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University of Sheffield

Department of Control Engineering

The structure of Kron's polyhedron model and the scattering problem

H. Nicholson

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Summary

Kron proposed a polyhedron network model or 'automaton' for representing a wide range of physical system problems. The model includes a sequence of higher-dimensional structures each formulated by the form of an orthogonal electrical network with closed- and open-paths. A set of electromagnetic waves is assumed to propagate across the polyhedron, and many other physical concepts are introduced intuitively into the model, including those concerned with multidimensional 'generalised' machines, thermodynamics, fluid flow and statistical phenomena. The present work highlights the significance of the orthogonal electrical network in the polyhedron structure, and it is shown that a similar structure exists in the general scattering representation of a flow process. The formulation of the scattering problem would then appear to provide an analytical basis for many of the physical concepts discussed by Kron. It is also shown to be associated with the system problems concerned with the multimachine power system, least-squares estimation and linear optimal control and filtering. Kron's polyhedron model and the scattering problem thus incorporate similar basic features which appear to be of fundamental importance in general system theory.

1. Introduction

The basic equations of electrical network theory are of fundamental importance and can be used analogously to represent many other physical system problems. Kron has made significant contributions[1-14], and has used the properties of electrical networks for representing distributed parameter systems including Maxwell's electromagnetic field equations.[1,15,16] They have also been used to represent heat flow, elastic field problems, fluid flow and neutron diffusion.[2,17] Branin[18] has also illustrated the correspondence between the algebraic properties of higher-dimensional electrical networks and the gradient, divergence and curl expressions associated with the operational form of the vector calculus.

The algebraic structure of the conventional electrical network is extended by Kron into a multidimensional polyhedron containing points, lines, surfaces and volume elements with superimposed electromagnetic parameters. Such a model is proposed for extending the conventional least-squares problem of curve fitting and interpolation, and improved fitting is obtained by introducing additional information in the vicinity of the data points based on the physical properties of the polyhedron model. Kron generalises the polyhedron by introducing the concept of multidimensional 'generalised' machines in an oscillatory self-organising structure. Such an 'automaton' containing superimposed electromagnetic properties was proposed for the solution of a wide variety of problems, ranging from multidimensional curve fitting to the phenomena in molecules and crystals. A close analogy exists between the polyhedral waves and the waves that propagate within a crystal,[12] and the possible application of crystals in computers and for the modelling of 'artificial brains' was also discussed. A further extension was considered by introducing an increased number of coupled polyhedra using additional parameters associated with fluid flow, thermodynamics and other physical phenomena.[12] A statistical interpretation was also given to the model containing random elements, with an ensemble of samples represented as superimposed electromagnetic quantities.[12] Although certain concepts were apparently introduced intuitively into the polyhedron the model does introduce a structure into system theory which now appears to be of considerable significance.

There appears to have been little attempt to explore or extend Kron's pioneering work on self-organising models and multidimensional space filters, possibly because some of the proposals appear to lack analytical principles in the published work. The present work attempts to illustrate the structure of Kron's basic model and to highlight, particularly, the significance of the electrical network problem in the polyhedron structure. It is also shown that the model can be identified with the general scattering representation of a flow process which can provide a framework for investigating the properties of an interconnected sequence of multidimensional networks, and thus provide a unifying theory for many of the concepts introduced by Kron into the polyhedron model. The correspondence is shown to extend to include the solution of the multimachine power system problem and also the linear optimal control and filtering problems. This supports certain aspects of Kron's proposals and provides further evidence of the significance of the electrical network and scattering problems in general system theory.

2. The structure of the polyhedron model

The concept of electromagnetic waves propagating across a polyhedron network is used by Kron as the basis of a multidimensional space filter.[19] This provides a physical structure which can be used to represent the surface of a function

$$y_i = f(x^i_{(q)})$$

and its higher-order divided differences associated with a set of experimental data $x^i_{(q)}$, $i = 1..n$, $q = 1..k$. A conventional linear 1-network is formed by spanning the space between the n vertices representing the data points which define a 0-dimensional network in a k-dimensional Euclidean space.[6,10] This can then be augmented by a 2-network with neighbouring branches forming planes, then by a 3-network with tetrahedrons formed from adjacent triangles, and so on until the k-dimensional space is spanned by a k-network. The resulting polyhedron[4,6,12] consisting of an interconnected set of k+1, q-networks represents a topological structure which can provide additional information concerning the geometrical properties of the surface passing through the given points. Each q-network is represented by the transformations associated with a conventional electrical network including closed- and open-paths, which Kron refers to as an orthogonal network. Similar paths appear in electromagnetic theory, fluid dynamics and in the topology of differentiable manifolds which supports the validity of the representation used for the q-network problem.[14]

2.1 The orthogonal electrical network[6,11-14,16,20,39] In the general electrical network problem, with individual branch variables defined in Fig. 1, the primitive network with impedance matrix Z is represented by

$$E + e = Z(I + i) \quad \text{or} \quad V = ZJ \; , \quad J = YV \tag{1}$$

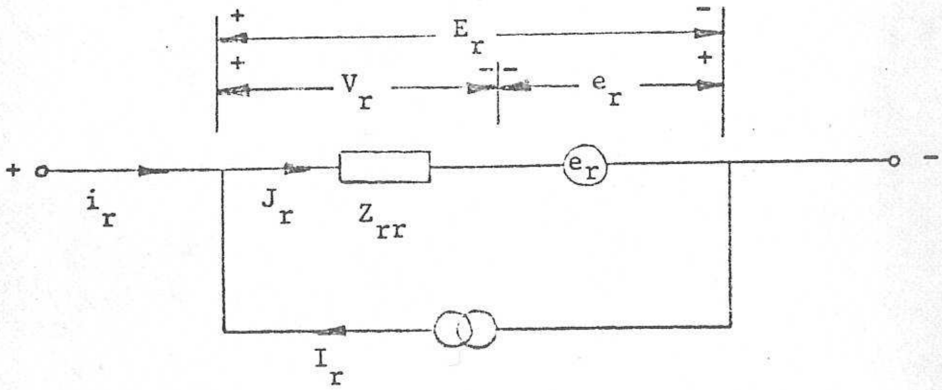

Fig. 1 rth primitive network branch

In the orthogonal formulation, the network structure is defined by square, nonsingular connection matrices

$$C = [C_c, C_o] \; , \quad A = [A^c, A^o] = (C^T)^{-1}$$

including components related to specified closed- and open-paths, with A^o, C_c representing the usual branch-node-pair and branch-mesh matrices respectively. Primed variables $i^{c'}, e_c{}'$ are then assumed to exist in the m closed-paths, and $I^{o'}, E_o{}'$ in the p(=b-m) open-paths containing arbitrarily connected branches. The notation (c,o) defines closed- and open-paths, with subscripts and superscripts signifying dual properties, as used by Kron. A set of paths and designated variables are illustrated in Section 9. The b 'coil' variables in the connected network are then given by

$$J = b \overset{\;\; m \quad p}{\begin{bmatrix} C_c & C_o \end{bmatrix}} \begin{bmatrix} i^{c'} \\ I^{o'} \end{bmatrix} \qquad V = b \overset{\;\; m \quad p}{\begin{bmatrix} A^c & A^o \end{bmatrix}} \begin{bmatrix} e_c{}' \\ E_o{}' \end{bmatrix} \tag{2}$$

The orthogonality condition $C^T A = \delta_b$ with partitioned components gives

$$C_c{}^T A^c = \delta_m, \quad C_o{}^T A^o = \delta_p \; , \quad C_c{}^T A^o = 0 \; , \quad C_o{}^T A^c = 0$$

where δ represents the unit matrix.

Trees and links can now be introduced as special cases of the orthogonal formulation,[14,21,22] with

$$C = \begin{array}{c} \\ T \\ L \end{array} \begin{array}{c} \begin{array}{cc} c & o \end{array} \\ \begin{bmatrix} C_T & B_T \\ C_L & O \end{bmatrix} \end{array} \qquad A = \begin{array}{c} \\ T \\ L \end{array} \begin{array}{c} \begin{array}{cc} c & o \end{array} \\ \begin{bmatrix} O & A_T \\ \delta_m & A_L \end{bmatrix} \end{array}$$

Matrix C_o defines the open-paths or tree branches and A^c the link branches in the closed-paths. For the unit-tree and unit-link case[22,23], with

$$B_T = \delta_p \ , \ C_L = \delta_m \ , \ C_T = -B_T A_L^{\ T} \ , \ i = \begin{bmatrix} i^T \\ i^L \end{bmatrix} \ , \ E = \begin{bmatrix} E_T \\ E_L \end{bmatrix}$$

eqns 2 give

$$i^L = i^{c'} \ , \ i^T = C_T i^{c'} \ , \ E_L = A_L E_o' \ , \ E_T = A_T E_o' = E_o'$$

Thus E_o' and $i^{c'}$ represent tree branch voltages and link currents respectively.

The branch variables are related by Kirchoff's laws defined by

$$C_c^{\ T} E = 0 \ , \ (A^o)^T i = 0$$

Also

$$E = A^o E_o' \ , \quad i = C_c i^{c'}$$

and equivalent induced mesh-voltage and nodal-current sources are given by

$$e_c' = C_c^{\ T} e \ , \quad I^{o'} = (A^o)^T I$$

In the orthogonal formulation, new equivalent sources e_L, I^T are referred to the links and open-paths or tree branches respectively, and specified by[21,24,25]

$$\tilde{e} = \begin{bmatrix} e_T \\ e_L \end{bmatrix} = A^c e_c' = \begin{bmatrix} 0 \\ C_c^{\ T} e \end{bmatrix} \ , \ \tilde{I} = \begin{bmatrix} I^T \\ I^L \end{bmatrix} = C_o I^{o'} = \begin{bmatrix} B_T (A^o)^T I \\ 0 \end{bmatrix}$$

Now from eqns 1 and 2

$$\begin{bmatrix} e_c' \\ E_o' \end{bmatrix} = C^T Z C \begin{bmatrix} i^{c'} \\ I^{o'} \end{bmatrix} \ , \quad \begin{bmatrix} i^{c'} \\ I^{o'} \end{bmatrix} = A^T Y A \begin{bmatrix} e_c' \\ E_o' \end{bmatrix} \tag{3}$$

where $C^T Z C = \begin{bmatrix} C_c^{\ T} Z C_c & C_c^{\ T} Z C_o \\ C_o^{\ T} Z C_c & C_o^{\ T} Z C_o \end{bmatrix} = \begin{bmatrix} Z_1 Z_2 \\ Z_3 Z_4 \end{bmatrix} \ , \ A^T Y A = \begin{bmatrix} (A^c)^T Y A^c & (A^c)^T Y A^o \\ (A^o)^T Y A^c & (A^o)^T Y A^o \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}$

