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THE THEORY OF SEPARABLE PROCESSES WITH APPLICATIONS TO THE IDENTIFICATION OF NON-LINEAR SYSTEMS

by

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ABSTRACT

By considering the class of separable random processes, a generalised Wiener-Hopf equation is derived for systems which can be described by a model consisting of a linear system in cascade with a static non-linear element, followed by another linear system. This result, together with a similar relationship for the second order cross-correlation function, is used to formulate an identification and structure testing algorithm for this class of non-linear system. The results of a simulation study are included to illustrate the validity of the algorithm.

1. INTRODUCTION

Although various techniques have been developed for identifying the terms in the Volterra series representation of non-linear systems, the generality of this description tends to ignor the structural features of the system under consideration. Ideally, system identification should provide a knowledge of the structure of the system as well as parameter values associated with each element. One way to try and achieve this goal is to work with specific classes of non-linear systems. For example, a great deal of attention has been directed towards bilinear systems[1,2], and systems which can be described by linear dynamic subsystems in cascade with static power non-linearities[3,4,5,6,7]. Thus by investigating non-linear systems which are simple extensions of the linear case, it may be possible to derive results which can be generalised to more complicated non-linear systems.

In the present study, previous results derived for the Wiener[8] and general model[9] are extended and an identification algorithm which inherently tests the structure of cascade connections of linear dynamic and non-linear subsystems is developed when the input belongs to the class of separable processes. The algorithm represents an extension of a result due to Nuttall[10] who showed that for a wide class of input signals the cross-correlation function between the input and output of a single-valued non-linearity is proportional to the autocorrelation function of the input. This result known as the invariance property is reviewed in the next section together with the definition and properties of the separable class of random processes. Nuttall's results are extended to include separability under linear transformation and the validity of the invariance property after a double non-linear transformation is investigated.

Using these results the general model which consists of a linear system in cascade with a no-memory non-linear element followed by a second linear system is investigated. It is shown that, when the input belongs to the class of separable processes, the input-output cross-correlation function is directly proportional to a second order integral involving only the impulse responses of the linear elements and the autocorrelation function of the input whatever the non-linear device. This result represents an extension of the Wiener-Hopf equation to a non-linear system. In a similar manner an expression relating the second order cross-correlation function to a multiple integral involving the fourth order moment of the input process and the product of the linear subsystem impulse responses is derived.

When the system input has the properties of a white noise process, these results can be simplified considerably, and an identification algorithm which provides information regarding the system structure is presented in Section 4. Simulated examples are included to illustrate the validity of the algorithm.

2. SEPARABLE RANDOM PROCESSES

2.1 Definition of Separability

The separable class of random processes is defined in terms of a property of partially integrated second order statistics[10]. Let $p(\alpha,\beta;\tau)$ be the joint probability density function for the two stationary random processes $\alpha(t)$ and $\beta(t)$, and define

$$g(\beta,\tau) = \int_{-\infty}^{\infty} \alpha p(\alpha,\beta;\tau)\,d\alpha \tag{1}$$

If the g-function separates as

$$g(\beta,\tau) = g_1(\beta)g_2(\tau) \quad \forall\ \beta,\tau \tag{2}$$

then the process $\alpha(t)$ is separable with respect to the process $\beta(t)$. Notice that, contrary to Nuttall's original definition[10] of separability, equation (2) includes both the ac and dc components of the signal $\alpha(t)$. This definition simplifies the results of Section 4.

Considering the cross-correlation function

$$\begin{aligned}\phi_{\alpha\beta}(\tau) &= \overline{\alpha(t)\beta(t+\tau)} \\ &= \int\int \alpha\beta\ p(\alpha,\beta;\tau)\,d\alpha d\beta \\ &= \int g(\beta,\tau).\beta.d\beta \end{aligned} \tag{3}$$

where all integrals will be from $-\infty$ to ∞ unless stated otherwise. If $\alpha(t)$ is separable with respect to $\beta(t)$, equation (2) holds and equation (3) reduces to

$$\phi_{\alpha\beta}(\tau) = g_2(\tau) \int g_1(\beta).\beta d\beta \tag{4}$$

hence

$$\begin{aligned} g_2(\tau) &= \frac{\phi_{\alpha\beta}(\tau)}{\int g_1(\beta).\beta d\beta} \\ &= q_2(0)\,\phi_{\alpha\beta}(\tau) \end{aligned} \tag{5}$$

where $$q_2(0) = \frac{g_2(0)}{\phi_{\alpha\beta}(0)} \tag{6}$$

Thus, if the g-function separates as in equation (2), the function $g_2(\tau)$ must be a constant multiplied by $\phi_{\alpha\beta}(\tau)$.

It can readily be shown that

$$\overline{\alpha(t)\beta^n(t+\tau)} = \int\int \alpha\beta^n p(\alpha,\beta;\tau)\,d\alpha d\beta$$

$$= b_n \phi_{\alpha\beta}(\tau) \;\forall\; n \qquad (7)$$

if the process $\alpha(t)$ is separable with respect to the process $\beta(t)$, where b_n is a real number independent of τ. Although equation (7) is a sufficient and not a necessary condition for separability, it is valuable as a test of separability when all the moments indicated on the left hand side of equation (7) exist.

Consider the special case when $\beta(t) = \alpha(t)$, and define $p(\alpha_1,\alpha_2;\tau)$ to be the second order probability density function of the stationary random process $\alpha(t)$. The process $\alpha(t)$ is defined as separable if the g-function, equation (1) separates as

$$g_\alpha(\alpha_2,\tau) = g_{\alpha,1}(\alpha_2)\,g_{\alpha,2}(\tau) \;\forall\; \tau,\ \alpha_2 \qquad (8)$$

and analogous to equation (5)

$$g_{\alpha,2}(\tau) = \frac{g_{\alpha,2}(0)\,\phi_{\alpha\alpha}(\tau)}{\phi_{\alpha\alpha}(0)} = q_{\alpha,2}(0)\,\phi_{\alpha\alpha}(\tau) \qquad (9)$$

From Wang and Uhlenbeck[11] the second order probability density function when $\tau = 0$ can be written as

$$p(\alpha_1,\alpha_2;0) = p(\alpha_1)\,\delta(\alpha_2-\alpha_1)$$

and hence

$$g_\alpha(\alpha_2,0) = \alpha_2 p(\alpha_2) = g_{\alpha,1}(\alpha_2)\,g_{\alpha,2}(0) \qquad (10)$$

where $p(\alpha_2)$ is the first order probability density function. Equation (10) yields

$$g_{\alpha,1}(\alpha_2) = \frac{\alpha_2 p(\alpha_2)}{g_{\alpha,2}(0)} \tag{11}$$

and from equation (8)

$$g_\alpha(\alpha_2,\tau) = \frac{\alpha_2 p(\alpha_2)\phi_{\alpha\alpha}(\tau)}{\phi_{\alpha\alpha}(0)} \tag{12}$$

Thus, if a process is separable the g-function must split up into the product of the autocorrelation function and a simple first order statistic involving only the first order probability density function of the process.

Notice that when $\alpha(t) \neq \beta(t)$ we are unable to say anything specific about the form of $g_1(\beta)$. Whereas the second order probability density function of a process has in it a delta function for zero time shift, no such relationship holds for the joint probability density.

Fortunately the separable class of random processes is fairly wide and includes the Gaussian process, sine-wave process, phase or frequency modulated process, squared Gaussian process etc.

Although in general the sum of two separable processes is non-separable, the product of two zero mean independent stationary separable processes is always separable irrespective of the particular statistics involved[10].

2.2 Separability under Linear Transformation

Consider the situation when the signal $\beta(t)$ is passed through a linear filter, with an impulse response $h(t)$, to produce the new process $\eta(t)$. We wish to prove that if $\alpha(t)$ is separable with respect to $\beta(t)$, then $\alpha(t)$ is also separable with respect to $\eta(t)$.

If $\alpha(t)$ is separable with respect to $\eta(t)$ then

$$g_\eta(\eta,\tau) = \int \alpha p(\alpha,\eta;\tau)\,d\alpha \qquad (13)$$

$$= g_{\eta,1}(\eta)\,g_{\eta,2}(\tau) \qquad (14)$$

Define the correlation function

$$\phi_{\alpha\eta}(\tau) = \int\int \alpha\eta p(\alpha,\eta;\tau)\,d\alpha d\eta$$

$$= \int\int\int h(\theta)\,\alpha\beta p(\alpha,\beta;\tau-\theta)\,d\alpha d\beta d\theta \qquad (15)$$

But from equations (1) and (2)

$$\phi_{\alpha\eta}(\tau) = \int\int h(\theta)\,g_1(\beta)\,g_2(\tau-\theta)\,d\beta d\theta$$

$$= \int \beta g_1(\beta)\,g_3(\tau)\,d\beta \qquad (16)$$

where $$g_3(\tau) = \int h(\theta)\,g_2(\tau-\theta)\,d\theta \qquad (17)$$

Using equation (14)

$$\phi_{\alpha\eta}(\tau) = \int \eta g_\eta(\eta,\tau)\,d\eta$$

$$= g_{\eta,2}(\tau)\int \eta g_{\eta,1}(\eta)\,d\eta \qquad (18)$$

Comparison of equations (16) and (18) gives

$$g_{\eta,2}(\tau) = g_3(\tau) \qquad (19)$$

$$\int \eta\, g_{\eta,1}(\eta)\,d\eta = \int \beta g_1(\beta)\,d\beta \qquad (20)$$

where by considering equations (5) and (17)

$$g_3(\tau) = g_{\eta,2}(\tau) = \frac{1}{\int \beta g_1(\beta)\,d\beta}\int h(\theta)\,\phi_{\alpha\beta}(\tau-\theta)\,d\theta$$

$$= \frac{1}{\int \eta g_{\eta,1}(\eta)\,d\eta}\int h(\theta)\,\overline{\alpha(t)\,\beta(t+\tau-\theta)}\,d\theta$$

$$= \frac{\overline{\alpha(t)\eta(t+\tau)}}{\int \eta g_{\eta,1}(\eta)\, d\eta} \tag{21}$$

Thus if $\alpha(t)$ is separable with respect to $\beta(t)$ it is also separable with respect to $\eta(t)$.

