A Model for Irreversible Investment with Construction and Revenue Uncertainty

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Abstract

This paper presents a model of investment in projects that are characterized by uncertainty over both the construction costs and revenues. Both processes are modeled as spectrally negative Lévy jump-diffusions. The optimal stopping problem that determines the value of the project is solved under fairly general assumptions. It is found that the current value of the benefit-to-cost ratio (BCR) decreases in the frequency of negative shocks to the construction process. This implies that the cost overruns that can be expected if one ignores such shocks are increasing in their frequency. Based on calibrated data, the model is applied to the proposed construction of high-speed rail in the UK and it is found that its economic case cannot currently be made and is unlikely to be met at any time in the next decade. In addition it is found that ignoring construction uncertainty leads to a substantial probability of an erroneous decision being taken.

Keywords: Investment under Uncertainty, Railway investment, Optimal stopping

1. Introduction

The theory of investment under uncertainty has been successful in the past few decades to help decision-makers understand how uncertainty over future payoffs influences the optimal timing of investment projects.[1] Many investment projects, however, are not only characterized by uncertainty over future payoffs, but also over the costs and time of construction.[2]

In this paper, a model is developed that can be used to value projects, such as large-scale infrastructure projects, that are characterized by three features: (i) there is uncertainty over the construction time, (ii) the decision to start construction is irreversible, and (iii) the benefits of the project only accrue after the

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[1] See, for example, McDonald and Siegel (1986), Brennan and Schwartz (1985), and Dixit and Pindyck (1994).

[2] Such models have been developed by, among others, Majd and Pindyck (1987), Alvarez and Keppo (2002), Pindyck (1993), Bar–Ilan and Strange (1996), Schwartz and Moon (2000), and Hsu and Schwartz (2008).

construction process is finalized. As an example of an investment project satisfying these characteristics one can think of (large) infrastructure investment, such as the construction of a new airport, a new rail link, or a new (nuclear) power station. From the point of view of investment appraisal the main issues with construction lags are that (i) costs may be borne for longer, depending on (often uncertain) construction speed and (ii) benefits may, consequently, accrue later if construction is delayed. Both effects reduce the value of a project: a delay in construction increases the expected present value of construction costs, while at the same time reducing the expected present value of the benefits.

These two factors, revenues and construction, are modeled as two (possibly correlated) Lévy jump-diffusions.[3] For example, if the project under consideration is a railway line, the process underlying the construction could represent the mileage of track that has been constructed up to a certain point in time. In our model the value of a project is the solution to a particular optimal stopping problem of a form not usually encountered in the literature on real options. In this problem, simplistically stated, the irreversible decision to start construction will be based on the current *prediction* of the revenues after construction is completed at an unknown time in the future. This prediction depends, therefore, on the properties of the construction process, which may be correlated with the revenue process. It turns out, and this is the main result of the paper, that an optimal trigger can be found for the current state of revenues (i.e. the revenues that would accrue if the project were operational immediately) beyond which it is optimal to take the irreversible decision to start construction.

It is common practice in project evaluation to base decisions on the *benefit-to-cost ratio* (BCR).[4] This is the ratio of the (estimated) present value of future revenues and the (estimated) present value of the construction costs. Orthodox theory teaches that a project is worthwhile if the BCR exceeds unity. Standard real options theory shows that this threshold should be increased in order to take into account revenue uncertainty. This paper argues that the threshold will be different when accounting for potential construction delays. Whether the threshold increases or decreases is ambiguous, as we will see. Evidence will be presented, however, that, on balance, investment will be delayed further.

The model is illustrated for a project where the construction process follows a spectrally negative geometric Lévy process and the revenue process follows a geometric Brownian motion. A 2013 report into the viability of a high-speed rail link between the UK cities of London and Birmingham (HS2) serves as the

[3]The advantage of modeling construction uncertainty as a stochastic process, rather than assuming a random time to completion is that in this way revenues and construction can be correlated. Correlating these sources of uncertainty may be attractive, because one could think, for example, that both demand for, say, high-speed rail and its construction costs are partly driven by a common macroeconomic factor. In a booming economy demand is high, but it is also more costly to construct in an environment of rising wages and dearer materials. If construction budgets per unit of time are fixed, this could slowdown construction.

[4]See, for example, Vickerman (2007) for an overview of project evaluation of infrastructure projects.

basis for a numerical illustration to estimate the current BCR and its threshold for this project. It is found that the report overestimates the current BCR and that it does not meet the threshold that arises from the methodology advocated in this paper. In fact, it will be shown that the probability that the economic case for HS2 is very unlikely to be met in the next 10 years.[5] In its focus on high-speed rail investment as a case study, the paper is related to Pimantel et al. (2012). That paper does identify time-to-build as an important factor in high-speed rail construction, but does not take it specifically into account.

The importance of the development of techniques dealing with construction uncertainty is well-established empirically. For example, Pohl and Mihaljek (1992) show that there tends to be a divergence between *ex ante* and *ex post* project evaluations, especially when construction times are long and uncertain. In particular, appraisal estimates tend to be too optimistic (i.e. the reported BCR is too high). A study by Flyvbjerg et al. (2002), using data on 258 transportation infrastructure projects worth US$90 billion, shows that almost 9 out of 10 projects have higher costs than estimated and that the average cost overrun is 28%. For rail projects this increases to 45%. The same authors, in Flyvbjerg et al. (2004), expand on these results and find evidence that cost overruns are more prominent the longer the implementation phase of the project. Even though the engineering profession continues to work on improving the methods used for cost-benefit analysis, typically these models are not explicitly dynamic.[6] In fact, our model leads to similar estimates of cost overruns as reported in Flyvbjerg et al. (2002, 2004) for reasonable parameter values.

The approach to construction uncertainty advocated here is, to the best of my knowledge, new in the literature. In the existing literature on real options, time to build is incorporated in several ways. For example, Bar–Ilan and Strange (1996) and Alvarez and Keppo (2002) consider a model of investment under uncertainty where the time to build is deterministic. They find that an increase in the investment lag increases the investment threshold and, thus, delays investment. In a recent paper, Sarkar and Zhang (2013) show that this result can be reversed if the project is sufficiently reversible and/or has a high enough growth rate. An alternative approach to investment lag is introduced by Majd and Pindyck (1987) who model the *remaining capital expenditure to completion* (RCEC) as a state variable and allow the decision-maker (DM) to vary construction intensity. In their model the evolution of the RCEC is deterministic and they find that the optimal construction intensity policy is of the bang-bang type: either construct at the maximum intensity or don't construct at all. Schwartz and Moon (2000) and Hsu and Schwartz (2008) extend this approach to the case where the RCEC evolves stochastically over time and apply it to R&D projects in the pharmaceutical industry (Schwartz and Moon, 2000) and the design of research incentives (Hsu and Schwartz, 2008). Finally, Pindyck

[5] Another advantage of the model presented here is that the probability that investment will take place in a given period of time can be explicitly computed.

[6] See, for example, Mills (2001), Molenaar (2005), and Touran and Lopez (2006).

(1993) distinguishes between technical uncertainty and input cost uncertainty for the construction process, but assumes that the value of the finished project is known and fixed, *ex ante*. The use of (spectrally negative) Lévy processes in real options analysis has been championed by, among others, Boyarchenko and Levendorskiĭ (2002) and Alvarez and Rakkolainen (2010).

Whereas the literature on RCEC tends to focus on the value of construction flexibility, the point of this paper is to analyse a model where construction time (and, thus, RCEC) is random, but the decision to start construction is irreversible. This type of model is particularly suited for investment in infrastructure, where, typically, construction takes place continuously until the project has finished. This feature makes the analytics of the model simpler than the aforementioned literature, because the RCEC is no longer a state variable. This allows for a characterization of the solution to the optimal investment timing problem in a large class of problems. In addition, for a simple case where revenue and construction uncertainty are driven by geometric Brownian motion (as is commonly assumed in the literature) my approach leads to closed form solutions, whereas the existing literature on RCEC relies on numerical methods.[7] In fact, as will be seen, even if construction uncertainty is augmented by a jump component, an analytical solution still exists.

The results of the model presented here differ in an important way from some of the literature on construction lags. Both Pindyck (1993) and Bar–Ilan and Strange (1996) conclude that construction lags speed up investment, in that it increases the RCEC below which investment is optimal (Pindyck, 1993) or decreases the value trigger above which investment is optimal (Bar–Ilan and Strange, 1996). In their models this happens because there is a possibility to (temporarily) stop construction. This reduces the effect of irreversibility on the project's value and, thus, speeds up investment. In many R&D situations, like pharmaceuticals this seems a reasonable assumption, but many large scale infrastructure projects are politically almost entirely irreversible. In such cases the risk of construction delays should lead to postponed investment.

The paper is organized as follows. In Section 2 the issues surrounding appraisal of investment projects under uncertainty are introduced. Section 3 presents the model and the main results.Section 4 provides a particular example, where the optimal BCR threshold is computed analytically when the stochastic processes follow spectrally negative geometric Lévy processes. A case study of high-speed rail in the UK is presented in Section 5 and Section 6 provides some concluding remarks.

2. An Informal Introduction to the Problem

The standard way of appraising investment projects is by conducting a cost-benefit analysis, resulting in a benefit-to-cost ratio (BCR). Typically, such an

[7] Another advantage is that the expected time to completion can be computed explicitly, which is important for planning purposes.

exercise consists of estimating the present value of the benefits, PV, and an estimate of the sunk costs, I, resulting in $BCR = PV/I$. Investment should take place if, and only if, $BCR > 1$.

