quad \dot{u}_{x\kappa} \quad \dot{u}_{y\kappa} \quad \dot{\theta}_\kappa \quad \dots \quad \dot{u}_{x\ nmast} \quad \dot{u}_{y\ nmast} \quad \dot{\theta}_{nmast}\ \}$$



$$\mathbf{f}_\kappa^{(ext)} = \mathbf{T}_\kappa^T \cdot \mathbf{f}^{(ext)}_{\kappa^- \ \kappa^+} \tag{31}$$

- Central Differences Method - variables: velocities at masters,

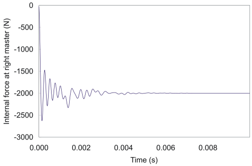
$$\mathbf{M}\,\ddot{\mathbf{u}}(t_n) + \mathbf{C}\,\dot{\mathbf{u}}(t_n) + \mathbf{f}^{int}(\mathbf{u}_n) = \mathbf{f}^{ext} \tag{32}$$

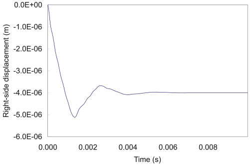
- Main differences with Belytschko CBE,
  - The velocity gradient update is made on the slaves rather than in the masters
  - CBE may be expanded or contracted on transversal direction
  - Stress computation considers convective terms of stress rate

Compression

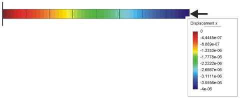
A bar, formed by 8 CBE elements, is compressed in this test. The external force (-2000 N) is applied distributed in the two slaves nodes on the right side and constrained at left side.

The results agree exactly with the solution, i.e. contraction $-4 \cdot 10^{-6}\,m$ and axial compressive stress $-20000 Pa$





Axial stresses $\sigma_{xx}[N/m^2]$ due to compression

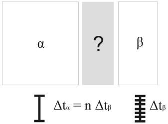


Horizontal displacements $u_x\,[m]$ due to compression

I/E-E SUBCYCLING ALGORITHM

Introduction of the proposed subcycling algorithm

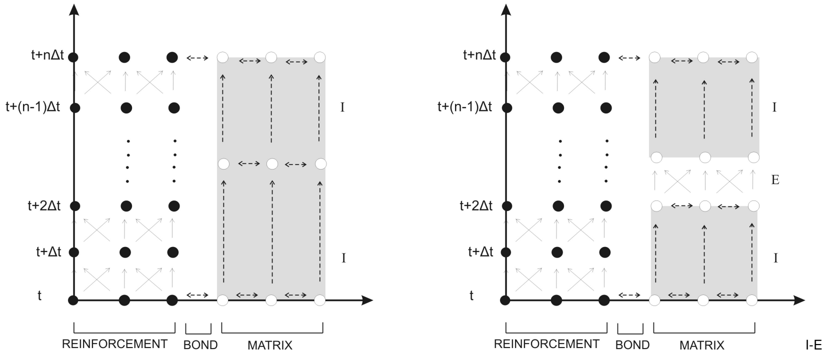
- Subcycling is the use of different 'TIMESTEP sizes' to integrate different degrees of freedom in a model



- In our case 'TIMESTEP/PSEUDOTIMESTEP' 'cos:
  1. A coupling of implicit N-R –matrix component– and explicit CDM –reinforcement– is performed
  2. Quasi-static solutions
- Main novelties of the proposed scheme:
  1. Implicit N-R/explicit CDM has never been reported in the literature
  2. Application to quasi-static simulations of reinforced materials – bond-slip constitutive approach

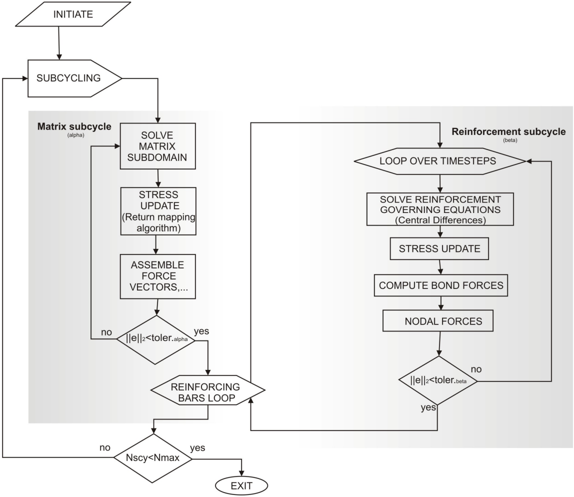
Additional characteristic

- The tangential displacements are used to calculate the slip between the matrix and reinforcement.
- The interface bond stress is a function of the slip.



