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SYSTEM TRANSMISSION ZEROS: A GEOMETRIC ANALYSIS

by

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Abstract

The open-loop transmission zeros of a linear multivariable system are defined geometrically in terms of state space properties of (A,B,C). A canonical form is derived which illustrates the physical source of zeros in terms of state feedback and system observability. A distinction is made between open and closed-loop zeros of the system, and sufficient conditions for both sets of zeros to coincide are derived.

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1. Introduction

This paper is concerned with a linear time-invariant system S(A,B,C) of the form

$$\dot{x} = Ax+Bu \quad , \qquad x \in R^n, \ u \in R^\ell$$

$$y = Cx \quad , \qquad y \in R^m \qquad \ldots(1)$$

and the geometric structure of S(A,B,C) giving rise to the presence of transmission zeros as defined by Davison and Wang (1974).

The transmission zeros are defined to be the set $Z_o(A,B,C)$ of complex numbers λ such that, if

$$P(s) = \begin{bmatrix} sI_n - A & -B \\ C & 0 \end{bmatrix} \qquad \ldots(2)$$

then

$$\text{rank } P(\lambda) < n+\min(m,\ell) \qquad \ldots(3)$$

Note that $Z_o(A,B,C)$ may be empty, contain a finite number of symmetric complex numbers or include the whole complex plane. Note also that $Z_o(A,B,C)$ contains no information on multiplicity.

Throughout the paper, the following assumptions are made

$$m \geq \ell \qquad \ldots(4)$$

$$\text{rank } B = \ell \qquad \ldots(5)$$

$$\text{rank } C = m \qquad \ldots(6)$$

$$Z_o(A,B,C) \text{ is non-empty and finite} \qquad \ldots(7)$$

Condition (4) is not restrictive as the case of $m<\ell$ can be analysed by applying the results to the system $S(A^T,C^T,B^T)$. Conditions (5) and (6) are valid in almost all applications. Condition (7) is crucial in the proof of all results and, apart from its validity in almost all applications, is related to the desirable system characteristic of functional controllability (Rosenbrock 1970).

In feedback design applications it is necessary to consider the possibility of the introduction of extra zeros due to squaring down. For example, if $m = n$, then, using equations (2), (3), (5), (6), it follows that $Z(A,B,C) = \phi$ (the empty set). If, however, $\ell<m$ and K is an ℓxm constant matrix, then, in general, $Z(A,B,KC) \neq \phi$. In general, for an ℓxm constant matrix K, and noting that $\ell \leq m$,

$$Z_o(A,B,C) \subset Z_o(A,B,KC) \qquad \ldots(8)$$

In order to distinguish between the transmission zeros of (A,B,C) and zeros induced by squaring the plant down, it is convenient to term $Z_o(A,B,C)$ the set of open-loop transmission zeros and $\lambda \in Z_o(A,B,C)$ an open-loop tranmission zero. In contrast, the closed-loop zero set is defined by

$$Z_c(A,B,C) \triangleq \bigcap_K Z_o(A,B,KC) \qquad \ldots(9)$$

where the intersection runs over all ℓxm constant matrices K. It follows directly from (8) that

$$Z_o(A,B,C) \subset Z_c(A,B,C) \qquad \ldots(10)$$

so that every open-loop transmission zero becomes a closed-loop zero.

In section 2, a geometric definition of open-loop transmission zero is proposed and a preliminary discussion of the concept of null transmission subspace is undertaken. In section 3, the concept is extended to construct a direct sum decomposition of the state space which is used to construct a system canonical form and to define multiplicity of open-loop transmission zeros in a geometric manner. In section 4, an analysis of closed-loop zeros is presented. In section 6 the canonical form is used to interpret the physical source of zeros in terms of state feedback. In section 5, an inductive construction of the state space decomposition is developed. Finally, in section 7, the concepts are illustrated by several examples.

2. A Geometric Definition of Transmission Zero

By assumption $\ell \leq m$ so that, if $\lambda \in Z_o(A,B,C)$, there exists $x \in R^n$, $y \in R^\ell$ such that $||x|| + ||y|| > 0$ and

$$\{\lambda I_n - A\}x - By = 0$$

$$Cx = 0 \qquad \ldots(11)$$

As rank $B=\ell$ then $x \neq 0$ so, if $N(c)$ denotes the null space of C and $R(B)$ denotes the range space of B, the following result is obtained.