It has been recognized for several decades now that this approach ignores the irreversibility of many investment decisions as well as the uncertainty surrounding benefits and/or costs. These give the decision-maker an *option value of waiting*: by delaying investment one can see how the probability of future losses evolves. A decision to invest should be made only when that probability is low enough. For example, consider the construction of a railway line. The future benefits of the line depend crucially on passenger numbers, Y. Suppose that the process $(Y_t)_{t\geq 0}$ follows a geometric Brownian motion, i.e.

$$\frac{dY}{Y} = \mu dt + \sigma dB_t, \quad Y_0 = y,$$

where $(B_t)_{t\geq 0}$ is a standard Wiener process, i.e. $B_t \sim \mathsf{N}(0, t)$. Then a railway line that runs forever at constant operating costs, o, and a constant ticket price, p, has a present value, discounted at the constant rate $r > \mu$, of

$$\overline{PV}(y) = \mathsf{E}_y \left[\int_0^\infty e^{-rt}(pY_t - o)dt \right] = \frac{py}{r - \mu} - \frac{o}{r}.$$

Throughout the paper we will normalize $p = 1$.

The optimal time of investment is determined by the solution to the *optimal stopping problem*

$$\overline{F}(y) = \sup_\tau \mathsf{E}_y \left[e^{-r\tau} \left(\overline{PV}(Y_\tau) - I \right) \right],$$

over the set of all stopping times. It is well-known (see, for example, Dixit and Pindyck, 1994) that the solution to this problem prescribes that one should invest as soon as passenger numbers exceed the threshold

$$\overline{Y} = \frac{\beta_1}{\beta_1 - 1}(r - \mu) \left(\frac{o}{r} + I \right),$$

where $\beta_1 > 1$ is the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2 \beta(\beta - 1) + \mu\beta - r = 0.$$

In terms of the BCR this means that investment should take place as soon as

$$BCR(y) = \frac{\overline{PV}(y)}{I} \geq \frac{\overline{PV}(\overline{Y})}{I} = 1 + \frac{1}{\beta_1 - 1} \left[1 + \frac{o/r}{I} \right] \equiv \overline{BCR}(> 1). \tag{1}$$

This is the familiar threshold that is determined by a balance of the cost of foregoing revenues and the benefits of waiting for more information and reducing the risk of encountering future losses.

Including a deterministic construction time-lag into this framework is straightforward. Suppose that construction takes T years and that the construction cost flow is c. Since $\mathsf{E}_y(Y_T) = ye^{\mu T}$, it follows immediately that in this case the present value of benefits equals

$$\widehat{PV}(y) = \mathsf{E}_y\left[\int_T^\infty e^{-rt}(Y_t - o)dt\right] = e^{-rT}\left(\frac{ye^{\mu T}}{r-\mu} - \frac{o}{r}\right).$$

The sunk costs of investment are obtained as

$$I = \int_0^T ce^{-rt}dt = \frac{c}{r}\left(1 - e^{-rT}\right),$$

so that the optimal investment trigger is

$$\widehat{Y} = \frac{\beta_1}{\beta_1 - 1}\frac{r-\mu}{e^{-(r-\mu)T}}\left(e^{-rT}\frac{o}{r} + \frac{c}{r}\left(1 - e^{-rT}\right)\right),$$

leading to a threshold BCR of

$$\widehat{BCR} = 1 + \frac{1}{\beta_1 - 1}\left[1 + \frac{o}{c}\frac{e^{-rT}}{1 - e^{-rT}}\right]. \tag{2}$$

Note that $\widehat{BCR} < \overline{BCR}$, reflecting the fact that one should start construction earlier because the revenues are expected to increase over the construction period. The next section introduces an extension to this basic model where the construction process is explicitly modeled as a stochastic differential equation.

3. The Model and Main Results

Consider a risk-neutral decision-maker (DM) who can invest in a project that requires costs to be incurred over an uncertain construction time and leads to an uncertain stream of payoffs once construction is completed. The two sources of uncertainty are represented by a stochastic process $(X_t)_{t\geq 0}$, taking values in $\mathcal{X} = (a_1, b_1) \subset \mathbb{R}$, for the construction process, and a stochastic process $(Y_t)_{t\geq 0}$, taking values in $\mathcal{Y} = (a_2, b_2) \subset \mathbb{R}$, for the profit stream. So, the state variable is $Z = (X, Y)$, taking values in $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$. Uncertainty is modeled by a family of probability measures $(\mathsf{P}_z)_{z\in\mathcal{Z}}$ on a measurable space $(\Omega, \mathscr{F})$, endowed with a filtration $(\mathscr{F}_t)_{t\geq 0}$. The restrictions of P_z to $\mathcal{X}$ and $\mathcal{Y}$ are denoted by P_x and P_y, respectively.[8]

The processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are assumed to be adapted to $(\mathscr{F}_t)_{t\geq 0}$ and to follow the time homogeneous Lévy processes

$$dX_t = \mu_1(X_{t-})dt + \sigma_1(X_{t-})dB_{1t} + \int_{\mathbb{R}} \kappa_1(u, X_{t-})\tilde{N}_1(dt, du), \quad \text{and}$$
$$dY_t = \mu_2(Y_{t-})dt + \sigma_2(Y_{t-})dB_{2t} + \int_{\mathbb{R}} \kappa_2(u, Y_{t-})\tilde{N}_2(dt, du),$$

[8]That is, P_z is the product measure of P_x and P_y.

respectively, where $(B_{i,t})_{t\geq 0}$ are standard Brownian motions with $\mathsf{E}_{(x,y)}(dB_{1t}dB_{2t}) = \rho dt$, $\tilde{N}_1$ and $\tilde{N}_2$ are independent compensated Poisson random measures, with Lévy measures m_1 and m_2, respectively, and $(X_0, Y_0) = (x, y)$, $\mathsf{P}_x \otimes \mathsf{P}_y$-a.s. It is assumed that both $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are spectrally negative, i.e. that $\kappa_1(\cdot) \leq 0$, P_x-a.s. and $\kappa_2(\cdot) \leq 0$, P_y-a.s.

For any $x, X^* \in \mathcal{X}$ and $y, Y^* \in \mathcal{Y}$, let

$$\tau_x(X^*) = \inf\{t \geq 0 | X_t \geq X^*\} \quad \text{and} \quad \tau_y(Y^*) = \inf\{t \geq 0 | Y_t \geq Y^*\},$$

under P_x and P_y, respectively, be the first hitting times of X^* and Y^*.[9] The *generators* of $(X_t)_{t\geq 0}$ (on $C^2(\mathcal{X})$), $(Y_t)_{t\geq 0}$ (on $C^2(\mathcal{Y})$) and $(Z_t)_{t\geq 0}$, (on $C^2(\mathcal{Z})$) are given by the partial integro-differential equations

$$\begin{aligned}
\mathscr{L}_X g =& \frac{1}{2}\sigma_1^2(x)\frac{\partial^2 g(\cdot)}{\partial x^2} + \mu_1(x)\frac{\partial g(\cdot)}{\partial x} + \int_{\mathbb{R}} [g(x+\kappa_1(u)) - g(x) - \frac{\partial g(\cdot)}{\partial x}\kappa_1(u)] m_1(du),\\
\mathscr{L}_Y g =& \frac{1}{2}\sigma_2^2(y)\frac{\partial^2 g(\cdot)}{\partial y^2} + \mu_2(y)\frac{\partial g(\cdot)}{\partial y} + \int_{\mathbb{R}} [g(y+\kappa_2(u)) - g(y) - \frac{\partial g(\cdot)}{\partial y}\kappa_2(u)] m_2(du),
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{L}_Z g =& \frac{1}{2}\sigma_1^2(x)\frac{\partial^2 g(\cdot)}{\partial x^2} + \frac{1}{2}\sigma_2^2(y)\frac{\partial^2 g(\cdot)}{\partial y^2} + \rho\sigma_1(x)\sigma_2(y)\frac{\partial^2 g(\cdot)}{\partial x \partial y}\\
&+ \mu_1(x)\frac{\partial g(\cdot)}{\partial x} + \mu_2(y)\frac{\partial g(\cdot)}{\partial y}\\
&+ \int_{\mathbb{R}} [g(x+\kappa_1(u)) - g(x) - \frac{\partial g(\cdot)}{\partial x}\kappa_1(u)] m_1(du)\\
&+ \int_{\mathbb{R}} [g(y+\kappa_2(u)) - g(y) - \frac{\partial g(\cdot)}{\partial y}\kappa_2(u)] m_2(du)\\
=& \mathscr{L}_X g + \mathscr{L}_Y g + \rho\sigma_1(x)\sigma_2(y)\frac{\partial^2 g(\cdot)}{\partial x \partial y},
\end{aligned}$$

respectively.