subcycling (LHS) and I/E-E subcycling (RHS)

Flowchart


INITIATE
SUBCYCLING
Matrix subcycle
(alpha)
SOLVE MATRIX SUBDOMAIN
STRESS UPDATE (Return mapping algorithm)
ASSEMBLE FORCE VECTORS,...
no
||e||2<toler.alpha
yes
REINFORCING BARS LOOP
Reinforcement subcycle
(beta)
LOOP OVER TIMESTEPS
SOLVE REINFORCEMENT GOVERNING EQUATIONS (Central Differences)
STRESS UPDATE
COMPUTE BOND FORCES
NODAL FORCES
no
||e||2<toler.beta
yes
no
Nscy<Nmax
yes
EXIT

Governing equations for the matrix

The problem reduces to find the displacements at t_n such that the equilibrium is satisfied,

$$\boldsymbol{R}(\boldsymbol{u}_n^\alpha) = \mathbf{f}_\alpha^{int}(\boldsymbol{u}_n^\alpha) - \mathbf{f}_\alpha^{ext} = 0 \tag{33}$$

...solving the linearised version of the residual for the incremental global displacement $\delta \boldsymbol{u}^{\alpha\,(k)}$,

$$\mathbf{K}_T \delta \boldsymbol{u}^{\alpha\,(k)} = -\boldsymbol{R}^{(k-1)}(\boldsymbol{u}_{n+1}^\alpha) \qquad where \qquad \mathbf{K}_T = \left. \frac{\partial \boldsymbol{R}}{\partial\, \boldsymbol{u}^\alpha} \right|_{\boldsymbol{u}_{n+1}^{(k-1)}}$$

and the nodal forces,

$$\mathbf{f}_\alpha^{int}(\boldsymbol{u}_n) = \bigwedge_{e \in \Im_\alpha} \left\{ \int_{\Omega^{(e)}} \mathbf{B}^T \, \hat{\boldsymbol{\sigma}}(\boldsymbol{u}_n^\alpha) \, \mathrm{d}v \right\}$$

The external force is updated at the beginning of the matrix-subcycle with the bond forces and the normal internal reactions to the rebars.

$$\mathbf{f}_\alpha^{ext\,(j+1)} = \mathbf{f}_\alpha^{ext\,(0)} + \mathbf{f}_{\beta \rightleftharpoons \alpha}^{link\,(j)} \tag{34}$$

Governing equations for the reinforcement

Explicit Method, Dynamic Relaxation ...

$$\mathbf{M}_\beta\, \ddot{\mathbf{u}}^\beta(t_m) + \mathbf{C}_\beta\, \dot{\mathbf{u}}^\beta(t_m) + \mathbf{f}_\beta^{int}(\mathbf{u}_m^\beta) = \mathbf{f}_\beta^{ext} + \mathbf{f}_{\alpha\rightleftharpoons\beta}^{link} \tag{35}$$

The static solution is obtained from the attenuation of the transient response, leaving the static solution for the applied load.

$$\mathbf{f}_\beta^{int}(\boldsymbol{u}_m^\beta) = \bigwedge_{e\,\in\,\Im_\beta} \left\{ \int_{\Omega^{(e)}} \mathbf{B}^T\, \hat{\boldsymbol{\sigma}}(\boldsymbol{u}_m^\beta)\,\mathrm{d}v \right\} \quad \mathbf{f}_\beta^{ext} = \bigwedge_{e\,\in\,\Im_\beta} \left\{ \int_{\Omega^{(e)}} \mathbf{N}^T\, \boldsymbol{b}\mathrm{d}v + \int_{\partial S^{(e)}} \mathbf{N}^T\, \boldsymbol{q}\,\mathrm{d}s \right\}$$

