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The Design of Optimum Regulator Controls for Multivariable, Class "O" Processes by Independent Optimisation of Steady State and Dynamic Performance

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1. Introduction

An important class of linear/quadratic optimal control problems, the solution of which can reduce plant operating costs enormously, is that concerned with the regulation of Class "O"* processes. Such processes are very common in the chemical, and process industries generally. Within wide limits, such a process will often run smoothly, no matter what values of process input or type of control strategy are employed. The economic performance of the plant can be very far from satisfactory however if it is inappropriately controlled. This is because the integral cost function for such a process usually involves penalties on the variance of certain states, (e.g. product compositions), from pre-specified constant targets, and also on the services, (inputs), consumed by the plant in approaching these targets. Because of the sustained demand for services in a steady-state situation, and because of sustained deviations from the target states, this cost does not converge to a constant value, (as usually happens with integrating type processes), but continues to integrate with time after the process itself has reached a steady state. Any deviation from the optimum cost is therefore very serious economically.

The term "regulator" is here employed to describe the control strategy appropriate to this problem rather than using the term "reference tracking system" since the reference signals in this particular exercise are constant, although non-zero. Many texts[1] restrict the term "regulator control" to the case of zero references. In the case of integrating processes, (i.e. processes of class greater than zero), the constant reference tracking problem can be reduced to the conventional regulator problem[2] by the use of error co-ordinates taken about the reference signals. This is a very convenient procedure but relies on the process integrators to ensure that steady-state and target values of the state variables approach equality as process time, T_f, approaches infinity. With class "O" systems this condition is not satisfied and the steady-state values

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*A Class "O" process has no pure integration terms in any of the elements of its transfer-function matrix.

of the states are therefore unknown at the problem outset. Furthermore, they depend, not only on the target values, but also on the optimal control strategy yet to be designed.

The problem can, of course, be solved by augmenting the process state-vector[3], x_1, by a vector, x_2, of reference quantities to yield an overall state-vector, x, and then applying the conventional methods of optimal control design, e.g. the matrix Riccati approach[3], to the enlarged system. Some problems arise in the case of class-0 processes however in that, as T_f tends to infinity, the optimised cost coefficient matrix*, P, does not converge to a steady solution because the optimised process cost, $x^T Px$, cannot settle at a constant value, (except in the trivial case of input vector $u = [0]$), for the reasons aforementioned. Thus, as state vector x reaches its steady state, the matrix P must continue to integrate, at least in some of its elements. More importantly, if the basic system has order n and is then augmented by m-constant reference, (and/or constant disturbance), states, then the number of equations to be solved, at each step of the integration of the matrix Riccati equation, becomes $(n+m)(n+m+1)/2.0$. In many problems, m may well approach n in magnitude, (particularly in the case of a multi-output system having relatively simple transfer-function matrix elements), and if n is say, 18, then solution times well exceeding 2000 seconds[4] can be expected, if a wide range of process eigenvalues exist.

A method of optimisation based on error co-ordinates is therefore highly desirable since a reduction of up to 75% in the problem solution is made

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* It can be shown[3] that the optimised cost of a linear process $\dot{x} = Ax + Bu$ with an integral quadratic performance index, $= \int_0^{T_f} (x^T Qx + u^T Ru)dt$, is given by $x^T Px$, where x is the state vector and P is a time varying cost coefficient matrix found by solving the matrix Riccati equation

$$\dot{P} = PBR^{-1}B^T P - Q - A'P - PA$$

and the optimal control vector is given by the equation -

$$u = -R^{-1}B^T P \quad .$$

possible by the consequential reduction in the number of co-state equations to $n(n+1)/2.0$.

The method to be developed is attractive as it reduces the exercise to two independent stages; firstly the steady-state process is optimised with respect to the steady-state process inputs to yield the optimal values of these signals and also of the steady process states, in terms of the known reference and disturbance signals, and, secondly, the dynamic deviations about these steady states are now formed, the process written in terms of these deviations and the appropriate matrix Riccati equation, (of order $n(n+1)/2.0$), solved to yield a control law in terms of the state deviations.

