Monograph:

Nicholson, H. (1971) Concepts of Scattering in the Linear Optimal Control Problem. Research Report. ACSE Research Report 7 . Department of Automatic Control and Systems Engineering

University of Sheffield

Department of Control Engineering

Concepts of scattering in the linear optimal control problem

H. Nicholson

Research report No. 7

August 1971

Summary

The scattering formulation characterising the effects of incident and reflected waves in a physical flow process which includes a series of obstacles is discussed. Similar concepts can be applied in the linear optimal control problem which includes terminal constraints, and the defining linear and quadratic matrix differential equations are shown to be analogous in both cases. The control problem is also shown to include a series of 'obstacles' which can be interconnected according to the properties of the star product which forms the basis of the scattering representation of the general physical system. The scattering matrix of electrical network theory and the associated power relations are also shown to exist analogously in the linear optimal regulator problem. The resulting 'incident' and 'reflected' variables possess similar properties to the scattering variables of network theory, and the particular form of scattering matrix is shown to be closely related to the matrix solution of the steady-state Riccati equation. More general energy considerations are extended to the optimal control problem and discussed in terms of inequality properties of a scattering-type matrix, and a particular solution of the Riccati equation is shown to be associated with the property defining the state of dissipation in the physical system.

1. Introduction

Many problems of mathematical physics concerned with propagation through media containing obstacles or distributed constants involve concepts of scattering and can be formulated in terms of a summation of the effects of incident and reflected waves. The broad basic concepts of scattering form a unified mathematical structure for a number of related disciplines, including particularly, circuit and transmission line theory, probability and neutron diffusion.[1] The wave concept and scattering representation would also appear to have application in the traffic flow problem in which a vehicle stream is represented by a continuous fluid density. The effect of a sudden change of vehicle speed or of a traffic signal on the traffic stream, may then be represented by the properties of an obstacle specified in terms of a scattering matrix. The flow process including series-connected obstacles may be characterised in terms of a combined scattering matrix which defines the incident and reflected coefficients for the overall process and relates the input and output waves at the system boundary. These properties lead to functional and differential equations which arise in many physical system problems and can be identified, particularly, with the state-adjoint variable relations and the matrix Riccati differential equation appearing in the linear optimal control problem. Also, in the dynamic theory of scattering applied to the wave equation, a scattering operator is defined in terms of forward and reverse wave operators ($S = W_+^{-1}W_-$)[20] which may be identified with the solution of the Riccati equation in the optimal control problem related to 'forward' and 'backward' transition-matrix components.[14]

The object of the paper is to illustrate particularly how the properties of the scattering matrix and the associated incident and reflected variables can be introduced into the linear optimal control problem which includes a terminal constraint. Problems of energy dissipation and matching have been studied in detail for the distributed-constant flow process represented in terms of the scattering matrix, and it is believed that consideration of similar concepts can give greater physical understanding of the optimal control problem.

2. The scattering process

The propagation of energy at a given frequency in a physical system, such as a transmission line, can be studied in terms of a 'disturbance' or 'wave' which propagates through a series of obstacles or segments specified by the scattering matrices $S_i = \begin{bmatrix} t_i & \rho_i \\ r_i & \tau_i \end{bmatrix}$, where t, τ and r, ρ represent complex transmission and reflection coefficients respectively.[1,2] With two adjacent obstacles, as in FIG 1, the resultant reflected and incident waves with complex amplitudes v_i are defined by

$$\begin{bmatrix} v_3 \\ v_2 \end{bmatrix} = S_1 \begin{bmatrix} v_1 \\ v_4 \end{bmatrix}, \quad \begin{bmatrix} v_5 \\ v_4 \end{bmatrix} = S_2 \begin{bmatrix} v_3 \\ v_6 \end{bmatrix} \tag{1}$$

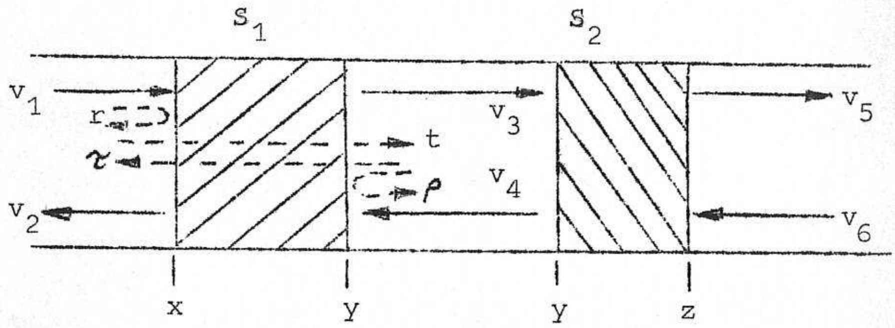

FIG 1 Propagation through adjacent obstacles

The scattering representation may be generalised to include nxn transmission and reflection matrix functions $S_i(x,y)$, $R_i(x,y)$, $U_i(x,y)$, $W_i(x,y)$ with spatial coordinates (x,y) and n-vectors V_i, which may be associated with an obstacle containing 2n-terminal pairs.[1,3] Thus

$$V_r = M_j V_i, \quad M_j = \begin{bmatrix} S_j & U_j \\ W_j & R_j \end{bmatrix} \tag{2}$$

For the two-obstacle system, the overall reflected and incident waves will be related by the form

$$\begin{bmatrix} V_5 \\ V_2 \end{bmatrix} = M_1 * M_2 \begin{bmatrix} V_1 \\ V_6 \end{bmatrix} \tag{3}$$

where
$$M_1 * M_2 = \begin{bmatrix} S_2(I-U_1W_2)^{-1}S_1, & U_2+S_2U_1(I-W_2U_1)^{-1}R_2 \\ W_1+R_1W_2(I-U_1W_2)^{-1}S_1, & R_1(I-W_2U_1)^{-1}R_2 \end{bmatrix} \tag{4}$$

Eqn 4 defines the star-product or combined scattering matrix for two adjacent obstacles. With a series of obstacles the overall scattering matrix is obtained by the continued star-product

$$M = M_1 * M_2 * \circ\circ * M_n \tag{5}$$

In contrast, the transformations $(V_2V_1) \to (V_4V_3) \to (V_6V_5)$ relating input-output variables across each obstacle are defined by the relations

$$\begin{bmatrix} V_2 \\ V_1 \end{bmatrix} = \begin{bmatrix} R_1 - W_1 S_1^{-1} U_1, & W_1 S_1^{-1} \\ -S_1^{-1} U_1, & S_1^{-1} \end{bmatrix} \begin{bmatrix} V_4 \\ V_3 \end{bmatrix} = L_1 \begin{bmatrix} V_4 \\ V_3 \end{bmatrix} \tag{6}$$

The overall transformation is then defined by the continued matrix product

$$L = L_1 L_2 \circ\circ\circ\circ\circ L_n \tag{7}$$

The star product plays a fundamental role in the scattering representation of energy transfer in the physical system. It can be extended to form a set of functional matrix equations in the functions $S(x,z)$, $R(x,z)$, $U(x,z)$, $W(x,z)$ which then leads to a matrix differential system[3,5] which is analogous to a similar formulation of the linear optimal control problem.