Initiation of the subcycle $(m = 0)$,

$$\dot{\mathbf{u}}_{\frac{1}{2}}^\beta \;=\; \mathbf{M}_\beta(\mathbf{f}_\beta^{ext}(t_0) + \mathbf{f}_{\alpha\rightleftharpoons\beta}^{link}(t_0) - \mathbf{f}_\beta^{int}(t_0))\,\frac{\Delta t_1}{2}\,; \quad \mathbf{u}_0^\beta \;=\; \mathbf{u}_n^\alpha\,; \quad \mathbf{u}_1^\beta \;=\; \mathbf{u}_0^\beta + \Delta t_{\frac{1}{2}}\,\dot{\mathbf{u}}_{\frac{1}{2}}^\beta$$

and following iterations,

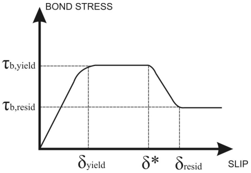
$$\dot{\mathbf{u}}_{m+\frac{1}{2}}^\beta = \frac{2\mathbf{M}_\beta - \Delta t_m \mathbf{C}_\beta}{2\mathbf{M}_\beta + \Delta t_m \mathbf{C}_\beta}\dot{\mathbf{u}}_{m-\frac{1}{2}}^\beta + \frac{\mathbf{f}_\beta^{ext}(t_m) + \mathbf{f}_{\alpha\rightleftharpoons\beta}^{link}(t_m) - \mathbf{f}_\beta^{int}(t_m)}{2\mathbf{M}_\beta + \Delta t_m \mathbf{C}_\beta}$$

$$\mathbf{u}_{m+1}^\beta = \mathbf{u}_m^\beta + \Delta t_{m+\frac{1}{2}}\,\dot{\mathbf{u}}_{m+\frac{1}{2}}^\beta; \quad \mathbf{C}_\beta = \sqrt{\frac{\mathbf{u}_m^{\beta\,T}\mathbf{K}_m^\beta \mathbf{u}_m^\beta}{\mathbf{u}_m^{\beta\,T}\mathbf{M}_\beta \mathbf{u}_m^\beta}}; \quad K_{ii}^\beta = \frac{f_i^\beta(\mathbf{u}_m^\beta) - f_i^\beta(\mathbf{u}_{m-1}^\beta)}{\Delta t_{m-\frac{1}{2}}\,\dot{u}_{m-\frac{1}{2}}^\beta}$$

Interface

The bond stress is calculated through the interface constitutive law ϑ,

$$\tau_b(\Im_\beta^{(j)}) = \vartheta(s_x(\Im_\alpha^{(j)})) \tag{36}$$



The slip is simply given by,

$$s_i(\Im_\alpha^{(j)}) = u_i^{\alpha\,(j-1)} - u_i^{\beta\,(j)} \tag{37}$$

The relative displacement, in a point x between nodes i and $i+1$, is interpolated as follows,

$$s_x(\Im_\alpha^{(j)}) = s_i(\Im_\alpha^{(j)})\,N_i(x) + s_{i+1}(\Im_\alpha^{(j)})\,N_{i+1}(x) \tag{38}$$

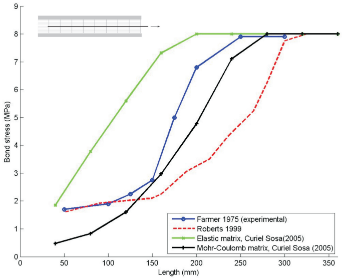
The external force for the next subcycle is calculated through equilibrium balance in a slide considering the reinforcing bar, the matrix and the interface. If A_p is the perimetric area surrounding the reinforcing bar between two nodes, the bond force is,

$$f_{\alpha\,i}^{link} = \tau_b(\Im_\beta^{(j)})\,A_p \tag{39}$$

Pullout tests

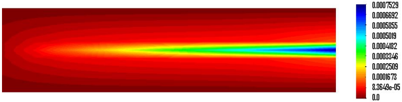
- A metallic bar, embedded in the matrix and loaded at the outside edge $(100\ KN)$.
- Matrix: 250 quadrilateral elements with a thickness of $78.5\ mm$. Reinforcing bar: 8 beam elements $A_{cs} = \pi \cdot 10^{-4}\ m^2$ and a total embedded length $l_{emb} = 0.35\ m$. The interface law: $\delta_{yield} = 0.00032\ m$ and $\tau_{b,yield} = 8\ MPa$.