The method is therefore intuitively acceptable and straightforward although it is in fact necessary to verify that the two optimisation exercises do yield a control strategy which is optimum overall. This is shown to be true in the development of the method which follows in section 2 of the report. A formal statement of the method is given in section 3 and in sections 4 and 5 the method is applied to illustrative problems.

2. Development of the Design Method

Processes of the type considered in this report are described by the matrix differential equation

$$\dot{x}_1 = A\, x_1 + B\, u \qquad \ldots(1)$$

where x_1 is an nxl state vector
u is an rxl input vector
A is an nxn plant matrix and is constant and non-singular
B is an nxr constant driving matrix.

The non-singularity condition on matrix A precludes the existence of any pure integrations in the system transfer-function matrix, and also demands the exclusion from state vector, x_1, of any constant reference or disturbance signals. These are accounted for later. The process need not necessarily be completely controllable in that x_1 may contain non-constant external disturbances although, as will be demonstrated, such signals must be stable, (i.e. tend to zero as T_f increases towards infinity).

It is required to control the process, by manipulation of the inputs, u, in such a manner as to minimise

$$\lim_{T_f \to \infty} J(T_f) \quad , \quad \text{where } J(T_f) = \int_0^{T_f} (x^T Q x + u^T R u) dt \qquad \ldots(2)$$

where $$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \ldots(3)$$

and $$Q = \begin{bmatrix} Q_1 , & Q_3^T \\ Q_3 , & Q_2 \end{bmatrix} \qquad \ldots(4)$$

Vector x_2 is a mxl column matrix of constant reference signals and constant process disturbances and therefore described by the matrix differential equation

$$\dot{x}_2 = [0] \qquad \ldots(5)$$

and the cost coefficient matrices Q_1, Q_2, Q_3 and R are constant and Q_1, Q_2 and R are also symmetric. The matrices have the following dimensions

Q_1 ..... nxn
Q_2 ..... mxm
Q_3 ..... mxn
R ..... rxr

If x_{1s} and u_s are the steady-states to which state and input vectors x_1 and u respectively converge as $T_f \to \infty$, then dynamic deviations from these steady states may be defined as follows

$$y = x_1 - x_{1s} \qquad \ldots(6)$$
$$z = u - u_s \qquad \ldots(7)$$

Thus if J^* denotes the minimised performance integral given in equation 2 then, from equations 2, 3 and 4,

$$J^* = \min_{u(t)} \left\{ \int_0^{T_f} (x_1^T Q_1 x_1 + 2x_1^T Q_3^T x_2 + x_2^T Q_2 x_2 + u^T R u) dt \right\}$$

and therefore, writing x_1 and u in terms of y, x_{1s}, z and u_s, using equations 6 and 7, this cost becomes

$$J^* = \min_{z(t), u_s} \left\{ \int_0^{T_f} (y^T Q_1 y + 2x_{1s}^T Q_1 y + x_{1s}^T Q_1 x_{1s} \right.$$

$$+ 2y^T Q_3^T x_2 + 2x_{1s}^T Q_3^T x_2 + x_2^T Q_2 x_2 + z^T Rz + 2z^T Ru_s + u_s^T Ru_s)dt\}$$

Noting that certain of the above integrand terms are dependent on only z, some only on u_s, some on both and some on neither, it is possible to rewrite the expression for J^* in the form

$$J^* = \min_{z(t)} \{\int_0^{T_f} (y^T Q_1 y + z^T Rz)dt + \left[\min_{u_s} \int_0^{T_f} (x_{1s}^T Q_1 x_{1s} + 2x_{1s}^T Q_3^T x_2 \right.$$

$$\left. + u_s^T Ru_s + 2y^T Q_3^T x_2 + 2x_{1s}^T Q_1 y + z^T Ru_s)dt \right]\} + \int_0^{T_f} x_2^T Q_2 x_2 dt \qquad ...(8)$$