The solutions for tree-branch voltage drops $E_o'(E_T)$ and mesh currents $i^{c'}(i^L)$ related to the source variables in the equivalent network are then given by

$$\begin{bmatrix} i^{c'} \\ E_o' \end{bmatrix} = \begin{bmatrix} -Z_1^{-1}Z_2 & Z_1^{-1} \\ Z_4 - Z_3Z_1^{-1}Z_2 & Z_3Z_1^{-1} \end{bmatrix}\begin{bmatrix} I^{o'} \\ e_c' \end{bmatrix} = \begin{bmatrix} Y_2Y_4^{-1} & Y_1 - Y_2Y_4^{-1}Y_3 \\ Y_4^{-1} & -Y_4^{-1}Y_3 \end{bmatrix}\begin{bmatrix} I^{o'} \\ e_c' \end{bmatrix} \quad (4)$$

Then assuming $Z = \mathrm{diag}(Z_T, Z_L)$, $Y = \mathrm{diag}(Y_T, Y_L)$

$$Y_4^{-1} = [(A^o)^T Y A^o]^{-1} = Z_4 - Z_4A_L^T(A_LZ_4A_L^T + Z_L)^{-1}A_LZ_4 = (Z_4^{-1} + A_L^TY_LA_L)^{-1} \quad (5)$$

where $Z_4 = B_T^TZ_TB_T$ and $Z_1^{-1} = (A_LZ_4A_L^T + Z_L)^{-1}$. Eqn 4 represents the orthogonal formulation of the electrical network problem, and with eqn 5 forms the basis of Kron's work on network tearing and interconnection with added links.[20]

The usual mesh- and node-method solutions are included in eqn 4 with

$$i = C_c i^{c'} = C_c Z_1^{-1}(e_c' - Z_2 I^{o'}) = L(e - Z\tilde{I})$$

where $$L = C_c(C_c^T Z C_c)^{-1}C_c^T = C_c Z_1^{-1} C_c^T = Y - YMY$$

is the branch-admittance matrix and

$$M = A^o [(A^o)^T Y A^o]^{-1}(A^o)^T = A^o Y_4^{-1}(A^o)^T = Z - ZLZ$$

is the branch-impedance matrix. The branch voltages are similarly given by

$$E = A^o E_o' = A^o Y_4^{-1}(I^{o'} - Y_3 e_c') = M(I - Y\breve{e})$$

Kron now assumes that each isolated network forming the polyhedron model can be represented by the form of solution for the conventional orthogonal 1-network problem, with sources $I_{(q)}$, $e_{(q)}$, induced variables $I'_{(q)}$, $i'_{(q)}$, $e^{(q)'}$, $E^{(q)'}$ and 'branch' variables $i_{(q)}$, $E_{(q)}$, representing electrical network or electromagnetic variables associated with the q-network in the general polyhedron model. In each q-network, matrices $C^{(q)}$, $A_{(q)}$ define the connection of volume elements,[6] and closed- and open-path currents and voltages satisfy Kirchoff's laws, with[11]

$$(C_c^{(q)})^T E_{(q)} = 0 , \quad (A^o_{(q)})^T i_{(q)} = 0$$

The transformations of eqn 4 associated with the orthogonal 1-network problem, and similarly with each q-network, are illustrated in Fig. 2. The basic diagram for the conventional electrical network[16] also includes the boundary operators or partial connection matrices $C_c^{(0)}$, $C_c^{(1)}$, $A^o_{(1)}$ and $A^o_{(2)}$. The rectangular incidence matrices M^i_{i+1} shown in Fig. 2 interconnect the simplexes or volume elements of a neighbouring i- and i+1-

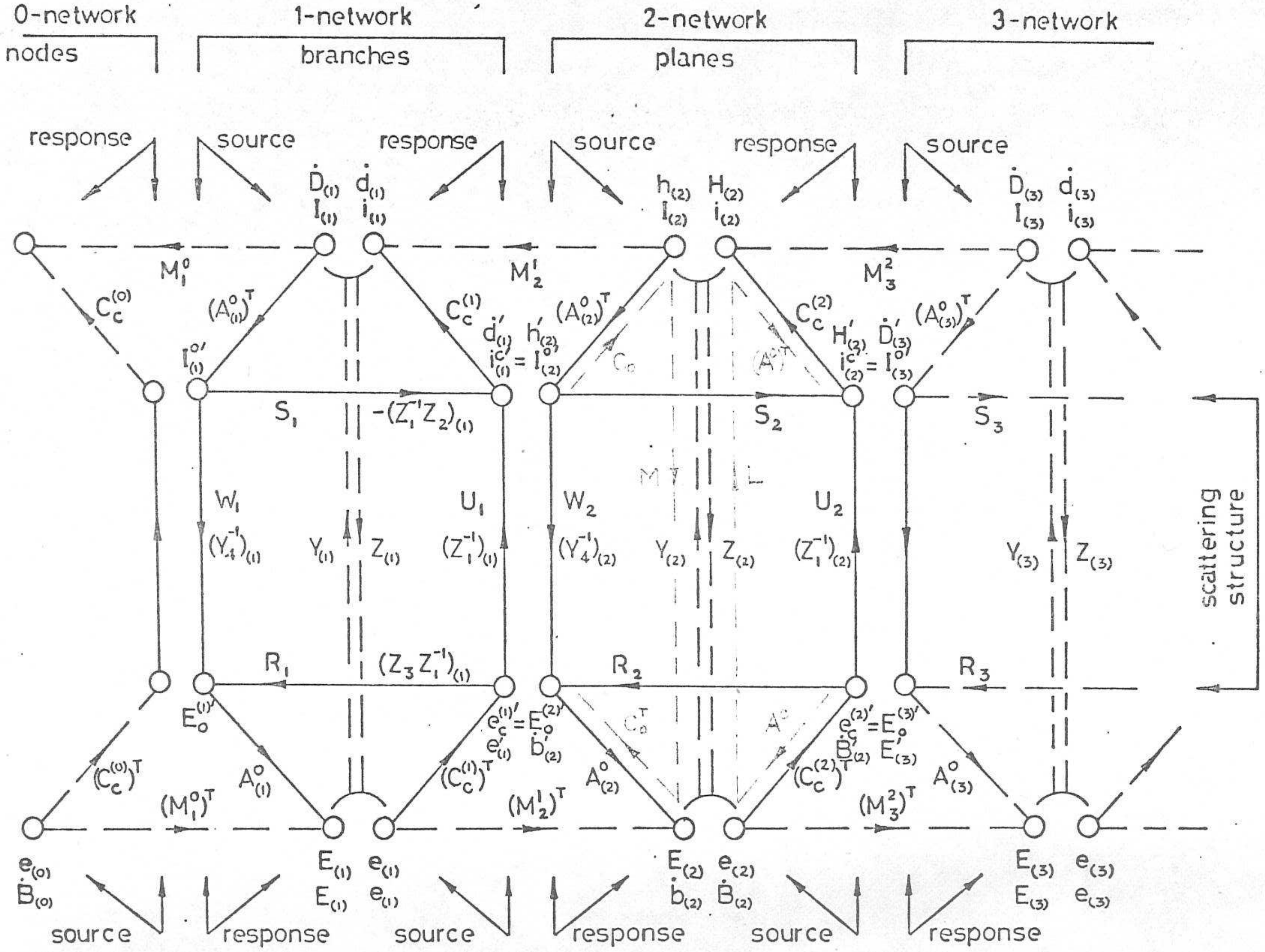

FIG 2 Transformation diagram for sequence of orthogonal networks

primitive network, such as the branches with nodes or planes, and relate two different-dimensional variables. They also possess properties of divergence and curl operators.[11-14] The significant feature of the polyhedron model is the use of the solution for the conventional orthogonal electrical network problem as a basis for each higher dimensional network. The assumption is supported physically by the properties of electromagnetic wave propagation which Kron considered could be associated with a connected network structure.

The transformation diagram includes the basic characteristics of Roth's diagram, with the impedance- and admittance-type operators directed vertically across the diagram and the dimensionless or connection-type operators directed horizontally between similar variables. The diagram illustrates particularly the properties of the projection-type matrices L,M and Z_1^{-1}, Y_4^{-1} and of the interacting 'residual'-type operators $-Z_1^{-1}Z_2$, $Z_3Z_1^{-1}$ linking closed- and open-path variables, and their effect in transferring the source components to the 'branch' variables.

The boundary operators and incidence matrices for the i-network are related by

$$C_c^{(i)}(A^o_{(i+1)})^T = M^i_{i+1}$$

and possess the orthogonality properties

$$(C_c^{(i)})^T A^o_{(i)} = 0\ , \quad M_i^{i-1} M^i_{i+1} = 0$$

similar to curl grad = 0 and div curl = 0.[6] Also $L_i(M_i^{i-1})^T = 0$. For the 1-network, the relationships

$$\begin{aligned} L &= C_c Z_1^{-1} C_c^T\ , \quad Z_1^{-1} = (A^c)^T L A^c \\ M &= A^o Y_4^{-1} (A^o)^T\ . \quad Y_4^{-1} = C_o^T M C_o \end{aligned} \tag{6}$$

indicate that the operators Z_1^{-1}, L and Y_4^{-1}, M are associated directly with the closed- and open-path variables respectively. The closed-path component A^c transforms L and isolates the admittance matrix Z_1^{-1}, and similarly, the open-path component C_o transforms M and isolates Y_4^{-1}. The correspondences $Z_1^{-1} \leftrightarrow L$, $L \leftrightarrow Z_1^{-1}$ and $Y_4^{-1} \leftrightarrow M$, $M \leftrightarrow Y_4^{-1}$ are illustrated in Fig. 2, and suggest a form of 'inverse' for $(A^o)^T$ and C_c defined by C_o and $(A^c)^T$ respectively. Similar forms are used by Kron to 'open up nonexistent pathways of propagation in the open-circuited polyhedron', and are considered to justify the requirement for an orthogonal network formulation.

Kron also introduces a '2-phase' (primal and dual) structure for obtaining a complete representation of the k-dimensional space, with the q-simplexes of the dual polyhedron each orthogonal in space to a (k-q)-simplex of the primal polyhedron.[11-14] The hyperplanes of the dual network form coupled closed- and open-paths that are orthogonal respectively to the open- and closed-paths of the primal network. The dual (k-1)-network acts as an external environment for the physical elements, and also provides an interpretation for the connection matrix A,[13] with columns defining closed- and open-2 paths of an 'invisible', dual 2-network.