2.1 The differential representation of a scattering process

A Riccati-type differential system can be associated with the complex matrix elements of the scattering matrix $S(x,y)$ and the star product $S(x,z) = S(x,y)*S(y,z)$.[1] The results can then be extended by relating arbitrary complex nxn matrices $A(y)$, $B(y)$, $C(y)$, $D(y)$ to the matrices $S(x,y)$, $R(x,y)$, $U(x,y)$, $W(x,y)$ which can be shown to be solutions of the matrix differential equations

$$\begin{bmatrix} S_y & U_y \\ W_y & R_y \end{bmatrix} = \begin{bmatrix} (D+UC)S, & A + DU + UB + UCU \\ RCS, & R(B + CU) \end{bmatrix}, \quad S_y \equiv \partial S/\partial y \tag{8}$$

with initial conditions $M(x,x) = I_{2n}$. If $D \equiv B^T$ then A, C, U, W are symmetric, and $S \equiv R^T$ corresponding to equivalent 'transmission' properties and conventional linearity. It is of interest to note that, as in the scalar case,[1] the differential system of eqn 8 may be represented as a single matrix Riccati-type equation

$$M_y = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} M + M \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} + M \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} M \tag{9}$$

and the off-diagonal reflection-type components of eqns 8 and 9 are of quadratic form.

Reed[5] also considers a linearisation of eqn 8 which is obtained by a change of variable associated with the transformation $(V_4V_3) \rightarrow (V_2V_1)$ of eqn 6. Thus define the 2nx2n matrices

$$H(x,y) \equiv L^{-1} = \begin{bmatrix} R^{-1}, & -R^{-1}W \\ UR^{-1}, & S-UR^{-1}W \end{bmatrix}, \quad G(y) = \begin{bmatrix} -B & -C \\ A & D \end{bmatrix} \tag{10}$$

Then H is a solution of the linear matrix differential equation

$$H_y = GH, \qquad H(x,x) = I_{2n} \tag{11}$$

Also
$$[H^{-1}]_y = -H^{-1}G \tag{12}$$

3. <u>Linear optimal control with quadratic criterion and linear terminal constraints</u>

The quadratic differential equation for U_y in eqn 8 can be identified with the matrix Riccati differential equation in the linear optimal control problem. The linear differential equations for S_y, R_y can also be shown to be analogous to similar sets of equations associated with the optimal tracking problem and with the optimal regulator problem incorporating a terminal constraint.

Thus consider the linear optimal control problem for the system described by the equations

$$\dot{x}(t) = Ax + Bu, \quad y = Cx, \quad x(t_o) = x_o \tag{13}$$

with performance index

$$J(u) = \tfrac{1}{2}\int_{t_o}^{t_1} (y^TQy + u^TRu + 2y^TWu)dt + \tfrac{1}{2}y^T(t_1)Fy(t_1) \tag{14}$$

and terminal constraint[7]

$$z = Zy(t_1) \tag{15}$$

where t_1 is specified, and x is the n-state vector, y the m-output vector ($m \leq n$), u the r-control vector ($r \leq n$) and Z,z represent a wxm-matrix and w-vector ($w \leq m$) of known constants respectively. Q is an mxm positive semi-definite matrix and G is an rxr positive definite matrix. A scalar product of the state and control variables is included with the mxr matrix W, which may be required to represent, say, the power functions of an electromechanical system as in Section 7. Similar cross-product weighting also appears in the model following problem and in the optimal control problem for the linear system containing input derivatives.[8] It also appears inherently in the general variational problem.[6]

Direct correspondence of the scattering process with the optimal control problem follows by considering the Hamiltonian form

$$H = \tfrac{1}{2}(y^TQy + u^TRu + 2y^TWu) + p^T(Ax + Bu) \tag{16}$$

The optimal trajectory of the state (x) and adjoint (p) variables is then given by

$$\mathring{x}(t) = \bar{A}x - \bar{B}p \; , \; \bar{A} = A - BR^{-1}W^TC \; , \; \bar{B} = BR^{-1}B^T \tag{17}$$

$$\mathring{p}(t) = -C^T\bar{Q}Cx - \bar{A}^Tp \; , \; \bar{Q} = Q - WR^{-1}W^T \tag{18}$$

with an optimal control law

$$u(t) = -R^{-1}[B^Tp(t) + W^TCx(t)] \tag{19}$$

The optimal trajectory is also required to intersect the terminal hyperplane of eqn 15.

At $t = t_1$, the boundary conditions are defined by the transversality condition

$$p(t_1) = \frac{\partial}{\partial x(t_1)}\left\{\tfrac{1}{2}\, y^T(t_1)Fy(t_1) + [Zy(t_1) - z]^T\, \lambda\right\}$$

$$= [C^TFCx(t_1) + C^TZ^T\lambda] \tag{20}$$

where λ is a w-vector of constant multipliers. A state-adjoint variable relation is then considered of the form

$$p(t) = P(t)\, x(t) + G(t)\lambda \tag{21}$$

with $$P(t_1) = C^TFC \; , \qquad G(t_1) = C^TZ^T \tag{22}$$

Combining eqns 13, 17, 18 and 21 then gives the matrix differential equations

$$\mathring{P} = -C^T\bar{Q}C - P\bar{A} - \bar{A}^TP + P\bar{B}P \tag{23}$$

$$\mathring{G} = (P\bar{B} - \bar{A}^T)G \tag{24}$$

Eqn 23 may also be written in the form

$$\mathring{P} = (PB + C^TW)R^{-1}(B^TP + W^TC) - PA - A^TP - C^TQC \tag{25}$$

which is equivalent to the 'Legendre' form of matrix differential equation associated with the general variational problem.[6] The solution of eqn 13 with eqns 19 and 21 can now be used to define the terminal constraint in terms of the state x and multipliers λ. Thus