It is therefore first necessary to minimise within the square brackets with respect to steady state control vector u_s before proceeding to the overall minimisation of J with respect to the control deviation vector z. If J_s^* denotes the contents of the square brackets in equation 8 then

$$J_s^* = \min_{u_s} \{ \int_0^{T_f} (x_{1s}^T Q_1 x_{1s} + 2x_{1s}^T Q_3^T x_2 + u_s^T Ru_s)dt + 2 \int_0^{T_f} (y^T Q_3^T x_2 + x_{1s}^T Q_1 y + z^T Ru_s)dt\} \qquad ...(9)$$

and, since x_{1s} and u_s are constants in time, then

$$J_s^* = \min_{u_s} \{(x_{1s}^T Q_1 x_{1s} + 2x_{1s}^T Q_3^T x_2 + u_s^T Ru_s)T_f + 2 \int_0^{T_f} (y^T Q_3^T x_2 + x_{1s}^T Q_1 y + z^T Ru_s)dt\} \qquad ...(10)$$

Now minimising J_s with respect to u_s must be performed subject to the steady-state process equation obtained by putting $\dot{x}_1$ to zero in equation (1), thus giving

$$x_{1s} = - [A^{-1}B] \; u_s \qquad ...(11)$$

and hence, differentiating (10) partially with respect to each of the r controls, equating the resulting expressions to zero and grouping these r equations into one matrix equation gives

$$(-2[A^{-1}B]^T Q_1 x_{1s} - 2[A^{-1}B]^T Q_3^T x_2 + 2Ru_s)T_f$$

$$+ 2 \int_0^{T_f} (-[A^{-1}B]^T Q_1 y + Rz)dt = 0 \qquad ...(12)$$

Now this report is concerned with long-term process optimisation, i.e. the case of $T_f \rightarrow \infty$, and as T_f becomes larger and larger so the first term of the L.H.S, of equation 12 dominates more and more the second, integral term since y and z, by definition, both tend to zero as T_f increases toward infinity.

Hence the optimised steady state control is given by

$$[A^{-1}B]^T Q_1 x_{1s} + [A^{-1}B]^T Q_3^T x_2 = Ru_s \qquad \ldots(13)$$

and from (11) and (13), eliminating x_{1s}, we get

$$[R + [A^{-1}B]^T Q_1 [A^{-1}B]]u_s = [A^{-1}B]^T Q_3^T x_2 \qquad \ldots(14)$$

The optimum u_s can therefore be calculated from equation 14 from a knowledge of the reference signals, x_2, and the problem parameters. The associated steady-state vector, x_{1s}, may then be calculated using equation 11.

It is important to note that equation (14) would have resulted from merely optimising the steady state cost-rate, $= x_{1s}^T Q\, x_{1s} + u_s^T R u_s$, subject to the steady-state process equation (11), due to the insignificance of the deviation contribution to J_s, as $T_f \to \infty$.

Equations 13 and 14 are however not feedback control laws since they do not completely specify the feedback coefficients. They merely specify certain relations between x_{1s}, u_s and x_2 which will hold once the controller reaches a steady state. For an optimum control law, even for the case of $T_f \to \infty$, we have still to optimise J by choice of z(t); (refer back to equation 8).

Substituting back in equation (8) our optimised solutions for u_s and x_s and noting that some of the resulting terms are independent of z(t) we are left with the task of minimising a dynamic cost J_g, where the optimised value, J_g^*, is given by

$$J_g^* = \min_{z(t)} \left\{ \int_0^{T_f} (y^T Q_1 y + z^T R z)dt + \int_0^{T_f} 2(y^T Q_3^T x_2 + x_{1s}^T Q_1 y + z^T R u_s)dt \right\} \qquad \ldots(15)$$

Focusing attention on the second integral, (here termed I_2), of the R.H.S. of equation 15 we have that

$$I_2 = \int_0^{T_f} 2[y^T(Q_1 x_{1s} + Q_3^T x_2) + z^T R u_s]dt$$

and using equation 13, which relates the optimum x_{1s} and u_s yields the result

$$I_2 = 2\int_0^{T_f} (y^T [[A^{-1}B]^T]^{-1} Ru_s + z^T R u_s)dt \qquad \ldots(16)$$