2.2 <u>The polyhedron model with electromagnetic variables</u> The structure of the polyhedron model can incorporate the concept of a combined transverse electromagnetic wave with closed-path dielectric and magnetic variables $(\dot{d}_{(i)}, e_{(i)})$ and $(h_{(i)}, \dot{b}_{(i)})$ respectively, and a longitudinal wave with open-path dielectric and magnetic variables $(\dot{D}_{(i)}, E_{(i)})$ and $(H_{(i)}, \dot{B}_{(i)})$ respectively, i = 0..k, propagating from the 0-dimensional points to the k-dimensional network elements.[5,6] The variables can then be incorporated in the polyhedron model as illustrated in Fig. 2. The propagation induces electrical and electromagnetic variables consistent with the form of Maxwell's field equations in all dimensions, represented by curl $H = J+\dot{D}$, div $D = \rho$, curl $E = -\dot{B}$, div $B = 0$, $B = \mu H$, $D = \varepsilon E$, $J = \sigma E$, with the usual notation.

The electrical network variables (i;E) and electromagnetic variables $(H,\dot{d};\dot{b},E)$ are considered as responses or 'effects' of the sources (I;e) and $(h,\dot{D};\dot{B},e)$ respectively. In this correspondence, $\dot{B}$ behaves as a voltage source although it exists essentially as a response variable in the operational form of the 3-network problem.[18] The even- and odd-dimensional networks are considered to possess magnetic and dielectric properties respectively, and the square of each volume element forming the isolated q-network is used to define an impedance as a component of the primitive impedance matrix $Z_{(q)}$. Thus the square of the length of each branch represents 'dielectric reluctance' and the 2-simplexes are considered as magnetic networkswith the square of a triangular area representing 'magnetic reluctance'.

For application in the estimation problem, dependent variables (y) are assigned to the vertices of the 0-network as rates of change of magnetic flux lines or generated voltages $\dot{B}_{(o)}$, and an open-path voltage or difference of potential then appears on each branch of the 1-network, given by

$$E_{(1)} = A^{o}_{(1)} E_{o}^{(1)'} = A^{o}_{(1)} \dot{B}'_{(o)} = A^{o}_{(1)} (C_{c}^{(o)})^{T} \dot{B}_{(o)} = (M_{1}^{o})^{T} \dot{B}_{(o)} \equiv \text{div } \dot{B}_{(o)}$$

Dielectric fluxes given by the total displacement current $\dot{D}_{(1)}+\dot{d}_{(1)}$ then flow in each branch and an mmf produced by the displacement current appears on planes of the magnetic network, and open- and closed-path magnetic flux lines $\dot{b}_{(2)}+\dot{B}_{(2)}$ appear across the planes. Propagation thus proceeds with

$$\dot{B}_{(o)} \rightarrow E_{(1)},\ e_{(1)} \rightarrow \dot{D}_{(1)},\ \dot{d}_{(1)} \rightarrow h_{(2)},\ H_{(2)} \rightarrow \dot{b}_{(2)},\ \dot{B}_{(2)}$$

and the cycle of 'open-circuit' propagation of the electromagnetic wave is assumed to continue similarly after each two dimensions, with the general steps, for $i = 1,3..k$,[6]

$$\dot{b}_{(i+1)} + \dot{B}_{(i+1)} = Z_{(i+1)} (M^{i}_{i+1})^{\dagger} Y_{(i)} (M^{i-1}_{i})^{T} \left[\dot{b}_{(i-1)} + \dot{B}_{(i-1)}\right] ,\ \dot{b}_{(o)} = 0 \tag{7}$$

where $(M^{i}_{i+1})^{\dagger}$ ($\equiv$curl^{-1}) represents an equivalent or generalised inverse of the rectangular incidence matrix M^{i}_{i+1}.

Kron[6] considers an equivalent inverse for the rectangular matrix M^{i}_{i+1} related to components of the nonsingular connection matrices A,C, given by

$$(M^{i}_{i+1})^{\dagger} = \left[C_{c}^{(i)} (A^{o}_{(i+1)})^{T}\right]^{\dagger} \equiv C_{o}^{(i+1)} (A^{c}_{(i)})^{T} \tag{8}$$

The form of the inverse has physical significance in terms of the current relationships in the orthogonal network problem given by

$$i = C_{c} i^{c'} ,\quad i^{c'} = (A^{c})^{T} i ,\quad I = C_{o} I^{o'} ,\quad I^{o'} = (A^{o})^{T} I$$

The connection matrix components A^{c}, C_{o} forming the equivalent inverse are illustrated in the transformation diagram of Fig. 3.

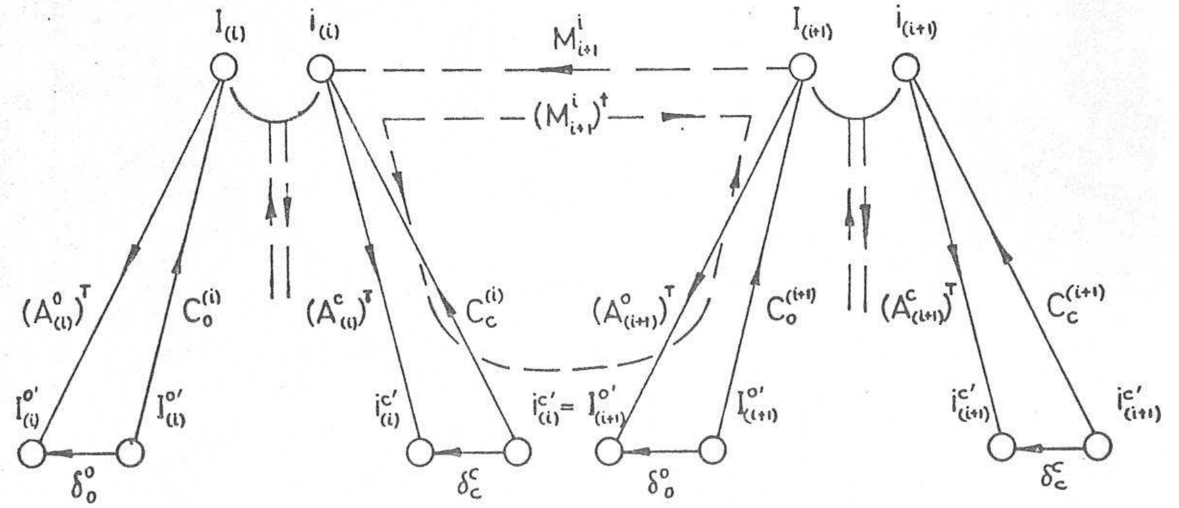

FIG 3 Current relationships for adjacent orthogonal networks

2.2.1 Divided differences Kron associates the induced variables resulting from the propagation of an electromagnetic wave across the polyhedron structure with higher-order divided differences, representing generalisations of similar quantities used in the calculus of finite differences. These are considered to provide additional information in terms of geometrical properties which can be used to obtain improved fitting in an estimation scheme. The divided differences are defined, for $i = 1,3,..,$ by the electromagnetic variables,

$$\dot{D}_{(i)},\ \dot{d}_{(i)} \equiv \Delta^{i}y/\Delta x^{i}\ ,\ \dot{b}_{(i+1)},\ \dot{B}_{(i+1)} \equiv \Delta^{i+1}y/\Delta x^{i+1}$$

which appear as combined 'response' and 'source' variables in the transformation diagram of Fig. 2, and are related to the previous differences[8] as a discrete single-stage process. The first divided difference can be identified with the conventional first-order difference defined for a function $f(x)$ specified at points x_p, $p = 1,2..k$. Thus

$$\dot{D}_{(1)} + \dot{d}_{(1)} = Y_{(1)}(M_1^o)^T\dot{B}_{(o)} = Y_{(1)}(M_1^o)^T f(x) \equiv \left[f(x_i) - f(x_j)\right]/d_{ij}$$

where $d_{ij}{}^2 = \sum_{p=1}^{k} (x^i_{(p)} - x^j_{(p)})^2$. Then

$$(M_1^o)^T f(x) \equiv f(x_i) - f(x_j)$$

The divided differences may also be derived using the scattering representation discussed in Section 4. This would inherently include the effects of interaction between the coupled networks, compared to Kron's 'open-circuit' propagation across the polyhedron which apparently avoids the necessity for considering such interaction.

The total stored electrostatic and magnetic energy in each network can be related to a weighted quadratic form of the respective divided differences, and the variables in each q-network will be associated with a minimum overall stored energy condition.[10,12]

3. The multimachine structure in the polyhedron model

Kron extends the polyhedron model by considering that each stationary q-network and its dual k-q-network forms the 2-phase reference axes of a k-dimensional 'generalised' rotating electrical machine, analogous to the orthogonal reference axes of a conventional electrical machine (with $k = 2$, $q = 1$).[12,13] Each generalised machine introduces electromagnetic variables and also speed and torque variables which can be assigned to the network branches in a dynamic polyhedron model. It is now shown that a direct analogy also exists between the conventional multimachine system problem and the propagation of an electromagnetic wave across the polyhedron structure. The solution also exists within the scattering representation of a flow process, as discussed in Section 5.3.

3.1 Multimachine power system Consider the connection of synchronous generators with an equivalent network of generator nodes represented by

$$i_N = Y_N v_N$$

where Y_N is a symmetrical matrix of driving point and transfer admittances. The machine voltages referred to direct- and quadrature-rotor axes are given by

$$v = e - Z_M i$$

where $Z_M = \mathrm{diag}(Z_{Mk})$ and Z_{Mk} represents the k-machine transient reactance matrix with components x'_{dk}, x'_{qk}. With m machines connected to n network nodes the machine-network voltages and currents are related by

$$v = A(\theta)v_N , \quad i_N = A^T(\theta)i$$

where θ represents a load angle between the machine field axes and the common network reference axes, and $A(\theta)$ is an mxn connection matrix with elements $A_{ki} = (e^{j\theta_k}, 0)$ defining the generator-node connections. The network and generator voltages are then given by[20,36]

$$v_N = \left[Y_N + A^T(\theta)Y_M A(\theta)\right]^{-1} A^T(\theta)Y_M e , \quad v = M(\theta)Y_M e$$

where $\quad M(\theta) = A(\theta)\left[Y_N + A^T(\theta)Y_M A(\theta)\right]^{-1} A^T(\theta)$

is a symmetrical mxm impedance matrix, which exists similarly in the electrical network and least-squares estimation problem incorporating a priori information. We may also define a dual transformation matrix

$$L = Y_M - Y_M M Y_M = (Z_M + AZ_N A^T)^{-1}$$

Machine current is then given by

$$i = Y_M(e-v) = Y_M(\delta - MY_M)e = Y_M F^{-1} e = Le$$

where $\quad F = (\delta - MY_M)^{-1} = \delta + AZ_N A^T Y_M$

represents a return-difference-type matrix. Also

$$i_N = A^T Le \quad , \qquad v = MY_M L^{-1} i = (AZ_N A^T) i$$

The general steps defining the propagation of an electromagnetic wave across adjacent networks in the polyhedron model can now be associated with the solution for the multimachine problem. The properties of the orthogonal network form a basis for the correspondence which highlights the importance and validity of the orthogonal concept as discussed by Kron. Each cycle of open-circuit propagation of the electromagnetic wave, as defined by eqn 7, may be stated in the form

$$Y_{(i+1)}\left[\dot{b}_{(i+1)} + \dot{B}_{(i+1)}\right] = h_{(i+1)} = \left[C_c^{(i)} (A^o_{(i+1)})^T\right]^{\dagger} Y_{(i)} \left[C_c^{(i-1)} (A^o_{(i)})^T\right]^T \left[\dot{b}_{(i-1)} + \dot{B}_{(i-1)}\right]$$