The process $(X_t)_{t\geq 0}$ represents the progress of construction. The state of construction at time t is given by the supremum process $\sup_{0\leq s\leq t} X_s$. Construction is assumed to start at some $\hat{x}$ and is finished as soon as some exogenously given $x^* > \hat{x}$ is reached. It is assumed that $[\hat{x}, x^*] \subset \mathcal{X}$, and that $\tau_{\hat{x}}(x^*) < \infty$, $\mathsf{P}_{\hat{x}}$-a.s. The latter assumption ensures that construction is completed in finite time a.s. The construction costs are given by a measurable function $c : \mathcal{X} \to \mathbb{R}_+$, where it is assumed that

$$\mathsf{E}_{\hat{x}}\left[\int_0^\infty e^{-rt} |c(\sup_{0\leq s\leq t} X_s)| dt\right] < \infty.$$

[9]When no confusion is possible, subscripts will be dropped.

This assumption ensures that discounted construction costs (if construction continues forever) are, in expectation, finite. Denote these by

$$I(x) = \mathsf{E}_x \left[\int_0^\infty e^{-rt} c(\sup_{0 \le s \le t} X_s) dt \right] > 0, \quad \hat{x} \le x \le x^*.$$

Since $(X_t)_{t \ge 0}$ is assumed to be spectrally negative, the supremum process has continuous sample paths, a.s., which implies that

$$\inf\{t \ge 0 | \sup_{0 \le s \le t} X_s \ge X^*\} = \tau_x(X^*).$$

On the revenue side it is assumed that, once construction is finished, the profit flow accruing from the project is given by some measurable function $f : \mathcal{Y} \to \mathbb{R}$, $f \in C^2(\mathcal{Y})$, with $f' > 0$ and $f'' \le 0$, where it is assumed that

$$\mathsf{E}_y \left[\int_0^\infty e^{-rt} |f(Y_t)| dt \right] < \infty, \quad \text{all } y \in \mathcal{Y}.$$

Denote the present value of revenues by

$$PV_R(y) = \mathsf{E}_y \left[\int_0^\infty e^{-rt} f(Y_t) dt \right], \quad y \in \mathcal{Y}.$$

The net present value of the project, under P_z, $z = (x, y) \in [\hat{x}, x^*] \times \mathcal{Y}$, then equals

$$F(x, y) = \mathsf{E}_z \left[- \int_0^{\tau_x(x^*)} e^{-rt} c(\sup_{0 \le s \le t} X_s) dt + \int_{\tau_x(x^*)}^\infty e^{-rt} f(Y_t) dt \right].$$

Note that, for $x \ge x^*$ it holds that $\tau_x(x^*) = 0$ and, thus, that

$$F(x, y) = \mathsf{E}_y \left[\int_0^\infty e^{-rt} f(Y_t) dt \right] = PV_R(y), \quad x \ge x^*.$$

For any project that has not started yet, the NPV of commencing when the current value of the process $(Y_t)_{t \ge 0}$ is y equals $F(\hat{x}, y)$.

The DM wishes to choose the investment time to maximize the project's value, i.e. to solve the optimal stopping problem

$$F^*(y) = \sup_{\tau \in \mathscr{M}} \mathsf{E}_y \left[e^{-r\tau} F(\hat{x}, Y_\tau) \right], \tag{3}$$

where $\mathscr{M}$ is the set of stopping times with respect to the filtration $(\mathscr{F}_t)_{t \ge 0}$.

Sufficient conditions for a solution to this problem will be given below. First, however, the net present value of the project is determined. The NPV can be used to compute the current estimate of the benefit-to-cost ratio of the project. The main difference with the standard real options literature is that the NPV function here depends on both the current state of the revenues as well as the construction process.

Proposition 1. *Suppose that*

(i) there exists an increasing solution $\zeta \in C^2(\mathcal{X})$ *to the equation* $\mathscr{L}_X \zeta = r\zeta$, *such that* $\zeta(a_1) = 0$;

(ii) there exists a solution $\varphi \in C^2(\mathcal{Z})$ *to the equation* $\mathscr{L}_{(X,Y)}\varphi = r\varphi$, *such that* $\varphi(a_1, y) = \varphi(x, a_2) = 0$, *all* $x \in \mathcal{X}$ *and* $y \in \mathcal{Y}$, *and* $\varphi(x^*, y) = PV_R(y)$, *all* $y \in \mathcal{Y}$.

Then the net present value of the investment project is given by,

$$F(\hat{x}, y) = \varphi(\hat{x}, y) - \left(1 - \frac{\zeta(\hat{x})}{\zeta(x^*)}\right) I(\hat{x}). \tag{4}$$

The proof of this proposition can be found in Appendix A.

Denote the expected present value of construction costs (under P_x) by

$$PV_I(x) = \left(1 - \frac{\zeta(x)}{\zeta(x^*)}\right) I(x) > 0,$$

and let $\bar{y}$ denote the traditional NPV threshold of the project, i.e. the smallest value that solves $\varphi(\hat{x}, \bar{y}) = PV_I(\hat{x})$ (provided it exists). Sufficient conditions for the existence of a solution to the optimal stopping problem (3) can now be established. The solution to this problem will provide the threshold benefit-to-cost ratio against which any current estimate should be compared.

Proposition 2. *Suppose, in addition to the assumptions of Proposition 1, that,*

(i) the function φ *is such that* $\varphi'_y > 0$ *and* $\varphi''_{yy} \leq 0$;

(ii) there exists an increasing and convex solution $\nu \in C^2(\mathcal{Y})$ *to the equation* $\mathscr{L}_Y \nu = r\nu$, *such that* $\nu(a_2) = 0$;

(iii) $\lim_{y \uparrow b_2} \varphi(\hat{x}, y) > PV_I(\hat{x})$; *and*

(iv) the function

$$\frac{1}{\nu(y)} \left[\varphi(\hat{x}, y) - \left(1 - \frac{\zeta(\hat{x})}{\zeta(x^*)}\right) I(\hat{x})\right], \tag{5}$$

has a stationary point $y^* \in \mathcal{Y}$.

Then y^* *is unique,* $\bar{y}$ *is unique,* $y^* > \bar{y}$, *and* $\tau(y^*)$ *is a solution to the optimal stopping problem* (3) *with*

$$F^*(y) = \begin{cases} \frac{\nu(y)}{\nu(y^*)} \left[\varphi(\hat{x}, y^*) - \left(1 - \frac{\zeta(\hat{x})}{\zeta(x^*)}\right) I(\hat{x})\right] & \text{if } y < y^* \\ \varphi(\hat{x}, y) - \left(1 - \frac{\zeta(\hat{x})}{\zeta(x^*)}\right) I(\hat{x}) & \text{if } y \geq y^*. \end{cases}$$

If (5) *has no stationary point, then the optimal stopping problem has no solution and investment is never optimal.*

The proof of this proposition can be found in Appendix B.

A question that remains is whether functions $\zeta(\cdot)$, $\varphi(\cdot)$, and $\nu(\cdot)$ as described in the propositions actually exist. Based on known results in the literature it can be shown that increasing functions $\zeta(\cdot)$ and $\nu(\cdot)$ always exist for any diffusion (cf. Borodin and Salminen, 1996). In addition, if the mapping $y \mapsto \mu_2(y) - ry$ is non-increasing, the increasing function $\nu(\cdot)$ is also convex (cf. Alvarez, 2003). The conditions on $\varphi(\cdot)$ are more difficult to establish in any generality. As will be seen in Section 4, for $\varphi(\cdot)$ to be concave in y, the expected growth of the revenues should not be higher than the rate at which revenues are discounted. This makes intuitive sense, for if this is not the case, then the expected discounted revenues will explode. At the same time, the growth rate of construction should exceed the discount rate, because otherwise the revenues will not be positively valued and only the construction costs matter.

The BCR is now easily computed as

$$BCR(x,y) = \frac{\varphi(x,y)}{[1-\zeta(x)/\zeta(x^*)]I(x)}.$$

For a project that has not started yet this can be reduced to

$$BCR(y) = \frac{\varphi(\hat{x},y)}{[1-\zeta(\hat{x})/\zeta(x^*)]I(\hat{x})}.$$

Standard practice prescribes that an investment should be undertaken when the BCR exceeds unity. Proposition 2, however, prescribes another threshold. The optimal stopping time (i.e. the optimal time of investment) is $\tau(y^*) = \inf\{t \geq 0 | Y_t \geq y^*\}$. Since y^* is the unique stationary point of (5), it satisfies

$$\nu(y^*)\varphi'_y(\hat{x},y^*) = [\varphi(\hat{x},y^*) - (1-\zeta(\hat{x})/\zeta(x^*))I(\hat{x})]\,\nu'(y^*),$$

it follows that investment is optimal if the BCR exceeds the threshold

$$BCR(y^*) = 1 + \frac{\nu(y^*)}{\nu'(y^*)}\frac{\varphi'_y(\hat{x},y^*)}{(1-\zeta(\hat{x})/\zeta(x^*))I(\hat{x})} \equiv BCR^*.$$

So, Proposition 2 shows that investment should take place only when the *current* (estimate of) benefits of the investment, y, is such that $BCR(y) \geq BCR^*$. From the assumptions it is obvious that $BCR^* > 1$. Policy makers should, therefore, increase the hurdle rate of investment, a result that is well-known and standard in the literature on real options (see, for example, Dixit and Pindyck, 1994).