	Matrix	Reinforcing bar
Young Modulus	$14\ GPa$	$180\ GPa$
Poisson Ratio	0.1	0.33
Yield stress		$300\ MPa$
Mass density	$2400\ Kg/m^3$	
Friction angle (degrees)	20.0	
Dilatancy angle (degrees)	20.0	

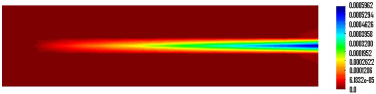


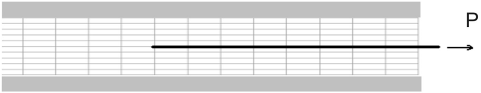
Contours of axial displacements (m)

Elastic matrix:



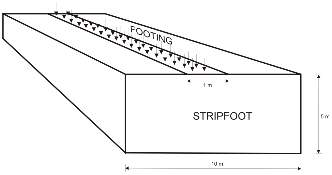
Mohr-Coulomb material model for the matrix:

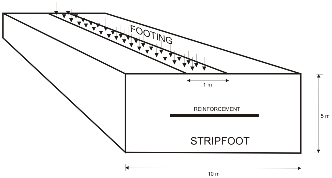




Stripfoot

- The present example analyses the interaction of a footing structure over a block. A stripfoot is the foundation for a wall.
- The problem consists of a long rectangular footing with width $100cm$ and depth $500cm$ lying on a block which is assumed sufficiently long to conduct a plane strain problem.
- 135 eight-nodes quadrilateral elements –four gauss points. Only one-half of the cross-section is discretised thanks to the symmetry. The finite element mesh for the block discretises a sufficiently long part of the domain with depth and half-width of $500\,cm$

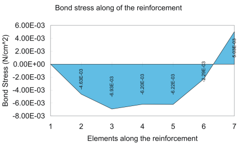


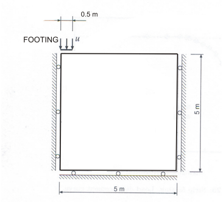


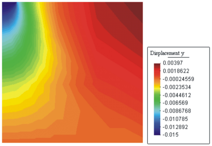
Stripfoot analyses

PLANE STRAIN ASSUMPTIONS:

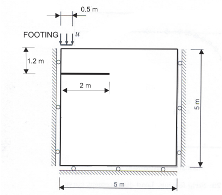
1. The contact between foot and block is assumed with no friction. This is equivalent to prescribing the vertical displacements of the nodes beneath the foot. The horizontal displacements of these nodes remain unconstrained. A total vertical displacement of $-0.015\ cm$ is applied.
2. Reinforcement: thickness $10\ cm$ and width $400\ cm$
3. The stripfoot is modelled as a perfectly-plastic Tresca material with Young modulus, $10^6\ N/cm^2, \nu = 0.48, \sigma_y = 98.0\ N/cm^2$

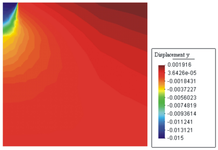






$u_y\ (cm)$ no-reinf.





$u_y\ (cm)$ reinforced

Reinforced stripfoot – last cycle error estimates

INCREMENT NUMBER 1 (Total loading $-0.005\ cm$)

Iteration	Relative Resid. Norm(%)	Max Resid. Norm
1	5.81303	225.507
2	7.58396	321.353
3	0.356725	16.1874
4	$0.171651 \cdot 10^{-01}$	0.589764
5	$0.146106 \cdot 10^{-03}$	$0.580985 \cdot 10^{-02}$
6	$0.105544 \cdot 10^{-07}$	$0.590794 \cdot 10^{-06}$

INCREMENT NUMBER 2 (Total loading $-0.010\ cm$)

Iteration	Relative Resid. Norm(%)	Max Resid. Norm
1	4.87752	307.582
2	0.698960	30.3034
3	0.106738	5.82527
4	$0.676782 \cdot 10^{-03}$	$0.337495 \cdot^{-01}$
5	$0.367806 \cdot 10^{-07}$	$0.156072 \cdot 10^{-05}$

INCREMENT NUMBER 3 (Total loading $-0.015\ cm$)

Iteration	Relative Resid. Norm(%)	Max Resid. Norm
1	3.13677	182.628
2	0.458456	25.6864
3	$0.194172 \cdot 10^{-01}$	1.28360
4	$0.410986 \cdot 10^{-02}$	0.264128
5	$0.321121 \cdot 10^{-06}$	$0.198041 \cdot 10^{-04}$
6	$0.149982 \cdot 10^{-11}$	$0.453042 \cdot 10^{-10}$