2.3 The Invariance Property

Consider the system illustrated in FIG 1, where $x_1(t)$ is the input to an all-pass network and $x_2(t)$ is the input to a non-linear no-memory device with a transfer chracteristic $F[\cdot]$

$$y_2(t) = F\{x_2(t)\} \tag{22}$$

Define the input cross-correlation function as

$$\phi_{x_1x_2}(\tau) = \overline{x_1(t)x_2(t+\tau)} = \iint x_1x_2p(x_1,x_2;\tau)\, dx_1, dx_2 \tag{23}$$

where $p(x_1,x_2;\tau)$ is the joint probability density function of the inputs $x_1(t)$ and $x_2(t)$. Similarly, for the output cross-correlation function

$$\phi^F_{y_1y_2}(\tau) = \overline{y_1(t)y_2(t+\tau)} = \overline{x_1(t)F\{x_2(t+\tau)\}}$$

$$= \iint x_1F[x_2]p(x_1,x_2;\tau)\, dx_1dx_2 \tag{24}$$

where the superscript 'F' is used to indicate the dependence of the correlation function on the particular device F that is used.

In general there is no relationship between $\phi_{x_1x_2}(\tau)$ and $\phi^F_{y_1y_2}(\tau)$, however when $x_1(t)$ is separable with respect to $x_2(t)$

$$g_x(x_2,\tau) = g_{x,1}(x_2)q_{x,2}(0)\phi_{x_1x_2}(\tau) \tag{25}$$

Hence, from equation (24) and the definition of the g-function

$$\phi^{F}_{y_1y_2}(\tau) = \int F[x_2]\, g_x(x_2,\tau)\, dx_2$$

$$= q_{x,2}(0)\, \phi_{x_1x_2}(\tau) \int F[x_2]\, g_{x,1}(x_2)\, dx_2$$

$$= C_F\ \phi_{x_1x_2}(\tau)\ \forall\ F \text{ and } \tau \tag{26}$$

where C_F is a number depending on the non-linear device and the input statistics. Thus regardless of the non-linear device used, the input and output correlation functions are identical, except for a scale factor, when the undistorted process is separable with respect to the input to the non-linear element.

This behaviour of the correlation functions is known as the invariance property[10]. Note that we include as a special case $x_1(t) = x_2(t)$, when the invariance property relates the input-output cross-correlation function to the input autocorrelation function

$$\phi^{F}_{xy}(\tau) = C_{F_x}\ \phi_{xx}(\tau)\ \forall\ F \text{ and } \tau \tag{27}$$

provided the input process x(t) is separable. Since $x_1(t) = x_2(t) = x(t)$, C_{F_x} can be readily evaluated from equation (26)

$$C_{F_x} = \frac{1}{\phi_{xx}(0)} \int x\ F[x]\, p(x)\, dx \tag{28}$$

2.4 Double Non-linear Transformations

Nuttall[10] investigated the problem of inserting a non-linear device in the top lead of FIG 1, but found that the invariance property held under very restrictive conditions. For the purpose of the present study, it is sufficient to investigate the situation when the top lead contains a squaring device, as illustrated in FIG 2.

Applying the results of Section 2.1, and setting $\alpha(t) = y_1(t) = x_1^2(t)$, $\beta(t) = x_2(t)$ in equation (1), the process $x_1^2(t)$ is defined as separable with respect to the process $x_2(t)$ if

$$g_d(x_2,\tau) = \int x_1^2\ p(x_1,x_2;\tau)\,dx_1 \tag{29}$$

can be expressed as

$$g_d(x_2,\tau) = g_{d,1}(x_2)\,g_{d,2}(\tau) \tag{30}$$

In general, equation (30) will not hold. However, when $x_1(t) = x_2(t)$ is a Gaussian process equation (30) can be shown to be valid, and this case will be studied in detail in section 4.

Define the correlation functions

$$\phi_{y_1y_2}(\tau) = \int\int x_1^2 F\{x_2\} p(x_1,x_2;\tau)\,dx_1dx_2 \tag{31}$$

$$\phi_{x_1^2x_2^2}(\tau) = \int\int x_1^2x_2^2\ p(x_1,x_2;\tau)\,dx_1dx_2 \tag{32}$$

If $x_1^2(t)$ is separable with respect to $x_2(t)$, equation (32) can be expressed as

$$\begin{aligned}\phi_{x_1^2x_2^2}(\tau) &= \int x_2^2\ g_d(x_2,\tau)\,dx_2\\ &= g_{d,2}(\tau) \int x_2^2\ g_{d,1}(x_2)\,dx_2\end{aligned} \tag{33}$$

and hence

$$g_{d,2}(\tau) = q_{d,2}(0)\,\phi_{x_1^2x_2^2}(\tau) \tag{34}$$

where $$q_{d,2}(0) = \frac{g_{d,2}(0)}{\phi_{x_1^2x_2^2}(0)} \tag{35}$$

From equation (31)

$$\phi_{y_1y_2}(\tau) = \int F[x_2]\, g_d(x_2,\tau)\, dx_2 \tag{36}$$

$$= \phi_{x_1^2 x_2^2}(\tau)\, q_{d,2}(0) \int g_{d,1}(x_2)\, F[x_2]\, dx_2$$

$$= C_{FF}\, \phi_{x_1^2 x_2^2}(\tau) \;\forall\; F \text{ and } \tau \tag{37}$$

which represents the invariance property when the upper network contains a squaring device. C_{FF} is a constant depending on the non-linear device and input statistics only.

For the special case $x_1(t) = x_2(t) = x(t)$

$$\phi_{x_1^2 y_2}(\tau) = C_{FF_x}\, \phi_{x^2 x^2}(\tau) \;\forall\; F \text{ and } \tau \tag{38}$$

provided $x^2(t)$ is separable with respect to $x(t)$ where

$$C_{FF_x} = \frac{1}{\phi_{x^2 x^2}(0)} \int x^2\, F[x]\, p(x)\, dx \tag{39}$$

3. ANALYSIS OF THE GENERAL MODEL

The class of non-linear systems considered in the present analysis consists of a linear system with impulse response $h_1(t)$ in cascade with a non-linear zero-memory element and a linear system with impulse response $h_2(t)$, as illustrated in FIG 3. It is assumed that the non-linear element can be represented by a transfer characteristic of the form

$$y(t) = \gamma_1 x(t) + \gamma_2 x^2(t) + \ldots.\gamma_k x^k(t) \tag{40}$$

3.1 First-Order Cross-Correlation Function

In order that we may develop an identification and structure testing algorithm for systems which can be represented by the general model a relationship between the correlation functions $\phi_{u_1 z}(\sigma)$ and $\phi_{u_1 u_2}(\sigma)$ must be established. This can be achieved by applying the results derived in previous sections to the system illustrated in FIG 3.

From the Convolution theorem

$$z_2(t) = \int h_2(\theta) y(t-\theta) d\theta \qquad (41)$$

and

$$y(t) = F\{\int h_1(\tau_1) u_2(t-\tau_1) d\tau_1\}$$

$$= \int Q(t,\tau_1) u_2(t-\tau_1) d\tau_1 \qquad (42)$$

where $Q(t,\tau_1)$ is a function of t and τ_1 only. An expression for $Q(t,\tau_1)$ can be obtained by expanding equation (42) in a Volterra series[9,12] assuming that the non-linear element can be represented by a finite polynomial of the form of equation (40). Thus

$$y(t) = \int \omega_1(\tau_1) u_2(t-\tau_1) d\tau_1 + \int\int \omega_2(\tau_1,\tau_2) u_2(t-\tau_1) u_2(t-\tau_2) d\tau_1 d\tau_2 + \ldots$$

$$\ldots + \underbrace{\int \quad \int}_{n \text{ integrals}} \omega_n(\tau_1,\tau_2 \ldots \tau_n) u_2(t-\tau_1) \ldots u_2(t-\tau_n) d\tau_1 \ldots d\tau_n \qquad (43)$$

where the function $\omega_n(\tau_1,\tau_2 \ldots \tau_n)$ is termed the Volterra kernal of order n. For the simple non-linear system relating $u_2(t)$ and y(t) in FIG 3, the m'th order Volterra kernal is given by[8]

$$\omega_m(\tau_1,\tau_2\ldots\tau_m) = \gamma_m \prod_{p=1}^{m} h_1(\tau_p) \tag{44}$$

and the Volterra expansion of equation (43) can be expressed as

$$\begin{aligned} y(t) &= \gamma_1 \int h_1(\tau_1)u_2(t-\tau_1)d\tau_1 \\ &+\gamma_2\int\int h_1(\tau_1)h_1(\tau_2)u_2(t-\tau_1)u_2(t-\tau_2)d\tau_1 d\tau_2+\ldots \\ &\ldots +\gamma_k\int\ldots\int h_1(\tau_1)\ldots h_1(\tau_k)u(t-\tau_1)\ \ldots \\ &\ldots\ u(t-\tau_k)d\tau_1\ldots d\tau_k \end{aligned} \tag{45}$$

Subtracting equation (45) from equation (42)

$$\begin{aligned} &\int\{Q(t_1\tau_1)-\gamma_1 h_1(\tau_1)-\gamma_2 h_1(\tau_1)\int h_1(\tau_2)u_2(t-\tau_2)d\tau_2 \\ &+\ \ldots\ +\gamma_k h_1(\tau_1)\int\ldots\int h_1(\tau_2)\ldots h_1(\tau_k)u_2(t-\tau_2)\ \ldots \\ &\ldots\ u_2(t-\tau_k)d\tau_2\ldots d\tau_k\}u_2(t-\tau_1)d\tau_1 = 0 \end{aligned} \tag{46}$$