Now from equations 1, 6 and 7 we have

$$\dot{y} = Ay + Bz + Ax_{1s} + Bu_s$$

and since $Ax_{1s} + Bu_s = 0$, (steady state process equation),

$$\dot{y} = Ay + Bz \qquad \ldots(17)$$

$$\therefore \quad y = A^{-1}\dot{y} - A^{-1}Bz$$

and hence

$$y^T = \dot{y}^T[A^{-1}]^T - z^T[A^{-1}B]^T \qquad \ldots(18)$$

Thus, eliminating y^T from equation 16, we get

$$I_2 = 2\int_o^{T_f} \dot{y}^T[A^{-1}]^T[[A^{-1}B]^T]^{-1}\, Ru_s\, dt$$

$$\therefore \quad I_2 = -2(y(0)^T[A^{-1}]^T[[A^{-1}B]^T]^{-1}\, Ru_s) \qquad \ldots(19)$$

$$\text{as } T_f \to \infty$$

I_2 is thus a function of variables u_s and $y(0)$ which are both independent of $z(t)$ and therefore, to minimise J_g with respect to $z(t)$, from inspection of equation (15), it is merely necessary to minimise J_d, where J_d^* is given by

$$J_d^* = \underset{T_f \to \infty}{\mathrm{Lim}}\ \min_{z(t)} \int_o^{T_f} (y^TQ_1y + z^TRz)dt \qquad \ldots(20)$$

, y and z being related by dynamic equation 17, i.e.,

$$\dot{y} = Ay + Bz \qquad \ldots(17)$$

3. <u>Summarised Design Procedure</u>

This may be briefly specified as follows:-

Given an n'th order, r-input, time-invariant, class "O" process

$$\dot{x}_1 = Ax_1 + Bu \qquad \text{(where A is non-singular)}$$

to be controlled so as to minimise

$$J = \underset{T_f \to \infty}{\mathrm{Lim}} \int_o^{T_f} \left([x_1^T, x_2^T] \begin{bmatrix} Q_1, & Q_3^T \\ Q_3, & Q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u^TRu\right)dt$$

where Q_1 has dimensions nxn, Q_2 - mxm, Q_3 - mxn and R - rxr, and x_2 is a column vector of m-constant reference signals and disturbances,

$$\text{define} \quad y = x_1 - x_{1s} \quad \text{and} \quad z = u - u_s$$

where x_{1s} and u_s are the vectors x_1 and u at steady state. The optimum control strategy may be found by the following procedure.

Firstly, calculate the optimum u_s and x_{1s} in terms of x_2 by minimising the integrand of J with respect to u_s, subject to the steady-state process equation

$$x_{1s} = -A^{-1}Bu_s \quad .$$

This procedure, in fact, yields the formula

$$[R + [A^{-1}B]^T Q_1 [A^{-1}B]]u_s = [A^{-1}B]^T Q_3^{\,T} x_2$$

Secondly, write the dynamic equation of the process in the form

$$\dot{y} = Ay + Bz$$

and minimise the integral

$$J_d = \lim_{T_f \to \infty} \int_0^{T_f} (y^T Q_1 y + z^T R z)dt$$

by appropriate choice of z(t), using the algebraic matrix Riccati equation

$$PBR^{-1}B^TP = Q_1 + A^TP + PA$$

, and hence find the optimum control law, in terms of state and input deviations, in the form

$$z = Ky$$

, where $K = -R^{-1}B^TP$

Hence find the overall optimum control law, in terms of states, inputs, and the constant references and disturbances, in the form

$$u = u_s + K(x_1 - x_{1s})$$

, (u_s and x_{1s} having been found in the first part of the exercise in terms of x_2).

The technique is now applied to two simple examples for which analytic solutions are obtainable. These examples illustrate the ease with which the method may be applied and the results interpreted. The real advantage of the method, over the "augmented state-space", approach, lies in the time saved in computing large non analytic, problems however.

4. Scalar Example

4.0 Problem

The class 0, scalar process

$$\dot{x}_1 = -\omega_s x_1 + \omega_s u$$

is to be controlled, by manipulation of the single input u(t) so as to minimise the performance index

$$\lim_{T_f \to \infty} J(T_f) \ , \quad \text{where } J(T_f) = \int_0^{T_f} \left[(x_1 - x_2)^2 + \lambda u^2\right] dt$$

, where x_2 is a constant reference and λ a constant weighting factor.

