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MULTIVARIABLE ROOT-LOCI AND THE INVERSE TRANSFER FUNCTION MATRIX

by

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Abstract

The orders, asymptotic directions and pivots of the root locus of an invertible, strictly proper system are shown to be invariant under a large class of dynamic output feedbacks. The results are related to the structure of the polynomial component of the inverse system and subsequently translated into a simple computational scheme.

1. Introduction

The concept of the root-locus of the linear time invariant system

$$\dot{x}(t) = A\,x(t) + B\,u(t) \qquad x(t) \in R^n$$
$$y(t) = C\,x(t) \qquad y(t),\ u(t) \in R^m \tag{1}$$

subjected to unity negative feedback with scalar gain $p \geq o$, is now well established in the sense that it has been placed on sound theoretical foundations (MacFarlane and Postlethwaite, 1977) and efficient computational procedures have been derived (Owens 1978a, Shaked and Kouvaritakis 1976). The analysis has been based on the properties of the open-loop transfer function matrix

$$Q(s) = C(sI_n - A)^{-1}B \tag{2}$$

and the relationship between the closed-loop polynomial $\rho_c(s)$ and open-loop polynomial $\rho_o(s)$,

$$\frac{\rho_c(s)}{\rho_o(s)} = |I_m + p\,Q(s)| \tag{3}$$

This paper is concerned with the invariance of the asymptotic directions and pivots of the root-locus of square, invertible systems subjected to dynamic minor loop feedback and the application of the results to the characterisation of these important parameters in terms of the polynomial component of the inverse system $Q^{-1}(s)$.

2. Invariance Relations and Minor loop feedback

It has been established (Owens, 1978b) that the orders and asymptotic directions of the infinite zeros of the system (1) are invariant under constant state feedback and output injection transformations $A \to A + BF + KC$. These concepts can be extended to consider the possibility of minor loop feedback illustrated in Fig. 1a where $H(s)$ is a rational (not necessarily proper) transfer function matrix. The closed-loop poles are defined by the relation

$$|I_m + p\,(I_m + Q(s)\,H(s))^{-1}Q(s)| = 0 \tag{4}$$

The following result indicates that the asymptotic properties of the root-locus are invariant under certain choices of H(s).

Theorem 1.

The orders, asymptotic directions and pivots of the infinite zeros of the system depicted in Fig. 1a are independent of H(s) if

$$\lim_{s \to \infty} s\, Q(s)\, H(s) = 0 \tag{5}$$

Proof

Noting that

$$|I_m + p(I_m + Q(s)H(s))^{-1}Q(s)| = \frac{|I_m + Q(s)H(s) + pQ(s)|}{|I_m + Q(s)\, H(s)|} \tag{6}$$

then the infinite zeros are described by the identity,

$$|I_m + Q(s)\, H(s) + p\, Q(s)| = 0 \tag{7}$$

Using previous results (Owens, 1978a) we can suppose the existence of strictly positive integers q , $d_j\,(1 \leqslant j \leqslant q)$ and k_j , $1 \leqslant j \leqslant q$, such that

$$k_1 < k_2 < \ldots < k_q$$

$$\sum_{j=1}^{q} d_j = m \tag{8}$$

and a real nonsingular transformation T_1 together with unimodular matrices

$$L(s) = \begin{pmatrix} I_{d_1} & 0 & \cdots & \cdots & 0 \\ O(s^{-1}) & I_{d_2} & & & \vdots \\ & & & & 0 \\ O(s^{-1}) & \cdots & \cdots & O(s^{-1}) & I_{d_q} \end{pmatrix}$$

$$M(s) = \begin{bmatrix} I_{d_1} & O(s^{-1}) & \cdots & O(s^{-1}) \\ \vdots & & & \vdots \\ 0 & I_{d_2} & & \vdots \\ \vdots & & & O(s^{-1}) \\ 0 & \cdots & 0 & I_{d_q} \end{bmatrix} \tag{9}$$

such that

$$L(s)\, T_1^{-1}\, Q(s)\, T_1\, M(s) = \text{block diag}\, \{G_j(s)\}_{1 \leq j \leq q} + O(s^{-(k_q + 2)}) \tag{10}$$

where the $d_j \times d_j$ transfer function matrices $G_j(s)$, $1 \leq j \leq q$, have uniform rank k_j , $1 \leq j \leq q$, and the notation $\psi(s) = O(s^{-\ell})$ indicates that $\lim_{s \to \infty} s^{\ell} \psi(s)$ is finite. Equation (7) now becomes