<u>Result One</u> (Geometric Definition of Transmission Zeros)

$$\lambda \in Z_o(A,B,C) \quad \text{iff} \quad \omega_1(\lambda) \triangleq \{(\lambda I_n - A)^{-1} R(B)\} \cap N(C) \neq \{0\} \qquad \ldots(12)$$

<u>Proof</u>

If $\lambda \in Z_o(A,B,C)$ then, from the previous discussion, there exists a solution vector $x \neq 0$ to equation (11) ie $x \in N(C)$ and $x \in (\lambda I_n - A)^{-1} R(B)$ so that $\omega_1(\lambda) \neq \{0\}$. Conversely, if $\omega_1(\lambda) \neq \{0\}$, there exists a non-zero $x \in N(C)$ satisfying $(\lambda I_n - A)x \in R(B)$ ie $\lambda \in Z_o(A,B,C)$.

QED

Under the stated conditions of section 1, this result gives a necessary and sufficient condition for any complex number λ to be an open-loop system transmission zero. Result 1 can be used to obtain an equivalent algebraic definition (Kouvaritakis and MacFarlane, 1975). Let N,M be full rank $(n-\ell)\mathrm{x}n$ and $n\mathrm{x}(n-m)$ constant matrices respectively, and

$$NB = 0 \quad , \quad CM = 0 \qquad \ldots(13)$$

then the following result is obtained,

Result Two (The NAM algorithm)

$$\lambda \in Z_o(A,B,C) \quad \text{iff} \quad \text{rank } N\{\lambda I_n - A\}M < n-m \qquad \ldots(14)$$

Proof

If rank $N(\lambda I_n - A)M < n-m$, there exists $z \neq 0$ such that $N(\lambda I_n - A)Mz = 0$. Let $x = Mz$, then, as M is full rank, $x \neq 0$. Also $x \in N(c)$ and $N(\lambda I_n - A)x = 0$ so that $(\lambda I_n - A)x \in R(B)$ ie $\omega_1(\lambda) \neq \{0\}$ and, by Result 1, $\lambda \in Z_o(A,B,C)$. Conversely, if $\lambda \in Z_o(A,B,C)$, there is $x \in N(c)$ such that $x \neq 0$ and $(\lambda I_n - A)x = b \in R(B)$. This implies that $N(\lambda I_n - A)x = Nb = 0$ and the result follows by writing $x = Mz$ and noting that $z \neq 0$.

QED

The following analysis represents a preliminary discussion of the concepts of the following sections, where the primary objective is the geometric characterization of $Z_o(A,B,C)$ and a geometric definition of zero multiplicity.

The subspace $\omega_1(\lambda) \subseteq N(c)$ characterizes all non-zero solution vectors of equation (11). Interpreting $xe^{\lambda t}$ as the state response to the driving function $ye^{\lambda t}$, then $xe^{\lambda t}$ produces zero output from the system for all time. Hence $\omega_1(\lambda)$ characterizes the set of solutions of the state equations which produce zero output when excited by exponential inputs of frequency λ. For this reason, it is natural to term $\omega_1(\lambda)$ the null transmission subspace corresponding to $\lambda\in Z_o(A,B,C)$. The following result identifies some useful properties of $\omega_1(\lambda)$.

Result Three

(a) $\omega_1(\lambda)\cap R(B) = \{0\}$ ...(15)

(b) $A\omega_1(\lambda)\subset\omega_1(\lambda) \oplus R(B)$ ...(16)

(c) $d_\lambda \triangleq \dim \omega_1(\lambda) \equiv$ rank defect of $P(\lambda)$ ...(17)

(d) $\omega_1(\lambda)$ is invariant under state feedback ...(18)

Proof

To prove (15), take $x\in\omega_1(\lambda)\cap R(B)$, then $x\in N(c)$, $x\in R(B)$ and $(\lambda I-A)x\in R(B)$. Hence $Ax\in R(B)$ so that $(sI_n-A)x\in R(B)$ for all complex numbers s. Hence $x\in\omega_1(s)$ and the result follows by taking $s\notin Z_o(A,B,C)$ and applying Result 1 to prove that $x = 0$.

Equation (16) follows directly as $x\in\omega_1(\lambda)$ implies $Ax\in\omega_1(\lambda)\oplus R(B)$, the sum being direct due to (a).