4. An Illustration: Building a High-Speed Rail Link

To illustrate how Propositions 1 and 2 can be used, consider the construction of a new high-speed rail link. We model the revenues as a geometric Brownian motion (GBM) on $\mathcal{Y} = \mathbb{R}_+$, and assume $(Y_t)_{t\geq 0}$ follows the stochastic differential equation

$$\frac{dY_t}{Y_t} = \mu_2 dt + \sigma_2 dB_2. \tag{6}$$

The construction progress is modeled as a jump-diffusion on $\mathcal{X} = \mathbb{R}_+$, solving the stochastic differential equation[10]

$$\frac{dX_t}{X_{t-}} = \mu_1 dt + \sigma_1 dB_2 - \int_{\mathbb{R}} u\tilde{N}(dt, du), \tag{7}$$

where $0 < u < 1$, P_x-a.s. We allow for possible correlation between the two processes: $\mathsf{E}[dB_1 dB_2] = \rho dt$, where $\rho \in (-1, 1)$. The stream of construction costs is assumed to be constant at $c > 0$.[11] The costs of operating the rail line are assumed to be constant and equal to $o > 0$ per period. The jumps in $(X_t)_{t\geq 0}$ are assumed to be Beta distributed with parameters a and b, i.e.[12]

$$m'(u) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^{a-1}(1-u)^{b-1}. \tag{8}$$

The increasing solution to $\mathscr{L}_Y \nu = r\nu$ with $\nu(0) = 0$ is easily obtained as

$$\nu(y) = A_2 y^{\beta_1},$$

where $\beta_1 > 0$ is the positive root of the quadratic equation

$$\mathscr{Q}_y(\beta) \equiv \frac{1}{2}\sigma_2^2 \beta(\beta - 1) + \mu_2 \beta - r = 0,$$

and A_2 is a positive constant. The function ν is convex only if $\beta_1 > 1$, i.e. if $r > \mu_2$.

The increasing solution to $\mathscr{L}_X \zeta = r\zeta$ with $\zeta(0) = 0$ is solved by (cf. Alvarez and Rakkolainen, 2010) $\zeta(x) = A_1 x^{\gamma_1}$, where $\gamma_1 > 0$ is the positive root of the equation

$$\begin{aligned}
&\mathscr{Q}_x(\gamma) \equiv \frac{1}{2}\sigma_1^2 \gamma(\gamma - 1)x^\gamma + \mu_1 \gamma x^\gamma - r + \int_{\mathbb{R}} [(x - xu)^\gamma - x^\gamma + \gamma u x^\gamma] m_1(du) = 0 \\
&\iff \frac{1}{2}\sigma_1^2 \gamma(\gamma - 1) + (\mu_1 + \lambda \mathsf{E}(U))\gamma - r - \lambda + \lambda \mathsf{E}[(1 - U)^\gamma] = 0 \\
&\iff \frac{1}{2}\sigma_1^2 \gamma(\gamma - 1) + \left(\mu_1 + \lambda \frac{a}{a+b}\right)\gamma - (r + \lambda) + \lambda \frac{\Gamma(a+b)\Gamma(b+\gamma)}{\Gamma(b)\Gamma(a+b+\gamma)} = 0,
\end{aligned}$$

[10] Negative jumps could also be included in $(Y_t)_{t\geq 0}$, albeit at the cost of more notation. It seems reasonable to assume that construction is more prone to jumps than revenues, so revenue jumps are ignored for simplicity.

[11] The assumption of a constant construction cost flow implies that an explicit solution can be found for $I(x)$.

[12] The assumption of Beta distributed jumps allows for analytical solutions. The Beta distribution allows for a wide variety of density shapes, so that this seems a reasonable assumption.

and A_1 is a positive constant.

In order to obtain φ, first note that

$$\varphi(x,y) = Bx^{\alpha}y^{1-\alpha},$$

solves the differential equation $\mathscr{L}_{(X,Y)}\varphi - r\varphi = 0$, but only if α solves the equation

$$\begin{aligned}\mathscr{Q}(\alpha) \equiv & \frac{1}{2}[(\sigma_1-\sigma_2)^2 + 2(1-\rho)\sigma_1\sigma_2]\alpha(\alpha-1) + \left(\mu_1 - \mu_2 + \lambda\frac{a}{a+b}\right)\alpha \\ & + \mu_2 - (r+\lambda) + \lambda\frac{\Gamma(a+b)\Gamma(b+\alpha)}{\Gamma(b)\Gamma(a+b+\alpha)} = 0.\end{aligned}$$

For $\rho \in (-1,1)$ this equation has two roots, α_1 and α_2, and, since $\mathscr{Q}(0) < 0$, it holds that $\alpha_1 > 0$ and $\alpha_2 < 0$. So, the general solution to the equation $\mathscr{L}_{(X,Y)}\varphi = r\varphi$ is

$$\varphi(x,y) = B_1x^{\alpha_1}y^{1-\alpha_1} + B_2x^{\alpha_2}y^{1-\alpha_2},$$

where B_1 and B_2 are constants.

In order to satisfy the boundary conditions $\varphi(x,0) = \varphi(0,y) = 0$, it needs to hold that $B_2 = 0$ and $\alpha_1 < 1$, respectively. The latter condition is fulfilled if $r < \mu_1$, which implies that the growth rate of the construction process should exceed the discount rate. This makes intuitive sense, for if $r > \mu_1$, then the expected revenues are discounted faster than the rate of progress on the construction, which implies that the construction costs fully drown out the revenues.

Note that $I(x) = c/r$, so that we find

$$PV_I(x) = \left[1 - \left(\frac{x}{x^*}\right)^{\gamma_1}\right]\frac{c}{r}.$$

It also follows that the present value of the profits of the project is

$$PV_R(y) = \mathsf{E}_y\left[\int_0^\infty e^{-rt}(Y_t - o)dt\right] = \frac{y}{r-\mu_2} - \frac{o}{r}.$$

The boundary condition $\varphi(x^*,y) = PV_R(y)$ then gives

$$B_1 = \left[\frac{y}{r-\mu_2} - \frac{o}{r}\right](x^*)^{-\alpha_1}y^{\alpha_1-1},$$

so that

$$\varphi(x,y) = \left[\frac{y}{r-\mu_2} - \frac{o}{r}\right]\left(\frac{x}{x^*}\right)^{\alpha_1}.$$

Note that $\varphi'_x > 0$, $\varphi'_y > 0$, and $\varphi''_{yy} \leq 0$.

Since all the conditions of Proposition 2 are met, the optimal value of the project can be obtained by finding a stationary point y^* of

$$\frac{1}{\nu(\hat{y})}\left[\varphi(\hat{x},\hat{y}) - \left(1 - \left(\frac{\hat{x}}{x^*}\right)^{\gamma_1}\right)\frac{c}{r}\right].$$

Standard computations yield that

$$y^* = \frac{\beta_1}{\beta_1 - 1}(r - \mu_2)\left[\frac{o}{r} + \frac{c}{r}\left(1 - \left(\frac{\hat{x}}{x^*}\right)^{\gamma_1}\right)\left(\frac{\hat{x}}{x^*}\right)^{-\alpha_1}\right].$$

The threshold BCR beyond which investment is optimal then can be computed as

$$BCR^* = 1 + \frac{1}{\beta_1 - 1}\left[1 + \frac{o}{c}\frac{(\hat{x}/x^*)^{\alpha_1}}{1 - (\hat{x}/x^*)^{\gamma_1}}\right]. \tag{9}$$

The specifics of the underlying stochastic construction process determine the expected construction time, provided it exists. For the model described in this section the following proposition, the proof of which can be found in Appendix C, describes this operator. The proposition uses the *digamma function*, ψ, defined by $\psi(x) = \Gamma'(x)/\Gamma(x)$.

Proposition 3. *Suppose that* $(X_t)_{t\geq 0}$ *follows the diffusion* (7) *with Lévy measure* (8)*. If*

$$\mu_1 > \frac{1}{2}\sigma_1^2 + \lambda\left(\psi(a+b) - \psi(b) - \frac{a}{a+b}\right), \tag{10}$$

then $\mathsf{E}_x[|\tau(x^*)|] < \infty$*, for any* $x < x^*$*, and*

$$\mathsf{E}_x[\tau(x^*)] = \frac{\log(x^*/x)}{\mu_1 - \frac{1}{2}\sigma_1^2 - \lambda\left(\psi(a+b) - \psi(b) - \frac{a}{a+b}\right)}.$$

Furthermore, $\psi(a+b) - \psi(b) - \frac{a}{a+b} > 0$.

The second part of Proposition 3 shows that the expected construction time is increasing in the jump intensity.

5. An Application: High-Speed Rail in the UK

As an application of the model presented in Section 4, this section will look at a particular case study: investment in Phase 1 of HS2, a high-speed rail link between the UK cities of London and Birmingham. A recently published "strategic case" provides the figures used below, which are taken at face value and used merely for illustrative purposes.[13] This report estimates the (present value in 2011 prices) benefits of this rail link to be £28bn (this includes £4.3bn in wider economic benefits), whereas the (present value) construction costs are estimated to be £15.65bn. Operating costs are estimated to have a present value of £8.2bn. The report includes capital spending such as replacement of

[13] All figures quoted in this paper are taken from "The Strategic Case for HS2", published on October 29, 2013 by the Department for Transport (DfT) and High Speed Two (HS2) Ltd. The report can be downloaded from `https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/254360/strategic-case.pdf`.

rolling stock, etc., which will be ignored here. The report then provides a BCR of 1.7, which renders this a "medium value" project in government parlance.