4.1 Solution

In this particular example, n = 1, m = 1, and r = 1 and the problem matrices are, therefore

$$\begin{array}{llll} A = -\omega_s & , & Q_2 & = 1 \\ B = \omega_s & , & Q_3 & = -1 \\ Q_1 = 1 & , & R & = \lambda \end{array}$$

4.1.1 Part 1 It is first necessary to optimise the steady-state cost-rate, $(x_{1s} - x_2)^2 + \lambda u_s^2$, with respect to u_s to determine optimum steady-state operating conditions. The minimisation must be carried out subject to the steady state process equation

$$x_{1s} = u_s$$

This may be achieved by direct substitution in the algebraic equation, (14),

$$\left[R + [A^{-1}B]^T Q_1 [A^{-1}B]\right] u_s = [A^{-1}B]^T Q_3^T x_2$$

giving, in this case

$$(\lambda + 1) u_s = (-1).(-1)\ x_2$$

$$\therefore \quad \underline{u_s = x_2/(1 + \lambda)}$$

and, from the steady state process equation,

$$\underline{x_s \;=\; x_2/1 + \lambda)}$$

Alternatively, rather than pre-supposing the general result given by equation 14, the steady-state minimisation may be carried out from first principles thus -

$$\frac{d\left[(x_{1s}-x_2)^2 + \lambda u_s^2\right]}{du_s} \;=\; 2(x_{1s}-x_2)\frac{dx_{1s}}{du_s} + 2\lambda u_s$$

$$= \;2(x_{1s}-x_2) + 2\lambda u_s \;=\; 0\,, \qquad \text{for optimum } u_s\,.$$

$\therefore$ since $u_s = x_{1s}$

$$2(u_s-x_2) + 2\lambda u_s \;=\; 0$$

$$\therefore \qquad \underline{u_s \;=\; x_2/(1 + \lambda)}$$

4.1.2 Part 2 It is now necessary to define the deviations of the process state and input from their steady state values -

$$y \;=\; x_1 - x_{1s} \qquad \text{and} \qquad z \;=\; u - u_s$$

and note that $\qquad \dot{y} \;=\; -\omega_s y + \omega_s z$

To optimise this dynamic system we substitute in the algebraic Riccati equation

$$PBR^{-1}B^TP \;=\; Q_1 + A'P + PA$$

which in this case yields the single scalar equation

$$\omega_s^2 p^2/\lambda \;=\; 1 - 2p\omega_s$$

the solution of which is

$$p \;=\; (\pm\sqrt{1 + 1/\lambda} \;-\; 1)\lambda/\omega_s$$

The optimum z(t) is, in general, given by

$$z \;=\; Ky \qquad \text{where} \qquad K = -R^{-1}B^TP$$

and therefore, in this example the single coefficient k is given by

$$k = -\omega_s p/\lambda \;=\; -(\sqrt{1 + 1/\lambda} - 1)$$

, the sign of the square root term being chosen to ensure negative feedback of y.

Hence, $\underline{z = -(\sqrt{1 + 1/\lambda} - 1)y}$ for optimum control.

4.1.3 <u>Complete Control Strategy</u> It is now necessary to combine the results of the steady state and dynamic optimisations to obtain the complete control law thus

$$u = z + u_s = ky + u_s$$

, hence, $u = k(x_1 - x_{1s}) + u_s = kx_1 + (1-k)u_s$

, (since $x_{1s} = u_s$ in this example).

Now from part 1, $u_s = x_2/(1 + \lambda)$ and therefore substituting for u_s in the equation above we get

$$\underline{\underline{u = -(\sqrt{1 + 1/\lambda} - 1)x_1 + x_2/(\lambda\sqrt{1 + 1/\lambda})}}$$

<u>Note</u>

The optimum control is thus obtained as a function of process state x_1 and reference x_2. The same result may be derived in the conventional manner by optimising the augmented system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\omega_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \omega_s \\ 0 \end{bmatrix} u$$

with the performance index

$$\lim_{T_f \to \infty} J(T_f) \text{ , } \quad \text{where } J(T_f) = \int_0^{T_f} (x^T Q x + \lambda u^2)\,dt$$

and, $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad$ and $\quad Q = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

realising that the element $\dot{p}_{22}$ of the matrix $\dot{P}$ in the matrix Riccati equation $\neq 0$ as $T_f \to \infty$.