To prove (18), let $P:R^n\to R^n$ be linear and $R(P)\subset R(B)$, then $x\in\omega_1(\lambda)$ implies $(\lambda I-A)x\in R(B)$ so that $(\lambda I-A-P)x\in R(B)$. Conversely, if $x\in N(c)$ and $(\lambda I-A-P)x\in R(B)$ then $(\lambda I-A)x\in R(B)$ and hence $x\in\omega_1(\lambda)$.

To prove (17) let r = rank defect of $P(\lambda)$. If $\lambda\notin Z_o(A,B,C)$ the result is trivial, so take $\lambda\in Z_o(A,B,C)$. To prove that $d_\lambda\leq r$, note that there exists d_λ linearly independent vectors $x_j\in\omega_1(\lambda)$, $1\leq j\leq d_\lambda$. If $(\lambda I_n-A)x_j=By_j$, define an $(n+\ell)x(n+\ell)$ matrix

$$\eta(\lambda) \triangleq \begin{bmatrix} x_1 & \cdots & x_{d_\lambda} & t_{d_\lambda+1} & \cdots & t_{n+\ell} \\ y_1 & \cdots & y_{d_\lambda} & & & \end{bmatrix}$$

where t_j, $d_\lambda+1\leq j\leq n+\ell$ are chosen such that rank $\eta(\lambda) = n+\ell$. It follows directly that $P(\lambda)\eta(\lambda) = [0....0\mu_{d_\lambda+1}....\mu_{n+\ell}]$ or, as rank $P(\lambda)\eta(\lambda)$ = rank $P(\lambda)$ we have $d_\lambda \leq r$. The result will follow trivially if $r \leq d_\lambda$, but, in a similar manner, there exists a matrix $\eta(\lambda)$ of the form

$$\eta(\lambda) = \begin{bmatrix} x_1 \;\cdots\cdots\; x_r & \\ & t_{r+1} \;\cdots\cdots\; t_{n+\ell} \\ y_1 \;\cdots\cdots\; y_r & \end{bmatrix}$$

such that rank $\eta(\lambda) = n+\ell$ and $P(\lambda)\eta(\lambda) = [0...0\ \mu_{r+1}...\mu_{n+\ell}]$. By multiplication it follows trivially that $x_j \in \omega_1(\lambda)$, $1 \leq j \leq r$. Moreover x_j, $1 \leq j \leq r$, are linearly independent for, if

$\sum_{j=1}^{r} \alpha_j x_j = 0$ then $(\lambda I_n - A) \sum_{j=1}^{r} \alpha_j x_j = B \sum_{j=1}^{r} \alpha_j y_j = 0$ and rank $B = \ell$ implies

$\sum_{j=1}^{r} \alpha_j x_j = 0$. That is $\eta(\lambda)\ [\alpha_1 ... \alpha_r 0 ... 0]^T = 0$ and rank $\eta(\lambda) = n+\ell$ implies $\alpha_j = 0$, $1 \leq j \leq r$. Hence $r \leq d_\lambda$ which proves the result.

QED

Corollary

There exists a similarity transformation E_λ such that

$$CE_\lambda = [C_\lambda \quad 0_{m,d_\lambda}] \qquad ...(19)$$

$$E_\lambda^{-1}B = \begin{bmatrix} I_\ell \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} B_\lambda \\ 0_{d_\lambda,\ell} \end{bmatrix} \qquad ...(20)$$

$$E_\lambda^{-1}AE_\lambda = \left[\begin{array}{c|c} A_{11} & \begin{array}{c} K_\lambda \\ \hline 0 \end{array} \\ \hline A_{21} & \lambda I_{d_\lambda} \end{array}\right] \qquad ...(21)$$

where K_λ is an $\ell \times d_\lambda$ matrix.

Proof

Let $E_\lambda = [B\ P\ Q]$ where the columns of the $n \times d_\lambda$ matrix Q span $\omega_1(\lambda)$. Equation (15), with suitable choice of P implies E_λ is invertible. Moreover, by construction,

$$E_\lambda \begin{bmatrix} I_\ell \\ 0 \end{bmatrix} = B$$

and $\omega_1(\lambda) \subset N(c)$ implies $CQ = 0$. Finally $x \in \omega_1(\lambda)$ implies $Ax - \lambda x \in R(B)$ which proves the result.