$$|I_m + (L(s)(I_m + O(s^{-2}))M(s))^{-1}\, L(s)\, T_1^{-1}\, Q(s)\, T_1\, M(s)| = 0 \tag{11}$$

where

$$L(s)(I_m + O(s^{-2}))\, M(s) = \begin{bmatrix} I_{d_1} + O(s^{-2}) & \cdots & O(s^{-1}) \\ \vdots & & \vdots \\ O(s^{-1}) & & \vdots \\ \vdots & & O(s^{-1}) \\ O(s^{-1}) & \cdots\; I_{d_q} & + \; O(s^{-2}) \end{bmatrix} \tag{12}$$

takes the same form as L(s) M(s). It follows directly (Owens, 1978a) that the system has $k_j\, d_j$ infinite zeros of order k_j , $1 \leq j \leq q$, described by the relations

$$|I_{d_j} + p\, G_j(s)| = 0 \tag{13}$$

The theorem is now proved as the $G_j(s)$, $1 \leq j \leq q$, depend only upon Q(s).

The theorem can be extended slightly by writing $Q(s) = G(s)\ K(s)$ where $G(s)$ and $K(s)$ are the plant and controller transfer function respectively and considering the configuration of Fig. 1(b). The following result is then obtained.

Theorem 2.

The orders, asymptotic directions and pivots of the root locus of the system of Fig. 1.b are independent of $H(s)$ if

$$\lim_{s\to\infty} s\ G(s)\ H(s) = 0 \tag{14}$$

Proof

Follows, as in the proof of theorem 1, from the identity,

$$\begin{aligned} &|I_m + p\ (I_m + G(s)\ H(s)^{-1}\ Q(s)| \\ &= \frac{|I_m + G(s)\ H(s) + p\ Q(s)|}{|I_m + G(s)\ H(s)|} \end{aligned} \tag{15}$$

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3. Multivariable Root-loci and the Inverse System

Consider the mxm invertible systems $Q(s)$ subjected to unity negative feedback with scalar gain $p \geqslant o$, and write the inverse system in the form,

$$Q^{-1}(s) = \sum_{j=o}^{k^*} s^j A_{k^*-j} + H_o(s)\ ,\ k^* \geqslant 1\ ,\ \ A_o \neq 0 \tag{16}$$

where $H_o(s)$ is strictly proper. It is convenient to define

$$P_\ell(s) = \sum_{j=\ell}^{k^*} s^j\ A_{k^*-j}\quad ,\quad H_\ell(s) = Q^{-1}(s) - P_\ell(s)\qquad o \leqslant \ell \leqslant k^* \tag{17}$$

when,

Theorem 3.

If $k \geqslant 1$ is the unique integer such that $\lim_{s\to\infty} s^k Q(s)$ exists and is nonzero, then

$$|P_\ell(s)| \not\equiv 0 \quad ,$$
$$\lim_{s\to\infty} s^k P_\ell^{-1}(s) = \lim_{s\to\infty} s^k Q(s) \quad , \quad o \leq \ell \leq k \qquad (18)$$

Proof

From the definitions $\lim_{s\to\infty} Q(s)\, H_\ell(s) = 0$, $o \leq \ell \leq k$, and hence $\lim_{s\to\infty} Q(s)\, P_\ell(s) = I_m$, indicating that $|P_\ell(s)| \not\equiv 0$, $o \leq \ell \leq k$, and $\lim_{s\to\infty} P_\ell^{-1}(s)\, H_\ell(s) = 0$, $o \leq \ell \leq k$.

The result now follows by writing

$$Q(s) = (P_\ell(s) + H_\ell(s))^{-1} = (I_m + P_\ell^{-1}(s)\, H_\ell(s))^{-1}\, P_\ell^{-1}(s) \qquad (19)$$

It follows directly that

$$\lim_{s\to\infty} s\, P_\ell^{-1}(s)\, H_\ell(s) = 0 \quad , \quad o \leq \ell \leq k-1 \qquad (20)$$

which leads to the result:

Theorem 4.

The orders, asymptotic directions and pivots of the invertible system $Q(s)$ subjected to unity negative feedback with scalar gain $p \geq o$ are identical to those of the invertible systems $P_\ell^{-1}(s)$, $o \leq \ell \leq k-1$.

