A NOTE ON COMPENSATION OF MULTIVARIABLE ROOT-LOCI

BY

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Abstract

A recently derived computational approach to the analysis of the asymptotic behaviour of the root locus of a square invertible system $Q_o(s)$ is used to construct easily realizable compensation elements. It is demonstrated that the asymptotic directions and pivots can be manipulated at will by the use of dyadic controllers with subsystem dynamics represented by classical compensation elements.

1. Introduction

In recent papers (Kouvaritakis and Shaked 1976, MacFarlane and Postlethwaite 1977, Owens 1976, 1977, Postlethwaite 1977) several techniques have been suggested for the calculation of the asymptotic directions and pivots of a square, invertible system represented by the mxm transfer function matrix $Q_o(s)$ when subjected to unity negative feedback with scalar gain $p \geqslant o$. The analyses represent an important step forward into the identification of control difficulties and provide useful information concerning the presence of oscillation and instability in the closed-loop system at high gains $p \to +\infty$. This paper considers the problem of systematic manipulation of asymptotic directions and pivots (regarded as design parameters) by the use of elementary controller compensation elements. The approach can be regarded as existence theorems or, as they are proved by construction, they can be regarded as synthesis procedures.

2. Compensation of Uniform Rank Systems

Consider an mxm strictly proper, invertible plant $G(s)$ subjected to unity negative feedback with forward path controller $K(s) = pK_1(s)$, where $K_1(s)$ is mxm, proper and invertible and $p \geqslant o$ is a real scalar gain. The transfer function matrix

$$Q_o(s) = G(s)\ K_1(s) \tag{1}$$

is said to have uniform rank k if its series expansion about the point at infinity takes the form (Owens, 1977)

$$Q_o(s) = s^{-k}Q_k + s^{-(k+1)}Q_{k+1} + \dots \tag{2}$$

$$|Q_k| \neq 0$$

If the spectrum $\{\lambda_j\}_{1 \leq j \leq m}$ of Q_k has ℓ distinct entries η_j, $1 \leq j \leq \ell$, of multiplicity d_j , $1 \leq j \leq \ell$, and Q_k has eigenvector matrix T_o,

$$T_o^{-1}\ Q_k\ T_o = \text{block diag}\ \{\eta_j\ I_{d_j}\}_{1 \leq j \leq \ell} \tag{3}$$

then writing

$$T_o^{-1} Q_{k+1} T_o = \begin{bmatrix} N_{11} & \cdots\cdots & N_{1\ell} \\ \vdots & & \\ \vdots & & \\ N_{\ell 1} & \cdots\cdots & N_{\ell\ell} \end{bmatrix} \qquad (4)$$

where N_{ij} has dimension $d_i \times d_j$, $1 \leq i, j \leq \ell$, it can be shown (Owens 1977) that the closed-loop system has km k^{th} - order infinite zeros of the form

$$s(p) = p^{1/k}\, \eta_{ij} + \frac{\alpha_{jr}}{k\eta_j} + \varepsilon_{ijr}(p)$$

$$\lim_{p\to\infty} \varepsilon_{ijr}(p) = 0$$

$$1 \leq i \leq k \quad , \quad 1 \leq r \leq d_j \quad , \quad 1 \leq j \leq \ell \qquad (5)$$

where $p^{1/k}$ is the positive-real k^{th} root of p , $\{\eta_{ij}\}_{1\leq i\leq k}$ are the distinct k^{th} roots of $-\eta_j$ and $\{\alpha_{jr}\}_{1\leq r\leq d_j}$ are the eigenvalues of N_{jj}.

Regarding system compensation as the restructuring of the root-locus plot, it is natural to ask whether the asymptotic directions and pivots of $Q_o(s)$ can be manipulated by suitable choice of control systems.

(a) Manipulation of Asymptotic Directions

The manipulation of the asymptotic directions η_{ij} is achieved by manipulation of the eigenvalues of Q_k. Let $\tilde{T}_o$ be a nonsingular mxm matrix and replace K(s) by p $K_1(s)$ $K_2(s)$ where the extra controller factor takes the form

$$K_2(s) = Q_k^{-1}\, \tilde{T}_o \text{ diag } \{\tilde{\lambda}_j\}_{1\leq j\leq m}\, \tilde{T}_o^{-1} \qquad (6)$$