Then using the equivalent inverse given by eqn 8,

$$(A^o_{(i+1)})^T h_{(i+1)} = \left[(A^c_{(i)})^T Y_{(i)} A^o_{(i)}\right] (C_c^{(i-1)})^T \left[\dot{b}_{(i-1)} + \dot{B}_{(i-1)}\right] \qquad (9)$$

Equation 9 now corresponds to a current relationship in the electrical network problem. It is interesting to note that the equivalent inverse for the rectangular incidence matrix suggested by Kron leads to a solution which can be identified with the network problem. Now with specified closed- and open-paths the term $(A^c_{(i)})^T Y_{(i)} A^o_{(i)}$, which appears as a component in eqn 3, can be stated in the form

$$Y_2 = -Z_1^{-1} Z_2 Y_4 = -Z_1^{-1} Z_2 (Z_4 - Z_3 Z_1^{-1} Z_2)^{-1}$$

Equation 9 then gives the equivalent form

$$(A^o_{(i+1)})^T h_{(i+1)} = -(Z_1^{-1} Z_2)_{(i)} (Z_4 - Z_3 Z_1^{-1} Z_2)^{-1}_{(i)} (C_c^{(i-1)})^T \left[\dot{b}_{(i-1)} + \dot{B}_{(i-1)}\right]$$

which can be identified with the solution for network current ($i_N = A^T Le$) in the multimachine problem. This gives a correspondence between the polyhedron wave and machine system variables represented by

$$(A^o_{(i+1)})^T h_{(i+1)} \equiv i_N \;, \quad (C_c^{(i-1)})^T \left[\dot{b}_{(i-1)} + \dot{B}_{(i-1)}\right] \equiv e$$

$$-(Z_1^{-1} Z_2)_{(i)} = -\left[(C_c^{(i)})^T Z_{(i)} C_c^{(i)}\right]^{-1} (C_c^{(i)})^T Z_{(i)} C_o^{(i)} \equiv A^T(\theta)$$

$$(Z_4)_{(i)} = (C_o^{(i)})^T Z_{(i)} C_o^{(i)} = (B_T^{(i)})^T Z_{(i)T} B_T^{(i)} \equiv Z_M$$

$$-(Z_1)_{(i)} = -(C_c^{(i)})^T Z_{(i)} C_c^{(i)} = -\left[(C_T^{(i)})^T Z_{(i)T} C_T^{(i)} + Z_{(i)L}\right] \equiv Z_N$$

The form of the machine impedance matrix Z_M thus suggests the existence of a machine structure associated with the polyhedron tree components. The network impedance Z_N includes a correspondence with both tree and link elements, and the connection operator $A^T(\theta)$ can be identified with a scattering-type transmission component $S_{(i)}$ (as discussed in Section 4). Kron refers to the concept of an electrical machine structure existing in the polyhedron model without illustrating the analytical details. The correspondence now shown to exist between the conventional multimachine problem and wave propagation in the polyhedron based on the solution of the orthogonal network would support the validity of this concept.

4. Interconnection of the multidimensional network sequence

In the overall polyhedron structure, the waves in the various spaces[4] and the induced variables in adjacent networks will interact, and it will be necessary to consider the effects of such coupling. The structure of the orthogonal electrical network solution and the interrelationships of the variables in adjacent networks, would now suggest that the physical variables appearing in the polyhedron will interact according to the general scattering representation of a flow process incorporating coupled obstacles or distributed constants. The interconnection of the network sequence can be formulated within the framework of the scattering problem, and Kron's proposals for introducing the properties of a wide range of physical systems and also statistical concepts into the polyhedron model would now appear to be particularly relevant to the general scattering problem. Thus Kron's network sequence and the scattering representation of a flow process are closely related, and both developments are of considerable importance. It is also of interest to note that the original mathematical work on scattering by Redheffer, Reid, Bellman and others, and Kron's independent study of multidimensional networks was being carried out during the same period, although the important concept of the orthogonal network which forms the basis of the polyhedron model was conceived by Kron much earlier, in 1937.

4.1 Scattering theory An overall flow process incorporating a series of obstacles or distributed parameters can be represented by an interconnected set of scattering matrices which relate the incident and reflected waves appearing at the boundaries of the obstacles. The resulting algebraic framework based on the combined scattering matrix has been used particularly to represent propagation through a stratified dielectric medium, radiative transfer, neutron diffusion and transmission line theory. A diffusion problem has also been interpreted as a probability model which includes a set of matrix differential equations similar to those associated with the scattering problem for the general flow process.[26,27] The problem of wave propagation in inhomogeneous media has also been studied using invariance principles and functional equation techniques.[28]

An obstacle containing 2n-terminal pairs, with incident and reflected n-vectors V_i, V_r, can be characterised by the nxn transmission and reflection matrix functions $S_j(x,y)$, $R_j(x,y)$, $U_j(x,y)$, $W_j(x,y)$ with spatial coordinates (x,y). Then

$$V_r = T_j V_i , \qquad T_j = \begin{bmatrix} S_j & U_j \\ W_j & R_j \end{bmatrix}$$

With two adjacent obstacles, as represented in FIG 4, the reflected and incident waves are related by

$$\begin{bmatrix} V_3 \\ V_2 \end{bmatrix} = T_1 \begin{bmatrix} V_1 \\ V_4 \end{bmatrix} \qquad \begin{bmatrix} V_5 \\ V_4 \end{bmatrix} = T_2 \begin{bmatrix} V_3 \\ V_6 \end{bmatrix}$$

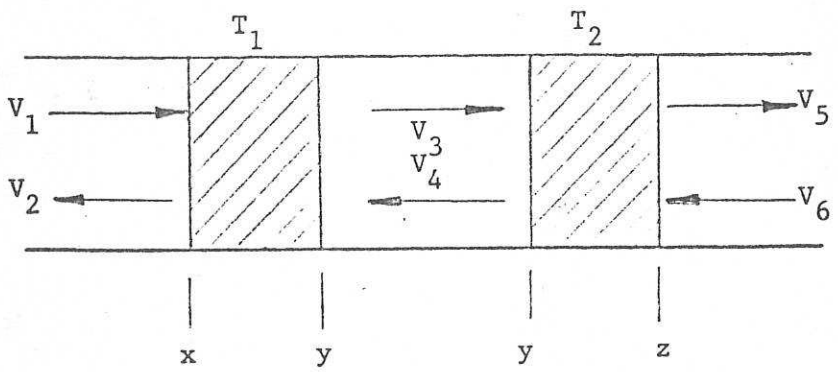

FIG 4 Propagation through adjacent obstacles

The cascaded process may also be represented by the transformation diagram or signal flow graph of FIG 5.

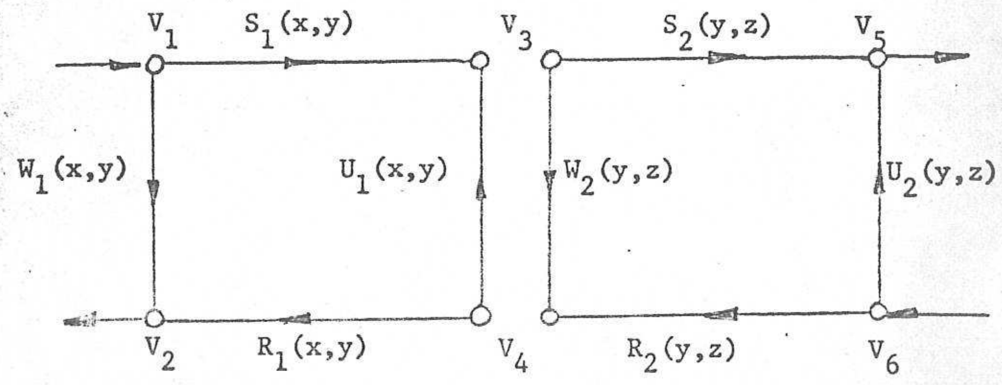

FIG 5 Transformation diagram representing the scattering process

Then for the combined structure

$$\begin{bmatrix} V_5 \\ V_2 \end{bmatrix} = \begin{bmatrix} S_2(\delta-U_1W_2)^{-1}S_1, & U_2+S_2U_1(\delta-W_2U_1)^{-1}R_2 \\ W_1+R_1W_2(\delta-U_1W_2)^{-1}S_1, & R_1(\delta-W_2U_1)^{-1}R_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_6 \end{bmatrix} = T_1{*}T_2 \begin{bmatrix} V_1 \\ V_6 \end{bmatrix} \qquad (10)$$

where $T_1{*}T_2$ defines the star product or combined scattering matrix for two adjacent obstacles. For a series of obstacles, the overall scattering matrix is given by the continued star product $T = T_1{*}T_2{*}\ldots{*}T_n$.

The orthogonal electrical network solution of eqn 4 corresponds to a transformation of 'input-output' variables similar to $(V_2V_1) \rightarrow (V_4V_3)$ across each 'obstacle'. However, if the solution is compared with the properties of a scattering-type matrix relating V_1,V_4 as 'source' variables $(I^{o'},e_c')$ and V_3,V_2 as 'response' variables $(i^{c'},E_o')$, the resulting transformation diagram corresponds to the form of Kron's sequence of q-networks, and possesses similar dual properties. This correspondence can then be used to develop an analytical basis for investigating the properties of the polyhedron structure, including particularly the effects of interaction between the coupled q-networks. The components of eqn 4 can also be identified with the form of eqn 10 and will partition according to the properties of the star product.

The transformation diagram of FIG 2, incorporating the scattering representation and components of the orthogonal network solution, is of the same general form as Kron's network sequence, with similar closed- and open-path dual relationships. It includes response variables $i^{c'}_{(q)}$, $E_o^{(q)'}$ as direct summations of transformed source components which do not appear in Kron's diagram and introduce the analogy with the scattering representation.

The properties of the combined scattering matrix could now be used to investigate the coupling effects between the isolated q-networks, which Kron referred to as a problem requiring solution.