Proof

Equation (19) indicates that $Q(s)$ can be regarded as a system $P_\ell^{-1}(s)$ with a dynamic feedback loop $H_\ell(s)$. The result follows directly from theorem 1 using equation 20.

In effect, the asymptotic properties of the root-locus can be deduced entirely from the higher order terms in the polynomial component of the inverse system (equation (16)). To illustrate the approach, consider the important case when $Q(s)$ has uniform rank k i.e. (Owens 1978a) $\lim_{s\to\infty} s^k\, Q(s)$ exists and is nonsingular. Equivalently,

$$|A_o| \neq 0 \quad , \quad k = k^* \qquad (21)$$

and theorem 4 states that the orders, asymptotic directions and pivots can be deduced from the system

$$P_{k-1}^{-1}(s) = \frac{1}{s^{k-1}}(s\,A_o + A_1)^{-1} \tag{22}$$

For example, the techniques of Owens (1976) could be applied to $P_{k-1}(s)$ or the methods of Owens (1978a) and Kouvaritakis and Shaked (1976) to the series expansion,

$$P_{k-1}^{-1}(s) = \frac{1}{s^k}A_o^{-1} - \frac{1}{s^{k+1}}A_o^{-1}A_1A_o^{-1} + o(s^{-(k+2)}) \tag{23}$$

Finally, if the decomposition (16)-(17) is applied to G(s) then theorem 3 remains valid with Q(s) replaced by G(s) and the following result is easily deduced from theorem 2.

Theorem 5.

The orders, asymptotic directions and pivots of the invertible system $Q(s) = G(s)\,K(s)$ subjected to unity negative feedback with scalar gain $p \geq o$ are identical to those of the invertible systems $P_\ell^{-1}(s)\,K(s)$, $o \leq \ell \leq k-1$.

In particular this indicates that compensation of the root-locus can be undertaken using the 'high frequency approximations' $P_\ell^{-1}(s)$, $o \leq \ell \leq k-1$, to the plant G(s).

4. A Computational Procedure

Gives the form of the inverse system (16) and the parameter $k \geq 1$, the results of the previous section indicate that the asymptotic properties of the root-locus can be deduced from the structure of the Markov matrices of $P_{k-1}^{-1}(s)$. If the parameter k is not known, then the results also indicate that analysis of $P_o^{-1}(s)$ will be sufficient. In the authors opinion, this approach could prove difficult in practice unless Q(s) has uniform rank. An alternative approach is described in this section.

Using an obvious parallel to previous analysis (Owens 1978a) suppose that there exists integers $q \geq 1$, $k_1 < k_2 < \ldots < k_q$, and d_j, $1 \leq j \leq q$, and a nonsingular transformation T_1 together with unimodular matrices $L(s)$, $M(s)$ of the form shown in equation (9) such that

$$L(s)\, T_1^{-1}\, Q^{-1}(s)\, T_1\, M(s) = \text{block diag}\,\{G_{q+1-j}^{-1}(s)\}_{1 \leqslant j \leqslant q} + 0(s^{k_1-2}) \tag{24}$$

where the $d_j x d_j$ transfer function matrices $G_j(s)$ have uniform rank k_j, $1 \leqslant j \leqslant q$. The structure of $G_j^{-1}(s)$, $1 \leqslant j \leqslant q$, can be obtained by direct application of a previously derived algorithm (Owens 1978a) to the matrix $[A_o, A_1, \ldots, A_{k*}]$ and is of the form

$$G_j^{-1}(s) = s^{k_j} A_o^{(j)} + s^{k_j-1} A_1^{(j)} + 0(s^{k_j-2})$$

$$|A_o^{(j)}| \neq 0\,, \quad 1 \leqslant j \leqslant q \tag{25}$$

The return-difference determinant

$$|I_m + p\, Q(s)| = p^m |Q(s)| \cdot |p^{-1} Q^{-1}(s) + I_m| \tag{26}$$

and hence the infinite zeros are the unbounded solutions of

$$0 = |I_m + p^{-1} Q^{-1}(s)| = |L(s)\, M(s) + p^{-1}\, L(s) T_1^{-1} Q^{-1}(s)\, T_1 M(s)|$$

$$= \left| \begin{pmatrix} I_{d_q} + 0(s^{-2}) + p^{-1} G_q^{-1}(s) & 0(s^{-1}) & \cdots & 0(s^{-1}) \\ 0(s^{-1}) & I_{d_{q-1}} + 0(s^{-2}) + p^{-1} G_{q-1}^{-1}(s) & & \vdots \\ \vdots & & \ddots & 0(s^{-1}) \\ 0(s^{-1}) & \cdots & & I_{d_1} + 0(s^{-2}) + p^{-1} G_1^{-1}(s) \end{pmatrix} + p^{-1}\, 0(s^{k_1-2}) \right| \tag{27}$$