Since this estimate of the BCR is obtained without using an explicitly dynamic model, the parameters for the model described here have to be calibrated and "guesstimated" based on the information provided. The estimate of the (present value of the) construction costs are reported with the upper bound of a 95% prediction interval of £21.4bn, and an estimated time to completion of 8 years. We could think of the completed track as the realization of (the supremum process of) an arithmetic Brownian motion. The volatility of this process can be found to be 9.94 miles p.a., assuming cost and mileage volatility are proportional.[14]

To keep the measures of volatility of construction and revenues at comparable levels (see below) it makes sense to think of the unit of measurement in batches of 10 miles, i.e. a volatility of $\sigma_1 = .994$. The trend of this process would then be $15/8 = 1.875$ miles p.a., since Birmingham is 150 miles from London. If this arithmetic Brownian motion is then converted into a geometric Brownian motion, this suggests a trend $\mu_1 \approx 2.37$, and a starting point $\hat{x} = 1$. Since it is estimated that this distance will be covered in 8 years, the inferred value for x^* (i.e. the value that gives $\mathsf{E}_{\hat{x}}\left[\tau(x^*)\right] = 8$) is $3.27 \cdot 10^6$. The cost flow is inferred to be $c = £2.24$bn p.a.

The discount rate used in the report is 3.5%, which is transformed to the continuous rate $r = .0344$. The present value of the benefits of the railway is estimated to be £28bn. No clear growth rate of revenues is mentioned in the report, so it will be assumed here that $\mu_2 = .022$, which is the report's assumed growth rate of passenger numbers. A present value of £8.2bn for operating costs leads to a constant operating cost flow of $o =$£.28bn p.a. With an estimated construction time of 8 years this implies that the report's authors assume that *8 years after starting construction* the present value of operations satisfies

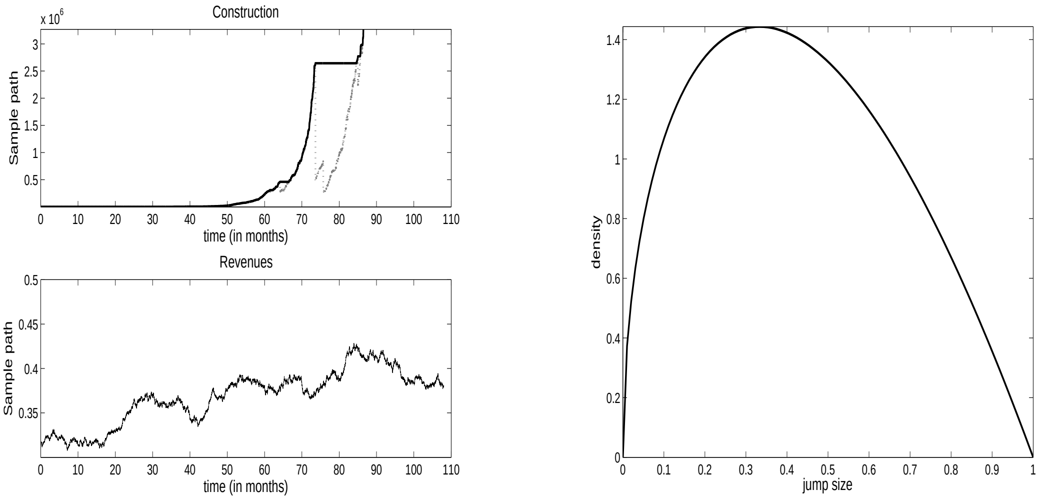
$$\frac{Y_8}{r - \mu_2} - \frac{o}{r} = 8.2 \cdot e^{8r} \iff Y_8 = £.3742\text{bn}.$$

Since $\mathsf{E}_{Y_0}(Y_t) = Y_0 e^{\mu_2 t}$, this implies that $Y_0 = e^{-8\mu_2} =$£.3138bn. The volatility of revenues accruing from HS2 is taken to be $\sigma_2 = .2$.[15]

Since we assume that the construction process can be modeled as a stochastic process $(X_t)_{t\geq 0}$ which follows a GBM with Beta distributed negative jumps, the jump component of the process $(X_t)_{t\geq 0}$ still needs to be determined. We take the expected jump rate to be 3/7 (i.e. $a = 1.5$ and $b = 2$). As a baseline case for the frequency of an unexpected delay we assume that they occur, on average, once a year, i.e. $\lambda = 1$. A sample path for both $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ is given in Figure 1a, whereas the assumed density for the jump sizes is depicted in Figure 1b.

[14] I assume that the report's authors used a normal distribution, even though no distributional assumptions are given in the report.

[15] This figure is not given in the report, but corresponds to a values used regularly in the



(a) sample paths (b) jump size

Figure 1: Left-panel top: Sample path for $(X_t)_{t\geq 0}$ (in dark grey) and the supremum process (in black). Left-panel bottom: Sample path for $(Y_t)_{t\geq 0}$. Right-panel: Density of jump sizes with base-case parameters.

The data for this investment project are summarized in Table 1. Using these we compute the current BCR, BCR_0, and the threshold BCR, BCR^*, as well as the expected time to completion, expected construction costs, and expected cost overrun. In addition, we compute the probability that investment will be optimal within the next 10 years. This probability can be explicitly computed, cf. Harrison (1985). Results are reported in Table 2.

Even if the construction process does not suffer from unexpected shocks, but just from day-to-day risk, (i.e. if $\lambda = 0$) two conclusions can be drawn. First, the report's estimated BCR of 1.7 is wide off the mark and, rather, comes in at .99. This happens because the revenues should be discounted much more than the report allows for. This, in turn, is due to the fact that there is uncertainty over the benefits while the railway line is active, but also *while construction is taking place*. Under current DfT practice a BCR of .99 would put the project just on the cusp of being "medium value for money". However, and this is the second conclusion, the BCR threshold beyond which a project can be called value for money is not unity, but, in this case, 6.62. In fact, the probability with which this threshold is reached in the next 10 years is very low: 1.37%.

Note that the expected construction costs are reported as £15.58bn, which is below the reported £15.65bn. This is due to the way we chose x^* as is clarified

literature, see Dixit and Pindyck (1994).

Description	**Parameter**	**Source**	**Value**
construction duration	T	report	8 years
present value of revenues	$PV_R(Y_0)$	report	£28.94bn
present value of operating costs	$e^{-rT}\frac{o}{r}$	report	£8.2bn
present value of construction costs	$PV_I(\hat{x})$	report	£15.65bn
expected construction time	$\mathsf{E}_{\hat{x}}[\tau(x^*)]$	report	8 years
discount rate	r	report	.0344
initial state of construction	$\hat{x}$	assumed	1
construction state at completion	x^*	inferred	$3.27 \cdot 10^6$
current revenues (p.a.)	Y_0	inferred	.3138
construction costs	c	inferred	£2.24bn p.a.
operating costs (p.a.)	o	inferred	£.28bn p.a.
expected construction growth rate	μ_1	inferred	2.3691
expected revenue growth rate	μ_2	assumed	.022
construction volatility	σ_1	inferred	.9941
revenue volatility	σ_2	assumed	.2
correlation coefficient	ρ	assumed	.2
jump intensity	λ	assumed	1
expected jump rate	$\frac{a}{a+b}$	assumed	3/7

Table 1: Data for the base-case of a numerical analysis of the HS2 project.

λ	BCR_0	BCR^*	$\mathsf{E}_{\hat{x}}[\tau(x^*)]$	$PV_I(\hat{x})$	Cost overrun (%)	Probability
0	.99	6.62	8	15.58	-.42	.0137
1	.87	6.39	9.24	17.55	12.11	.0090

Table 2: Some quantities of interest for different jump intensities. Other parameter values are taken as in Table 1.

in the following proposition.

Proposition 4. *Let $T > 0$. Suppose that $\mu_1 > \frac{1}{2}\sigma_1^2$ and $\lambda = 0$. If x^* is chosen such that $\mathsf{E}_{\hat{x}}[\tau(x^*)] = T$, then*

$$PV_I(\hat{x}) < \left(1 - e^{-rT}\right)\frac{c}{r}.$$

The proof of can be found in Appendix D. Of course, we could have chosen x^* to equate expected construction costs, but then expected construction times would not have been equal. This simply shows the inherent incompatibility between the dynamic approach advocated here and the static approach that underlies the data.

As is to be expected, the picture is worse if there are unexpected negative shocks to the construction process. If, on average, there is one such event per year, the current BCR estimate drops further to .87, while the BCR threshold decreases to 6.39. The latter decrease might seem surprising, but is due to the fact that we are working with a *compensated* Poisson process. Here the compensator is positive, because the shocks are negative. The decrease in the current estimate is bigger, however, so that, on balance, the effect of unexpected shocks is negative. This can be seen from the expected construction time which goes up to 9.24 years, whereas the present value of expected costs goes up to £17.55bn, i.e. an expected cost overrun of 12.11%. The probability that the BCR threshold is reached within the next 10 years also goes down, to .90%.

Note, by the way, that even if one does not take construction uncertainty into account and accepts $BCR_0 = 1.7$, then the threshold used in Section 2 still implies that the project is value for money only if the threshold $BCR^* = 6.81$ is reached.

5.1. Comparative statics

It is important to study the robustness of the model against parameter changes due to the fairly *ad hoc* way in which some parameters have, by necessity, been chosen. In this subsection we present the most interesting findings.