$$z = S(t)x(t) + N(t)\lambda , \quad S(t_1) = ZC , \quad N(t_1) = 0 \tag{26}$$

Then $$\lambda = N^{-1}(t)z - N^{-1}(t)S(t)x(t) \tag{27}$$

and from eqn 21 we then obtain the adjoint-state variable relation

$$p(t) = P^*x(t) + g(t) , \quad g = GN^{-1}z \tag{28}$$

where $$P^* = P - GN^{-1}S \tag{29}$$

The optimal control law of eqn 19 will then be given by

$$u(t) = -R^{-1}B^TGN^{-1}z - R^{-1}(B^TP^* + W^TC)x \tag{30}$$

and is related to the state x, terminal constraint z and to the time-varying solution matrices $P^*(t)$, $G(t)$ and $N(t)$. Relations for $\mathring{S}$ and $\mathring{N}$ can be obtained by differentiation of eqn 26. The optimal control problem including a linear terminal constraint can then be defined by the set of matrix differential equations

$$\begin{bmatrix} \mathring{G} & \mathring{P} \\ \mathring{N} & \mathring{S} \end{bmatrix} = \begin{bmatrix} (P\bar{B}-\bar{A}^T)G , & -C^T\bar{Q}C - P\bar{A} - \bar{A}^TP + P\bar{B}P \\ S\bar{B}G , & S(\bar{B}P - \bar{A}) \end{bmatrix} \tag{31}$$

with terminal conditions

$$\begin{bmatrix} G(t_1) & P(t_1) \\ N(t_1) & S(t_1) \end{bmatrix} = \begin{bmatrix} C^TZ^T , & C^TFC \\ 0 & ZC \end{bmatrix} \tag{32}$$

Also, $S(t) = G^T(t)$. A similar set of differential equations for $\mathring{P}$, $\mathring{g}$, $\mathring{n}$ results by using the Hamilton-Jacobi-Bellman equation with the assumed form of performance index[10]

$$J(x,t) = \tfrac{1}{2}x^TPx + g^T(t)x + n(t) \tag{33}$$

The matrix P^* of eqn 29 will also satisfy the Riccati equation

$$\dot{P}^* = -C^T\bar{Q}C - P^*\bar{A} - \bar{A}^TP^* + P^*\bar{B}P^* \tag{34}$$

Substitution of eqn 29 in eqn 34 then illustrates that any solution P is related to P^* by the form of eqn 29, where the matrices G and N satisfy the differential equations of eqn 31.[9] The optimal tracking problem without terminal constraint also introduces a matrix differential set of the form of eqn 31.[11]

4. Concepts of scattering in the linear optimal control problem

The scattering formulation for propagation in the physical system is based on concepts of incident and reflected waves appearing at the boundaries of an obstacle. The overall effects produced by the interconnection of adjacent obstacles can then be characterised by the star product of the associated scattering matrices, which leads to a set of linear and nonlinear matrix Riccati-type differential equations. The existence of such similar sets of differential equations suggests that concepts of scattering and a star product representation of interconnected 'obstacles' occurring in the time domain may also be relevant in the formulation of the linear optimal control problem.

The differential equation components of eqns 8 and 31 are directly analogous, with the space argument (x,y) corresponding to the time interval (t_o,t). Similar symmetrical properties also apply to the coefficient matrices. Also, by analogy, the matrix G(y) of eqn 10 associated with the linearisation of eqn 8 can be identified with the coefficient matrix defining the optimal trajectory in eqns 17 and 18. The differential equations of eqn 31 may then be transformed to the equivalent linear set

$$\overset{\circ}{H}_c = \begin{bmatrix} \bar{A}, & -\bar{B} \\ -C^T\bar{Q}C, & -\bar{A}^T \end{bmatrix} H_c , \quad H_c = \begin{bmatrix} (G^T)^{-1}, & -(G^T)^{-1}N \\ P(G^T)^{-1}, & G-P(G^T)^{-1}N \end{bmatrix} \tag{35}$$

Note that the relation $P \equiv U = H_{21}H_{11}^{-1}$ in both the scattering and optimal control problems defines the solution of the corresponding matrix Riccati differential equation.

The equations relating the state and adjoint variables, and the equations including the terminal constraint and constant multipliers in the optimal control problem may now be used to form an analogy with the scattering representation of a physical system. The terminal conditions in the regulator problem can be associated with a spatial boundary or obstacle containing (m+w) terminal pairs in the flow process which is characterised by a scattering matrix. Similarly, the conditions existing at time t in the control problem may be associated with a scattering matrix, and properties of the star product may be used to interconnect the time-variant 'obstacles'. Thus, we may define a scattering representation using eqns 27 and 28 and eqns 15 and 20 to give

$$\begin{bmatrix} \lambda \\ p(t) \end{bmatrix} = \begin{bmatrix} -N^{-1}(t)G^T(t), & N^{-1}(t) \\ P^*(t)\ , & G(t)N^{-1}(t) \end{bmatrix} \begin{bmatrix} x(t) \\ z \end{bmatrix} = M_1 \begin{bmatrix} x(t) \\ z \end{bmatrix} \tag{36}$$

$$\begin{bmatrix} p(t_1) \\ z \end{bmatrix} = \begin{bmatrix} C^TZ^T\ , & C^TFC \\ 0\ , & ZC \end{bmatrix} \begin{bmatrix} \lambda \\ x(t_1) \end{bmatrix} = M_2 \begin{bmatrix} \lambda \\ x(t_1) \end{bmatrix} \tag{37}$$

It is of interest to note that the components of eqns 36 and 37 possess similar symmetrical properties which can be identified with the transmission and reflection coefficients and vectors in the scattering matrices of eqn 2. The 'reflection' coefficients are symmetric and the 'transmission' coefficients possess dual-type properties as in the general scattering representation which is analogous to the bilateral property of objects satisfying the reciprocity theorem with $t = \tau$. The composite object also possesses similar properties.

The star product of eqn 4 may now be used to obtain a relation between the external 'reflected' and 'incident' variables in the control problem. Thus

$$\begin{bmatrix} p(t_1) \\ p(t) \end{bmatrix} = M_1{*}M_2 \begin{bmatrix} x(t) \\ x(t_1) \end{bmatrix} \tag{38}$$

where $$M_1{*}M_2 = \begin{bmatrix} -C^TZ^TN^{-1}(t)G^T(t)\ , & C^T(F+Z^TN^{-1}(t)Z)C \\ P^*(t)\ , & G(t)N^{-1}(t)ZC \end{bmatrix} \tag{39}$$

The variables associated with the scattering representation and the existence of two adjacent 'obstacles' may be illustrated in the form of FIG 2.

