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Noise Analysis for Quadratic Equation Error Parameter Tracking

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Equation error parameter tracking is a particularly simple analogue technique for estimating parameters of a plant of known structure. A brief description of the method is given below and further details may be found in refs 1,2.

Although parameter tracking can be applied to both linear and nonlinear processes it will be assumed for simplicity that the plant is described by the equation

$$(s^{\ell} + \alpha_1 s^{\ell-1} + \dots \alpha_{\ell})x = (\alpha_{n+1} s^{m-\ell-1} + \dots \alpha_m)u \qquad (1)$$

where n of the α_i are unknown. Writing

$$\left.\begin{aligned} y_i &= s^{\ell-i}x \quad ; \quad 0 \leqslant i \leqslant \ell \\ y_i &= -s^{m-i}u \quad ; \quad \ell+1 \leqslant i \leqslant m \end{aligned}\right) \qquad (2)$$

the plant equation becomes

$$y_o + \alpha_1 y_1 + \dots \alpha_m y_m = 0 \qquad (3)$$

In order to generate the equation error all the y_i must be available for measurement. When some of them are not available a generalised equation error may be generated by using matched "state variable filters" (ref 1). In this report we are concerned with the case in which all the y_i are available for measurement, but these measurements, z_i are corrupted with zero mean white noise η_i ie

$$z_i = y_i + \eta_i \qquad (4)$$

Using these signals the equation error

$$e = z_o + \hat{\alpha}_1 z_1 + \hat{\alpha}_2 z_2 + \dots \hat{\alpha}_m z_m \qquad (5)$$

is generated where $\hat{\alpha}_i$ are the estimates (or the known values) of the α_i.

For quadratic equation error parameter tracking the continuous adjustment laws for the n unknown parameters are

$$\frac{d\hat{\alpha}_i}{dt} = -K_i \frac{\partial e^2}{\partial \hat{\alpha}_i} \qquad (6)$$

$$= -K_i e z_i \quad ; \quad i = 1,\dots,n. \qquad (7)$$

For the special case in which the plant parameters are constant this system of equations becomes

$$\left.\begin{aligned} \mathring{\tilde{\alpha}}_1 &= K_1 e z_1 \\ \mathring{\tilde{\alpha}}_2 &= K_2 e z_2 \\ &\vdots \\ \mathring{\tilde{\alpha}}_n &= K_n e z_n \\ e &= -\tilde{\alpha}_1 z_1 - \tilde{\alpha}_2 z_2 \ldots\ldots - \tilde{\alpha}_n z_n + (\eta_o + \alpha_1 \eta_1 + \ldots \alpha_m \eta_m) \end{aligned}\right\} \tag{8}$$

where

$$\tilde{\alpha}_i = \alpha_i - \hat{\alpha}_i \tag{9}$$

and where, simply for convenience of notation, it has been assumed that the first n of the α_i are unknown. Since all the α_i are constant the noise terms in brackets may be replaced by a single white noise η with zero mean. The spectral density of this noise will be taken as b. Equation (8) may be simplified further by making the transformations

$$\left.\begin{aligned} a_i &= \frac{\tilde{\alpha}_i}{\sqrt{K_i}} \quad ; \quad i = 1,\ldots,n. \\ \omega_i &= z_i \sqrt{K_i} \quad ; \quad i = 1,\ldots,n. \end{aligned}\right\} \tag{10}$$

to give

$$\dot{a}_i = -\omega_i \left(\sum_{k=1}^{n} (a_k \omega_k) + \eta \right) \quad ; \quad i = 1,\ldots,n. \tag{11}$$

It is convenient at this stage to investigate the stability of equations (11) for the case in which $\eta = 0$. Choosing the Lyapunov function

$$L = \tfrac{1}{2} \sum_{i=1}^{n} (a_i^2) \tag{12}$$

we find that the total derivative

$$\dot{L} = -\left(\sum_{i=1}^{n} a_i \omega_i \right)^2 \tag{13}$$

which is negative semi-definite. Hence equations (11) are stable.

If $\dot{L} = 0$ continuously, then from equation (11) all the a_i are constant ie

$$a_i = c_i \quad ; \quad i = 1,\dots,n. \tag{14}$$

and thus

$$\sum_{i=1}^{n} c_i\omega_i = 0 \tag{15}$$

This implies that the ω_i are linearly dependent. Hence if the signals ω_i are linearly independent $\dot{L} = 0$ is not a trajectory of the system and the equilibrium point $a_i = 0$ is asymptotically stable in the large. This result is more complete than those given in refs 1,3.