The effect of the frequency of unexpected negative shocks on several quantities of interest can be found in Figure 2. The panel labeled "stochastic discount factor" gives the ratio of the expected discount factors with which revenues and costs are multiplied, respectively. So, for $\lambda = 0$, the benefits are discounted by a factor that is 4.3 times higher than the factor with which the construction costs are discounted. This is due to the fact that costs precede benefits and, as can be seen, the effect is substantial. Also note that the expected cost overrun (relative to the DfT estimate of £15.65bn) is steeply increasing in λ. In fact, the average rate of 28% found by Flyvbjerg et al. (2002) corresponds to $\lambda \approx 2$, whereas the 45% rate reported for railways corresponds to $\lambda \approx 2.85$. Counter-intuitively, the BCR threshold is actually decreasing in the average number of negative construction shocks. An explanation for this result is that, because an increase in λ increases the average length of construction time, the expected value of benefits at the time of completion is higher as well (because the trend

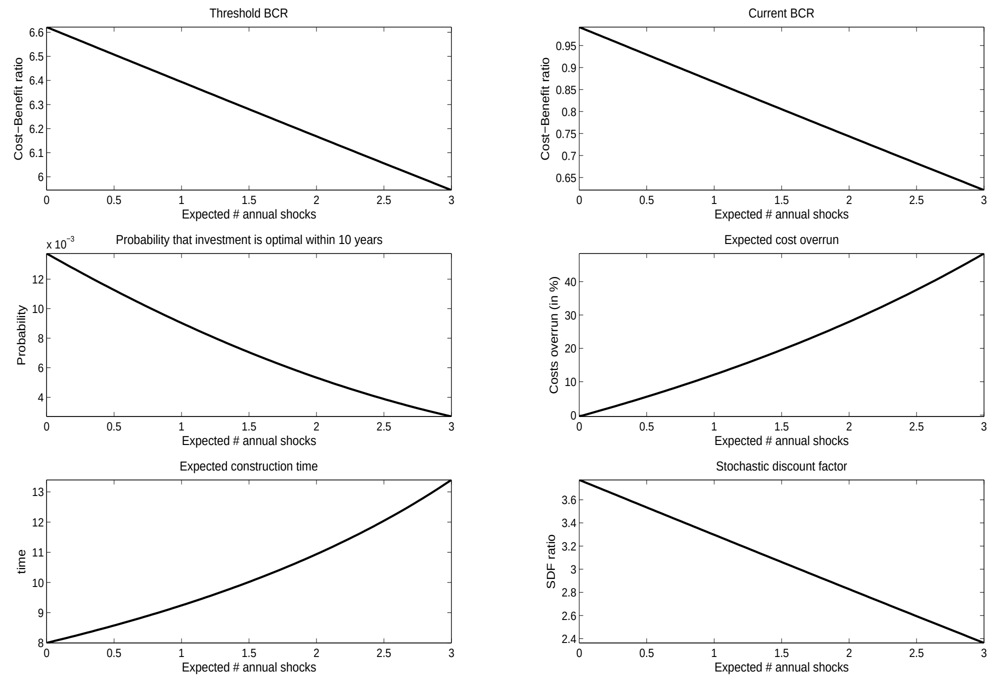


Figure 2: Comparative statics for frequency of unexpected shocks, λ.

in benefits is positive). This implies that one could start construction earlier. However, the current estimate of the BCR decreases as well, which is an unsurprising result. Combined, these two opposing effects lead to a decrease in the probability of investment becoming optimal in the next decade.

Comparative statics for σ_2 are given in Figure 3. The threshold BCR is increasing in σ_2. This happens because an increase in the volatility of the revenues increases the option value of the project. As is well known (see Sarkar, 2000) this does not necessarily imply that the probability of investment is also increasing. The non-monotonicity of the probability of the threshold being reached within the next 10 years can be seen in the bottom-left panel of Figure 3.

Figure 4 plots comparative statics for the *convenience yield*, $\delta = r - \mu_2$. This rate can be thought of as the “dividend” rate that one forgoes while not investing in the project. Note that the probability that investment will be optimal in the next 10 years in fairly sensitive to this parameter. This is due to the sensitivity of results to the discount rate, as can be seen in Figure 5. This sensitivity is well-documented and shows the importance of a careful study into its effects.

Finally, Figure 6 gives the comparative statics for the present value of the

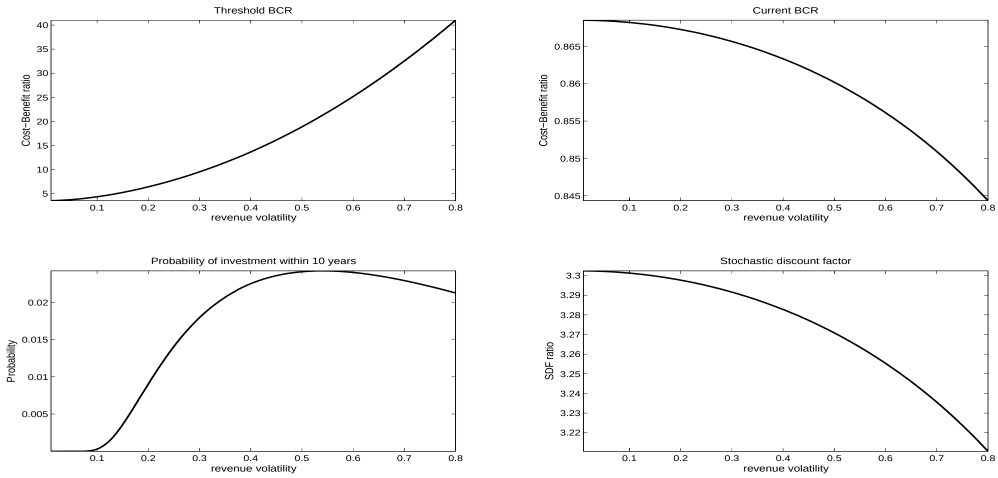


Figure 3: Comparative statics for revenue volatility, σ_2.

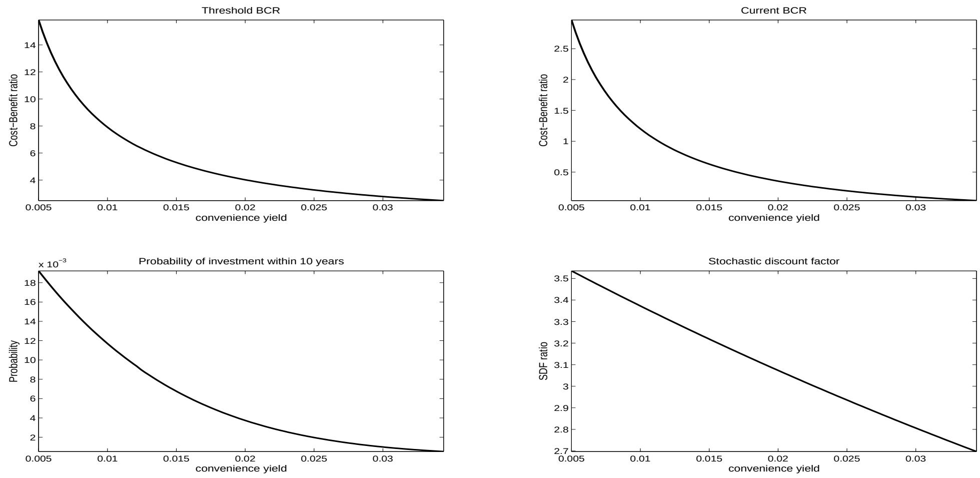


Figure 4: Comparative statics for convenience yield, δ.

wider economic benefits. The threshold BCR is insensitive to this value and equals $BCR^* = 6.62$. The current BCR estimate and probability that the threshold will be reached within 10 years are fairly sensitive to this value, even though the probability remains low in absolute terms.

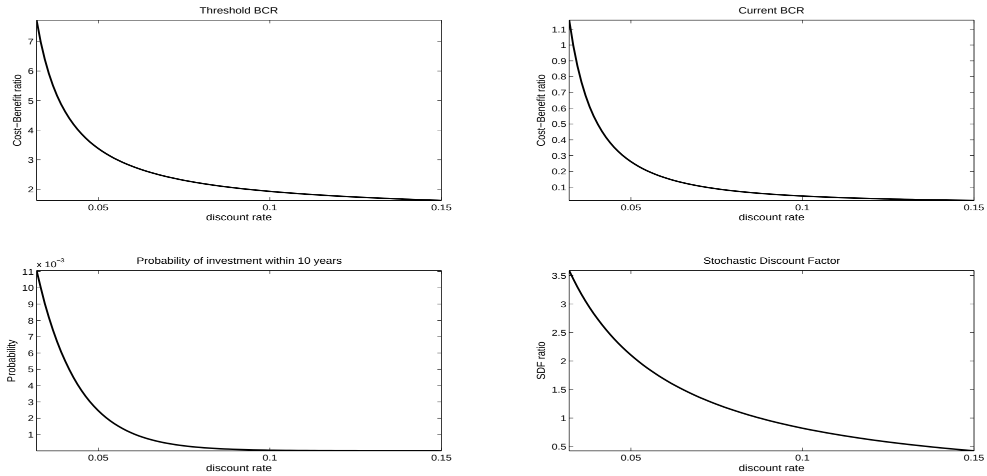


Figure 5: Comparative statics for discount rate, r.