The dimensionless transformation introduced by the components of the orthogonal network solution, given by

$$\delta - U_1 W_2 \equiv \delta - (Z_1^{-1})_{(i)} (Y_4^{-1})_{(i+1)} = \delta - \left[(A^c)^T L A^c\right]_{(i)} \left[C_o^T M C_o\right]_{(i+1)}$$

represents a residue effect appearing at the junction between adjacent networks and introduces coupling between the orthogonal networks or scattering obstacles. It has the characteristics of a return difference matrix resulting from a feedback-type connection. It does not appear to exist in Kron's analysis of wave propagation across the 'open-circuited' polyhedron, although it must be closely associated with the concept of an 'interconnection' network introduced into the process of tearing. The condition $\delta = U_1 W_2$, in which the star product for two connected obstacles does not exist, has been related to a state of resonance.[26,29]

The significance of the components of the combined star product, including the existence of the residue-type transformation δ-UW, and the corresponding structural properties of adjacent orthogonal networks, is illustrated in FIG 6. The response variables result from the direct horizontal transformations $S_2(\delta - U_1 W_2)^{-1} S_1$, $R_1(\delta - W_2 U_1)^{-1} R_2$ together with direct vertical components including a transformed source contribution directed through one of the interconnecting residual blocks. The direction of the incidence matrix transformation $M^1{}_2$ illustrates the 'pathway' of propagation used by Kron, which was assumed to avoid essentially the effects of interaction existing between adjacent networks. The connected zone between adjacent obstacles is characterised by the unit-matrix block δ and by the transformation $(\delta - UW)^{-1}$, depending upon the source of propagation. It thus introduces a return-difference matrix operator which is included inherently in the components of eqn 10. Kron also refers to a concept of feedback in relation to the sequence of orthogonal networks, which thus possibly relates to the properties of the return difference matrix associated with the operators U_1, W_2.

Equation 10 may also be considered as a lattice-type structure with unsymmetrical elements, relating outputs V_2, V_5 to inputs V_1, V_6. The resulting internal crossover connection could then be replaced by an ideal (-1:1) transformer which would correspond to Kron's application of an ideal

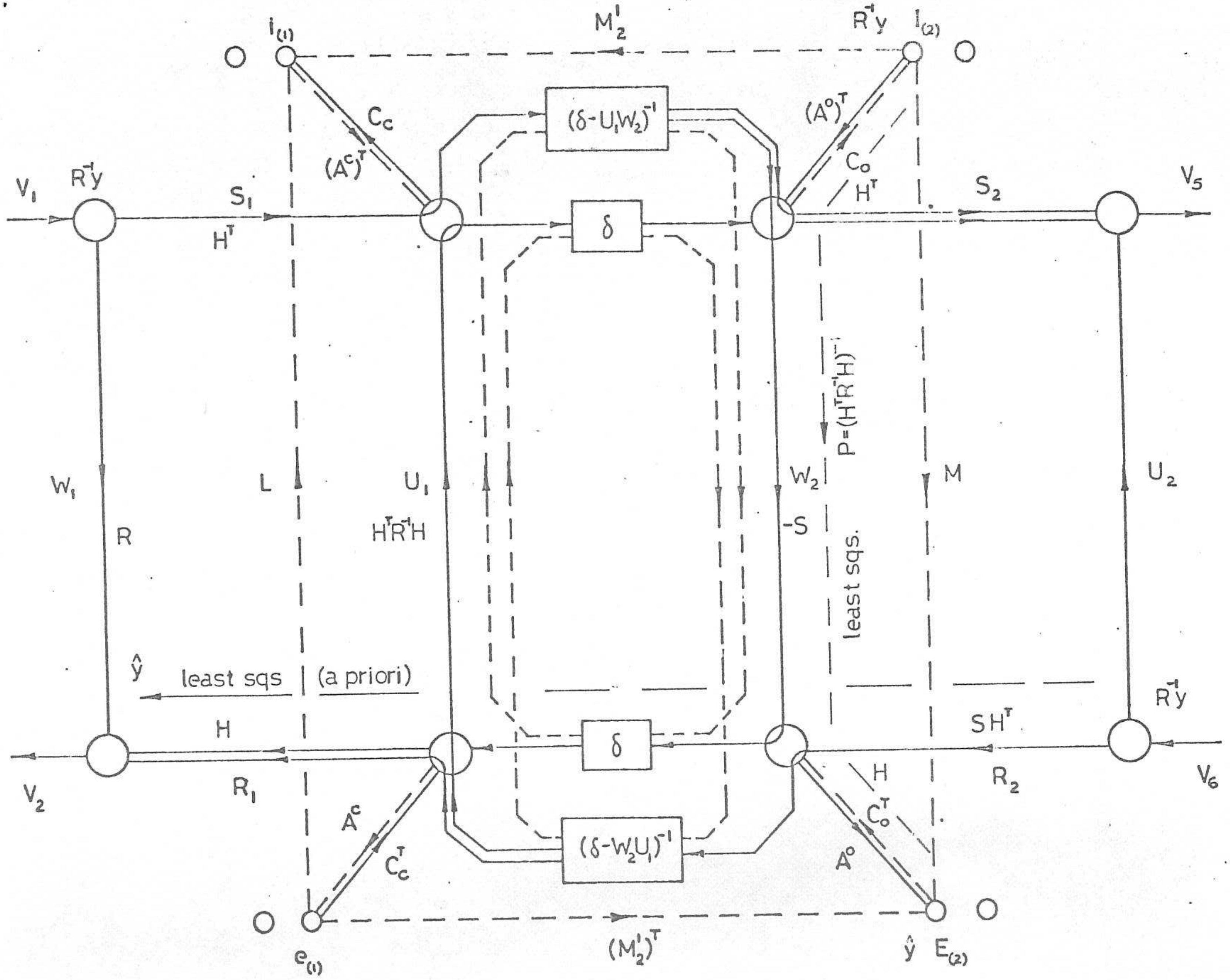

FIG 6 Internal structure of combined scattering matrix

transformer between adjacent orthogonal networks and between the primal and dual polyhedra.[12] A transposition of the transformation diagram will also produce a generalised 2-input, 2-output system with cross connections or interaction which would indicate the importance of this type of system structure.

The star product with a condition of matching including a lossless tuner represented by a unitary scattering matrix ($SS^* = \delta$), is given by[29]

$$\begin{bmatrix} S_1 & U_1 \\ W_1 & R_1 \end{bmatrix} * \begin{bmatrix} S_2 & -U_1^* \\ U_1^* & S_2 \end{bmatrix} = \begin{bmatrix} S_2(\delta-U_1U_1^*)^{-1}S_1, & 0 \\ W_1+R_1U_1^*(\delta-U_1U_1^*)^{-1}S_1, & R_1(\delta-U_1^*U_1)^{-1}S_2 \end{bmatrix}$$

with $\delta-U_1^*U_1 = S_2S_2^*$, $S_2U_1 = U_1^*S_2^*$. Such a condition introduced into the interconnecting 'network' induces a cancellation of the contribution from the adjacent incident variable in the response or reflected variable. A similar matching-type condition with zero reflection appears in the terminal scattering matrices which can be associated with the linear optimal control problem incorporating a terminal constraint.[35] A scattering matrix can also be diagonalised to include zero reflections using unitary transformation matrices.[29]

The algebraic structure of the star product extends to form a set of functional equations, for arbitrary points (x,y), (y,z), (x,z) in the flow process.[30] The results then generalise and give the matrix differential system[29-31]

$$T_y(x,y) = \begin{bmatrix} S_y & U_y \\ W_y & R_y \end{bmatrix} = \begin{bmatrix} (D+UC)S, & A+DU+UB+UCU \\ RCS & R(B+CU) \end{bmatrix}, \quad T_y \equiv \partial T/\partial y \tag{11}$$

where A,B,C,D are complex nxn matrix functions of the real variable y given by

$$(T_y)_o = \begin{bmatrix} D & A \\ C & B \end{bmatrix} \quad \text{and} \quad T(x,x) = \delta_{2n}$$

since no coupling or reflection occurs for a zero thickness medium. The transformation diagram of FIG 6 illustrates that the vertical admittance- and impedance-type operators U and W, which also possess properties of projection and covariance operators,[39] combine with the connection matrices C_c and A^o and provide a pathway related to the respective closed- and open-paths, from source to response variables. They represent solutions of quadratic Riccati-type equations, and the dimensionless-type operators S, R in the continuum represent solutions for the diagonal incident components of the differential equation system of eqn 11. This would also suggest that Kron's sequence of orthogonal networks would extend with coupling to include transformations which could be associated with a differential system including a matrix Riccati equation.

A linear matrix differential equation also exists in the scattering problem with[31,32]

$$H_y = GH \ , \quad G(y) = \begin{bmatrix} -B & -C \\ A & D \end{bmatrix} \ , \quad H(x,y) = \begin{bmatrix} R^{-1} \ , & -R^{-1}W \\ UR^{-1} \ , & S-UR^{-1}W \end{bmatrix} \ , \quad H(x,x) = \delta_{2n} \ .$$

Also

$$L = H^{-1} = \begin{bmatrix} R-WS^{-1}U, & WS^{-1} \\ -S^{-1}U, & S^{-1} \end{bmatrix}$$

satisfies

$$L(x,y)L(y,z) = L(x,z) \quad \text{with} \quad \begin{bmatrix} V_2 \\ V_1 \end{bmatrix} = L\begin{bmatrix} V_4 \\ V_3 \end{bmatrix}$$

The matrix L transforms the disturbance along the line in terms of input-output variables,[29] whereas the scattering matrix specifies the outputs at each end in terms of the inputs, and the correspondence represents an isomorphism.[30] It is interesting to note that the matrices H,L can be identified with the orthogonal network solution of eqn 4, and the scattering matrix components S,U,W,R then correspond to the impedance and admittance matrix components Z_i and Y_i in eqn 3. Such a correspondence indicates the significant role of the orthogonal network as a basic structure associated with all q-networks in the polyhedron model. The same structure and also the form of the matrix G appear similarly in a variational problem which includes an integral of an hermitian form in the dependent variable and its derivative, given by[33,34]

$$\mathcal{J}(\eta;y_1,y_2) = \int_{y_1}^{y_2} J\,dy \ , \quad J = \begin{bmatrix} \eta' \\ \eta \end{bmatrix}^* \begin{bmatrix} F & H \\ H^* & G \end{bmatrix} \begin{bmatrix} \eta' \\ \eta \end{bmatrix} = \begin{bmatrix} \eta' \\ \eta \end{bmatrix}^* \begin{bmatrix} \zeta \\ \zeta' \end{bmatrix} \qquad (12)$$

where * denotes conjugate transpose and F(y), G(y), H(y) are complex nxn matrix functions of real y, with F(y) non-singular and F(y), G(y) hermitian. The definition of variables ζ,ζ' in eqn 12 represents an L-type operator which can be transformed to a scattering T-form to give the Euler differential equation in the canonical variables η,ζ associated with the variational integral, thus

$$\begin{bmatrix}\eta' \\ \zeta'\end{bmatrix} = \begin{bmatrix}-B & -C \\ A & D\end{bmatrix}\begin{bmatrix}\eta \\ \zeta\end{bmatrix} = G(y)\begin{bmatrix}\eta \\ \zeta\end{bmatrix} \quad , \quad \begin{bmatrix}A & B \\ C & D\end{bmatrix} = \begin{bmatrix}G-H*F^{-1}H, & F^{-1}H \\ -F^{-1}, & H*F^{-1}\end{bmatrix}$$