QED

The corollary provides some insight into the state space properties producing system zeros and the main result provides special cases of relationships which prove to be crucial in the developments of later sections. Of particular interest at this stage is the application of the corollary to the investigation of zero multiplicity. For example, take $m=\ell$, then the multiplicity of $\lambda \in Z_o(A,B,C)$ can be defined to be the multiplicity of λ as a zero of $|P(s)|$ (see equation (2)). After suitable manipulation,

$$|P(s)| = \begin{vmatrix} sI_n - E_\lambda^{-1} A E_\lambda & -E_\lambda^{-1} B \\ CE_\lambda & 0 \end{vmatrix}$$

$$= (-1)^q (s-\lambda)^{d_\lambda} \begin{vmatrix} sI_{n-d_\lambda} - A_{11} & -B_\lambda \\ C_\lambda & 0 \end{vmatrix} \qquad \ldots(22)$$

for some integer q. Hence d_λ is less than or equal to the multiplicity of λ. To prove that inequality can hold, consider a system with transfer function

$$g(s) = (s+1)^2/s^3$$

and state space model

$$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad , \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

then $Z_o(A,B,C) = \{-1\}$ and rank $P(-1) = 3$. Applying Result 3, we obtain $d_{-1} = 1<2$.

3. Multiplicity of Open-loop Zeros and Decomposition of the State Space

An investigation of the geometric source of transmission zeros and zero multiplicity requires a generalization of the results of section 2. The analysis of this section is abstract, but interpretations of the results are included as suitable points in the analysis.

The starting point of the analysis is a generalization of relation (16). Define

$$\omega_B \triangleq \{X;\ X\subset N(c),\ X \text{ is a linear subspace of } C^n,\ AX\subset X+R(B)\} \quad \ldots(23)$$

then $Z_o(A,B,C) \neq \phi$ implies ω_B is non-trivial as $\lambda\in Z_o(A,B,C)$ gives $\omega_1(\lambda)\in\omega_B$ (by Result 3). The constraint $X\subset C^n$ is included to allow complex vectors corresponding to complex zeros of $S(A,B,C)$.

Now ω_B is partially ordered by inclusion and, as $N(c)$ is finite dimensional, every totally ordered subset of ω_B has an upper bound in ω_B. Hence, by Zorn's Lemma, ω_B contains a maximal element ω such that $V\in\omega_B$ and $\omega\subset V$ together imply $V = \omega$. Also, as $\omega\in\omega_B$,

$$A\omega \subset \omega+R(B) \quad \ldots(24)$$

The following result defines some properties of ω.

Result Four

(a) If $\hat{\omega}\in\omega_B$, then $\hat{\omega} \subset \omega$ ...(25)

(b) ω is invariant under state feedback ...(26)

(c) $\omega\cap R(B) = \{0\}$ ...(27)

(d) $n_z \triangleq \dim \omega \leqslant (n-m) - \dim \{R(B)\cap N(c)\}$ ...(28)

Moreover, sufficient conditions for equality to hold in (d) is $m=\ell$ and

$$N(c)\cap R(B) = \{0\} \qquad ...(29)$$

Proof

To prove (a), assume there exists $\hat{\omega}\in\omega_B$ such that $\hat{\omega}\not\subset\omega$. It is easily shown that $\omega+\hat{\omega}\in\omega_B$ which, together with the obvious relation $\omega\subset\omega+\hat{\omega}$ contradicts the maximality of ω ie $\hat{\omega}\subset\omega$.

To prove (b), let $\hat{\omega}$ be the maximal subspace corresponding to the triple $(A+P,B,C)$ where $P:R^n\to R^n$ is linear and $R(P)\subset R(B)$. If $x\in\hat{\omega}$, then $(A+P)x\in\hat{\omega}+R(B)$ so that $Ax\in\hat{\omega}+R(B)$ and hence $A\hat{\omega}\in\hat{\omega}+R(B)$. Hence, by (a), $\hat{\omega}\subset\omega$ and by symmetry $\omega\subset\hat{\omega}$ so that $\omega = \hat{\omega}$.