Noting that $p^{-1} s^{k_j}$, $1 \leqslant j \leqslant q$, cannot all tend to zero as $p \to +\infty$, suppose that $p^{-1} s^{k_q}$ has a finite cluster point. Application of Schur's formula to (27) yields the relation

$$|I_{d_q} + 0(s^{-2}) + p^{-1}\, G_q^{-1}(s) + p^{-1}\, 0(s^{k_1-2})| = 0 \tag{28}$$

After some manipulation this relation can be reduced to the form

$$|I_{d_q} + (I_{d_q} + G_q(s)\, O(s^{k_1-2}))^{-1}\, p\, G_q(s)\, (I_{d_q} + O(s^{-2}))|$$

$$= 0 \qquad (29)$$

Noting that $G_q(s)\, O(s^{k_1-2}) = O(s^{-2})$ then theorem 2 indicates that (29) can be replaced by

$$|I_{d_q} + p\, G_q(s)\, (I_{d_q} + O(s^{-2}))| = 0 \qquad (30)$$

if attention is restricted to asymptotic directions and pivots only. Using the fact (Owens 1978a) that the asymptotic directions and pivots of a uniform rank system can be determined from the first two terms in its series expansion about the point at infinity, it follows immediately that the system has $d_q k_q$ k_q^{th} order infinite zeros with asymptotic directions and pivots obtained from solution of the uniform rank problem,

$$|I_{d_q} + p\, G_q(s)| = 0 \qquad (31)$$

Suppose now that $p^{-1} s^{k_q}$ is unbounded, then application of Schurs formula to (27) yields

$$\left| \begin{pmatrix} I_{d_{q-1}} + O(s^{-2}) + p^{-1} G_{q-1}^{-1}(s) & & O(s^{-1}) \\ & & \vdots \\ O(s^{-1}) & & \vdots \\ \vdots & & O(s^{-1}) \\ O(s^{-1}) & \cdots\cdots & I_{d_1} + O(s^{-2}) + p^{-1} G_1^{-1}(s) \end{pmatrix} + p^{-1}\, O(s^{k_1-2}) \right| = 0 \qquad (32)$$

It follows immediately by induction (compare (32) and (27)) that the system has $k_j d_j$ k_j^{th} order infinite zero, $1 \leq j \leq q$, where asymptotic directions and pivots are defined by the uniform rank problems

$$|I_{d_j} + p\, G_j(s)| = 0\ ,\ 1 \leq j \leq q \qquad (33)$$

The calculation of the asymptotic directions and pivots can be completed by noting (equation 25)) that

$$G_j(s) = s^{-k_j}(A_o^{(j)})^{-1} - s^{-(k_j+1)}(A_o^{(j)})^{-1}A_1^{(j)}(A_o^{(j)})^{-1} + O(s^{-(k_j+2)}) \tag{34}$$

and application of known techniques (Owens 1978a).

Finally, it is easily verified from (24) that

$$k = k_1 < k_2 < \ldots . < k_q = k^* \tag{35}$$

and hence the system can only have integer order infinite zeros in the range $k \leqslant v \leqslant k^*$.

5. Conclusions

The paper has extended previous work (Owens, 1978a,b) by demonstrating that the orders, asymptotic directions and pivots of the infinite zeros of a square system Q(s) are invariant under a large class of dynamic minor loop feedbacks. The main application of these results has been to the investigation of the relationship of root-locus structure to the structure of the inverse system $Q^{-1}(s)$. In particular, it has been demonstrated that one need only consider a well-defined high frequency component of the polynomial part of $Q^{-1}(s)$. Of particular significance is the observations that this approximation is sufficient for the purposes of compensation studies and hence that the analysis of systems with polynomial matrix inverses (Owens 1976, 1978c) could be of great value in theoretical study of compensation schemes. Finally the results have been translated into a systematic computational procedure paralleling that recently derived (Owens 1978a) from direct consideration of Q(s).