The matrix differential system of eqn 11 also exists in the optimal tracking problem and in the optimal regulator problem incorporating a terminal constraint.[35] The form of the scattering matrix and star product and the orthogonal network solution, and also the L-form transformation in the Euler differential equation, appear similarly in the corresponding state-adjoint variable relationships. Thus for optimal control of the system represented by

$$\dot{x}(t) = Ax + Bu \;, \quad y = Cx \;, \quad x(t_o) = x_o$$

with performance index

$$J(u) = \tfrac{1}{2}\int_{t_o}^{t_1} (y^TQy + u^TRu)dt + \tfrac{1}{2}y^T(t_1)Fy(t_1)$$

and terminal constraint, $z = Zy(t_1)$, the boundary conditions are defined by the transversality condition[35]

$$p(t_1) = C^TFC\,x(t_1) + C^TZ^T\lambda$$

where λ is a vector multiplier. The state-adjoint variable relationship and the terminal constraint are then given by

$$p(t) = P(t)x(t) + G(t)\lambda \quad , \quad z = G^T(t)x(t) + N(t)\lambda$$

giving

$$\begin{bmatrix}\lambda \\ p(t)\end{bmatrix} = \begin{bmatrix}N^{-1} & -N^{-1}G^T \\ GN^{-1} & P-GN^{-1}G^T\end{bmatrix}\begin{bmatrix}z \\ x(t)\end{bmatrix}, \quad \begin{bmatrix}p(t_1) \\ z\end{bmatrix} = \begin{bmatrix}C^TZ^T & C^TFC \\ 0 & ZC\end{bmatrix}\begin{bmatrix}\lambda \\ x(t_1)\end{bmatrix}$$

By analogy with the scattering formulation, the interconnecting 'network' is represented by a unit matrix with $W_2 = 0$. The relationships can also be illustrated in a time-domain transformation diagram,[36] the structure of which appears as a component of Kron's polyhedron model. The variables $\lambda, p(t)$ can be identified with the closed- and open-path response variables $i^{c'}, E_o'$, and $x(t), z$ with the closed- and open-path source variables $e_c', I^{o'}$ respectively. A form of the optimal linear feedback control problem thus appears within the structure of the orthogonal network and scattering problems, and thus similarly in Kron's polyhedron model.

5. System concepts in the scattering problem

The scattering problem includes a connection of adjacent obstacles which incorporate the effects of a priori information into each successive stage of a multi-stage process, in contrast to the connection of adjacent networks in the polyhedron model based on the incidence matrix M^{i}_{i+1}. The representation also provides a unifying framework for a wide range of physical system problems which can be associated with a least power or minimum quadratic function condition, and include the properties of a return difference operator.

5.1 Least-squares estimation The classical least-squares problem concerned with obtaining the 'best' estimate of a parameter vector associated with a set of overdetermined linear equations is defined in terms of a measurement equation

$$y = Hx + v$$

where v is a residual error or zero-mean random noise vector with covariance matrix R. The minimum-variance estimate for

$$\min \left(J = E\left[\|y-Hx\|^2_{R^{-1}}\right]\right)$$

where $\|v\|^2_{R^{-1}} \equiv v^T R^{-1} v$, is given by

$$\hat{x} = (H^T R^{-1} H)^{-1} H^T R^{-1} y \quad \text{and} \quad \hat{y} = H\hat{x} = MR^{-1} \tag{13}$$

where $M = H(H^T R^{-1} H)^{-1} H^T$. Also

$$y - \hat{y} = (\delta - MR^{-1})y = RLy, \quad J = y^T Ly$$

The error covariance matrix is given by

$$P = E\left[(x-\hat{x})(x-\hat{x})^T\right] = (H^T R^{-1} H)^{-1}$$

which has a correspondence with the matrix Y_4^{-1} in the orthogonal network and W in the scattering problem. With a priori information available for the vector x, represented by $E\left[xx^T\right] = S$, $E\left[vx^T\right] = 0$, the linear estimate for

$$\min \left(J = \|y-Hx\|^2_{R^{-1}} + \|x\|^2_{S^{-1}}\right)$$

is given by

$$\hat{x} = (S^{-1} + H^TR^{-1}H)^{-1}H^TR^{-1}y = PH^TR^{-1}y$$

and $$\hat{y} = H\hat{x} = MR^{-1}y \;,\quad M = H(S^{-1} + H^TR^{-1}H)^{-1}H^T$$

We can also define $$L = R^{-1} - R^{-1}MR^{-1} = (R + HSH^T)^{-1} \;,\quad M = R - RLR$$

and $$P = (S^{-1} + H^TR^{-1}H)^{-1} = S - SH^TLHS$$

The least-squares solution of eqn 13 incorporating the operators M,L corresponds to the solution of the general electrical network problem[20], and can be identified as a vertical transformation existing within each of the component blocks of FIG 6. Thus Kron's vertical propagation across the polyhedron model, without additional source terms, and directed towards the induced variables representing divided differences, effectively introduces a least-squares operation, with the measurement and covariance matrices defined by the properties of the higher-dimensional elements and connection matrices. The physical variables representing divided differences in the polyhedron model will thus inherently introduce regression-type properties within each component of the network sequence.

It is now significant to note that the solution of the least-squares problem incorporating a priori information (S) cannot be identified with the solution of the conventional electrical network problem. A similar property does however appear to be introduced with the connection of adjacent orthogonal networks, as illustrated by the correspondence of the multimachine problem with the wave propagation discussed in Section 3. It also exists inherently within the framework of the scattering problem, and is introduced with the inter-connection of adjacent networks or obstacles at the level of the induced variables in FIG 2. The scattering representation includes vertical operators with covariance properties, as shown in FIG 6, together with a return-difference contribution from the adjacent stage which introduces the effects of a priori information. The scattering representation would appear to provide a more basic structure than Kron's polyhedron model with regard to its ability to incorporate previous-stage covariance-type information in terms of a feedback-type operator.

The correspondence of the a priori least-squares problem with the scattering formulation is illustrated in FIG 6, and exists with the analogy between the estimated measurement $\hat{y}$ and error $y-\hat{y}$ and the components of the response term V_2 in eqn 10, given by

$$\hat{y} = H(\delta + SH^T R^{-1} H)^{-1} (SH^T)(R^{-1} y) \equiv R_1(\delta - W_2 U_1)^{-1} R_2 V_6$$

$$y - \hat{y} = [R - HS(\delta + H^T R^{-1} HS)^{-1} H^T] (R^{-1} y) \equiv [W_1 + R_1 W_2 (\delta - U_1 W_2)^{-1} S_1] V_1$$

The elements of the least-squares solution can then be represented in a scattering form with

$$\begin{bmatrix} S_1 & U_1 \\ W_1 & R_1 \end{bmatrix} = \begin{bmatrix} H^T & H^T R^{-1} H \\ R & H \end{bmatrix} = \begin{bmatrix} \delta & R^{-1} \\ R & \delta \end{bmatrix} * \begin{bmatrix} H^T & O \\ O & H \end{bmatrix}, \quad \begin{bmatrix} S_2 & U_2 \\ W_2 & R_2 \end{bmatrix} = \begin{bmatrix} - & - \\ -S & SH^T \end{bmatrix} \tag{14}$$

Similar reflection relationships can be derived with $\hat{y}$ identified with the response V_5. The star product of the scattering problem thus updates a least-squares solution by incorporating a priori or previous-stage information.

5.2 Discrete optimal control Optimal control of the linear discrete dynamic system

$$x_k = \Phi x_{k-1} + \Delta u_{k-1}$$

for minimising the N-stage index

$$J_N = \sum_{i=1}^{N} (x_i^T Q x_i + u^T_{i-1} R u_{i-1})$$

by dynamic programming is given by the standard recursive relationships

$$u_{N-r} = K_r x_{N-r}, \quad K_r = -(\Delta^T P_{r-1} \Delta + R)^{-1} \Delta^T P_{r-1} \Phi, \quad r = 1,2..N$$

$$P_{r-1} = \Phi^T (P^{-1}_{r-2} + \Delta R^{-1} \Delta^T)^{-1} \Phi + Q, \quad P_o = Q$$

The covariance matrix P_{r-1} can now be identified with the component of the star product associated with the transformation $V_1 \rightarrow V_2$ in eqn 10. Thus

$$P_{r-1} = Q + \Phi^T P_{r-2} (\delta + \Delta R^{-1} \Delta^T P_{r-2})^{-1} \Phi \equiv W_1 + R_1 W_2 (\delta - U_1 W_2)^{-1} S_1$$

We can then define the scattering-type matrices

$$\begin{bmatrix} S_1 & U_1 \\ W_1 & R_1 \end{bmatrix} = \begin{bmatrix} \Phi & -\Delta R^{-1} \Delta^T \\ Q & \Phi^T \end{bmatrix} = \begin{bmatrix} \delta & O \\ Q & \delta \end{bmatrix} * \begin{bmatrix} \delta & -\Delta R^{-1} \Delta^T \\ O & \Phi^T \end{bmatrix}, \quad \begin{bmatrix} S_2 & U_2 \\ W_2 & R_2 \end{bmatrix} = \begin{bmatrix} - & - \\ P_{r-2} & - \end{bmatrix}$$

The covariance matrix P_{r-1} includes a current-stage component (Q) and a contribution from the previous (right) stage (P_{r-2}) which is updated by the effects of a return-difference-type operator. A similar correspondence exists in terms of the previous (left) stage components. The form of the control law also fits within the scattering representation and can be associated with a propagation, say from right to left ($V_6 \to V_2$), with

$$\Delta K_r = -\Delta R^{-1}\Delta^T(\delta + P_{r-1}\Delta R^{-1}\Delta^T)^{-1}P_{r-1}\Phi \equiv R_1(\delta - W_2U_1)^{-1}R_2$$

The dynamic programming algorithm thus operates according to the properties of the star product in the scattering process, which essentially effects a summation of the effects of a priori information by updating the previous-stage covariance operators.