Conclusion and Further Research

1. In-time I/E algorithm
   - Substantial reduction of CPU-time respect to the explicit simulation alone.
   - Convergence problems – critical points
   - A practical application of I/E might be non-smooth contact where implicit iterative methods fail to converge (including arc-length methods)
   - The critical part of the mesh –expected to diverge– might executed with I/E, whereas the remaining might be solved with the most convenient method
2. Development of a CBE
3. Subcycling algorithm – reinforced materials
   - Spatial adoption of timesteps/pseudotimesteps
   - Computational efficiency implied by executing reinforcement and continuum at two different velocities of execution and solve the jump associated to the interface.
   - Components of the RM are treated as isolated entities that are separately stepped in time
   - Development of better interface constitutive models
   - This technique remains a great wealth of opportunity for theory, physical modelling, algorithms and simulation.

Thank You!

Questions...

IBVP...

Given body forces, prescribed displacements and traction in the boundary, the problem (weak form) reduces to find the displacement field $\mathbf{u}$ that solve,

$$\int_{\partial\Omega} \mathbf{t} \cdot \bar{\mathbf{u}} \; ds - \int_{\Omega} \boldsymbol{\sigma} : \bar{\boldsymbol{\varepsilon}} \; dv + \int_{\Omega} \mathbf{b} \cdot \bar{\mathbf{u}} \; dv = 0 \tag{40}$$

where $\mathbf{t} = \mathbf{P}\mathbf{n}$ and $\bar{\varepsilon} = \nabla\bar{\mathbf{u}}$ is the the virtual strain. The principle of virtual works presented by Eq (40) becomes after introduction of the interpolation ($\bar{\mathbf{u}} = \mathbf{N}(\mathbf{x}) \; \tilde{\mathbf{u}}$) functions,

$$\int_{\partial\Omega} \mathbf{t} \cdot \mathbf{N}(\mathbf{x}) \; \tilde{\mathbf{u}} \; ds - \int_{\Omega} \boldsymbol{\sigma} : \mathbf{B} \cdot \tilde{\mathbf{u}} \; dv + \int_{\Omega} \mathbf{b} \cdot \mathbf{N}(\mathbf{x}) \; \tilde{\mathbf{u}} \; dv \; = \; 0 \tag{41}$$

Rearranging,

$$\left\{ -\int_{\Omega} \mathbf{B}^T \cdot \boldsymbol{\sigma} \; dv + \int_{\Omega} \mathbf{N}^T(\mathbf{x}) \cdot \mathbf{b} \; dv + \int_{\partial\Omega} \mathbf{N}^T(\mathbf{x}) \cdot \mathbf{t} \; ds \right\} \cdot \tilde{\mathbf{u}} \; = \; 0 \tag{42}$$

As the Eq (42) must be satisfied for any virtual displacement, it follows that,

$$-\int_{\Omega} \mathbf{B}^T \cdot \boldsymbol{\sigma} \; dv + \int_{\Omega} \mathbf{N}^T(\mathbf{x}) \cdot \mathbf{b} \; dv + \int_{\partial\Omega} \mathbf{N}^T(\mathbf{x}) \cdot \mathbf{t} \; ds \; = \; 0 \tag{43}$$

Discretisation

The problem reduces to find the global vector of nodal displacements that obeys the boundary conditions,

$$-\,\mathbf{f}^{(int)}(\mathbf{u}) + \mathbf{f}^{(ext)} \;=\; 0 \tag{44}$$

Global force associated to a node is obtained as the sum of the corresponding contributions from the element force vectors of all elements that share that node.

$$\mathbf{f}^{int}(\boldsymbol{u}) = \int_{\Omega} \mathbf{B}^{T}\!\cdot\!\boldsymbol{\sigma}\, dv \;=\; \bigwedge_{e=1}^{nelem} \left\{ \int_{\Omega^{(e)}} \mathbf{B}^{T}\!\cdot\!\boldsymbol{\sigma}\, dv \right\} = \bigwedge_{e=1}^{nelem} \left\{ \sum_{k=1}^{n_{gauss}} w_k\, \mathbf{B}_k^{T}\!\cdot\!\boldsymbol{\sigma}_k\, J_k \right\}^{(e)} \tag{45}$$