Since equation (46) must exist for arbitrary inputs $u_2(t)$, the function $Q(t,\tau_1)$ is given by

$$\begin{aligned} Q(t,\tau_1) &= \gamma_1 h_1(\tau_1)+\gamma_2 h_1(\tau_1)\int h_1(\tau_2)u_2(t-\tau_2)d\tau_2 \\ &+\ \ldots\gamma_k h_1(\tau_1)\int\ldots\int h_1(\tau_2)\ldots h_1(\tau_k)u_2(t-\tau_2) \\ &\ldots\ u_2(t-\tau_k)d\tau_2\ \ldots\ d\tau_k \end{aligned} \tag{47}$$

Combining equations (41) and (42), the output of the general model $z_2(t)$ can be expressed as

$$z_2(t) = \int\int h_2(\theta)Q(t-\theta,\tau)u_2(t-\theta-\tau_1)d\theta d\tau_1 \tag{48}$$

and the output correlation function can be defined as

$$\phi_{z_1z_2}(\varepsilon) = \overline{u_1(t-\varepsilon)z_2(t)}$$

$$= \iint h_2(\theta)\overline{Q(t-\theta,\tau_1)u_2(t-\theta-\tau_1)u_1(t-\varepsilon)d\tau_1 d\theta} \qquad (49)$$

The validity of the invariance property for the non-linear element in FIG 3 can be established using the results derived in section 2.2. Thus providing $u_1(t)$ is separable with respect to $u_2(t)$, then $u_1(t)$ is separable with respect to $x(t)$ and from equation (26)

$$\phi_{z_1y}(\sigma) = C_{FG}\phi_{u_1x}(\sigma) \;\forall\; F \text{ and } \sigma \qquad (50)$$

where C_{FG} is a constant depending on the non-linear device and the input statistics only. An expression for $\phi_{z_1y}(\sigma)$ can be obtained from equation (42) as

$$\phi_{z_1y}(\sigma) = \overline{u_1(t-\sigma)y(t)}$$

$$= \overline{\int Q(t,\tau_1)u_2(t-\tau_1)u_1(t-\sigma)d\tau_1} \qquad (51)$$

and similarly

$$\phi_{u_1x}(\sigma) = \overline{u_1(t-\sigma)x(t)}$$

$$= \int h_1(\tau_1)\overline{u_2(t-\tau_1)u_1(t-\sigma)d\tau_1} \qquad (52)$$

Combining equations (50), (51) and (52)

$$\overline{\int Q(t,\tau_1)u_2(t-\tau_1)u_1(t-\sigma)d\tau_1}$$

$$= C_{FG} \int h_1(\tau_1)\overline{u_2(t-\tau_1)u_1(t-\sigma)d\tau_1} \qquad (53)$$

and hence from equation (49)

$$\phi_{z_1 z_2}(\varepsilon) = \phi_{u_1 z_2}(\varepsilon) = C_{FG}\int\int h_2(\theta)h_1(\tau_1)\phi_{u_1 u_2}(\varepsilon-\theta-\tau_1)d\theta d\tau_1 \tag{53}$$

For the special case when $u_1(t) = u_2(t) = u(t)$, equation (53) relates the input-output cross-correlation function to an integral involving the autocorrelation function of the input

$$\phi_{uz_2}(\varepsilon) = C_{FG}\int\int h_2(\theta)h_1(\tau_1)\phi_{uu}(\varepsilon-\theta-\tau_1)d\theta d\tau_1 \tag{54}$$

where C_{FG} is a constant. Equation (54), which will be referred to as the generalised Wiener-Hopf equation, holds for any non-linear device $F[\cdot]$, providing the input process $u(t)$ is separable. This result, which represents an extension of the Wiener-Hopf equation to systems which can be described by a general model, forms the basis of the identification algorithm described in section 4.

3.2 Second-Order Cross-Correlation Function

Following the analysis of the previous section an expression for the second order correlation function $\phi_{u_1^2 z_2}(\varepsilon)$ is derived. By definition

$$\phi_{u_1^2 z_2}(\varepsilon) = \phi_{u_1 u_1 z_2}(\varepsilon,\varepsilon) = \overline{u_1(t-\varepsilon)u_1(t-\varepsilon)z_2(t)} \tag{55}$$

and from equation (48)

$$\phi_{u_1^2 z_2}(\varepsilon) = \int\int h_2(\theta)\overline{Q(t-\theta,\tau_1)u_2(t-\theta-\tau_1)u_1^2(t-\varepsilon)}d\theta d\tau_1 \tag{56}$$

Provided $u_1^2(t)$ is separable with respect to $u_2(t)$ is is also separable with respect to $x(t)$ and from the invariance property for double non-linear transformations, equation (38)

$$\phi_{u_1^2 y}(\sigma) = C_{FFG}\phi_{u_1^2 x^2}(\sigma) \;\forall\; F \text{ and } \sigma \tag{57}$$

An expression for $\phi_{u_1^2 y}(\sigma)$ can be obtained from equation (42) as

$$\phi_{u_1^2 y}(\sigma) = \overline{u_1^2(t-\sigma)y(t)}$$

$$= \overline{\int Q(t,\tau_1)u_2(t-\tau_1)u_1^2(t-\sigma)d\tau_1} \qquad (58)$$

and similarly

$$\phi_{u_1^2 x^2}(\sigma) = \overline{u_1^2(t-\sigma)x^2(t)}$$

$$= \int\int h_1(\tau_1)h_1(\tau_2)\overline{u_2(t-\tau_1)u_2(t-\tau_2)u_1^2(t-\sigma)}d\tau_1 d\tau_2 \qquad (59)$$

Combining equations (57), (58) and (59)

$$\overline{\int Q(t,\tau_1)u_2(t-\tau_1)u_1^2(t-\sigma)d\tau_1}$$

$$= C_{FFG}\int\int h_1(\tau_1)h_1(\tau_2)\overline{u_2(t-\tau_1)u_2(t-\tau_2)u_1^2(t-\sigma)}d\tau_1 d\tau_2 \qquad (60)$$

and hence from equation (56)

$$\phi_{u_1^2 z_2}(\varepsilon) = C_{FFG}\int\int\int h_2(\theta)h_1(\tau_1)h_1(\tau_2)\overline{u_2(t-\theta-\tau_1)}$$

$$\overline{u_2(t-\theta-\tau_2)u_1^2(t-\varepsilon)}d\tau_1 d\tau_2 d\theta \qquad (61)$$

For the special case when $u_1(t) = u_2(t) = u(t)$, equation (61) relates the second-order cross-correlation function to an integral involving the fourth order moment of the input process.

4. STRUCTURE TESTING FOR NON-LINEAR SYSTEM IDENTIFICATION

The results derived for the general model can now be formulated into an identification and structure testing algorithm. For the class of non-linear systems considered, notably cascade connections of linear dynamic and static non-linear systems, system structure refers to the position of the non-linear device in relation to the linear subsystems. Because the linear[13], Hammerstein[14,15] and Wiener[8] models are all subclasses of the general model, an algorithm that can determine which if any of these models coincides with the structure of the plant, and provides an estimate of each component subsystem, is highly desirable.

4.1 The General Model

Consider the system illustrated in FIG 3 when the input to the general model comprises the practical realization of a Gaussian white process u(t) and a non-zero mean level b. It can readily be shown that the Gaussian process is separable and hence all the results derived in previous sections are applicable.

Define

$$z'(t) = z_2(t) - \overline{z_2(t)}$$

where $z_2(t)$ is the system response to an input u(t)+b, and the superscript ' will be used throughout to indicate that a signal has zero mean. The first order cross-correlation function is defined as

$$\phi_{uz'}(\sigma) = \overline{z'(t)u(t-\sigma)} = \overline{z_2(t)u(t-\sigma)}$$

Thus, referring to FIG 3, and setting

$$u_1(t) = u(t)$$
$$u_2(t) = u(t)+b$$

where u(t) is a zero mean white Gaussian process, we may write from equation (53)

$$\phi_{uz'}(\sigma) = C_{FG}\int\int h_2(\theta)h_1(\tau_1)\overline{u(t-\sigma)\{u(t-\theta-\tau_1)+b\}}d\tau_1 d\theta$$
$$= C_{FG}\int\int h_2(\theta)h_1(\tau_1)\phi_{uu}(\sigma-\theta-\tau_1)d\tau_1 d\theta \tag{62}$$

Provided the signal u(t) has the properties of a white Gaussian process[16], then $\phi_{uu}(\sigma-\theta-\tau_1)$ approximates to a delta function at $\theta = \sigma-\tau_1$ and equation (62) reduces to

$$\phi_{uz'}(\sigma) = C_{FG}\int h_1(\tau_1)h_2(\sigma-\tau_1)d\tau_1 \tag{63}$$

where C_{FG} is a constant. If the output $z_2(t)$ is corrupted by a noise process n(t), this will not affect the estimate of equation (63), provided the input signal and noise are statistically independent (i.e. $\overline{u(t-\sigma)\,n(t)} = 0$).

The constant C_{FG} can be evaluated by considering equation (50)

$$\phi_{u_1y}(\sigma) = C_{FG}\phi_{u_1x}(\sigma) \;\forall\; F \text{ and } \sigma \tag{64}$$

and defining

$$x(t) = \int h_1(\theta)u(t-\theta)d\theta+b\int h_1(\theta)d\theta$$
$$= x'(t)+\mu_x \tag{65}$$

The left hand side of equation (64) can now be evaluated by considering the polynomial representation of the non-linear element, equation (40)

$$\phi_{u_1y}(\sigma) = \overline{u(t-\sigma)y(t)}$$

$$= \overline{u(t-\sigma)\{\gamma_1(x'(t)+\mu_x)+\gamma_2(x'(t)+\mu_x)^2+\ldots}$$

$$\ldots \overline{+\gamma_k(x'(t)+\mu_x)^k\}}$$

$$= \gamma_1\phi_{ux'}(\sigma)+2\gamma_2\mu_x\phi_{ux'}(\sigma)$$

$$+3\gamma_3\phi_{ux'}(\sigma)\int h_1^2(\theta)\,d\theta+3\mu_x^2\gamma_3\phi_{ux'}(\sigma)+\ldots \qquad (66)$$

assuming that $\int\phi_{uu}(t)\,dt = 1$.