5.3 Multimachine power system The solution of the multimachine power system problem in Section 3.1 corresponds directly with the least-squares solution incorporating a priori information. It also includes properties of a return-difference matrix, with the matrix L defining the effect of implicit feedback introduced by the machine-network interconnection and the machine voltage-current relationship. A correspondence then exists with the scattering problem, with machine terminal voltage v and voltage difference e-v analogous with the estimated measurement $\hat{y}$ and error $y-\hat{y}$ respectively in the least-squares solution. Thus

$$v = MY_Me \equiv \hat{y} \quad \text{and} \quad e-v = (Z_M - M)(Y_Me) \equiv y-\hat{y}$$

and from eqn 14 we can define

$$T_1 = \begin{bmatrix}\delta & Y_M\\ Z_M & \delta\end{bmatrix} * \begin{bmatrix}A^T(\Theta) & 0\\ 0 & A(\Theta)\end{bmatrix}, \quad T_2 = \begin{bmatrix}- & -\\ -Z_N & Z_NA^T(\Theta)\end{bmatrix}$$

The machine-network system may thus be considered as a 'flow' process with the properties of the star product producing an interconnection of the machine and network parameters by means of the connection matrix $A(\Theta)$.

5.4 Multivariable control problem For the general multivariable control system including a plant transfer function matrix G(s), forward controller C(s) and feedback controller K(s), the closed-loop response of output y(s) to a reference input r(s) is given by

$$y(s) = (\delta + GCK)^{-1}GCr(s)$$

By analogy with the least-squares problem, transformation operators may now be defined by

$$M(s) = K(\delta + GCK)^{-1}G\,, \qquad L(s) = C - CMC = CF^{-1}$$

where $\quad F(s) = \delta + KGC\,, \qquad F^{-1}(s) = \delta - MC$

represents a dimensionless return-difference matrix. A correspondence with the components of the star product can then be identified by analogy with the least-squares solution, with

$$Ky(s) = MCr(s) \equiv \hat{y} \quad \text{and} \quad r - Ky(s) = (C^{-1} - M)Cr(s) \equiv y - \hat{y}$$

provided C^{-1} exists. A scattering structure can then be defined for the multivariable feedback control problem with

$$T_1 = \begin{bmatrix} G & GCK \\ C^{-1} & K \end{bmatrix} = \begin{bmatrix} \delta & C \\ C^{-1} & \delta \end{bmatrix} * \begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix}\,, \quad T_2 = \begin{bmatrix} - & - \\ -\delta & G \end{bmatrix} \tag{15}$$

The scattering formulation could now possibly be used to investigate conditions of matching specified by an overall zero reflection produced by a control design say for the forward controller $C(s)$.

It is also of interest to note that the transfer function matrix for the linear state variable system can be identified with the components of the star product. The Laplace transformed output for the system

$$\dot{x}(t) = Ax(t) + Bu(t) + \Gamma w(t), \; y(t) = Cx(t) + Du(t)$$

where $w(t)$ represents a disturbance input, is given by

$$y(s) = \{C(s\delta - A)^{-1}B + D\}u(s) + C(s\delta - A)^{-1}\{\Gamma w(s) + x(o)\}$$

A scattering formulation can then be obtained, say in terms of the transformation $V_1, V_6 \rightarrow V_2$ in eqn 10, with

$$T_1 = \begin{bmatrix} B & A \\ D & C \end{bmatrix}\,, \qquad T_2 = \begin{bmatrix} - & - \\ \delta s^{-1} & \delta s^{-1} \end{bmatrix}$$

The transformation diagram is illustrated in FIG 7.

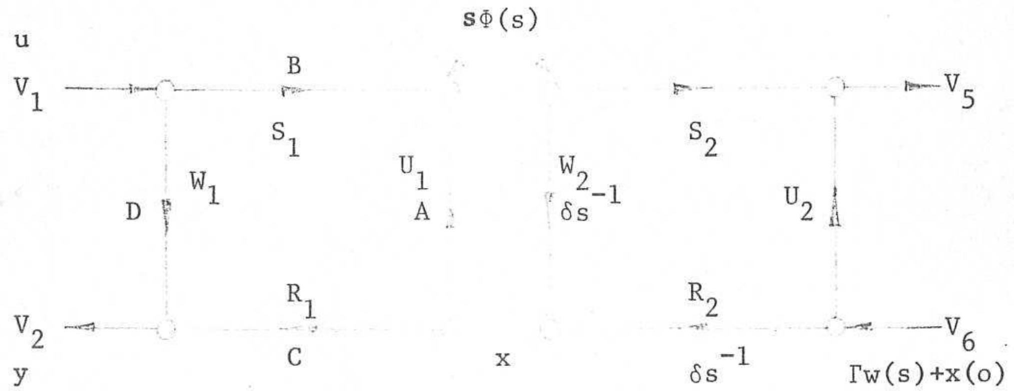

FIG 7 Transfer function matrix as a scattering process

The component δs^{-1} appears as a variance-type operator introducing a priori information into the following stage or obstacle represented by the system matrices $\{A,B,C,D\}$. The interconnecting structure is defined by $s\Phi(s)$ where resolvent

$$\Phi(s) \quad = \quad (s\delta - A)^{-1} \quad = \quad \psi(s)/\Delta_o(s)$$

$\psi(s)$ is the adjoint matrix of $s\delta$-A and $\Delta_o(s) = |s\delta - A|$ is the open-loop characteristic function. The poles of the open-loop transfer function matrix are the zeros of $\Delta_o(s)$ and are thus associated with the interconnecting component of the star product. The form of the interconnecting structure will thus determine the stability properties of the coupled system.

5.5 <u>The stationary Kalman-Bucy filter</u> The transfer function representation of the stationary Kalman-Bucy filter problem includes a feedback structure which exists similarly in the multimachine system problem, and also in the orthogonal network and scattering problems. In the filter problem, a random process x(t) is generated by the system

$$\dot{x}(t) \quad = \quad Ax(t) + Bu(t)$$

and includes an observation

$$z(t) \quad = \quad Hx(t) + v(t)$$

where u(t), v(t) represent zero-mean random noise vectors with covariance matrices $Q\,\delta(t-\tau)$ and $R\delta(t-\tau)$ respectively. The minimum-variance estimate of the state $\hat{x}(t)$ is generated by the Kalman-Bucy filter equations

$$\dot{\hat{x}}(t) \quad = \quad A\hat{x}(t) + K(t)\left[z(t) - H\hat{x}(t)\right] \quad , \quad K(t) = P(t)H^T R^{-1}$$

where the covariance matrix $P(t) = \left[x(t) - \hat{x}(t)\right]\left[x(t) - \hat{x}(t)\right]^T$ is given by a solution of the matrix Riccati differential equation

$$\dot{P} \quad = \quad AP + PA^T - PH^T R^{-1} HP + BQB^T$$

For the stationary filter problem, the Laplace transformed solution is represented by

$$\hat{x}(s) \quad = \quad \left[\delta + \Phi(s)PH^T R^{-1} H\right]^{-1} \Phi(s)PH^T (R^{-1} z(s))$$

$$\hat{y}(s) \quad = \quad H\hat{x}(s) \quad = \quad MR^{-1} z(s)$$

where $\quad M = H\left[\delta + \Phi(s)PH^T R^{-1} H\right]^{-1} \Phi(s)PH^T = H\left[P^{-1}\Phi^{-1}(s) + H^T R^{-1} H\right]^{-1} H^T$

and $\quad L = R^{-1} - R^{-1} MR^{-1} = (R + H\Phi PH^T)^{-1}$

The machine system relationship i = Le can then be identified in the filter problem, with

$$R^{-1} e(s) \quad = \quad Lz(s) \quad \text{or} \quad e(s) \quad = \quad RLz(s) \quad = \quad F^{-1}(s)z(s)$$

where e(s) represents the 'tracking-error' and F(s) is the return-difference matrix given by

$$F^{-1}(s) \;=\; \delta - MR^{-1} \quad \text{or} \quad F(s) \;=\; \delta + H\Phi(s)K$$

The solution of the stationary filter problem can also be associated with the scattering representation and may be considered as a flow process from observation to state estimate. With the covariance operator ΦP representing a priori information in the 'previous' stage, the estimate $\hat{y}$ and $z-\hat{y}$ can be identified with the transformations $V_6 \rightarrow V_2$ and $V_1 \rightarrow V_2$ in the scattering problem giving, similar to the form of eqn 14

$$T_1 \;=\; \begin{bmatrix} H^T & H^TR^{-1}H \\ R & H \end{bmatrix} \;=\; \begin{bmatrix} \delta & R^{-1} \\ R & \delta \end{bmatrix} * \begin{bmatrix} H^T & 0 \\ 0 & H \end{bmatrix}, \quad T_2 \;=\; \begin{bmatrix} - & - \\ -\Phi P & \Phi PH^T \end{bmatrix} \;=\; \begin{bmatrix} \delta & (P^{-1}-\delta) \\ 0 & \Phi \end{bmatrix} * \begin{bmatrix} - & - \\ -\delta & H^T \end{bmatrix}$$

The scattering representation now incorporates a priori information from the second-stage 'obstacle', represented by the transformed error covariance matrix ΦP. This corresponds with the unit-matrix a priori information introduced into the multivariable control problem in eqn 15. The condition of optimality for the filter problem in the frequency domain is represented by the spectral density for the observation, given by[40]

$$F(s)RF^T(-s) \;=\; R + G(s)QG^T(-s) \;, \qquad G(s) \;=\; C\Phi(s)B$$

It is also significant that this quadratic condition of optimality may now be considered as a component of a star product given by

$$\begin{bmatrix} - \;, & R+G(s)QG^T(-s) \\ - & - \end{bmatrix} \;=\; \begin{bmatrix} - & \delta \\ - & - \end{bmatrix} * \begin{bmatrix} G(s) & R \\ (\delta-Q^{-1}) & G^T(-s) \end{bmatrix} \;=\; \begin{bmatrix} - & \delta \\ - & - \end{bmatrix} * \begin{bmatrix} \delta & 0 \\ \delta & \delta \end{bmatrix} * \begin{bmatrix} G(s) & R \\ -Q^{-1} & G^T(-s) \end{bmatrix}$$

The structure of the scattering problem thus incorporates a condition of optimality in the frequency domain. The optimal filtering and control problems introduce a constraint of minimum quadratic performance which exists inherently in the electrical network problem as a minimum power condition, and such a condition also appears in the formulation of the scattering problem.

The correspondence of the multimachine system with other system problems concerned with least-squares estimation, linear control and filtering exists within the unifying framework of the scattering problem. The correspondence is also illustrated in the general feedback arrangement of FIG 8. The dynamic elements in the control problem are seen to be included within the structure of the system compared to the power system problem in which the machine units appear external to the structure of the interconnected network and includes a diagonal machine admittance matrix Y_M compared to the more general forward controller matrix C(s).

The scattering formulation, and similarly Kron's network sequence, includes a connection process associated with the properties of the reflection operators U,W appearing between the adjacent units of a connected system. This forms a basis for decomposition, which can be applied to the above system problems, and the subunits will appear as components of a scattering matrix in a star product representing the overall system. It is believed that the scattering formulation and Kron's polyhedron model will thus play a significant role in the decomposition of general large scale dynamic system problems which presently lack an adequate theoretical basis.