From the above it is clear that equation (11) represents a linear time-varying stable system excited by white noise. The Fokker-Planck equation (ref.4) of this set of equations is

$$\frac{\partial p}{\partial t} = \sum_{j=1}^{n} \frac{\partial}{\partial a_j} (p\omega_j \sum_{k=1}^{n} a_k\omega_k) + \frac{b}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2}{\partial a_j \partial a_k} (\omega_j\omega_k p) \tag{16}$$

where p is the probability density function $p(a_1, a_2 .. a_n, t)$. Multiplying equation (16) by $a_1^{A_1}.a_2^{A_2}.....a_n^{A_n}$ and integrating by parts with respect to $a_1, \dots a_n$ over the whole range $-\infty$ to $+\infty$ (ref.5) produces the following equation for the moment

$$M(A_1, A_2 \dots, A_n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} a_1^{A_1}.a_2^{A_2} \dots a_n^{A_n}.p \; da_1 da_2 \dots da_n \tag{17}$$

of the probability distribution

$$\frac{d}{dt} M(A_1, A_2, \dots A_n) = - \sum_{j=1}^{n} A_j\omega_j \sum_{k=1}^{n} \omega_k M(A_1, \dots A_j - 1, \dots A_k + 1, \dots, A_n)$$

$$+ \frac{b}{2} \sum_{j=1}^{n} A_j\omega_j \sum_{k=1}^{n} (A_k - \delta_{jk}) M(A_1, \dots A_j - 1, \dots A_k - 1, \dots A_n) \tag{18}$$

where δ_{jk} is the Krondecker delta.

Substituting for A_i in equation (18) gives the following set of equations for the ensemble (not time) averages of the distribution

$$\dot{\bar{a}}_i = -\omega_i \sum_{k=1}^{n} \bar{a}_k\omega_k \quad ; \quad i = 1,\dots,n. \tag{19}$$

This equation is similar to equation (11) with $\eta = 0$. Hence the equilibrium point $\bar{a}_i = 0$ is asymptotically stable in the large so long as the input signals are linearly independent.

Substituting for A_i in equation (18) also gives the following set of equations for the covariances of the distribution

$$\dot{\overline{a_i a_j}} = -\omega_i \sum_{k=1}^{n} (\omega_k \overline{a_k a_j}) - \omega_j \sum_{k=1}^{n} (\omega_k \overline{a_k a_i}) + \omega_i \omega_j b \qquad (20)$$

To investigate the stability of these equations about the equilibrium point $\overline{a_i a_j} = \frac{b}{2} \delta_{ij}$; $i,j = 1,\ldots n$ we define

$$\varepsilon_{ij} = \overline{a_i a_j} - \frac{b}{2} \delta_{ij} \qquad ; \quad i,j = 1,\ldots,n. \qquad (21)$$

and equation (20) becomes

$$\dot{\varepsilon}_{ij} = -\sum_{k=1}^{n} \omega_k (\omega_i \varepsilon_{kj} + \omega_j \varepsilon_{ki}) \; ; \quad i,j = 1,\ldots,n. \qquad (22)$$

We choose the Lyapunov function

$$L = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_{ij}^{2} \qquad (23)$$

which has the derivative

$$\dot{L} = -\tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \varepsilon_{ij} \omega_k (\omega_i \varepsilon_{kj} + \omega_j \varepsilon_{ki}) \qquad (24)$$

$$= -\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \varepsilon_{ij} \varepsilon_{ik} \omega_j \omega_k \qquad (25)$$

$$= -\sum_{i=1}^{n} \left(\sum_{k=1}^{n} \varepsilon_{ik} \omega_k \right)^2 \qquad (26)$$

which is negative semi-definite. Again if the ω_i are linearly independent it follows that the equilibrium point $\overline{a_i a_j} = \frac{b}{2} \delta_{ij}$ is asymptotically stable in the large. Using equations (10) this equilibrium point corresponds to $\overline{\tilde{\alpha}_i \tilde{\alpha}_j} = \sqrt{K_i K_j} \delta_{ij}$; $i,j = 1,\ldots n$.

The results of the above analysis may be summarized as follows. If the inputs, z_i to the parameter tracking system are linearly independent then the means and co-variances of the parameter deviations $\tilde{\alpha}_i$ tend to zero and the variances tend to $\frac{K_i b}{2}$. Since the response of any estimation scheme to noisy measurements is fundamental it is clear that these results are of great importance to our understanding of quadratic parameter tracking. The result that the variance of the deviations is independent of the form of the input signals is particularly interesting. It would appear that this no longer holds if there is coloured measurement noise, or if "state-variable filters" are used. It is also worth noting that although it is easily shown that at any time the parameter deviations are normally distributed, no information has been obtained on either the second order probability distribution or the autocorrelation and crosscorrelation functions of the $\tilde{\alpha}_i$. It would appear that the mathematics involved in this is much more complex.

References

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