It follows that $Q_o(s)$ $K_2(s)$ has uniform rank k if $\tilde{\lambda}_j \neq 0$, $1 \leq j \leq m$, and

$$Q_o(s)\, K_2(s) = s^{-k}\tilde{T}_o \text{ diag}\{\tilde{\lambda}_j\}_{1\leq j\leq m}\, \tilde{T}_o^{-1} + O(s^{-(k+1)}) \qquad (7)$$

i.e. the inclusion of $K_2(s)$ replaces T_o by $\tilde{T}_o$ and the eigenvalues

$\{\lambda_j\}_{1\leq j\leq m}$ by $\{\tilde{\lambda}_j\}_{1\leq j\leq m}$. In particular, writing $\tilde{T}_o = [x_1, x_2, \ldots, x_m]$ and choosing $\tilde{\lambda}_j = \bar{\tilde{\lambda}}_r$ whenever $x_j = \bar{x}_r$ ensures that $K_2(s)$ is real and hence physically realizable.

(b) Manipulation of Pivots

Suppose that the eigenvalues $\{\lambda_j\}_{1\leq j\leq m}$ and eigenvector matrix T_o of Q_k are as desired, and that T_o is specified such that N_{jj}, $1 \leq j \leq \ell$, is in triangular form or Jordan canonical form. Write

$$K_2(s) = T_o \text{ block diag } \{K_2^{(j)}(s)\}_{1\leq j\leq \ell}\, T_o^{-1}$$

$$K_2^{(j)}(s) = \text{diag } \{k_{jr}(s)\}_{1\leq r\leq d_j} \quad , \quad 1 \leq j \leq \ell \tag{8}$$

Suppose that

$$\lim_{s\to\infty} k_{jr}(s) = 1 \quad , \quad 1 \leq r \leq d_j \; , \; 1 \leq j \leq \ell \tag{9}$$

and write

$$K_2(s) = I_m + s^{-1} R + 0\,(s^{-2})$$

$$k_{jr}(s) = 1 + s^{-1}\beta_{jr} + 0(s^{-2})$$

where

$$R = T_o \text{ block diag } \{R_j\}_{1\leq j\leq \ell} T_o^{-1}$$

$$R_j = \text{diag } \{\beta_{jr}\}_{1\leq r\leq d_j} \quad , \quad 1 \leq j \leq \ell \tag{10}$$

It follows directly from the identity

$$Q_o(s)K_2(s) = s^{-k}Q_k + s^{-(k+1)} \{Q_{k+1} + Q_k R\} + \ldots \tag{11}$$

that the composite system has uniform rank k with mk k^{th}-order infinite zero of the form

$$s(p) = p^{1/k}\, \eta_{ij} + k^{-1}\{\frac{\alpha_{jr}}{\eta_j} + \beta_{jr}\} + \varepsilon_{ijr}(p)$$

$$\lim_{p\to\infty} \varepsilon_{ijr}(p) = 0 \; , \; 1 \leq i \leq k \; , \; 1 \leq r \leq d_j \; , \; 1 \leq j \leq \ell \tag{12}$$

A comparison of (12) and (5) indicates that the inclusion of the extra controller factor enables the systematic and independent manipulation of the pivots by suitable choice of compensation elements in (8). For example, if

$$k_{jr}(s) = \frac{(s+a_{jr})}{(s+b_{jr})}\left(1 + \frac{c_{jr}}{s}\right) \quad , \quad 1 \leqslant r \leqslant d_j \; , \; 1 \leqslant j \leqslant \ell \tag{13}$$

then the asymptotes with pivot $k^{-1}\eta_j^{-1}\alpha_{jr}$ is shifted by $k^{-1}\beta_{jr}$, where

$$\beta_{jr} = a_{jr} - b_{jr} + c_{jr} \quad , \quad 1 \leqslant r \leqslant d_j \; , \; 1 \leqslant j \leqslant \ell \tag{14}$$

In particular, if $K_2(s) = (s+a)/(s+b)I_m$, all pivots are shifted by the amount $k^{-1}(a-b)$.