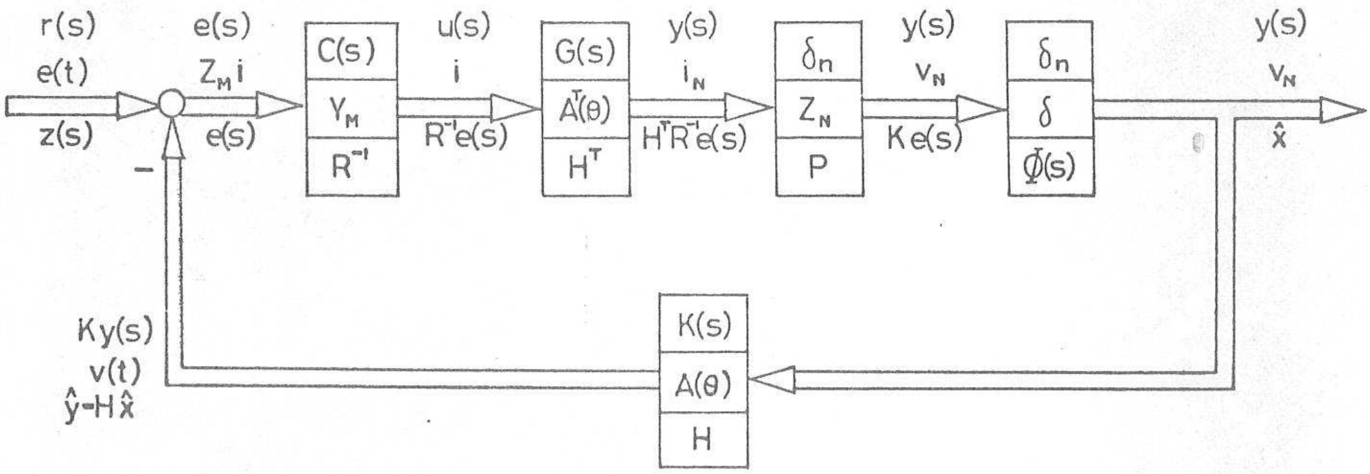

FIG 8 Basic feedback structure representing the multivariable control system, multimachine system and the stationary K-B filter

6. Conclusions

Certain aspects of Kron's pioneering work concerned with the polyhedron model which have important applications in many system problems have been highlighted. A unifying algebraic framework is provided by the orthogonal electrical network which forms the basis of the polyhedron model and also by the general scattering representation of a flow process. The scattering problem and the properties of the star product would now appear to have a direct application for interconnecting adjacent networks incorporating electrical and electromagnetic variables in the polyhedron model. Such concepts and relationships originating in electrical network theory and in the general theory of flow processes are thus of fundamental importance and can provide greater physical insight and understanding of Kron's polyhedron model. The application of the model in nonlinear estimation, based on the concept of introducing the structural properties of a physical system into an abstract data fitting problem, appears to be of considerable significance. However, Kron's published work is difficult to comprehend and appears to lack an adequate analytical basis which is required for extending the application of the work, although all the concepts appear to be particularly relevant and valid.

The formulation of the system problems concerned with linear optimal control, filtering, least-squares estimation and the multimachine power system requires a framework which incorporates the effects of a priori information. This exists inherently within the scattering representation of a flow process including a sequence of obstacles, and the resulting star product introduces an implicit feedback or return-difference effect associated with the interaction between obstacles. A similar type of structure exists in Kron's polyhedron model consisting of a sequence of orthogonal networks which are connected using the properties of incidence matrices associated with a physical topological structure. A correspondence then exists with the stationary Kalman-Bucy filter problem which would appear to support the proposed application of the polyhedron in the statistical estimation problem. Both representations incorporate basic features which are fundamental in system theory, and provide a unifying framework of dimensionless- and impedance/admittance- or variance-type operators required to obtain a stage-wise solution procedure which evolves by updating a priori information. It is evident that such evolution is inherent in the concept of dynamic programming and also in the theory of statistical estimation. Such a procedure will also be relevant to the flow of material, energy and information, and no doubt Kron appreciated the significance of this phenomena in proposing the application of the polyhedron for brain modelling. The scattering formulation now also appears to provide a suitable analytical framework for the decomposition of large scale dynamic systems which may include nonlinear elements. It incorporates a functional equation representation for a connected system related to the behaviour of the subunits, which also appears as an underlying concept in all of Kron's work.

The theory of scattering may also be applied to the interaction of waves of different frequency and also to the interaction between waves of the same frequency associated with different forms of energy, such as electromagnetic and pressure waves in a plasma.[37] The n-mode process can be formulated similar to the 1-mode problem and the overall system can be represented by similar sets of functional equations. The multimode problem may then suggest a structure for extending the polyhedron model to accommodate many other system concepts with additional physical variables, such as mechanical, elastic, hydrodynamical, chemical, thermodynamical and transport phenomena propagating within a transmission-type system as envisaged by Kron.[7,12,38] The scattering representation and the polyhedron model thus possess certain similar basic characteristics which provide a unifying structure which is of considerable significance and importance in general system theory.

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9. Appendix

In the electrical network of FIG 9, five branch currents can be related to the two closed- and three open-path currents by

$$J = \begin{bmatrix} C_c & C_o \end{bmatrix} \begin{bmatrix} i^{c'} \\ I^{o'} \end{bmatrix}, \qquad \begin{bmatrix} J^1 \\ J^2 \\ J^3 \\ J^4 \\ J^5 \end{bmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 4' & 5' & A' & B' & C' \end{array} \\ \begin{bmatrix} -1 & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & 1 \\ \cdot & -1 & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix} \end{array} \begin{bmatrix} i^{4'} \\ i^{5'} \\ I^{A'} \\ I^{B'} \\ I^{C'} \end{bmatrix}$$

The 'coil' and closed- and open-path voltages are also given by

$$V = \begin{bmatrix} A^c A^o \end{bmatrix} \begin{bmatrix} e_c{}' \\ E_o{}' \end{bmatrix}, \qquad \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 4' & 5' & A' & B' & C' \end{array} \\ \begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 \\ 1 & \cdot & 1 & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot & -1 \end{bmatrix} \end{array} \begin{bmatrix} e_{4'} \\ e_{5'} \\ E_{A'} \\ E_{B'} \\ E_{C'} \end{bmatrix}$$

The connection matrix $C_c^{(0)}$ relating the 0-cells and the node pairs, may be defined by selecting three independent node pairs, say D-A, D-B, D-C.[16] Then

$$C_c^{(0)} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccc} D\text{-}A & D\text{-}B & D\text{-}C \end{array} \\ \begin{bmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

The incidence matrix relating the 0- and 1-cells, with columns indicating that each branch extends between two nodes, is then given by

$$M_1^{\ 0} = C_c^{(0)} (A^{\ 0}_{(1)})^T = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccc} D\text{-}A & D\text{-}B & D\text{-}C \end{array} \\ \begin{bmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ A' \\ B' \\ C' \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & 1 & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 \end{bmatrix} \end{array} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} -1 & \cdot & \cdot & -1 & \cdot \\ \cdot & -1 & -1 & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & -1 \end{bmatrix} \end{array}$$

For the connection matrix $A^{o}_{(2)}$, with meshes 4',5' bounding areas 4,5 with clockwise orientation,

$$A^{o}_{(2)} = \begin{array}{c} \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} 4' & 5' \end{array} \\ \begin{bmatrix} 1 & \cdot \\ \cdot & -1 \end{bmatrix} \end{array}$$

Then the incidence matrix relating lines to planes, with columns indicating that each plane extends between three branches, is given by

$$M_2^{\ 1} = C_c^{(1)} (A^{o}_{(2)})^T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} 4' & 5' \end{array} \\ \begin{bmatrix} -1 & \cdot \\ 1 & 1 \\ \cdot & -1 \\ 1 & \cdot \\ \cdot & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 4' \\ 5' \end{array} \begin{array}{c} \begin{array}{cc} 4 & 5 \end{array} \\ \begin{bmatrix} 1 & \cdot \\ \cdot & -1 \end{bmatrix} \end{array} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} 4 & 5 \end{array} \\ \begin{bmatrix} -1 & \cdot \\ 1 & -1 \\ \cdot & 1 \\ 1 & \cdot \\ \cdot & -1 \end{bmatrix} \end{array}$$

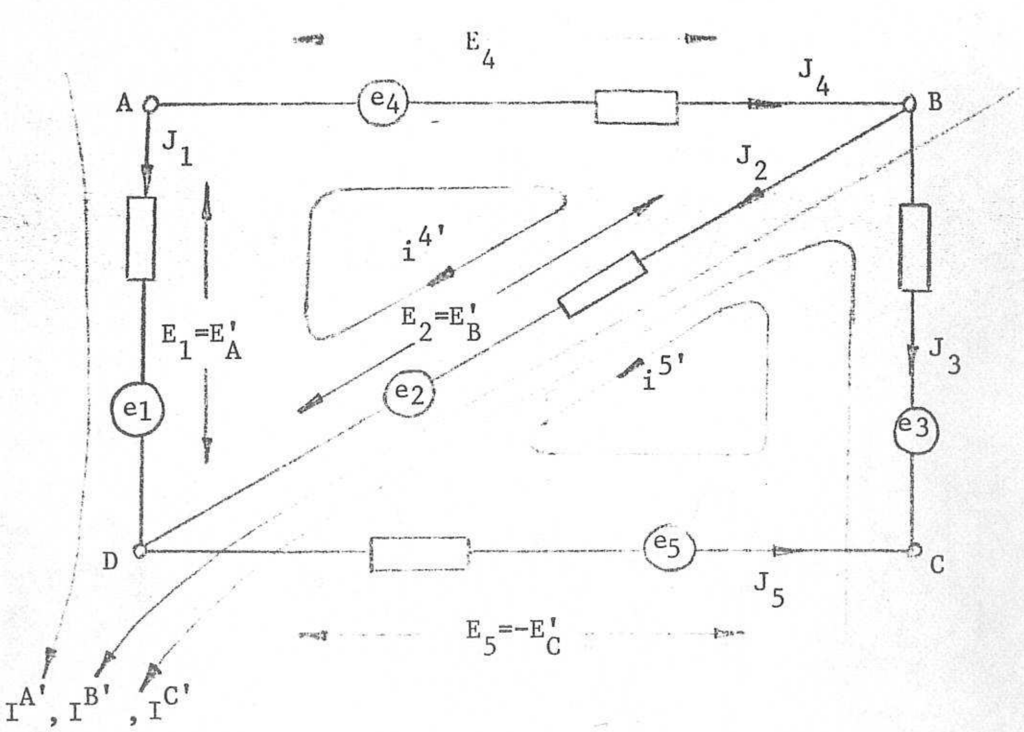

FIG 9 Closed- and open-paths

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