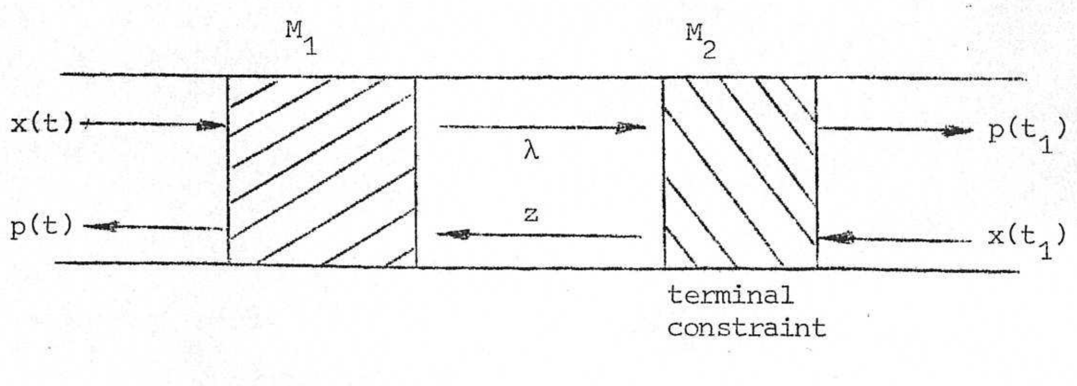

FIG 2. Adjacent 'obstacles' in the optimal control problem

A star-product decomposition of eqns 36 and 37 may also be used to illustrate the existence of a series of adjacent 'obstacles'. Thus

$$M_1{*}M_2 = \begin{bmatrix} I & O \\ P & I \end{bmatrix} * \begin{bmatrix} -Z^TN^{-1}G^T, & Z^TN^{-1}Z \\ -GN^{-1}G^T, & GN^{-1}Z \end{bmatrix} * \begin{bmatrix} C^T & C^TFC \\ O & C \end{bmatrix} \tag{40}$$

$$= N_1{*}N_2{*}N_3$$

where the scattering matrices are associated with the variables

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = N_1 \begin{bmatrix} x(t) \\ p(t) - Px(t) \end{bmatrix} \tag{41}$$

$$\begin{bmatrix} Z^T\lambda \\ p(t)-Px(t) \end{bmatrix} = N_2 \begin{bmatrix} x(t) \\ y(t_1) \end{bmatrix} \tag{42}$$

$$\begin{bmatrix} p(t_1) \\ y(t_1) \end{bmatrix} = N_3 \begin{bmatrix} Z^T\lambda \\ x(t_1) \end{bmatrix} \tag{43}$$

The interconnection of adjacent 'obstacles' may then be illustrated in the form of FIG 3, in which the state variables are seen to act as 'incident' variables in the forward direction to the left of the obstacle and the adjoint variables act as left reflected variables in the reverse direction.

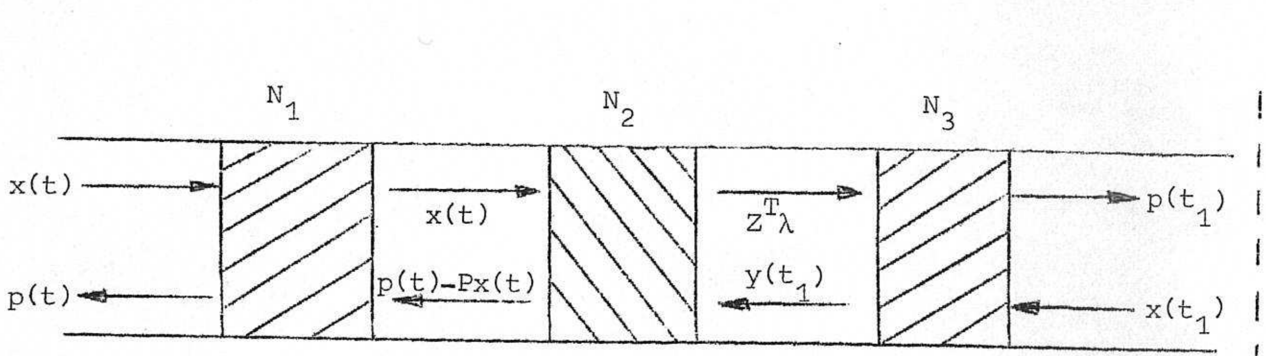

FIG 3. Decomposition of 'obstacles' in the optimal control problem

Additional decomposition will produce the continued star-product

$$M_1{*}M_2 = \begin{bmatrix} I & O \\ P & I \end{bmatrix} * \begin{bmatrix} G^T(t) & O \\ O & G(t) \end{bmatrix} * \begin{bmatrix} -N^{-1}(t) & N^{-1}(t) \\ -N^{-1}(t) & N^{-1}(t) \end{bmatrix} * \begin{bmatrix} Z^T & O \\ O & Z \end{bmatrix} * \begin{bmatrix} C^T & C^TFC \\ O & C \end{bmatrix} \tag{44}$$

The reflection properties are seen to be associated with the off-diagonal components P, N^{-1} and C^TFC, and P^*, $C^T(F+Z^TN^{-1}Z)C$ define the overall reflection in the composite representation. The terminal scattering matrices of eqn 44 which affect the matching properties of the star product exhibit zero right- and left-hand reflections in the variables $x(t)$ and $y(t_1)$ respectively, and are analogous with the matching properties of lossless tuners in transmission line theory.[2] In the infinite-time regulator problem, without terminal constraints, the matched scattering formulation simply defines the reflection-type properties of the matrix P.

In the transmission line problem, a length of line x separating two obstacles can be represented by a scattering matrix diag (e^{jk_1x}, e^{jk_2x}) where k_1, k_2 represent propagation constants in the forward and reverse directions respectively. In the optimal control problem, the matrix function $G(t)$ may be related to terminal conditions by

$$G(t) = \psi^{-1}(t_1,t)G(t_1) = \psi^{-1}(t_1,t)C^TZ^T \tag{45}$$

where $\psi(t_1,t)$ represents the fundamental matrix associated with the matrix $(P\bar{B}-\bar{A}^T)$. Then

$$\begin{bmatrix} G^T(t) & 0 \\ 0 & G(t) \end{bmatrix} = \begin{bmatrix} (\psi^{-1})^T & 0 \\ 0 & \psi^{-1} \end{bmatrix} * \begin{bmatrix} C & 0 \\ 0 & C^T \end{bmatrix} * \begin{bmatrix} Z & 0 \\ 0 & Z^T \end{bmatrix} \tag{46}$$

The intervening time interval $t - t_1$ may thus be considered as an 'obstacle' represented by the scattering matrix diag$((\psi^{-1})^T, \psi^{-1})$ which, in concept, is analogous to the scattering representation of the obstacle-free transmission line.