However,

$$\phi_{ux}(\sigma) = \overline{u(t-\sigma)\{x'(t)+\mu_x\}} = \phi_{ux'}(\sigma) \qquad (67)$$

and hence by comparison of equations (64), (66) and (67)

$$C_{FG} = \gamma_1+2\gamma_2 b\int h_1(\theta)\,d\theta+3\gamma_3\int h_1^2(\theta)\,d\theta$$

$$+3\gamma_3 b^2\int\int h_1(\tau_1)h_1(\tau_2)\,d\tau_1 d\tau_2+\ldots \qquad (68)$$

Providing the linear subsystem $h_1(t)$ is stable, bounded inputs bounded outputs, C_{FG} is a finite constant and equation (63) is valid.

Thus, by applying a white Gaussian process with mean level b to the system illustrated in FIG 3 and computing the first order cross-correlation function $\phi_{uz'}(\sigma)$, an estimate of the convolution of the linear subsystem impulse responses is obtained. A Gaussian signal with a mean level b is used to ensure that all terms in equation (68) contribute to C_{FG}. If the input had zero mean, all the even terms in equation (68) would be zero because of the symmetry of the Gaussian density function.

The second order cross-correlation function $\phi_{u_1^2 z'}(\sigma)$ can be evaluated in a similar manner. The invariance property of equation (57) can be expressed as

$$\phi_{u_1^2 y'}(\sigma) = C'_{FFG}\phi_{u_1^2 \omega'}(\sigma) \; \forall \; F \text{ and } \sigma \tag{69}$$

where $\omega'(t) = (x^2(t))'$; and the superscipt is used to indicate a zero mean process. Combining equations (61) and (69)

$$\phi_{u^2 z'}(\sigma) = C'_{FFG}\int\int\int h_2(\theta)h_1(\tau_1)h_1(\tau_2)\overline{u(t-\theta-\tau_1)}$$

$$\overline{u(t-\theta-\tau_2)}u^2(t-\sigma)d\tau_1 d\tau_2 d\theta$$

$$-C'_{FFG}\phi_{uu}(0)\int h_2(\theta)\phi_{x'x'}(0)d\theta \tag{70}$$

Utilizing the properties of a zero mean Gaussian process[16], equation (70) can be written as

$$\phi_{u^2 z'}(\sigma) = C'_{FFG}\int\int\int h_2(\theta)h_1(\tau_1)h_1(\tau_2)\{\phi_{uu}(\tau_1-\tau_2)\phi_{uu}(0)$$

$$+\phi_{uu}(\sigma-\theta-\tau_2)\phi_{uu}(\sigma-\theta-\tau_1)$$

$$+\phi_{uu}(\sigma-\theta-\tau_1)\phi_{uu}(\sigma-\theta-\tau_2)\}d\tau_1 d\tau_2 d\theta$$

$$-C'_{FFG}\phi_{uu}(0)\int h_2(\theta)\phi_{x'x'}(0)d\theta \tag{71}$$

Since the input u(t) represents the realization of a white Gaussian signal, the autocorrelation function $\phi_{uu}(\cdot)$ approximates to a delta function and equation (71) reduces to

$$\phi_{u^2z'}(\sigma) = 2C'_{FFG}\int h_2(\theta)h_1{}^2(\sigma-\theta)d\theta \tag{72}$$

where C'_{FFG} is a constant.

C'_{FFG} can be evaluated by expanding equation (69). From equations (40) and (65)

$$\begin{aligned} y'(t) &= y(t)-\overline{y(t)} \\ &= \gamma_1 x'(t)+\gamma_2\{(x'(t))^2-\phi_{x'x'}(0)\} \\ &\quad +2\gamma_2\mu_x x'(t)+\gamma_3(x'(t))^3+3\mu_x\gamma_3 \\ &\quad +\{(x'(t))^2-\phi_{x'x'}(0)\}+3x'(t)\mu_x{}^2\gamma_3+\ \ldots \end{aligned} \tag{73}$$

and hence

$$\phi_{u^2y'}(\sigma) = \{\gamma_2+3\mu_x\gamma_3+\ldots\}\overline{\{(x'(t))^2-\phi_{x'x'}(0)\}\ u^2(t-\sigma)} \tag{74}$$

Consideration of equation (65) shows that

$$\begin{aligned} \overline{((x'(t))^2-\phi_{x'x'}(0))u^2(t-\sigma)} &= \overline{\omega'(t)u^2(t-\sigma)} \\ &= \phi_{u^2\omega'}(\sigma) \end{aligned} \tag{75}$$

and equation (74) can be written in the form

$$\phi_{u^2y'}(\sigma) = \{\gamma_2+3\gamma_3 b\int h_1(\tau_1)d\tau_1+\ldots\}\phi_{u^2\omega'}(\sigma) \tag{76}$$

Inspection of equations (69) and (76) gives the final result

$$C'_{FFG} = \gamma_2+3\gamma_2 b\int h_1(\tau)d\tau+\ldots \tag{77}$$

Providing the linear subsystem $h_1(t)$ is stable, bounded inputs bounded outputs, C'_{FFG} is a finite constant and equation (72) is valid. If the output $z_2(t)$ in FIG 3 is corrupted by a noise signal $n(t)$, provided this is independent of the input $u(t)$ then $\overline{n(t)u^2(t-\sigma)} = 0 \forall \sigma$, and hence the estimate of equation (72) is unaffected by this noise.

The results for the general model when the input consists of a Gaussian white process with a mean level can be summarised as

1ST ORDER CCF:

$$\phi_{uz'}(\sigma) = C_{FG}\int h_1(\tau_1)h_2(\sigma-\tau_1)d\tau_1 \tag{78}$$

where
$$C_{FG} = \gamma_1+2\gamma_2 b\int h_1(\theta)d\theta+3\gamma_3\int h_1{}^2(\theta)d\theta$$
$$+3\gamma_3 b^2\int\int h_1(\tau_1)h_1(\tau_2)d\tau_1 d\tau_2+\ldots \tag{79}$$

2ND ORDER CCF:

$$\phi_{u^2z'}(\sigma) = 2C'_{FFG}\int h_2(\tau_1)h_1{}^2(\sigma-\tau_1)d\tau_1 \tag{80}$$

where
$$C'_{FFG} = \gamma_2+3\gamma_3 b\int h_1(\tau_1)d\tau_1+\ldots \tag{81}$$

Inspection of equations (78) and (80) shows that correlation analysis effectively decouples the identification problem into two distinct steps, identification of the linear subsystems and characterization of the non-linear element. A least squares algorithm which provides unbiased estimates of the individual linear subsystem impulse responses $\mu_1 h_1(t)$ and $\mu_2 h_2(t)$ and hence the associated pulse transfer functions, where μ_1 and μ_2 are constants, has been derived in a previous publication[9]. Once the linear systems have been identified estimates of the coefficients in the polynomial representation of the non-linearity can be readily computed.

4.2 The Wiener Model

The Wiener model consists of a linear system followed by a continuous non-linear element. The model is a much simplified version of Wiener's original non-linear system characterization[17] and belongs to the class of models studied by Cameron and Martin[18], and Bose[19]. Thus by setting $h_2(t) = \delta(t)$ in equations (78) to (81), comparable results for the Wiener model can be summarised as

1ST ORDER CCF:

$$\phi_{uz'}(\sigma) = C_{FW}h_1(\sigma) \tag{82}$$

where
$$C_{FW} = C_{FG} = \gamma_1+2\gamma_2 b\int h_1(\theta)\,d\theta$$
$$+3\gamma_3\int h_1^{\,2}(\theta)\,d\theta+3\gamma_3 b^2\int\int h_1(\tau_1)h_1(\tau_2)\,d\tau_1 d\tau_2+\ldots \tag{83}$$

2ND ORDER CCF:

$$\phi_{u^2z'}(\sigma) = 2C_{FFW}h_1^{\,2}(\sigma) \tag{84}$$

where
$$C_{FFW} = C'_{FFG} = \gamma_2+3\gamma_3 b\int h_1(\tau_1)\,d\tau_1+\ldots \tag{85}$$

The first order correlation function is therefore directly proportional to the linear system impulse response[8] and the second order correlation function is proportional to the square of $h_1(\sigma)$.

4.3 The Hammerstein Model

The Hammerstein model consists of a zero-memory non-linear element followed by linear dynamics. The model represents a realization of the Hammerstein operator, and was originally proposed by Narendra and Gallman[14]. Setting $h_1(t) = \delta(t)$ in equations (78) to (81), the results for the Hammerstein model can be summarised as

1ST ORDER CCF:

$$\phi_{uz'}(\sigma) = C_{FH}h_2(\sigma) \tag{86}$$

where $$C_{FH} = \gamma_1+2b\gamma_2+3\gamma_3\{\phi_{uu}(0)+b^2\}+\ldots \tag{87}$$

Although, theoretically $\phi_{uu}(0)$ would be infinite, in practice $u(t)$ can only approximate to a white noise process and $\phi_{uu}(0) = \rho^2$, the variance of $u(t)$, which is finite. Equation (87) can therefore be written as

$$C_{FH} = \gamma_1+2b\gamma_2+3\gamma_3\{\rho^2+b^2\} + \ldots \tag{88}$$

2ND ORDER CCF:

$$\phi_{u^2z'}(\sigma) = 2C_{FFH}\lambda h_2(\sigma)$$

where $$C_{FFH} = \gamma_2+3b\gamma_3 + \ldots \tag{90}$$

and $$\lambda = \int\phi^2_{uu}(t)\,dt$$

The first and second order correlation functions are therefore directly proportional to the impulse response of the linear element.