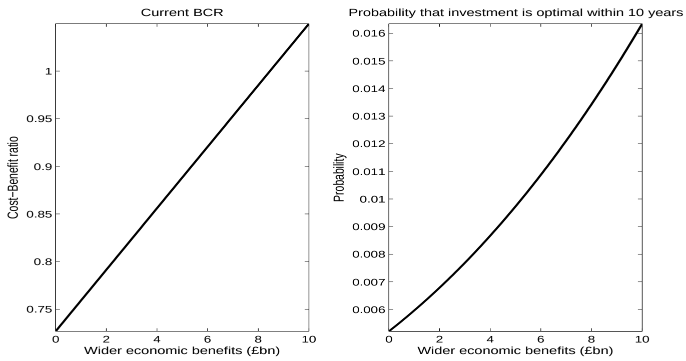
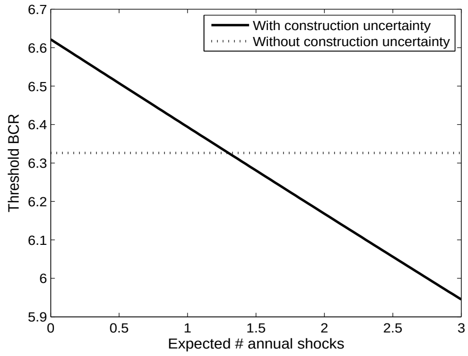


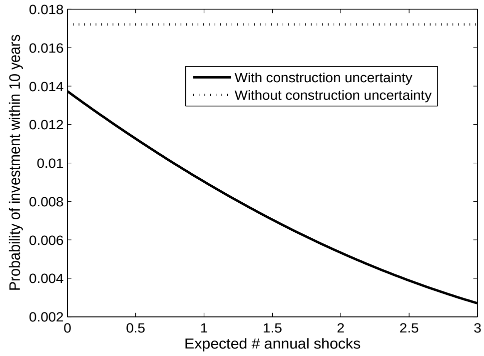
Figure 6: Comparative statics for wider economic benefits.

5.2. Comparing certain and uncertain construction processes

In Section 2 we discussed a simpler real options approach to railway investment where the time to completion was exogenously given. In that case the threshold BCR was given by $\widehat{BCR}$ in (2). In light of Proposition 4 it is to be expected that no clear result will emerge as to the ordering of $\widehat{BCR}$ and BCR^*. This is confirmed by a comparative statics analyses for the expected number of jumps (λ), as depicted in Figure 7a. For small values of λ the BCR under construction uncertainty is higher than the BCR without construction uncertainty; for higher values of λ the picture is reversed. As we have seen before, however, comparing BCR thresholds is not necessarily very informative, because construction uncertainty also affects the value of the current BCR. So, it may be better to compare, for example, the probability of investment becoming optimal anytime in the next, say, 10 years. This exercise is reported in Figure 7b, from which an unambiguous picture emerges: ignoring construction uncertainty leads to investment taking place too soon.



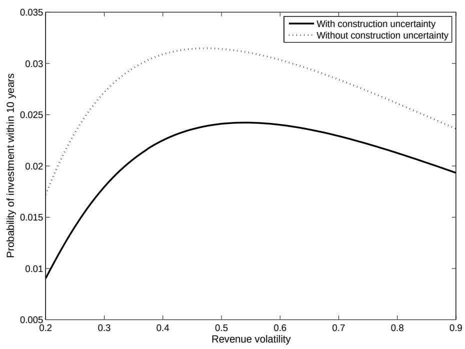
(a) Threshold BCR



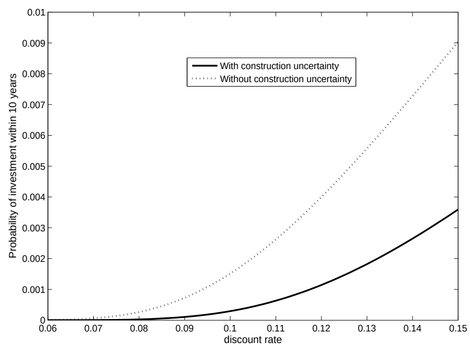
(b) Investment probability

Figure 7: Comparing construction certainty and construction uncertainty: comparative statics for the jump intensity. Base-case parameters are as in Table 1.

As was discussed in the previous subsection, the probability of investment becoming optimal within a fixed time horizon is rather sensitive to the parameters σ_2 and r in particular. In this case, however, the difference between taking construction uncertainty into account or not remains unambiguous: investment takes place too soon if construction uncertainty is ignored; see Figure 8.



(a) Threshold BCR



(b) Investment probability

Figure 8: Comparing construction certainty and construction uncertainty: comparative statics for the revenue volatility (left panel) and the discount rate (right panel). Base-case parameters are as in Table 1.

6. Concluding Remarks

This paper presented a model of investment under uncertainty where the time of construction is influenced by a stochastic process and revenues only start accruing when the construction process is finalized. As a result the expected discount factor applied to revenues is higher than the expected discount factor applied to costs. This, in turn, increases the threshold BCR beyond which investment is optimal. The paper differs from most real options models with construction uncertainty because construction is not assumed to be flexible. As a consequence, construction uncertainty destroys rather than creates value in this paper.

A case study using data from a report on the development of high-speed rail in the UK points to a few effects. Most importantly, the threshold BCR is increasing and the current estimate of the BCR is decreasing in the volatility of the construction process. The former result is typical in the real options literature and is due to the fact that an increase in the volatility increases the option value of the project. The latter result, however, usually does not occur for risk-neutral decision makers, because, typically, the NPV of investment does not depend on revenue volatility. Here, however, because construction and revenue uncertainty are correlated, the NPV of investment explicitly depends on revenue volatility in a negative way. Still, however, the probability of the BCR reaching the threshold within 10 years can be both increasing and decreasing in revenue volatility. This implies that different projects may lead to different investment advice depending on the particulars.

In addition, the presence of unpredictable negative shocks to the construction process reduces the current estimate of the BCR as well as the threshold BCR. The reduction in the current estimate is higher than in the threshold, which implies a lower probability of investment being optimal within a certain period. It has been shown numerically that this reduction in BCR can be dramatic.

Finally, it is important to realise that these effects are not due to risk aversion nor the application of a precautionary principle (as in models with ambiguity, such as, for example, Trojanowska and Kort, 2010); the decision-maker in this paper has been assumed to be risk-neutral. The results are entirely due to the dynamic uncertainty in both construction costs and operational revenues. This shows unpredictable construction delays are ignored at some peril.

Acknowledgements

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Appendix A. Proof of Proposition 1

First note that F can be written as

$$\begin{aligned} F(x,y) &= -\mathsf{E}_x\left[\int_0^\infty e^{-rt}c(\sup_{0\le s\le t} X_s)dt\right] + \mathsf{E}_{(x,y)}\left[\int_{\tau_x(x^*)}^\infty e^{-rt}\left(f(Y_t)+c(\sup_{0\le s\le t} X_s)\right)dt\right] \\ &= \mathsf{E}_{(x,y)}\left[e^{-r\tau(x^*)}PV_R(Y_{\tau(x^*)})\right] - \left(1-\mathsf{E}_x\left[e^{-r\tau(x^*)}\right]\right)I(x). \end{aligned}$$

Since $\mathsf{E}_x[\tau(x^*)] < \infty$ by assumption, and, since $(X_t)_{t\ge 0}$ is spectrally negative, the fact that

$$\mathsf{E}_x\left[\sup_{0\le s\le \tau(x^*)} X_s\right] = \mathsf{E}_x\left[x_{\tau(x^*)}\right] = x^*, \quad x < x^*,$$

an application of Dynkin's formula gives

$$\mathsf{E}_x\left[e^{-r\tau(x^*)}\zeta(X_{\tau(x^*)})\right] = \zeta(x) + \mathsf{E}_x\left[\int_0^{\tau(x^*)} e^{-rt}\left(\mathscr{L}_X\zeta(X_t) - r\zeta(X_t)\right)dt\right] = \zeta(x).$$

So,

$$\mathsf{E}_x\left[e^{-r\tau(x^*)}\right] = \frac{\zeta(x)}{\zeta(x^*)}.$$

Therefore,

$$\left(1-\mathsf{E}_x\left[e^{-r\tau(x^*)}\right]\right)I(x) = \left(1-\frac{\zeta(x)}{\zeta(x^*)}\right)I(x).$$

Another application of Dynkin's formula gives that

$$\begin{aligned} \mathsf{E}_{(x,y)}\left[e^{-r\tau(x^*)}\varphi(X_{\tau(x^*)},Y_{\tau(x^*)})\right] =& \varphi(x,y) \\ &+ \mathsf{E}_{(x,y)}\left[\int_0^{\tau(x^*)} e^{-rt}\left(\mathscr{L}_Z\varphi(X_t,Y_t) - r\varphi(X_t,Y_t)\right)dt\right] \\ =& \varphi(x,y). \end{aligned}$$

Since $\varphi(X_{\tau(x^*)},Y_{\tau(x^*)}) = PV_R(Y_{\tau(x^*)})$, $\mathsf{P}_{x,y}$-a.s., it holds that

$$\mathsf{E}_{(x,y)}\left[e^{-r\tau(x^*)}PV_R(Y_{\tau(x^*)})\right] = \varphi(x,y).$$

This establishes F. ■

Appendix B. Proof of Proposition 2

The proof is established in several steps.