5. Application to a 2-Input, 2-Output Process

5.0 Problem

The class 0, multivariable process

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1, & 0\\ 0, & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}2, & 1\\ 1, & 2\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

is to be controlled by manipulation of inputs $u_1(t)$ and $u_2(t)$ so that the performance index

$$\lim_{T_f\to\infty} J(T_f)\ , \quad \text{where } J(T_f) = \int_0^{T_f} \left[(x_1-x_3)^2 + (x_2-x_4)^2 + u_1^2 + u_2^2\right]dt$$

is minimised. x_3 and x_4 are constant reference signals.

5.1 Solution

The relevant problem matrices for this example are

$$A = \begin{bmatrix}-1, & 0\\ 0, & -1\end{bmatrix} \qquad B = \begin{bmatrix}2, & 1\\ 1, & 2\end{bmatrix}$$

$$Q_1 = \begin{bmatrix}1, & 0\\ 0, & 1\end{bmatrix} \qquad Q_2 = \begin{bmatrix}1, & 0\\ 0, & 1\end{bmatrix}$$

$$Q_3 = \begin{bmatrix}-1, & 0\\ 0, & -1\end{bmatrix} \qquad R = \begin{bmatrix}1, & 0\\ 0, & 1\end{bmatrix}$$

5.1.1 Steady State Optimisation

The steady state optimal inputs are given by

$$[R + [A^{-1}B]^T Q_1 [A^{-1}B]]u_s = [A^{-1}B]^T Q_3^T \underline{x}_2$$

where $\underline{x}_2$ is the constant reference vector, in this example equal to $[x_3,x_4]^T$

Now in this example $A^{-1}B = -\begin{bmatrix}2, & 1\\ 1, & 2\end{bmatrix}$ and therefore substituting in the above algebraic matrix equation we get

$$\left[\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right]\begin{bmatrix}u_{1s}\\ u_{2s}\end{bmatrix} = -\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_3\\ x_4\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix} = \begin{bmatrix} 2x_3 + x_4 \\ x_3 + 2x_4 \end{bmatrix}$$

and hence $$\begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix} = \left(\frac{1}{10}\right) \begin{bmatrix} 4x_3 - x_4 \\ -x_3 + 4x_4 \end{bmatrix}$$

The steady-state process equation is

$$\begin{bmatrix} x_{1s} \\ x_{2s} \end{bmatrix} = - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x_{1s} \\ x_{2s} \end{bmatrix} = \begin{bmatrix} 2u_{1s} + u_{2s} \\ u_{1s} + 2u_{2s} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 7x_3 + 2x_4 \\ 7x_4 + 2x_3 \end{bmatrix}$$

5.1.2 Dynamic Optimisation

Defining deviations of the states and inputs from their steady state values we have

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_{1s} \\ x_2 - x_{2s} \end{bmatrix}$$

and $$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} u_1 - u_{1s} \\ u_2 - u_{2s} \end{bmatrix}$$

and we note that $$\dot{y} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} y + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} z$$

It is now necessary to substitute the known matrices A, B, Q_1 and R in the algebraic matrix Riccati equation

$$PBR^{-1}B^TP = Q_1 + A^TP + PA$$

This gives

$$\begin{bmatrix} 5p_{11}^2 + 8p_{11}p_{12} + 5p_{12}^2 , & 5p_{11}p_{12} + 4p_{11}p_{22} + 4p_{12}^2 + 5p_{12}p_{22} \\ 5p_{11}p_{12} + 4p_{12}^2 + 4p_{11}p_{22} + 5p_{12}p_{22} , & 5p_{22}^2 + 8p_{22}p_{12} + 5p_{12}^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2p_{11} & , & -2p_{12} \\ -2p_{12} & , & 1-p_{22} \end{bmatrix}$$

$$\left.\begin{array}{ll} \text{From the symmetry of the problem,} \quad p_{11} = p_{22} & \dots(1) \\ \text{and therefore, from the Riccati eqn.,} & \\ 5p_{11}^2 + 8p_{11}p_{12} + 5p_{12}^2 = 1-2p_{11} & \dots(2) \\ 4p_{11}^2 + 10p_{11}p_{12} + 4p_{12}^2 = -2p_{12} & \dots(3) \end{array}\right\} \text{Costate Equations}$$