The matrix of coefficients obtained from eqns 17 and 18 also illustrate properties of a scattering matrix. The relations may be used to relate 'reflected' and 'incident' variables, provided $\bar{B}$ is nonsingular, by the form

$$\begin{bmatrix} p(t) \\ \mathring{p}(t) \end{bmatrix} = \begin{bmatrix} \bar{B}^{-1}\bar{A}, & -\bar{B}^{-1} \\ -(C^T\bar{Q}C+\bar{A}^T\bar{B}^{-1}\bar{A}), & \bar{A}^T\bar{B}^{-1} \end{bmatrix} \begin{bmatrix} x(t) \\ \mathring{x}(t) \end{bmatrix} \tag{47}$$

However, although eqn 47 may suggest properties of a scattering matrix, the component forms will not permit a star-product decomposition. Thus the concepts of scattering and the existence of 'obstacles' in the optimal control problem would appear to be relevant only to the constraint and state-adjoint variable relations. Similar concepts do not appear to exist in the equations defining the optimal trajectory.

Eqn 38 will, however, permit a continued-matrix decomposition of eqns 17 and 18 of the form

$$\begin{bmatrix}\mathring{x}(t)\\ \mathring{p}(t)\end{bmatrix} = \begin{bmatrix}\bar{A} & -\bar{B} \\ -C^T\bar{Q}C, & -\bar{A}^T\end{bmatrix}\begin{bmatrix} I & 0 \\ P^*, & G(t)N^{-1}(t)ZC\end{bmatrix}\begin{bmatrix}x(t)\\ x(t_1)\end{bmatrix} \qquad (48)$$

It is a remarkable fact that the equations defining the state-adjoint relations and the terminal constraints can be formulated according to the properties of the scattering matrix and related star product which are associated with the propagation of energy in the physical system. On this basis it may be considered appropriate to extend the associated mathematical principles of energy dissipation and matching to the linear optimal control problem.

5. The scattering matrix of electrical network theory in the linear optimal regulator problem

The scattering parameters of an electrical network are related to a transformation between linear combinations of network voltages and currents, and have an important application in the definition of energy constraints and power transfer in passive networks.[12,13] The parameters are analogous to the reflection and transmission coefficients used to describe wave propagation in transmission line theory. Parameters with similar properties can also be established in the linear optimal regulator problem.

For the n-port passive network $\underline{N}$ of FIG 4, a scattering matrix S is defined by the transformation

$$v_r = Sv_i \qquad \text{or} \qquad (v - i)/2 = S(v+i)/2 \qquad (49)$$

where v_r, v_i represent vectors of normalised 'reflected' and 'incident' voltages or scattering variables at the network ports.

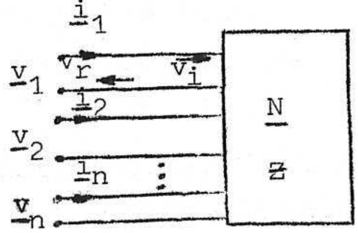

FIG 4. n-port network

The normalised voltage and current variables v,i are related to actual voltage and current vectors $\underline{v}$, $\underline{i}$ by

$$v = R_o^{-\frac{1}{2}}\underline{v}, \quad i = R_o^{\frac{1}{2}}\underline{i}, \quad \underline{v} = \mathcal{Z}\,\underline{i} \qquad (50)$$

Combining eqns 49 and 50 gives the scattering matrix related to the normalised impedance matrix $Z(= R_o^{-\frac{1}{2}}\mathcal{Z} R_o^{-\frac{1}{2}})$,

$$S = (Z+I_n)^{-1}(Z-I_n) = (Z-I_n)(Z+I_n)^{-1} \tag{51}$$

and

$$Z = (I_n-S)^{-1}(I_n+S) = (I_n+S)(I_n-S)^{-1} \tag{52}$$

The total network power is defined by

$$P_T = P_d+jQ = \underline{v}^{*T}\underline{i} = (\underline{v}^{*T}\underline{i} + \underline{i}^{*T}\underline{v})/2 + (\underline{v}^{*T}\underline{i} - \underline{i}^{*T}\underline{v})/2 \tag{53}$$

with real dissipated power

$$P_d = P_i - P_r = v_i^{*T} v_i - v_r^{*T} v_r = v_i^{*T} F v_i, \quad F = I_n - S^{*T}S \tag{54}$$

and reactive power

$$jQ = v_r^{*T} v_i - v_i^{*T} v_r = v_i^{*T} L v_i, \quad L = S^{*T} - S \tag{55}$$

where F is a nonsingular positive hermitian dissipation matrix, and L is a skew-hermitian matrix. For the reactive lossless network $P_d = 0$ and $S^{-1} = S^{*T}$, as a unitary matrix. For maximum delivered power, $S = 0$, $P_d = v_i^{*T} v_i$, $Z = I_n$, $\mathcal{Z} = R_o$, where P_d is equal to the sum of the available powers at the network ports with the network matched to the source impedance R_o. The coefficients of S thus measure the deviation of the circuit impedance or load from the normalising number R_o or from matched maximum power transfer conditions.

The linear optimal regulator problem, without terminal constraints, may also be formulated in terms of a transformation matrix S which is analogous to the scattering operator of electrical network theory. Thus we may consider defining new 'state' (incident) and 'adjoint' (reflected) variables related by a transformation or 'scattering' matrix S,

$$x_i = (x+p)/2 = (I+P)\,x/2, \quad x_r = (x-p)/2 = (I-P)x/2, \quad x_r = Sx_i \tag{56}$$

Then

$$S = (I - P)(I + P)^{-1} \tag{57}$$

$$P = (I + S)^{-1}(I - S) \tag{58}$$

Thus the P-matrix of the Riccati solution in the optimal regulator problem may be considered to possess properties analogous to the scattering matrix of electrical network theory. Similarly, the assumed transformation or symmetrical 'scattering' matrix in the control problem corresponds with the normalised network impedance matrix Z. The form of eqn 58 can also be associated with the standing-wave ratio in the study of lossless objects defined by $S = (1+|r|)/(1-|r|)$[15]. Similar forms also play a basic role in operator theory and represent a Cayley transform.[16] With asymptotic stability ($t_1 \rightarrow \infty$) a solution of the algebraic matrix Riccati equation is given by[14]

$$P = U_{21} U_{11}^{-1} \tag{59}$$

where U_{21}, U_{11} represent partitioned eigenvector components associated with the stable modes of the Hamiltonian matrix of eqn 35. Then

$$S = (U_{11} - U_{21})(U_{11} + U_{21})^{-1} \tag{60}$$

Other scattering variables may be defined using combinations of the state and control variables together with an interrelating transformation matrix. By analogy with the network problem, a 'power' function may also be defined by the product

$$P_1 \equiv x^T p = x_i^T x_i - x_r^T x_r = x_i^T (I - S^T S) x_i = x_i^T F x_i \tag{61}$$

and 'injected power'

$$P_i \equiv x^T Bu = x_i^T (S^T \bar{B} S - \bar{B}) x_i \tag{62}$$

The free system is then associated with the condition $S = I$. A concept of maximum 'delivered power' may be considered with the condition $S = 0$, $P = I$. Thus it may be possible to relate the coefficients of S to a deviation from matched or maximum 'power transfer' conditions. From the algebraic matrix Riccati equation such a condition would define the relation

$$Q = (B + W)R^{-1}(B^T + W^T) - (A + A^T) \tag{63}$$

and is associated with an incident variable $x_i = x$ and zero reflected variable x_r. In general, the solution of the algebraic matrix Riccati equation for a positive-definite matrix P requires an assumed positive-definite Q matrix. Thus Q must be constrained to be a positive-definite form, which may not always exist in eqn 63. For example, such a condition is not ensured for the system represented in companion matrix form with a single non-zero B-matrix element, as illustrated in Section 7.