4.4 The Linear Model

If the system were linear, then from FIG 3 $\gamma_n = 0$, $n = 2,3\ldots k$ in the polynomial representation of the non-linear element equation

(40), and the system weighting function would consist of the convolution of the impulse responses $h_1(t)$ and $h_2(t)$. From equations (78) and (81), the results for the linear model can be summarised as

1ST ORDER CCF:

$$\phi_{uz'}(\sigma) = C_{FG}\int h_1(\tau_1)h_2(\sigma-\tau_1)\,d\tau_1 \qquad (91)$$

where $\qquad C_{FG} = \gamma_1 \qquad (92)$

2ND ORDER CCF:

$$\phi_{u^2z'}(\sigma) = 0 \;\forall\; \sigma \qquad (93)$$

Thus, the first order correlation function provides an estimate of the system weighting function $h_1(t) * h_2(t)$, and the second order correlation function provides a convenient test for linearity.

4.5 A Structure Testing Algorithm

The results derived in sections (4.1) to (4.4) can be used directly to identify the component linear and non-linear subsystems in all the models considered. However, the relationship between the first order and second order cross-correlation functions provides valuable information regarding the system structure.

Thus, if the second order correlation function is zero for all time shifts then the system must be linear, and once a pulse transfer function model is fitted to the first order correlation function the identification is complete. If the first order and second order correlation functions are equal, except for a constant of proportionality, then the system must have the structure of a

Hammerstein model. However, if the second order correlation function is the square of the first order correlation function, except for a constant of proportionality then the system must have the structure of a Wiener model. Finally, if none of the above conditions hold then the system may have the structure of the general model. However, this is a necessary but not a sufficient condition which must be confirmed by parameterising the linear systems[9] $h_1(t)$, $h_2(t)$ and the non-linear element and examining the residuals.

Identification of cascade connections of linear dynamic and static non-linear systems using correlation analysis thus inherently provides information regarding the structure of these systems.

5. SIMULATION RESULTS

The identification procedure outlined above was used to test the structure of both linear and non-linear models. All the models were simulated on an ICL 1906S digital computer, and in each case 10,000 data points were generated by recording the response to a Gaussian white input sequence $N\{0.7, 1.2\}$.

Experimental and theoretical values of the first order correlation function of a linear model with pulse transfer functions $H_2(z^{-1}) = 1$ and

$$H_1(z^{-1}) = \frac{0.216\,z^{-1}}{1-1.579z^{-1}+0.67z^{-2}} \tag{94}$$

is illustrated in FIG 4. As expected, the second order correlation function was identically zero for all time, indicating that the system was linear.

A comparison of the estimated and theroetical values of the first order correlation function for a Wiener model is illustrated in FIG 5(a). The Wiener model consisted of the pulse transfer function of equation (94) in cascade with a non-linear element of the form

$$y(t) = 5.0x(t)+20.0x^2(t)+50.0x^3(t) \tag{95}$$

Inspection of FIG 5(b), showing the second order correlation function and the square of the first order correlation function indicates that the system has the structure of a Wiener model.

The first and second order correlation functions for a Hammerstein model, consisting of a non-linear element

$$y(t) = x(t)+0.5x^2(t)+0.8x^3(t) \tag{96}$$

in cascade with the linear system of equation (94), are illustrated in FIG 6. The similarity of the correlation functions clearly indicates that the system has the structure of a Hammerstein model.

Finally, a general model consisting of a linear system with pulse transfer function of equation (94) in cascade with the non-linear device equation (95) and a linear system with a pulse transfer function

$$H_2(z^{-1}) = \frac{5.0z^{-1}}{1-0.875z^{-1}} \tag{97}$$

was simulated. A comparison of the estimated impulse responses and the theoretical weighting sequences of the linear subsystems are illustrated in FIG 7(a) and 7(b).

Inspection of the estimated system parameters[9], summarised in Table 1(a), (b) and (c) for all the models, clearly demonstrates the effectiveness of the algorithm.

6. CONCLUSIONS

By considering the separable class of processes it has been shown that the Wiener-Hopf equation can be generalised to systems which can be described by a cascade correction of a linear dynamic, static non-linear and a linear system. A similar result involving the fourth order moment of the input process can be derived for the second order cross-correlation function. Both these results, which are invariant of the non-linear device except for a constant scale factor, form the basis for an identification algorithm which, when the input is white Gaussian, effectively decouples the identification procedure for this class of non-linear systems into two distinct steps; identification of the linear subsystems and characterisation of the non-linear element. The relationship between the first and second order correlation functions also provides information regarding the system structure, notably the position of the non-linear element with respect to the linear subsystems. This simplifies considerably the identification of this class of non-linear systems.

A least squares algorithm which provides unbiased estimates of the parameters associated with the pulse transfer functions of the linear systems and the polynomial representation of the non-linear device has been developed previously[9]. All the results can be applied, with only slight modification, even if the input is non-white[9], providing it is separable and has a non-zero mean.

The results have been extended to include feedback systems and other common system structures, and it is hoped that these will be published at a later date together with an analysis of the variance of the estimates.

7. REFERENCES

1. BROCKETT, R.W.: On the algebraic structure of bilinear systems; Theory and Applications of Variable Structure Systems, R.R.Mohler and A.Ruberti, Eds., New York, Academic Press, 1972.

2. BRUNI, C., DIPILLO, G., KOCH, G.: Bilinear systems: An appealing class of nearly linear systems in theory and applications; IEEE Trans. Auto. Cont., AC-19, 1974, pp.334-348.

3. COOPER, B., FALKNER, A.H.: Identification of simple non-linear systems; Proc.IEE, 122, 7, 1975, pp.753-755.

4. GARDINER, A.B.: Elimination of the effects of non-linearities on process cross-correlations; Electronics Letters, 2, 5, 1966, pp.164-165.

5. SMITH, W.W., RUGH, W.J.: On the structure of a class of non-linear systems; IEEE Trans. Auto. Cont., AC-19, 1974, pp.701-706.

6. WEBB, R.V.: Identification of the Volterra kernals of a process containing single-valued non-linearities; Electronics Letters, 10, 16, 1974, pp.344-346.

7. ECONOMAKOS, E.: Identification of a group of internal signals of zero memory non-linear systems, Electronics Letters, 7, 4, 1971, pp.99-100.

8. BILLINGS, S.A., FAKHOURI, S.Y.: Identification of non-linear systems using the Wiener model; Electronics Letters, 13, 17, 1977, pp.502-504.

9. BILLINGS, S.A., FAKHOURI, S.Y.: Identification of a class of non-linear systems using correlation analysis; submitted for publication.

10. NUTTALL, A.H.: Theory and application of the separable class of random processes; Tech. Report 343, Research Lab. of Electronics, M.I.T., 1958.

11. WANG, M.C., UHLENBECK, G.E.: On the theory of Brownian Motion; Revs. Modern Phys., 17, 1945, pp.323-342.

12. VOLTERRA, V.: Theory of Functionals; Blackie, 1930.

13. LEE, Y.W.: Statistical Theory of Communication; Wiley, 1960.

14. NARENDRA, K.S., GALLMAN, P.: An iterative method for the identification of non-linear systems using a Hammerstein model; IEEE Trans. Auto. Contr., AC-11, 1966, pp.546-550.

15. CHANG, F.H.I., LUUS, R.: A noniterative method for identification using the Hammerstein model; IEEE Trans. Auto. Contr., AC-16, 1971, pp.464-468.

16. LANING, J.H., BATTIN, R.H.: Random processes in automatic control; McGraw-Hill, 1956.

17. WIENER, N.: Nonlinear Problems in Random Theory; John Wiley, 1958.

18. CAMERON, R.H., MARTIN, W.T.: The orthogonal development of non-linear functions in series of Fourier-Hermite functionals; Ann. Math., 1947, 48, pp.385-389.

19. BOSE, A.G.: A theory of non-linear systems; Tech. Report 309, Research Lab. of Electronics, M.I.T., 1956.

PARAMETER	$n_{1,1}$	$d_{1,2}$	$d_{2,2}$	γ_1	γ_2	γ_3
Theoretical Values	0.216	-1.579	0.67	5.0	20.0	50.0
Parameter estimates from $\hat{\phi}_{uz'}(\tau)$ for the linear model	0.220	-1.577	0.665	-	-	-
Parameter estimates from $\hat{\phi}_{uz'}(\tau)$ for the Wiener model	0.221	-1.575	0.66	4.7	20.86	50.24

(a) The Linear and Wiener Models

PARAMETER	$n_{1,1}$	$d_{1,2}$	$d_{2,2}$	γ_1	γ_2	γ_3
Theoretical Values	0.216	-1.579	0.67	1.0	0.5	0.8
Parameter estimates from $\hat{\phi}_{uz'}(\tau)$	0.222	-1.58	0.669	0.95	0.47	0.827
Parameter estimates from $\hat{\phi}_{u^2z'}(\tau)$	0.196	-1.60	0.68	0.99	0.507	0.800

(b) The Hammerstein Model

PARAMETER	$n_{1,1}$	$d_{1,2}$	$d_{2,2}$	γ_1	γ_2	γ_3	$n_{2,1}$	$d_{2,1}$
Theoretical Values	0.216	-1.579	0.67	5.0	20.0	50.0	5.0	-0.875
Estimated Values	0.253	-1.576	0.669	4.743	20.731	56.06	4.76	-0.887

(c) The General Model

TABLE 1. SUMMARY OF THE IDENTIFICATION RESULTS

FIGURE CAPTIONS

FIG 1 Non-linear no-memory system

FIG 2 Double non-linear transformation

FIG 3 The General Model

$* * *$ Theoretical response $h_1(k)$

$- - -$ Experimental values $\hat{\phi}_{u_1 z'}(\tau)$

FIG 4 A comparison of impulse responses for the linear model

$* * *$ $h_1(k)$

$- - -$ $\hat{\phi}_{uz'}(\tau)$

o o o $\{\hat{\phi}_{uz'}(\tau)\}^2$

— - — $\hat{\phi}_{u^2 z'}(\tau)$

FIG 5 A comparison of impulse responses for the Wiener model

$* * *$ $\hat{\phi}_{uz'}(\tau)$

$- - -$ $\hat{\phi}_{u^2 z'}(\tau)$

FIG 6 A comparison of impulse responses for the Hammerstein model

$- - -$ Theoretical response $h_1(k)$

o o o Estimated values $\hat{h}_1(k)$

— - — Theoretical response $h_2(k)$

$* * *$ Estimated values $\hat{h}_2(k)$

FIG 7 A comparison of impulse responses for the general model.