$$\mathbf{f}^{ext} = \int_{\Omega} \mathbf{N}^{T}\cdot\boldsymbol{b}\, dv + \int_{\partial\Omega} \mathbf{N}^{T}\cdot\boldsymbol{t}\, ds \;=\; \bigwedge_{e=1}^{nelem} \left\{ \int_{\Omega^{(e)}} \mathbf{N}^{T}\cdot\boldsymbol{b}\, dv + \int_{\partial\Omega^{(e)}} \mathbf{N}^{T}\cdot\boldsymbol{t}\, ds \right\} \tag{46}$$

$$= \bigwedge_{e=1}^{nelem} \left\{ \sum_{k=1}^{n_{gauss}} w_k \mathbf{N}_k^{T}\cdot\boldsymbol{b}_k\, J_k + \sum_{i=1}^{n_{gaussb}} w_i\, \mathbf{N}_i^{T}\cdot\boldsymbol{t}_i\, J_i^{b} \right\}^{(e)} \tag{47}$$

where w_k are the weights associated to each gaussian or quadrature point and $J(\boldsymbol{\xi}) = det[\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}]$, $J_k = J(\boldsymbol{\xi}_k)$.

Elasto-plastic model

The basic constituents of the elasto-plastic model are,

1. Decomposition of strain in elastic and plastic parts.
2. Elastic evolution.
3. Yield criterion which is geometrically expressed by a yield surface,

$$\Phi = \{\boldsymbol{\sigma} \,\|\, Y(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = 0\} \tag{48}$$

4. Evolution of the plastic strain: plastic flow rule.

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\, \mathbf{N}(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = \dot{\gamma}\, \frac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{\sigma}} \tag{49}$$

5. Evolution of the yield border: hardening law.

$$\dot{\boldsymbol{\alpha}} = \dot{\gamma}\, \mathbf{H}(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = -\dot{\gamma}\, \frac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{\alpha}} \tag{50}$$

   Associative ($\Psi \equiv \Phi$) – metals models

6. Evolution of the plastic flow and hardening is complemented with the loading/unloading criterion,

$$\Phi \leq 0 \qquad \dot{\gamma} \geq 0 \qquad \Phi\, \dot{\gamma} = 0 \tag{51}$$

Integration Algorithm

In a time increment $[t_n, t_{n+1}]$, the increment of strain is given by Eq (52). The states variables are the elastic strain $\boldsymbol{\varepsilon}_n^e$ and the accumulated plastic strain $\bar{\boldsymbol{\varepsilon}}_n^p$ at the beginning of the time interval $[t_n, t_{n+1}]$. The trial state is defined in Eq (53).

$$\Delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n \tag{52}$$

$$\boldsymbol{\varepsilon}_{n+1}^{e\ trial} = \boldsymbol{\varepsilon}_n^e + \Delta\boldsymbol{\varepsilon} \tag{53}$$

$$\bar{\boldsymbol{\varepsilon}}_{n+1}^{p\ trial} = \bar{\boldsymbol{\varepsilon}}_n^p \tag{54}$$

The trial stress tensor is computed assuming elastic evolution in Eq (55). The trial yield stress depends on the accumulated plastic strain at t_n Eq (56).

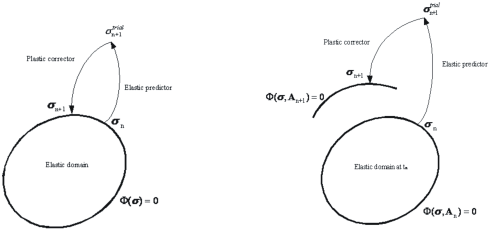
$$\boldsymbol{\sigma}_{n+1}^{trial} = \mathbf{D}^e : \boldsymbol{\varepsilon}_{n+1}^{e\ trial} \tag{55}$$

$$\sigma_{y\ n+1}^{trial} = \sigma_y(\bar{\boldsymbol{\varepsilon}}_n^p) \tag{56}$$

If $\boldsymbol{\sigma}_{n+1}^{trial}$ lies inside of the trial yield surface Eq (57), the evolution is purely elastic within the time interval $[t_n, t_{n+1}]$ and the trial state is the solution.

$$\Phi(\boldsymbol{\sigma}_{n+1}^{trial}, \sigma_{y\ n}) \leq 0 \tag{57}$$

Return Mapping I



The computational update is simply performed as follows,

$$\boldsymbol{\varepsilon}^e_{n+1} = \boldsymbol{\varepsilon}^{e\ trial}_{n+1} \tag{58}$$

$$\bar{\boldsymbol{\varepsilon}}^p_{n+1} = \bar{\boldsymbol{\varepsilon}}^{p\ trial}_{n+1} = \bar{\boldsymbol{\varepsilon}}^p_n \tag{59}$$