If the variables x_i are constrained by the condition $x_i^T x_i = 1$ then the power function of eqn 61 will be bounded by the inequalities $\lambda_{min}(F) \leqslant P \leqslant \lambda_{max}(F)$. This follows, as in the electrical network problem,[17,18] using the transformation $F = T^T \Lambda T$, where T is a real orthogonal matrix and Λ is a diagonal matrix of eigenvalues. Then

$$P_1 = (Tx_i)^T \Lambda (Tx_i) = \sum_{j=1}^{n} (Tx_i)_j^2 \lambda_j(F) \tag{64}$$

The 'injected power' function of eqn 62 will possess similar properties.

The optimal regulator problem may also be formulated in terms of the 'incident' and 'reflected' variables of eqn 56. Thus from eqn 17 and 18,

$$\begin{bmatrix} \mathring{x}_i \\ \mathring{x}_r \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} \bar{A}-\bar{A}^T-\bar{B}-C^T\bar{Q}C\ , & \bar{A}+\bar{A}^T+\bar{B}-C^T\bar{Q}C \\ \bar{A}+\bar{A}^T-\bar{B}+C^T\bar{Q}C\ , & \bar{A}-\bar{A}^T+\bar{B}+C^T\bar{Q}C \end{bmatrix}\begin{bmatrix} x_i \\ x_r \end{bmatrix} \tag{65}$$

This representation retains the same form as previously with symmetrical off-diagonal components. The overall system may also be associated with a matrix Riccati differential equation

$$2\mathring{S} = -SM_{12}S - SM_{11} + M_{22}S + M_{21} \tag{66}$$

where M_{ij} represent the matrix components of eqn 65. It is interesting to note that the Riccati differential eqn 66 may now be decomposed directly into components associated with skew-symmetrical and symmetrical matrices.

6. General energy concepts

Energy dissipation in the physical system, such as a transmission line, has been discussed in terms of inequality properties of the scattering matrix.[1,3,5] Thus, the system specified by the matrix $\begin{bmatrix} S & U \\ W & R \end{bmatrix}$ with $D(y) \equiv B^T(y)$ and $A(y)$, $C(y)$ symmetric is considered to be dissipative if the hermitian form

$$\mathcal{A}(y;r,q) \equiv q^*(D+D^*)q + q^*(A+C^*)r + r^*(A^*+C)q + r^*(B+B^*)r \tag{67}$$

is non-positive for all n-vectors r,q,[5,6] where * denotes the conjugate transpose. In terms of the notation of Section 2, if the scattering matrix is dissipative, then

$$|V_3|^2 + |V_2|^2 \leqslant |V_1|^2 + |V_4|^2$$

Eqn 67 may also be considered in the form

$$\mathcal{A} = (V_3{}^* V_4{}^*)(M+M^*)\begin{pmatrix} V_3 \\ V_4 \end{pmatrix}, \quad M = \begin{bmatrix} D & A \\ C & B \end{bmatrix} \tag{68}$$

By analogy with the linear optimal control problem, we consider the quadratic form

$$\mathcal{A}(t;r,q) \equiv -\left\{ q^T(\bar{A}+\bar{A}^T)q + q^T(C^T\bar{Q}C-\bar{B})r + r^T(C^T\bar{Q}C-\bar{B})q + r^T(\bar{A}+\bar{A}^T)r \right\} \tag{69}$$

The system will then be dissipative if

$$\begin{bmatrix} D+D^*, & A+C^* \\ A^*+C, & B+B^* \end{bmatrix} \equiv -\begin{bmatrix} \bar{A}+\bar{A}^T, & C^T\bar{Q}C-\bar{B} \\ C^T\bar{Q}C-\bar{B}, & \bar{A}+\bar{A}^T \end{bmatrix} \leqslant 0 \tag{70}$$

The non-positive condition of the hermitian form $\mathcal{N}(y;r,q)$ for all n-vectors r,q is, by analogy with the general variational problem[6] and from the Legendre form of eqn 25, equivalent in the control problem to the positive- and negative-definite conditions

$$\bar{B} - (\bar{A}+\bar{A}^T) \leqslant C^T\bar{Q}C \leqslant \bar{B} + (\bar{A}+\bar{A}^T) \tag{71}$$

or $$(B+C^TW)R^{-1}(B^T+W^TC) - (A+A^T) \leqslant C^TQC \leqslant (-B+C^TW)R^{-1}(-B^T+W^TC)+(A+A^T)$$

which is associated with the conditions

$$K[I] = (\bar{A}+\bar{A}^T) + (C^T\bar{Q}C - \bar{B}) \geqslant 0 \tag{72}$$

$$K[-I] = -(\bar{A}+\bar{A}^T) + (C^T\bar{Q}C - \bar{B}) \leqslant 0 \tag{73}$$

where $$K[P] = \mathring{P} + C^T\bar{Q}C + P\bar{A} + \bar{A}^TP - P\bar{B}P \tag{74}$$

For the self-adjoint system A is skew-symmetric ($= -A^T$) and the inequality conditions will simplify accordingly.

In the optimal control problem, the quadratic form of eqn 69 is equivalent to the difference of the 'active' incident and reflected powers, or to the difference of the rate of change of 'stored' incident and reflected energy. Thus with the n-vectors r,q identified with the state and adjoint variables,

$$\mathcal{N}(t;x,p) = -\frac{d}{dt}(x^Tx - p^Tp) \tag{75}$$

Eqn 75 may also be considered to represent the active power transferred from left to right across the system state-adjoint 'boundary', and is analogous to the form of power transfer across boundaries in the electrical network problem. In the regulator problem, without terminal constraint, we have

$$\mathcal{N}(t,x) = -\frac{d}{dt}[x^T(I - PP)x] \tag{76}$$

which corresponds with the form of the network power forms of Section 5 related to the scattering matrix. Similarly, with the condition P = I, the incident and reflected powers will be equal. Eqn 75 may also be stated in the form

$$-\mathcal{N}/2 = x^TAx + x^TBu - p^T\mathring{p} = P_a + P_i - p^T\mathring{p} \tag{77}$$

which includes components of 'absorbed power' P_a and 'injected power' P_i which have been considered with reference to energy exchanges between a system and its controller.[21]

The existence of the inequality conditions and the quadratic power form Λ thus illustrate the significance of the Riccati equation in establishing energy concepts in the linear optimal control problem. The condition of maximum 'delivered power' which resulted from introducing the scattering matrix of electrical network theory into the optimal control problem in Section 5 now appears as a particular condition of the above inequality constraints. In this case, the weighting matrix Q is defined with P = I which is directly associated with a matching condition and also forms a basis for the definition of the state of dissipation in a physical system.