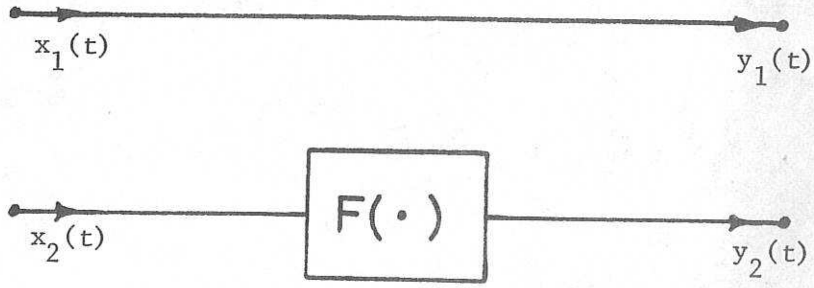

FIG 1 Non-linear no-memory system

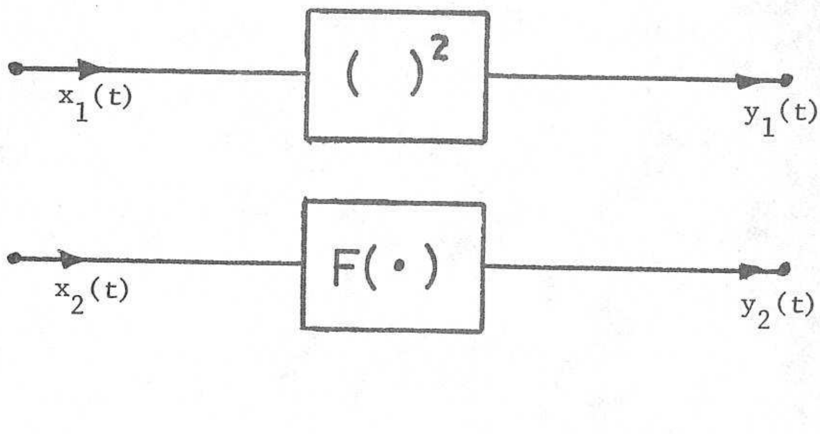

FIG 2 Double non-linear transformation

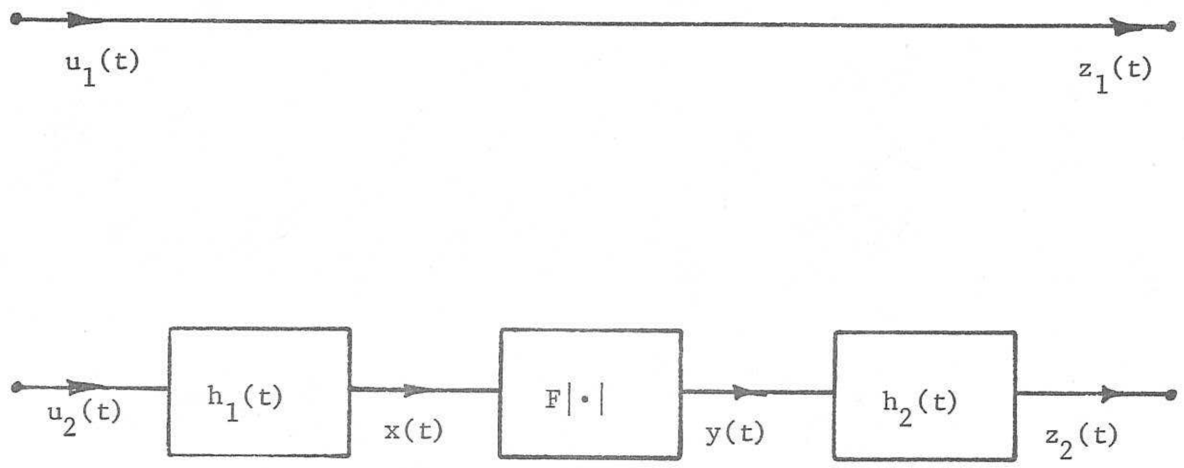

FIG 3 The general model

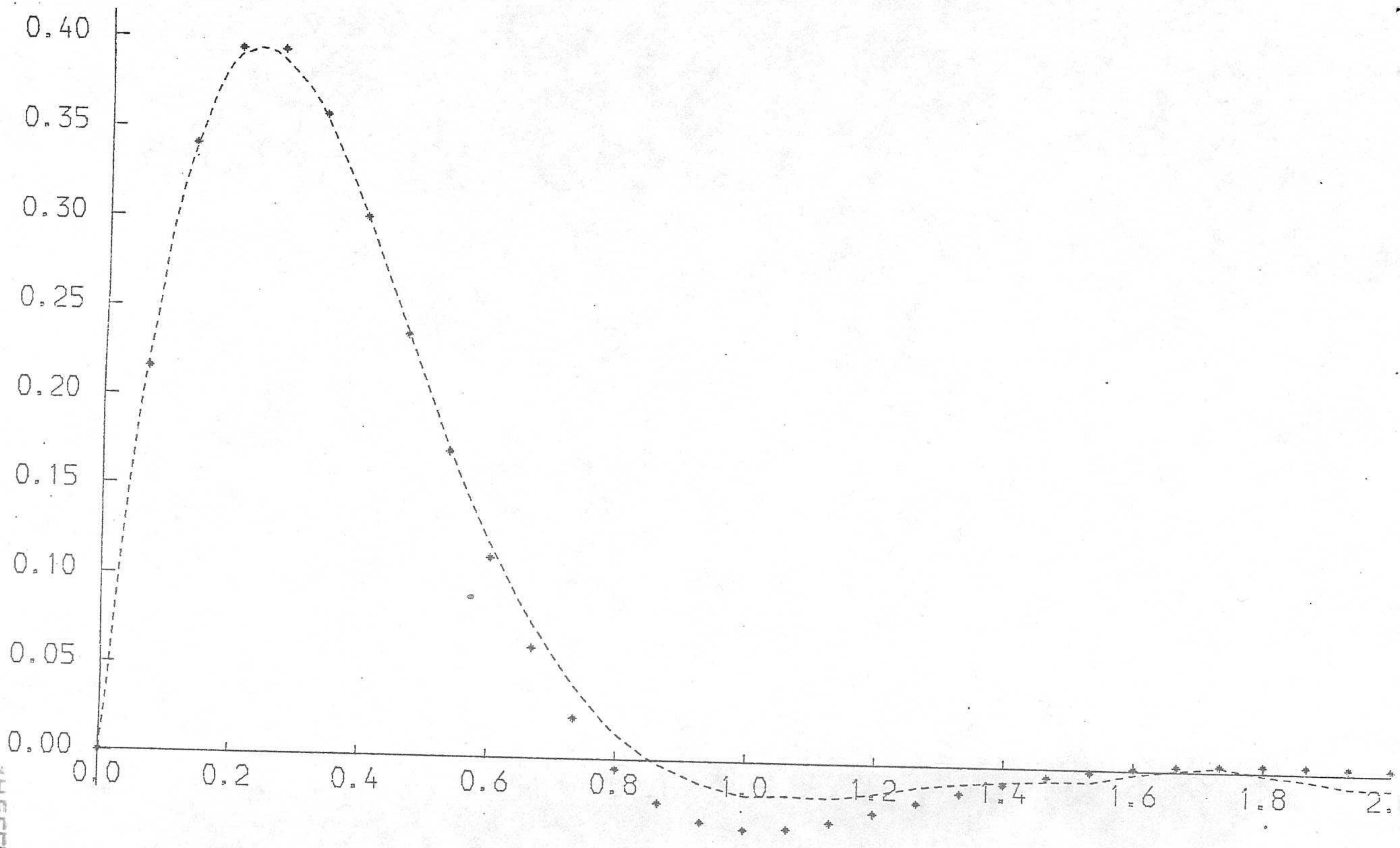

FIG 4 A comparison of impulse responses for the linear model

* * * Theoretical response $h_1(k)$

\- - - Experimental values $\hat{\phi}_{u_1 z'}(\tau)$

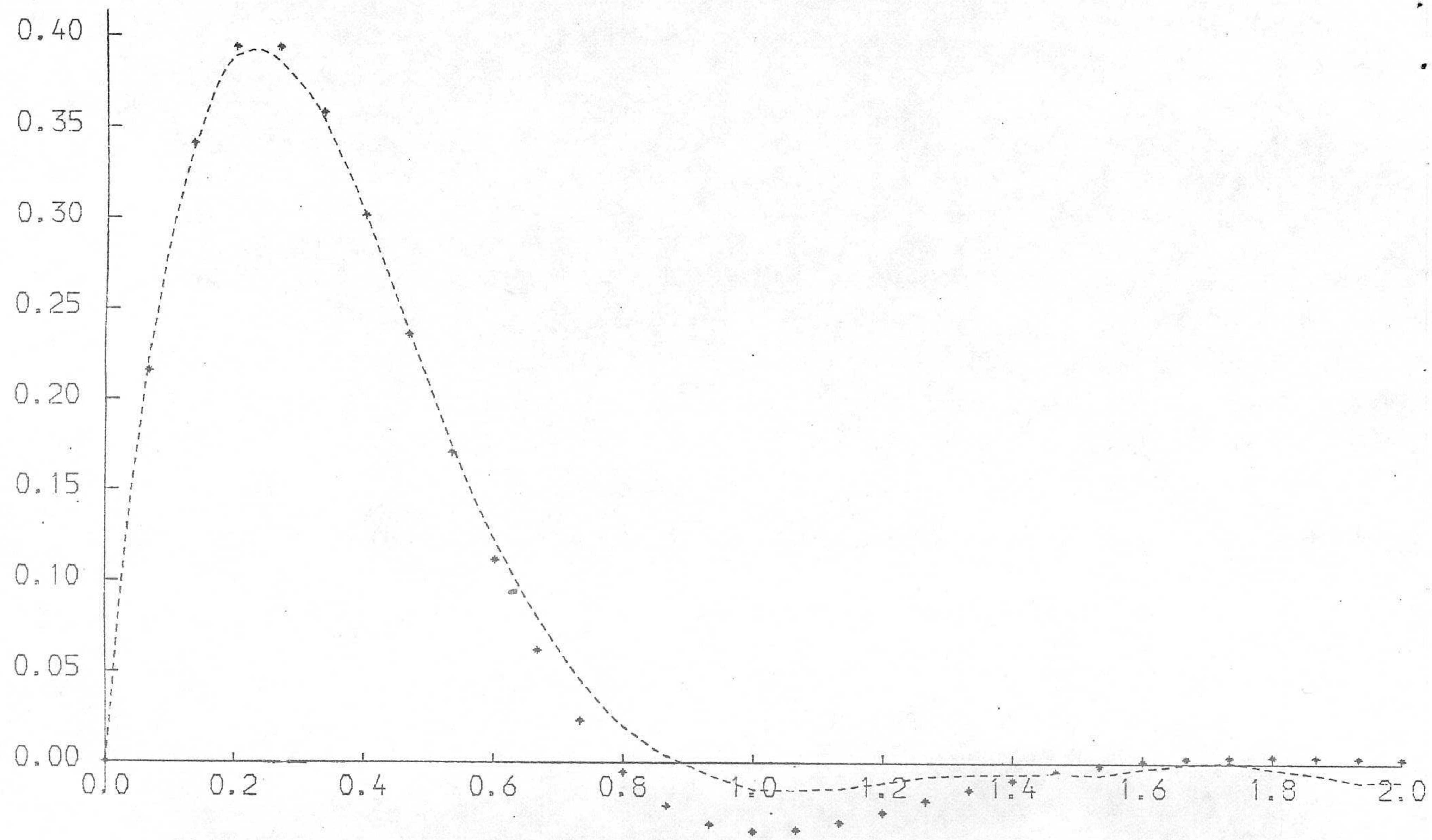

FIG 5(a) A comparison of impulse responses for the Wiener model