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}^{trial}_{n+1} \tag{60}$$

$$\sigma_{y\ n+1} = \sigma^{trial}_{y\ n+1} = \sigma_{y\ n} \tag{61}$$

Return Mapping II

Otherwise, the evolution is elasto-plastic and the trial state lies outside of the elastic domain defined by the yield surface. Therefore, a return mapping, is conducted. The implicit return mapping equations for the Von-Mises model are,

$$\boldsymbol{\varepsilon}^e_{n+1} = \boldsymbol{\varepsilon}^{e\;trial}_{n+1} - \Delta\gamma\sqrt{\frac{2}{3}}\frac{\mathbf{s}_{n+1}}{\|\mathbf{s}_{n+1}\|} \tag{62}$$

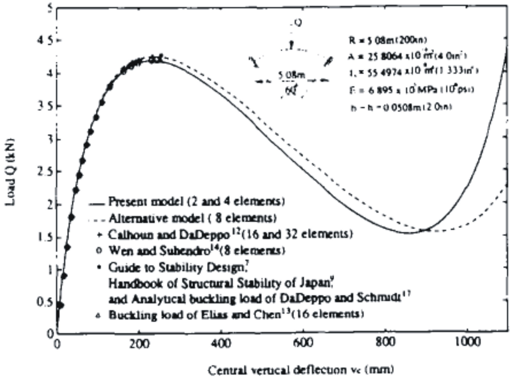
$$\bar{\boldsymbol{\varepsilon}}^p_{n+1} = \bar{\boldsymbol{\varepsilon}}^p_n + \Delta\gamma \tag{63}$$

$$0 = \sqrt{3\,J_2(\mathbf{s}_{n+1})} - \sigma_y(\bar{\boldsymbol{\varepsilon}}^p_{n+1}) \tag{64}$$

This set of algebraic non-linear equations has to be solved for $\boldsymbol{\varepsilon}^e_{n+1}$, $\bar{\boldsymbol{\varepsilon}}^p_{n+1}$ and $\Delta\gamma$. $\mathbf{s}_{n+1}$ is the deviatoric stress tensor Eq (65).

$$\mathbf{s}_{n+1} = 2G dev[\boldsymbol{\varepsilon}^e_{n+1}] \tag{65}$$

Results from the literature for the Trahair's Arch



Updating master nodal internal forces from slaves

1. For each master node of the current element, do:
   - i Extract displacements and velocities in the master of the current element from global arrays
   - ii Update position of master nodes:

$$\mathbf{x}_\kappa(t_{n+1}) = \mathbf{x}_\kappa(t_n) + \mathbf{u}_\kappa(t_n)$$

   - iii Director vectors at masters,

$$\mathbf{p}_\kappa(t_n) = cos(\theta_\kappa(t_n))\, \mathbf{e}_x \;+\; sin(\theta_\kappa(t_n))\, \mathbf{e}_y$$

   - iv Coordinates and velocities of slaves nodes through the director vector and the updated position of master nodes:

$$\mathbf{x}_{_{\kappa}\pm}(t_n) = \mathbf{x}_\kappa(t_n) \;\pm\; \frac{1}{2}\, h_\kappa\, \mathbf{p}_\kappa(t_n)$$

$$\mathbf{v}_{_{\kappa}\pm} = \mathbf{v}_\kappa \;\pm\; \frac{1}{2}\, h_{_{\kappa}0}\, \mathbf{\Omega}_\kappa \;\times\; \mathbf{p}_\kappa$$

2. Reset slave nodal internal force array
3. Loop over quadrature points – stress computation
4. Compute master nodal forces and return

$$\mathbf{f}_\kappa^{(int)} = \mathbf{T}_\kappa^T \cdot \mathbf{f}^{(int)}_{\kappa^- \; \kappa^+}$$

Stress computation

1. Compute jacobian at current quadrature point:

$$\mathbf{x},_{\xi} = \sum_{\kappa^*=1}^{nslave} \mathbf{x}_{\kappa^*} \frac{\partial N_{\kappa^*}}{\partial \xi}$$