6. References

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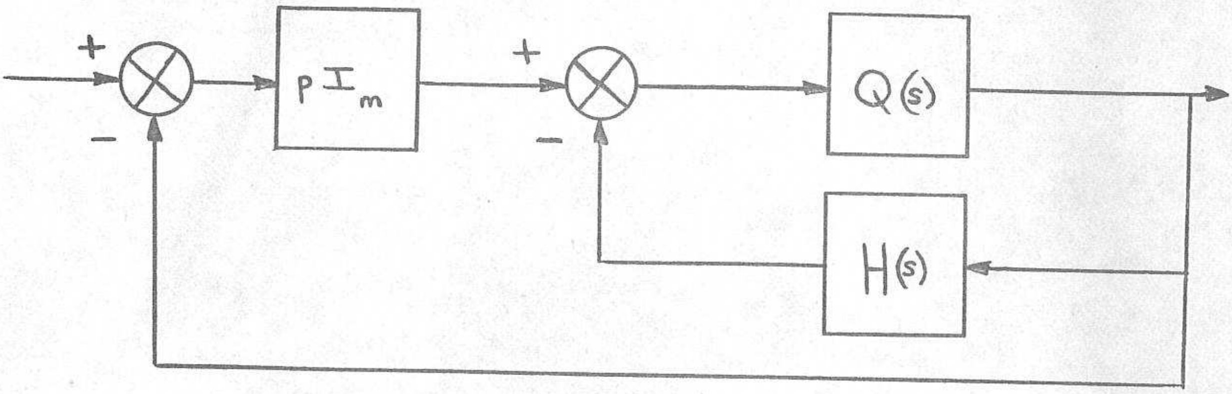
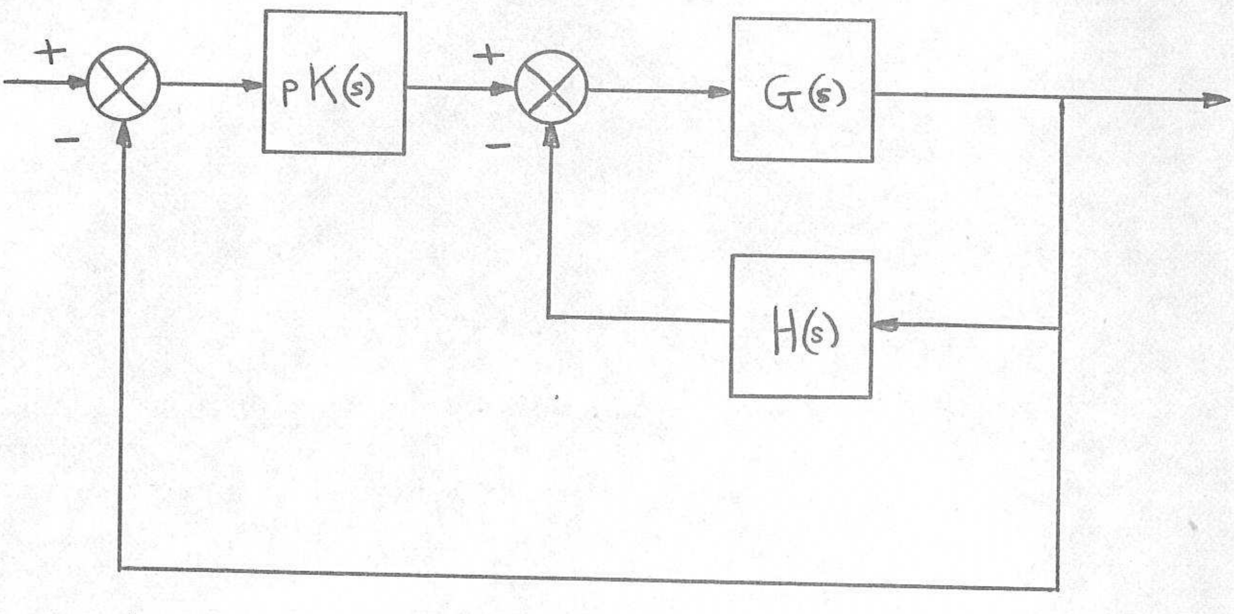


(a)



(b)

Fig. 1 Feedback Systems with Minor Loop Feedback.