* * * $\hat{h}_1(k)$ o o o $\{\hat{\phi}_{uz'}(\tau)\}^2$

\- - - $\phi_{uz'}(\tau)$ — - — $\hat{\phi}_{u^2z'}(\tau)$

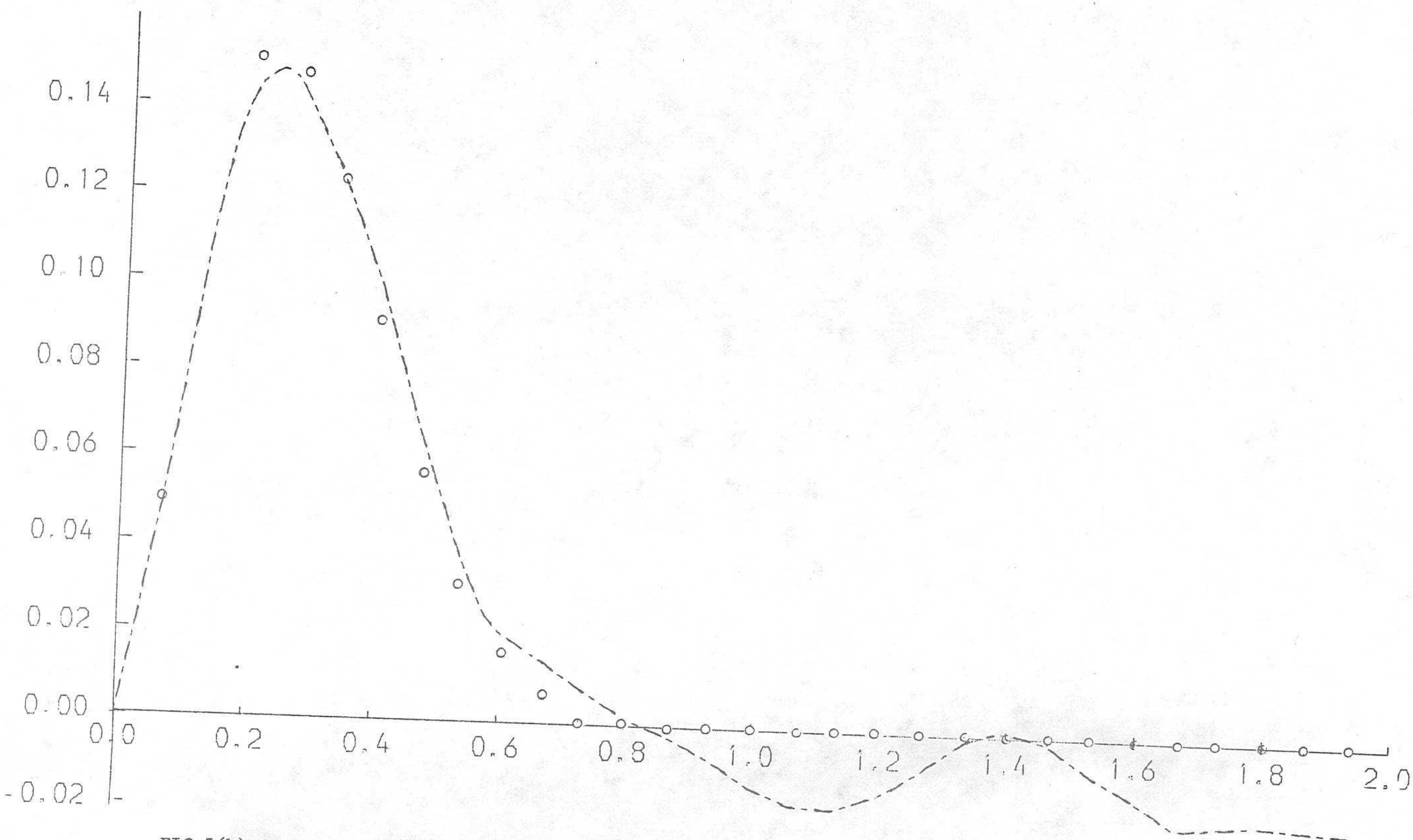

FIG 5(b) A comparison of impulse responses for the Wiener model

* * * $\hat{h}_1(k)$ o o o $\{\hat{\phi}_{uz'}(\tau)\}^2$

- - - $\hat{\phi}_{uz'}(\tau)$ —— - - —— $\phi_{u^2z'}(\tau)$

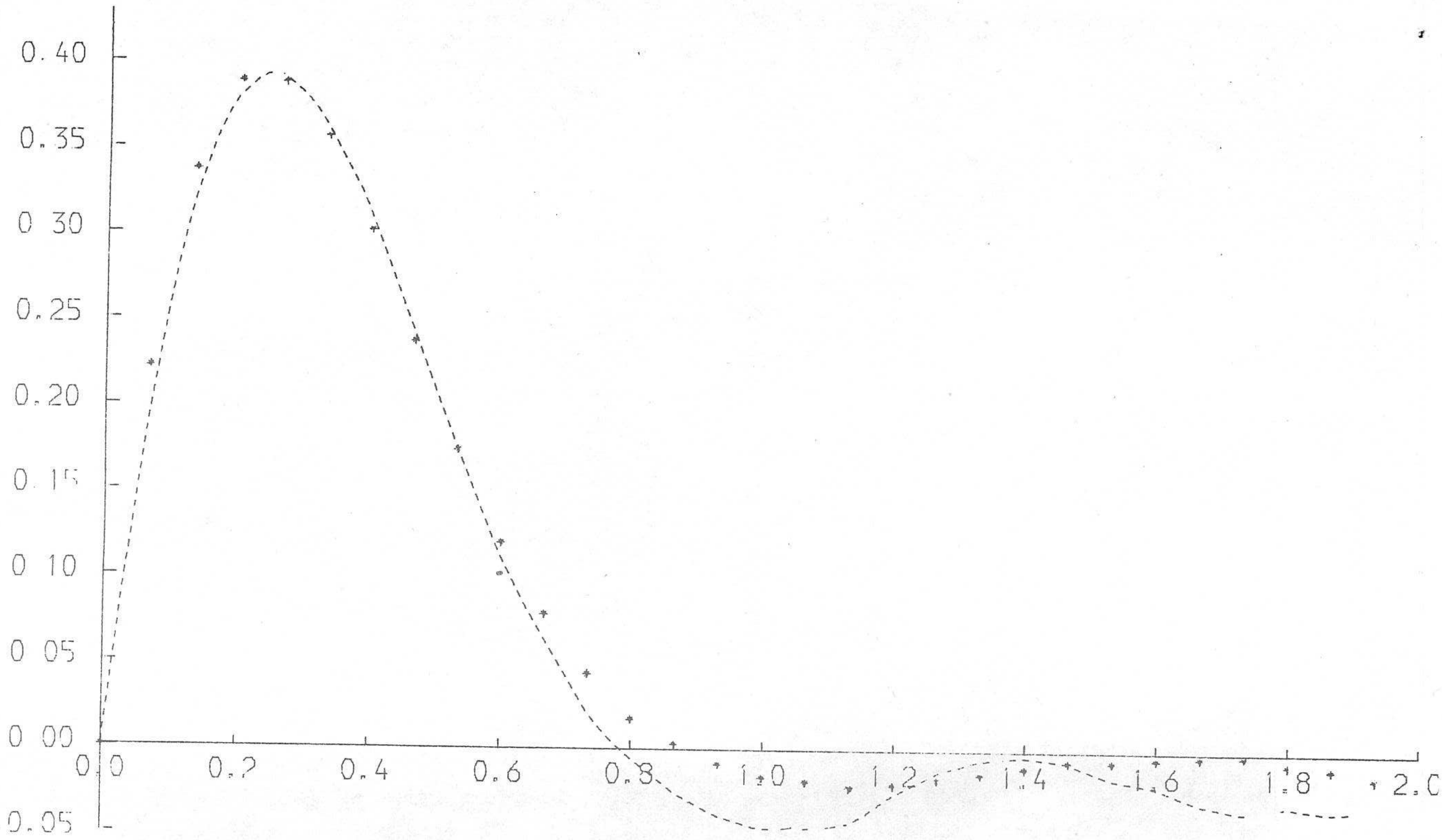

FIG 6 A comparison of impulse responses for the Hammerstein model

* * * $\hat{\phi}_{uz'}(\tau)$

\- - - $\hat{\phi}_{u^2z'}(\tau)$