6.1 Power concepts in an electromechanical system

A separately excited dc motor connected to a variable voltage supply and driving an inertia load with damping, as represented in FIG 5, is considered for illustrating power forms in the optimal regulator problem.

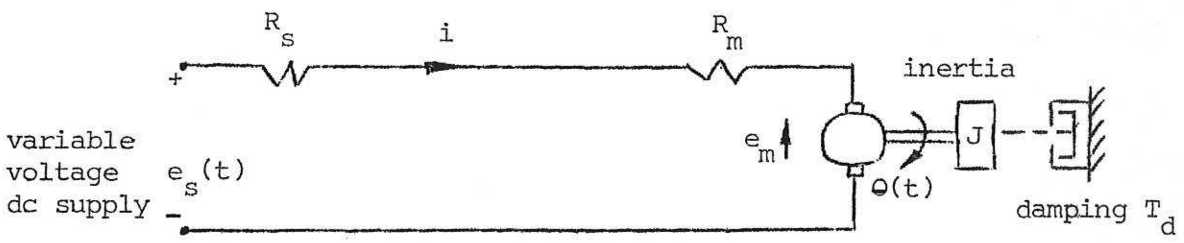

FIG 5. Electromechanical system with damping and inertia loading

System equations: $e_s(t) = (R_s+R_m)i + e_m$, neglecting field inductance (78)

$$e_m = k\omega \;, \quad k \text{ constant} \tag{79}$$

$$T = ki = J\ddot{\Theta} + T_d\,\dot{\Theta} \;, \quad T \text{ is developed motor torque} \tag{80}$$

State variable representation:

$$\begin{bmatrix} \dot{\Theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} \Theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} e_s \;, \quad \dot{x} = Ax + Bu \tag{81}$$

where $a_{22} = -[T_d + k^2/(R_s + R_m)]/J$, $b_2 = k/J(R_s + R_m)$

Input power to system P_i, to motor armature P_m as active power and total dissipated power P_d referred to units of the state equations are represented in the form

$$P = u^T Ru + x^T Qx + x^T Wu \ , \quad u = e_s \ , \quad x = (\Theta,\omega)^T \tag{82}$$

Thus
$$P_i = e_s i/J = (e_s^2 - e_s k\omega)/J(R_s + R_m) \tag{83}$$

$$R_i = 1/J(R_s + R_m) \ , \quad Q_i = 0 \ , \quad W_i = -B \tag{84}$$

$$P_d = [T_d\omega^2 + (e_s - e_m)^2/(R_s + R_m)]/J \tag{85}$$

$$R_d = R_i \ , \quad Q_d = \begin{bmatrix} 0 & 0 \\ 0 & -a_{22} \end{bmatrix} \ , \quad W_d = 2\,W_i \tag{86}$$

$$P_m = (e_m i - T_d\omega^2)/J = P_i - P_d \tag{87}$$

$$R_m = 0 \ , \quad Q_m = -Q_d \ , \quad W_m = B \tag{88}$$

A state function may also be defined in terms of power transferred to the environment and related to scalar products of the state and input variables. Thus

$$P = \tfrac{1}{2}\, x^T Qx + x^T Wu \ , \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \end{bmatrix} \ , \quad W = B \tag{89}$$

The condition $\partial P/\partial\omega = 0$ then defines the steady-state system equations. The state function is associated with active power stored in the magnetic field and transferred to the mechanical load (environment) as the difference of equivalent injected and dissipated power, and corresponds to the form of active power P_m. The state function includes no weighting of the system input cost as would be required in the optimal control problem, say for determining the supply voltage $e_s(t)$ for control of motor shaft position. In this case a performance criterion associated with output motor position and dissipated power may be considered in the form

$$J = \int_0^{t_1} (\gamma\Theta^2 + \lambda P_d)dt \tag{90}$$

The incident power forms of Section 6 resulting from the optimal control formulation do not correspond directly with the conventional power relations of eqns 83-89, particularly with regard to input and cross-product weighting. The active power or rate of change of stored energy

($\mathring{P}_s = P_a + P_i = x^T\mathring{x}$) is, however, similar to the active power delivered to the motor armature (P_m), obtained as the difference between input and dissipated power.

Optimal control of the electromechanical system for min J based on the form of eqn 90 with infinite-time settling may be obtained by direct solution of the steady-state matrix Riccati equation. The optimal control input is then given by

$$e_s = -R^{-1}(b_2p_{12}, b_2p_{22}+w_2)x = -\pi\Theta + k\omega(1 + \alpha - \psi) \tag{91}$$

and $$P_d = [\alpha + (\frac{-\pi\Theta}{k\omega} + \alpha - \psi)^2]\omega^2/((1 + \alpha)\tau) \tag{92}$$

where $$\pi = k[(1 + \alpha)\tau\gamma/\lambda]^{\frac{1}{2}} , \quad \alpha = \bar{R}T_d/k^2$$

$$\tau = -1/a_{22} = (J/T_d)\alpha/(1 + \alpha) , \quad \bar{R} = R_s + R_m$$

$$\psi^2 = \alpha^2 + \alpha + \beta , \quad \beta = 2[(1 + \alpha)\tau]^{3/2}(\gamma/\lambda)^{\frac{1}{2}}$$

The coefficient α represents the ratio of mechanical to electrical power dissipation, and also defines the ratio of voltage drop to back emf ($(e_s - e_m)/e_m$) in the steady-state equations. The coefficient β is associated with the measurement of position and is predominant in the coefficient ψ.