2. Set laminar basis at point $\mathbf{x}(\xi, \eta)$,

$$\hat{\mathbf{e}}_x = \frac{x,_{\xi}\ \mathbf{e}_x\ +\ y,_{\xi}\ \mathbf{e}_y}{\sqrt{x,_{\xi}^2 + y,_{\xi}^2}}\ ;\ \hat{\mathbf{e}}_y = \frac{-y,_{\xi}\ \mathbf{e}_x\ +\ x,_{\xi}\ \mathbf{e}_y}{\sqrt{x,_{\xi}^2 + y,_{\xi}^2}}$$

3. Rotation matrix $R(\mathbf{x}(\xi, \eta))$,

$$\mathbf{R} = \begin{bmatrix} \mathbf{e}_x \cdot \hat{\mathbf{e}}_x & \mathbf{e}_x \cdot \hat{\mathbf{e}}_y \\ \mathbf{e}_y \cdot \hat{\mathbf{e}}_x & \mathbf{e}_y \cdot \hat{\mathbf{e}}_y \end{bmatrix}$$

4. New positions and velocities of the slave nodes κ^* in laminar basis:

$$\hat{\mathbf{v}}_{\kappa^*} = \mathbf{R}^T \cdot \mathbf{v}_{\kappa^*}$$

$$\hat{\mathbf{x}}_{\kappa^*} = \mathbf{R}^T \cdot \mathbf{x}_{\kappa^*}$$

5. Jacobian in laminar components,

$$\hat{\mathbf{x}},_{\xi} = \mathbf{R}^T \cdot \mathbf{x},_{\xi} \cdot \mathbf{R}$$

Stress computation (cont.)

6. Calculate jacobian inverse $\hat{\mathbf{x}},_{\xi}^{-1}$
7. The shape function gradient is obtained as follows,

$$\mathbf{N}^T_{\kappa^*,\hat{x}} = \mathbf{N}^T_{\kappa^*,\xi} \cdot \hat{\mathbf{x}},_{\xi}^{-1}$$

8. Velocity gradient,

$$\hat{\mathbf{L}} = \hat{\mathbf{v}}_{\kappa^*} \cdot \mathbf{N}^T_{\kappa^*,\hat{x}}$$

9. Rate of deformation,

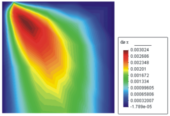
$$\hat{\mathbf{D}} = \frac{1}{2}\,(\hat{\mathbf{L}}^T + \hat{\mathbf{L}})$$

10. Update rate of stress

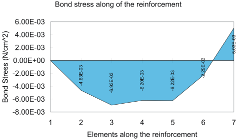
$$\frac{D\hat{\boldsymbol{\sigma}}}{Dt} = \frac{\partial\hat{\boldsymbol{\sigma}}}{\partial t} + \hat{\mathbf{v}} \cdot \nabla\hat{\boldsymbol{\sigma}}$$

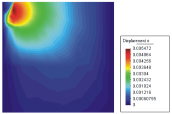
11. Add force-contribution to the nodal slave forces vector
12. Go to next quadrature point or end if quadrature points for this element completed.

Stripfoot analyses



$u_y \ (cm)$ no-reinf.

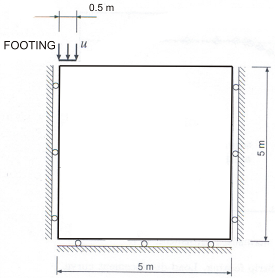


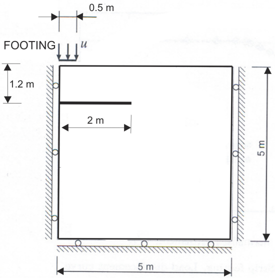


$u_y \ (cm)$ reinforced

Stripfoot - plane strain assumption

- The contact between foot and block is assumed with no friction. This is equivalent to prescribing the vertical displacements of the nodes beneath the foot. The horizontal displacements of these nodes remain unconstrained. A total vertical displacement of $-0.015\,cm$ is applied.
- Reinforcement: thickness $10\,cm$ and width $400\,cm$.
- The stripfoot is modelled as a perfectly-plastic Tresca material with Young modulus, $10^6\,N/cm^2$, $\nu = 0.48$, $\sigma_y = 98.0\,N/cm^2$





Bridging effect

- Assessment of the subcycling performance in flexure behaviour of reinforced notched-beam.
- The material properties of the matrix are: elastic modulus $35GPa$ and poisson ratio 0.2. And for the reinforcing bar: 200 GPa and 0.33 respectively