To prove (c), note that $(sI-A)\omega\subset\omega+R(B)$ for any complex number s and construct $\hat{\omega}$ such that

$$\omega = \hat{\omega} \oplus \{\omega\cap R(B)\}$$

It follows directly that $(sI_n-A)\omega\subset\hat{\omega}\oplus R(B)$. Suppose $q \triangleq \dim \hat{\omega}$, then $q \leqslant n_z$. Let x_j, $1\leqslant j\leqslant n_z$, be a basis for ω. That is, $1\leqslant j\leqslant n_z$,

$$(sI_n-A)x_j = \hat{x}_j+b_j \quad , \qquad \hat{x}_j\in\omega$$
$$b_j\in R(B)$$

Let α_j, $1\leqslant j\leqslant n_z$, be scalars such that $\sum_{j=1}^{n_z} \alpha_j\hat{x}_j = 0$ ie

$$(sI_n-A) \sum_{j=1}^{n_z} \alpha_j x_j \in R(B)$$

Choosing $s \notin Z(A,B,C)$ and applying Result 1, it follows that $\sum_{j=1}^{n_z} \alpha_j x_j = 0$ from which, by the linear independence of $\{x_j\}$, $\alpha_j = 0$, $1 \leq j \leq n_z$.
It follows directly that $\{\hat{x}_j\}_{1 \leq j \leq n_z}$ are linearly independent and $q = n_z$ ie

$$\omega \cap R(B) = \{0\}$$

which proves the result.

To prove (d), note that (c) implies $\omega \cap \{R(B) \cap N(c)\} = \{0\}$ so that the sum $\omega + \{R(B) \cap N(c)\}$ is direct. Moreover $\omega \cap \{R(B) \cap N(c)\} \subset N(c)$ and hence,

$$\dim \omega + \dim R(B) \cap N(c) \leq n-m$$

which proves equation (28). Finally, if $m = \ell$ and $N(c) \cap R(B) = \{0\}$, then

$$C^n = N(c) \oplus R(B)$$

Noting that $AN(c) \subset C^n$ it follows that $N(c) \in \omega_B$. By (a), $N(c) \subset \omega$ so that $\omega = N(c)$, and the result follows.

QED

The above result has a direct interpretation in terms of a state space canonical form defined by the following result.

Result Five (A Zero Canonical Form)

There exists a similarity transformation E such that

$$CE = \begin{bmatrix} \hat{C} & 0_{m,n_z} \end{bmatrix} \qquad \ldots(30)$$

$$E^{-1}B = \begin{bmatrix} I_\ell \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} \hat{B} \\ 0_{n_z,\ell} \end{bmatrix} \qquad \ldots(31)$$

$$E^{-1}AE = \left[\begin{array}{c|c} A_{11} & \begin{array}{c} K \\ \hline 0 \end{array} \\ \hline A_{21} & A_{22} \end{array}\right] \qquad \ldots(32)$$

where K is an $\ell x n_z$ matrix and A_{22} is an $n_z x n_z$ matrix. Moreover, the system $S(A_{11},\hat{B},\hat{C})$ is observable and

$$Z_o(A_{11},\hat{B},\hat{C}) = \phi \qquad \ldots(33)$$

Proof

Let $E = [B,P,Q]$ where the columns of the $n x n_z$ matrix Q span ω. Equation (27), with suitable choice of P, implies that E is invertible. Moreover, by construction,

$$E \begin{bmatrix} I_\ell \\ 0 \end{bmatrix} = B$$

and $\omega \subset N(c)$ implies $CQ = 0$. Also $x \in \omega$ implies (equations (24),(27)) $Ax \in \omega \oplus R(B)$ which proves the first part of the result.

To prove the second part of the result note that $Z_o(A_{11},\hat{B},\hat{C})$ is finite or empty (as $Z_o(A,B,C)$ is finite) and assume there exists $\lambda \in Z_o(A_{11},\hat{B},\hat{C})$. Then, by the corollary of Result 3, there is a transformation T such that

$$T^{-1}\hat{B} = \hat{B}$$

$$\hat{C}T = [\hat{C}_1 \;\; 0_{m,1}]$$

$$T^{-1}A_{11}T = \left[\begin{array}{c|c} \hat{A}_{11} & \hat{k} \\ \hline & 0 \\ \hat{A}_{21} & \lambda \end{array}\right]$$

for some $\ell x 1$ vector $\hat{k}$. Applying the transformation $\hat{T} \triangleq \begin{bmatrix} T & 0 \\ 0 & I_{n_z} \end{bmatrix}$ to the canonical form of equations (30)-(32), it follows that the last n_z+1 columns of $E\hat{T}$ span an n_z+1 dimensional subspace $\hat{\omega} \subset N(c)$ satisfying $\omega \subset \hat{\omega}$ and

$$A\hat{\omega} \subset \hat{\omega} + R(B)$$

which contradicts the definition of ω. Hence $Z_o(A_{11},\hat{B},\hat{C}) = \phi$.