3. <u>Compensation in the General Case</u>

If $Q_o(s)$ is not of uniform rank, the procedure described above can still be applied by a consideration of the decomposition of $Q_o(s)$. It has been shown (Owens, 1977) that, with only weak assumptions, there exists strictly positive integers q , d_j ($1 \leqslant j \leqslant q$) and k_j, $1 \leqslant j \leqslant q$, such that

$$\begin{gathered} k_1 < k_2 < \; . \; . \; . \; . \; < k_q \\ \sum_{j=1}^{q} d_j \;=\; m \end{gathered} \tag{15}$$

and a real nonsingular transformation T_1 together with unimodular matrices

$$L(s) = \begin{pmatrix} I_{d_1} & 0 & \cdots & 0 \\ O(s^{-1}) & I_{d_2} & & \vdots \\ \vdots & & \ddots & 0 \\ O(s^{-1}) & \cdots & O(s^{-1}) & I_{d_q} \end{pmatrix}$$

$$M(s) = \begin{pmatrix} I_{d_1} & O(s^{-1}) & \cdots & O(s^{-1}) \\ 0 & I_{d_2} & & \vdots \\ \vdots & & \ddots & O(s^{-1}) \\ 0 & \cdots & 0 & I_{d_q} \end{pmatrix} \tag{16}$$

such that

$$L(s)T_1^{-1}Q_o(s)T_1M(s) = \text{block diag}\,\{G_j(s)\}_{1\leq j\leq q} + O(s^{-(k_q+2)}) \quad (17)$$

where the $d_j x d_j$ transfer function matrices $G_j(s)$, $1 \leq j \leq q$, have uniform rank k_j , $1 \leq j \leq q$. An important observation is that the assumptions underlying the decomposition (17) are valid for 'almost all' choices of $K_1(s)$ and that it is always possible to choose $K_1(s)$ to assure their validity. Also, the transformation T_1 can be constructed from the computational procedure derived by the author (Owens, 1977)

The significance of (17) to the structure of the root-locus lies in the result (Owens, 1977) that $Q_o(s)$ generates $k_j d_j$ k_j^{th}-order infinite zeros, $1 \leq j \leq q$, whose asymptotic directions and pivots are identical to those obtained from the uniform rank systems $G_j(s)$, $1 \leq j \leq q$. Consider now the inclusion of the second controller factor

$$K_2(s) = T_1 \text{ block diag } \{K_2^{(j)}(s)\}_{1\leq j\leq q} T_1^{-1} \quad (18)$$

where the $d_j x d_j$ transfer function matrices $K_2^{(j)}(s)$, $1 \leq j \leq q$, are proper, minimum-phase and $\lim_{s\to\infty} K_2^{(j)}(s)$ is finite and nonsingular, $1 \leq j \leq q$. Defining the mxm matrix $\tilde{M}(s)$ by the identity

$$\text{block diag } \{K_2^{(j)}(s)\}_{1\leq j\leq q}\tilde{M}(s) = M(s) \text{ block diag } \{K_2^{(j)}(s)\}_{1\leq j\leq q} \quad (19)$$

it follows that $\tilde{M}(s)$ has the same structure as M(s), and

$$L(s)T_1^{-1}Q_o(s)K_2(s)T_1\tilde{M}(s) = \text{block diag } \{G_j(s)K_2^{(j)}(s)\}_{1\leq j\leq q} + O(s^{-(k_q+2)}) \quad (20)$$

Comparing (20) with (17) and noting that $G_j(s)\ K_2^{(j)}(s)$ has uniform rank k_j, $1 \leq j \leq q$, it follows directly from previous work (Owens, 1977) that the plant G(s) with controller $K(s) = p\ K_1(s)\ K_2(s)$ has $k_j d_j$ k_j^{th}-order infinite zeros, $1 \leq j \leq q$, whose asymptotic directions and pivots are identical to those obtained from the uniform rank systems $G_j(s)\ K_2^{(j)}(s)$, $1 \leq j \leq q$. In practice the compensation of $Q_o(s)$ by choice of $K_2(s)$ can be regarded as the compensation of the uniform rank subsystems $G_j(s)$, $1 \leq j \leq q$, followed by construction of $K_2(s)$ as in (18).

4. Summary and Conclusions

It has been shown that the parameters used in the calculation (Owens, 1977) of the asymptotic behaviour of the root-locus of linear multivariable systems can be used to construct compensation elements enabling the systematic and independent manipulation of all the asymptotic directions and pivots. The results are presented in the form of explicit synthesis procedures by direct construction of suitable control systems. Alternatively, they can be regarded as existence results motivating the search for alternative and perhaps superior control schemes. It is not claimed that the above analysis is a solution to the design problem. There are many remaining problems concerning

(i) the unification of root-locus concepts with other design techniques,

(ii) the physical interpretation of the Matrices T_o, T_1 and their use in design,

(iii) the relationship of the structure of the root-locus plot to transient performance,

(iv) the implications of previously noted sensitivity problems (Owens, 1977) on the ultimate viability of multivariable root-loci techniques, and

(v) the relevance of system zeros to the structure of the compensation elements.

5. References

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