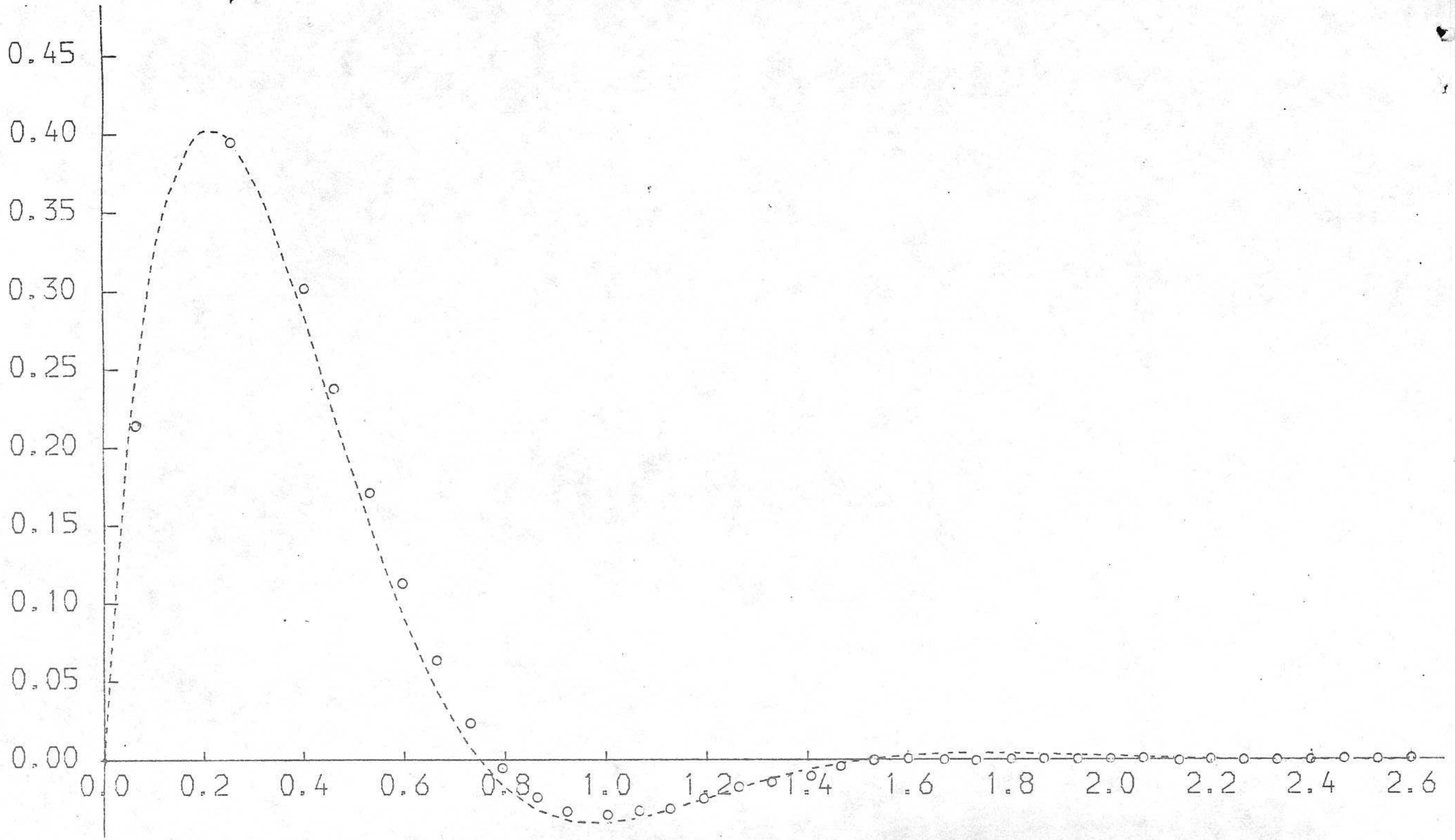

FIG 7(a) A comparison of impulse responses for the general model

- - - Theoretical response $h_1(k)$ —— - —— Theoretical response $h_2(k)$

o o o Estimated values $\hat{h}_1(k)$ * * * Estimated values $\hat{h}_2(k)$

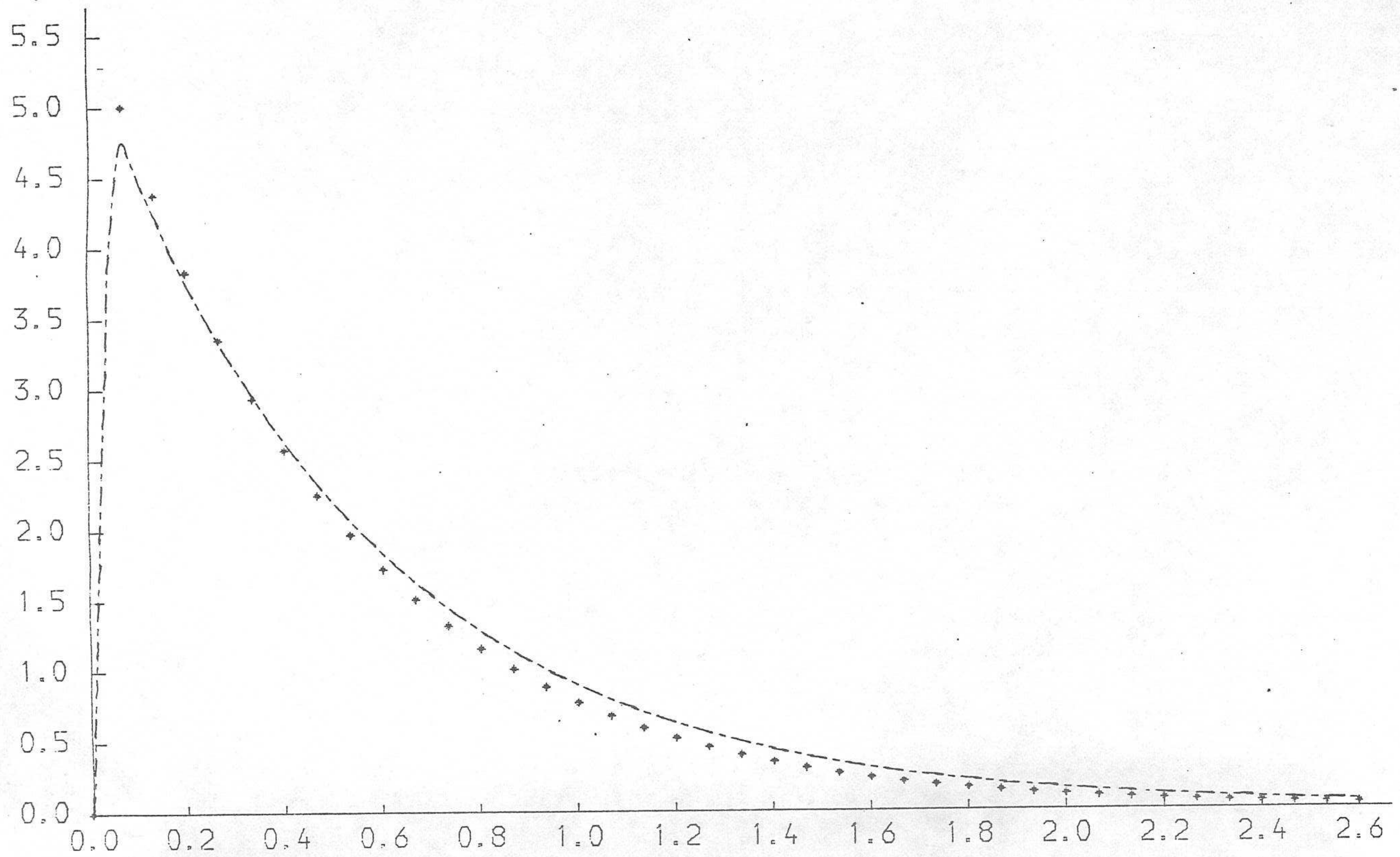

FIG 7(b) A comparison of impulse responses for the general model

- - - Theoretical response $h_1(k)$ —— - —— Theoretical response $h_2(k)$

o o o Estimated values $\hat{h}_1(k)$ * * * Estimated values $\hat{h}_2(k)$