1. Recall from Proposition 1 that $F(x,y) = \varphi(x,y) - PV_I(x)$. Since $\varphi(\hat{x},a_2) = 0 < PV_I(\hat{x})$ and $\zeta' > 0$, assumption (iii) implies that there is a unique $\bar{y} \in \mathcal{Y}$

such that $F(\hat{x}, \bar{y}) = 0$.
2. On $[y^*, b_2)$ it holds that $F^*(\cdot) = F(\hat{x}, \cdot)$. Since $\nu(a_2) = 0 > F(\hat{x}, a_2) = -PV_I(\hat{x})$, $\varphi'_y > 0$, $\nu' > 0$, and

$$\lim_{y \uparrow y^*} F^*(y) = F(\hat{x}, y),$$

it holds that $F^*(\cdot) > F(\hat{x}, \cdot)$ on $[y^*, b_2)$. So, $F^*(\cdot) \geq F(\hat{x}, \cdot)$ on $\mathcal{Y}$.

Denote

$$C = \{\, y \in \mathcal{Y} \mid F^*(y) > F(\hat{x}, y) \,\}.$$

This set is also called the *continuation region* where waiting is optimal.
3. We show that C is a connected set, such that $(a_2, \bar{y}] \subset C$. Suppose that (3) has a solution. From Peskir and Shiryaev (2006, Theorem 2.4) we know that $F^*(\cdot)$ is the least superharmonic majorant of $F(\hat{x}, \cdot)$ on $\mathcal{Y}$ and that the first exit time of C,

$$\tau_C = \inf\{\, t \geq 0 \mid Y_t \notin C \,\},$$

is the optimal stopping time.

We first show that $(a_2, \bar{y}) \subset C$. Let $y \leq \bar{y}$ and let

$$\tau = \inf\{\, t \geq 0 \mid F(\hat{x}, Y_t) \geq 0 \,\}.$$

Note that it is possible that $\mathsf{P}_y(\tau = \infty) > 0$. It holds that

$$\mathsf{E}_y \left[e^{-r\tau} F(\hat{x}, Y_\tau) \right] \geq 0 > F(\hat{x}, \bar{y}).$$

So, it cannot be optimal to stop at y and, hence, $(a_2, \bar{y}] \in C$.

We now show that C is a connected set. Suppose not. Then there exist points $y_1 > \bar{y}$ and $y_2 > y_1$, such that $y_1 \in \mathcal{Y} \setminus C$, and $y_2 \in C$. Let $\tau = \inf\{\, t \geq 0 \mid Y_t \geq y_2, Y_t \in \mathcal{Y} \setminus C \,\}$. Since $F^*(\cdot)$ is a superharmonic majorant of $F(\hat{x}, \cdot)$ it holds that

$$F(\hat{x}, y_1) = F^*(y_1) \geq \mathsf{E}_{y_1}[F^*(Y_\tau)] = \mathsf{E}_{y_1}[F(\hat{x}, Y_\tau)] > \mathsf{E}_{y_1}[F(\hat{x}, y_2)] = F(\hat{x}, y_2).$$

But this contradicts the fact that $F(\hat{x}, \cdot)$ is an increasing function.
4. Since the continuation set is a connected set, problem (3) can now be reduced to a maximization problem over thresholds:

$$\begin{aligned} F^*(y) &= \sup_\tau \mathsf{E}_y \left[e^{-r\tau} F(\hat{x}, Y_\tau) \right] \\ &= \sup_{\hat{y} \in \mathcal{Y}} \mathsf{E}_y \left[e^{-r\tau(\hat{y})} F(\hat{x}, \hat{y}) \right] \\ &= \sup_{\hat{y} \in \mathcal{Y}} \mathsf{E}_y \left[e^{-r\tau(\hat{y})} \right] F(\hat{x}, \hat{y}), \end{aligned}$$

where the last equality follows from the spectral negativity of $(Y_t)_{t\geq 0}$.
5. From Dynkin's formula and spectral negativity of $(Y_t)_{t\geq 0}$ it follows that

$$\mathsf{E}_y\left[e^{-r\tau(\hat{y})}\right] = \frac{\nu(y)}{\nu(\hat{y})}.$$

Therefore, problem (3) can be rewritten as

$$F^*(y) = \nu(y) \sup_{\hat{y}\in\mathcal{Y}} \frac{1}{\nu(\hat{y})} F(\hat{x}, \hat{y}). \tag{B.1}$$

6. If (B.1) has a solution it must be a stationary point y^* of (5), i.e. it should solve $f(y) = 0$, where

$$f(y) = \varphi'_y(\hat{x}, y^*)\nu(y^*) - \nu'(y^*)\left[\varphi(\hat{x}, y^*) - PV_I(\hat{x})\right] = 0. \tag{B.2}$$

Since $y^* > \bar{y}$ it holds that $\varphi(\hat{x}, y^*) > PV_I(\hat{x})$. Note that $f(\bar{y}) > 0$ and

$$f'(y) = \nu(y)\varphi''_{yy}(\hat{x}, y) - \nu''(y)F(\hat{x}, y) < 0,$$

on $[\bar{y}, b_2)$. So, if $f(y) = 0$ has a solution it is unique and is a maximum location of (5) and, hence, solves (3).
7. If $f(y) = 0$ has no solution than the maximum for (B.1) is not attained on

$\mathcal{Y}$ and, thus, $C = \mathcal{Y}$. ■

Appendix C. Proof of Proposition 3

Applying the characteristic operator of $(X_t)_{t\geq 0}$ to the function $f(x) = \log(x)$ gives

$$\begin{aligned}
\mathscr{L}_X f(x) &= -\frac{1}{2}\sigma_1^2 + \mu_1 + \int_{\mathbb{R}} \left[\log(x - ux) - \log(x) + u\right] m_1(du) \\
&= -\frac{1}{2}\sigma_1^2 + \mu_1 + \lambda\frac{a}{a+b} + \lambda\mathsf{E}[\log(1-U)].
\end{aligned}$$

Let B denote the Beta function. A straightforward computation yields that

$$\begin{aligned}
\mathsf{E}[\log(1-U)] &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \int_0^1 \frac{\partial}{\partial b} u^{a-1}(1-u)^{b-1} du \\
&= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial}{\partial b} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \\
&= \frac{1}{B(a,b)} \frac{\partial}{\partial b} B(a,b) = \frac{\partial}{\partial b} \log[B(a,b)] \\
&= \frac{\partial}{\partial b} \log[\Gamma(b)] - \frac{\partial}{\partial b} \log[\Gamma(a,b)] = \psi(b) - \psi(a+b).
\end{aligned}$$

Therefore,

$$\mathscr{L}_X f(x) = -\frac{1}{2}\sigma_1^2 + \mu_1 + \lambda\frac{a}{a+b} + \lambda[\psi(b) - \psi(a+b)].$$

Under (10) it holds that $\mathscr{L}_X f \leq 0$, so that Dynkin's formula gives that

$$\begin{aligned}\log(x^*) &= \log(x) + \mathsf{E}_x\left[\int_0^{\tau(x^*)} \mathscr{L}_X \log(X_t)dt\right]\\ &= \log(x) + \left[\mu_1 - \frac{1}{2}\sigma_1^2 + \lambda\left(\frac{a}{a+b} + \psi(b) - \psi(a+b)\right)\right]\mathsf{E}_x[\tau(x^*)],\end{aligned}$$

from which the result on $\mathsf{E}_x[\tau(x^*)]$ follows.

From the mean-value theorem it follows that there exists $c \in (b, a+b)$ such that

$$\psi(b) - \psi(a+b) = -a\psi'(c),$$

where ψ' is the trigamma function. From Abramowitz and Stegun (1972, Section 6.4) we find that

$$\psi'(a+b) \sim \frac{1}{a+b} + \frac{1}{2(a+b)^2} + \frac{1}{6(a+b)^3} + \cdots > \frac{1}{a+b}, \quad \text{as } a+b \to \infty.$$

Since the trigamma function is decreasing it, thus, follows that

$$\psi'(c) \geq \psi(a+b) \geq \frac{1}{a+b}.$$

So, it now follows that

$$\frac{1}{a+b} + \psi(b) - \psi(a+b) = a\left(\frac{1}{a+b} - \psi'(c)\right) < a\left(\frac{1}{a+b} - \psi'(a+b)\right) \leq 0.$$

■

Appendix D. Proof of Proposition 4

Under the assumptions it holds that

$$\mathsf{E}_{\hat{x}}\left[\tau(x^*)\right] = \frac{\log(x^*/\hat{x})}{\mu_1 - .5\sigma_1^2} = T.$$

It, therefore, follows that

$$\mathsf{E}_{\hat{x}}\left[e^{-r\tau(x^*)}\right] = \left(\frac{\hat{x}}{x^*}\right)^{\gamma_1} = e^{-\gamma_1(\mu_1 - .5\sigma_1^2)T}.$$

Since

$$\mathscr{Q}_x(r/(\mu_1 - .5\sigma_1^2)) = \frac{1}{2}\sigma_1^2\left(\frac{r}{\mu_1 - .5\sigma_1^2}\right)^2 > 0,$$

it follows that $r/(\mu_1 - .5\sigma_1^2) > \gamma_1$, from which the result immediately follows.
■

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