In terms of the components of the power function of eqn 69, the system will be dissipative, with a stable Riccati solution matrix P = I and $K[I] \geqslant 0$, with the condition $C^T\bar{Q}C \geqslant \bar{B} - (\bar{A}+\bar{A}^T)$, which reduces to the form

$$Q \geqslant \begin{bmatrix} R^{-1}w_1^2 , & R^{-1}w_1(b_2+w_2)-1 \\ R^{-1}w_1(b_2+w_2)-1 , & R^{-1}(b_2+w_2)^2-2a_{22} \end{bmatrix} \tag{93}$$

A positive-definite condition is then ensured with

$$(w_1^2/\tau) + w_1(b_2+w_2) > R/2$$

With W = 0 , $$Q \geqslant \begin{bmatrix} 0 & -1 \\ -1 , & R^{-1}b_2^2-2a_{22} \end{bmatrix} \tag{94}$$

and the solution of the steady-state Riccati equation with the equality condition gives $p_{11} = p_{22} = 1$, $p_{12} = 0$. However, Q is not positive definite and includes zero weighting on position Θ, and thus cannot be used to define the position control problem. The form of Q associated with a companion A-matrix will only permit control of the single variable corresponding to the nonzero element of matrix B. In this case, Q introduces positive weighting only on motor speed ω, as in the quadratic power functions, with $Q = \mathrm{diag}(0,q_{22})$, $P = \mathrm{diag}(0,1)$. For control of position θ a term q_{11} may be introduced, which will lead to control of the form of eqn 91 based on calculation of the elements p_{12} and p_{11}. A weighting of the variable θ will be introduced, however, by the cross-product matrix W under the conditions of eqn 93. In this case a positive-definite condition is ensured with $W \neq 0$. With $w_1 = 0$, as in the minimisation of the power function P_d, Q is not positive definite and weighting is restricted to the component q_{22}. With $w_2 = 0$, $w_1 \neq 0$, Q is positive definite and the condition appears to be less restrictive with $x^T Wu = w_1 \theta e_s$ which does not, however, appear in the power function forms. With equality in eqn 93 associated with the condition $P = I$, the optimal control law redcues to the form

$$u = -R^{-1}[w_1, b_2+w_2]x = -(w_1 \bar{R}J/\lambda)\theta - k\omega(1+w_2/b_2)/\lambda, \quad R = \lambda/(\bar{R}J) \tag{95}$$

and the controlled system matrix

$$A_c = A - BR^{-1}(B^T+W^T) = \begin{bmatrix} 0 & 1 \\ -b_2 w_1 R^{-1} , & a_{22}-b_2(b_2+w_2)R^{-1} \end{bmatrix} \tag{96}$$

Eigenvalue locations and a dissipated power function may then be determined based, say, on the sensitivity to the cross-product coefficients w_1, w_2. If the form of W is not restricted by physical considerations then choice of particular components may be used to ensure a positive-definite condition of the matrix Q with $P = I$.

Optimal control of an electric arc furnace model has also been investigated using various methods of solution of the Riccati equation.[19] Improved performance of electrode tip displacement with regard to maximum levels of overshoot and acceleration were obtained with the unit P-matrix control solution incorporating cross-product weighting, compared to the results obtained using an eigenvector solution P-matrix with an assumed positive-definite Q matrix. An equivalent positive-definite condition was obtained in the former case by deleting appropriate rows and columns to obtain $\lambda_i(Q) \geqslant 0$, with $P = \mathrm{diag}(0,I)$. However, the extent of the improved performance compared to the condition $P = I$ was limited, which might suggest some degree of redundancy in the effects of the Q-matrix components on the controlled response of the particular system.

7. Conclusions

Concepts of scattering and energy propagation in the distributed flow process have been associated analogously with the formulation of the linear optimal control problem incorporating terminal constraints. In particular, the state-adjoint variable relations and the terminal constraint equations have been shown to include incident and reflection-type variables and to possess properties directly associated with the star-product of scattering theory. Consideration of such relations and concepts will lead to a greater understanding of the optimal control problem, particularly with regard to energy transfer. They can also be used to form a unified basis for investigating the significance of the quadratic form weighting matrices, particularly for the more complex problem involving terminal constraints.

8. Acknowledgments

Computing assistance given by Mr. R. Roebuck is gratefully acknowledged.

9. References

1. BECKENBACH, E.F.: 'Modern mathematics for the engineer', Second series, McGraw-Hill, NY, 1961, Ch.12.
2. REDHEFFER, R.M.: 'On the relation of transmission-line theory to scattering and transfer', Jl. of Maths and Physics, 1962, 41, pp.1-41.
3. REDHEFFER, R.M.: 'Inequalities for a matrix Riccati equation', Jl. of Maths and Mechs., 1959, 8, pp.349-367.
4. REDHEFFER, R.M.: 'Supplementary note on matrix Riccati equations', ibid, 1960, 9, pp.745-748.
5. REID, W.T.: 'Solutions of a Riccati matrix differential equation as functions of initial values', ibid, 1959, 8, pp.221-230.
6. REID, W.T.: 'Properties of solutions of a Riccati matrix differential equation', ibid, 1960, 9, pp.749-770.
7. CITRON, S.J.: 'Elements of optimal control', Holt, Rinehart and Winston, 1969, Ch.8.
8. NICHOLSON, H.: 'Optimality with input derivatives', Electronics Letters, 1971, 7, 14, pp.387-389.
9. FRIEDLAND, B.: 'On solutions of the Riccati equation in optimization problems', Trans. IEEE AC-12, 1967, 3, pp.303-304.

10. DREYFUS, S.E.: 'Control problems with linear dynamics, quadratic criterion, and linear terminal constraints', ibid, AC-12, 1967, 3, pp.323-324.

11. ATHANS, M., and FALB, P.L.: 'Optimal control', McGraw-Hill, NY, 1966, p.793.

12. CARLIN, H.J., and GIORDANO, A.B.: 'Network theory', Prentice Hall, 1964.

13. OONO, J.: 'Application of scattering matrices to the synthesis of n ports', Trans. IEEE CT-3, 1956, 2, pp.111-120.

14. MARSHALL, S.A., and NICHOLSON, H.: 'Optimal control of linear multivariable systems with quadratic performance criteria', Proc. IEE, 1970, 117, 8, pp.1705-1713.

15. REDHEFFER, R.M.: 'Remarks on the basis of network theory', Jl. of Maths and Physics, 1949, 28, pp.237-258.

16. FILLMORE, P.A.: 'Notes on operator theory', Van Nostrand Reinhold, NY, 1970, p.88.

17. MARTINELLI, G.: 'Upper and lower bounds for the power consumption of passive networks', Electronics Letters, 1967, 3, 12, pp.553-554.

18. REZA, F.M.: 'A limitation of power consumption', Proc. IEEE Letters, 1967, 55, 8, p.1505.

19. NICHOLSON, H., and ROEBUCK, R.: 'Online optimal control of an electric arc furnace', 4th UKAC Control convention: Multivariable control system design and applications, Manchester, Sept. 1971.

20. LAX, P.D., and PHILLIPS, R.S.: 'Scattering theory', Acad. Press, 1967.

21. MACFARLANE, A.G.J.: 'Use of power and energy concepts in the analysis of multivariable feedback controllers', Proc. IEE, 1969, 116, 8, pp.1449-1452.