The solution of these equations, (which yields the essential "positive-definite" property of the P matrix) is

$$p_{11} = (-10 + \sqrt{10} + 9\sqrt{2})/18 = p_{22}$$

and $$p_{12} = (8 + \sqrt{10} - 9\sqrt{2})/18$$

The control law, in terms of the deviations z and y is $z = Ky$ where,

$$K = -R^{-1}B^TP$$

and in the case of this problem we get, knowing $p_{11} = p_{12}$,

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = -\begin{bmatrix} 2p_{11}+p_{12} & , & 2p_{12}+p_{11} \\ 2p_{12}+p_{11} & , & 2p_{11}+p_{12} \end{bmatrix}$$

The feedback coefficients acting on y_1 and y_2 to produce z_1 and z_2 are therefore

$$k_{11} = k_{22} = (4 - \sqrt{10} - 3\sqrt{2})/6$$

$$k_{12} = k_{21} = (-2 - \sqrt{10} + 3\sqrt{2})/6$$

5.1.3 The Complete Control Law

Now $z = Ky$

and therefore, $$u = u_s + Ky = u_s + K\begin{bmatrix} x_1 - x_{1s} \\ x_2 - x_{2s} \end{bmatrix}$$

$$= \left(\frac{1}{10}\right)\begin{bmatrix} 4x_3 - x_4 \\ -x_3 + 4x_4 \end{bmatrix} + K\begin{bmatrix} x_1 - 0.7x_3 - 0.2x_4 \\ x_2 - 0.2x_3 - 0.7x_4 \end{bmatrix}$$

$$= K\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} .4 - 0.7k_{11} - 0.2k_{12} & , & -.1 - 0.2k_{11} - 0.7k_{12} \\ -.1 - 0.7k_{12} - 0.2k_{22} & , & .4 - 0.2k_{12} - 0.7k_{22} \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

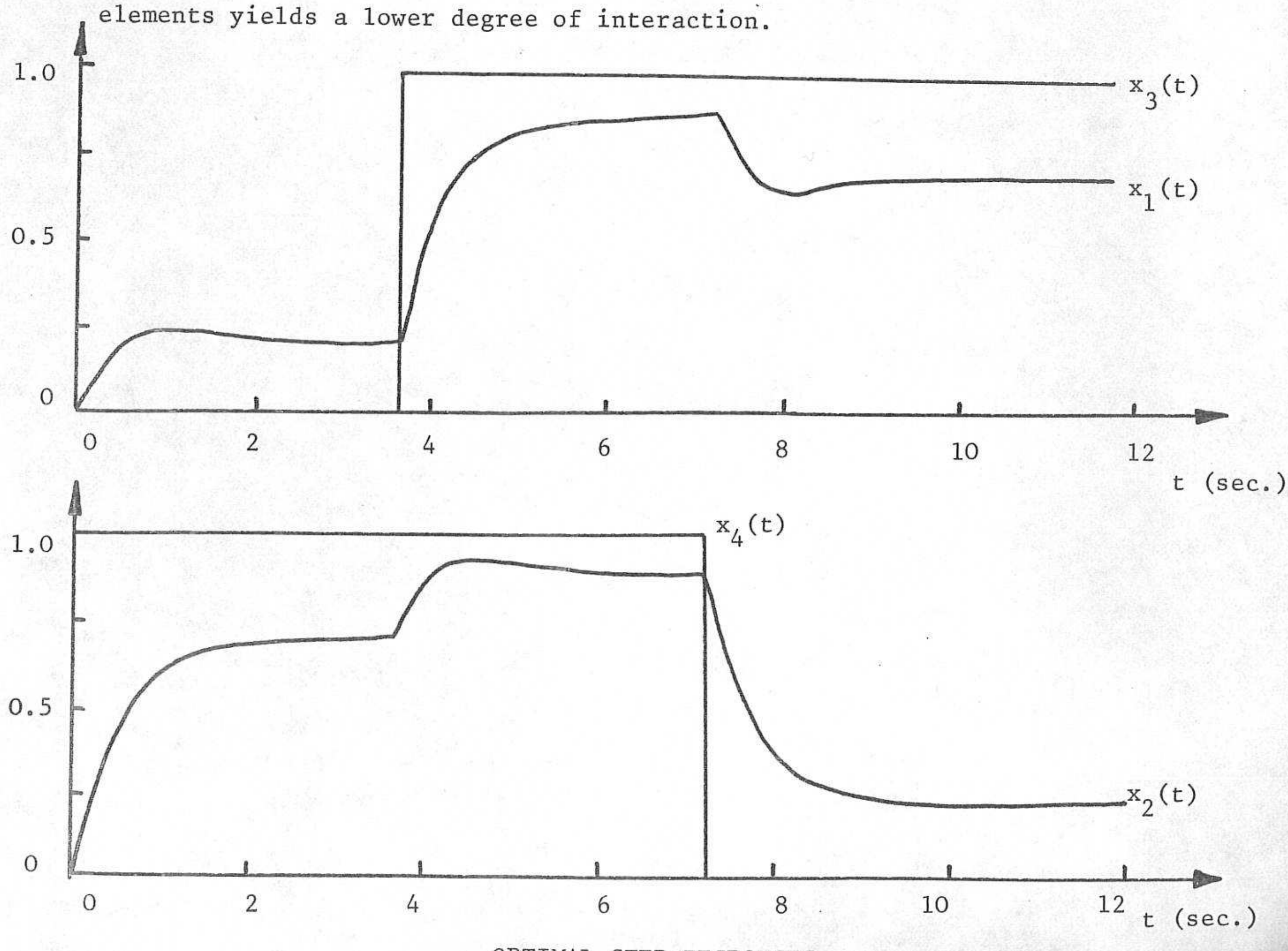
and substituting the values for the elements of K calculated in 5.1.2, we get

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 4 - \sqrt{10} - 3\sqrt{2}\ , & -2 - \sqrt{10} + 3\sqrt{2} \\ -2 - \sqrt{10} + 3\sqrt{2}\ , & 4 - \sqrt{10} - 3\sqrt{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ \frac{1}{2}\begin{bmatrix} 0.3\sqrt{10} + 0.5\sqrt{2}\ , & 0.3\sqrt{10} - 0.5\sqrt{2} \\ 0.3\sqrt{10} - 0.5\sqrt{2}\ , & 0.3\sqrt{10} + 0.5\sqrt{2} \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

and therefore, $$\underline{\underline{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} 0.566\ , & 0.153 \\ 0.153\ , & 0.566 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.826\ , & 0.122 \\ 0.122\ , & 0.826 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}}}$$

This is the complete optimum control law for the system and simulation of the process controlled in this manner produces the responses given below to step changes in the references x_3 and x_4. The relatively low degree of interaction between x_1 and x_4 and between x_2 and x_3 verifies qualitatively the validity of the optimum control law derived. Reduction of the R matrix elements yields a lower degree of interaction.



OPTIMAL STEP RESPONSES

6. Conclusions

A convenient method for optimising the regulator control of multivariable class-0 systems has been derived theoretically, which requires the separate optimisation of the steady state and dynamic performance of the system. The latter part of the exercise is based on error co-ordinates and therefore requires the solution of only an nxn matrix Riccati equation, where n is the number of basic process states disregarding any constant references and disturbances which appear in the problem. As demonstrated in sections 4 and 5, the technique may be applied with ease to analytical problems which would be more laboriously handled by the conventional method of augmenting the state space to n+m co-ordinates, (where m is the number of constant reference and disturbance signals). The technique really scores in the area of larger problems however where computer time prohibits the increase in the number of co-state equations to be solved by state augmentation. In some problems a time saving of up to 75% is possible where m approaches n and n is very large. The method presented also obviates the problem caused by the non-convergence of the P matrix in the state augmentation method.

The results presented are in fact consistent with somewhat more general results derived by Athens and Falb[1] for reference tracking problems. The merit claimed for this particular contribution lies not so much in the advancement of control theory as in the development, from this theory, of a practically convenient, readily understandable and intuitively acceptable design technique, applicable to a wide range of practical processes which hitherto presented considerable difficulty in control system design.

7. References

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