Finally, if $S(A_{11},\hat{B},\hat{C})$ is not observable, there exists a scalar λ and a non-zero vector x such that $x \in N(c)$ and $(\lambda I - A)x = 0 \in R(B)$ ie $\omega_1(\lambda) \neq \{0\}$ which, by Result 1, contradicts equation (33).

QED

The following result identifies the properties of A_{22} which lead to a natural definition of the multiplicity of $\lambda \in Z_o(A,B,C)$.

Result Six (Zeros and the Eigenvalues of A_{22})

λ is an eigenvalue of A_{22} if, and only if, $\lambda \in Z_o(A,B,C)$.

Proof

By inspection of the canonical form, if λ is an eigenvalue of A_{22}, there exists a non-zero vector $x \in \omega$ such that $Ax = \lambda x + b$ where $b \in R(B)$. Noting that $\omega \subset N(c)$ then (by Result 1) $\omega_1(\lambda) \neq \{0\}$ and hence $\lambda \in Z_o(A,B,C)$. Conversely, if $\lambda \in Z_o(A,B,C)$ there exists $x \in \omega_1(\lambda)$ such that $x \neq 0$ and $Ax = \lambda x + b$ for some $b \in R(B)$. However, by Result 4(a), noting that $\omega_1(\lambda) \subseteq \omega_B$, then $x \in \omega$ and hence, by inspection of the canonical form, λ is an eigenvalue of A_{22}.

QED

Result Six implies that,

$$|sI_{n_z} - A_{22}| = \prod_{\lambda \in Z_o(A,B,C)} (s-\lambda)^{n_\lambda} \qquad \ldots(34)$$

where n_λ is the multiplicity of λ as an eigenvalue of A_{22}, and

$$\sum_{\lambda \in Z_o(A,B,C)} n_\lambda = n_z \qquad \ldots(35)$$

Bearing in mind the result that $Z_o(A_{11},\hat{B},\hat{C}) = \phi$ (Result 5), it is natural to define

$n_\lambda \triangleq$ open-loop multiplicity of $\lambda \in Z_o(A,B,C)$ ...(36)

$n_z \triangleq$ total number of open-loop transmission zeros ...(37)

The relationship of these definitions to closed-loop behaviour and the squaring-down problem is discussed in the next section. The geometric characterization of n_λ is obtained by choosing a direct sum decomposition of ω of the form

$$\omega = \bigoplus_{j=1}^{p} I(\lambda_j) \qquad \ldots(38)$$

where, if $Z_o(A,B,C) = \{\lambda_j\}_{1 \leqslant j \leqslant p}$, then A_{22} takes the Jordan form,

$$A_{22} = \begin{bmatrix} J(\lambda_1) & 0 & \cdots\cdots & 0 \\ 0 & J(\lambda_2) & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots\cdots & 0 & J(\lambda_p) \end{bmatrix} \qquad \ldots(39)$$

Here $J(\lambda_j)$ is the Jordan block corresponding to the zero λ_j. Hence,

$$n_\lambda = \dim I(\lambda) \quad , \qquad \lambda \in Z_o(A,B,C) \qquad \ldots(40)$$

Finally, with the interpretation of definition (37), noting that if

$$r_1 \triangleq \text{rank } CB \qquad \ldots(41)$$

then

$$\dim R(B) \cap N(c) = \ell - r_1 \qquad \ldots(42)$$

and Result 4 provides an estimate of the number of open-loop transmission zeros,

$$\begin{aligned} n_z &\leqslant (n-m) - \dim R(B) \cap N(c) \\ &= (n-m) - (\ell - r_1) \end{aligned} \qquad \ldots(43)$$

equality holding if $m=\ell$ and $r_1 = \ell$. A better estimate is obtained using the following result.

Result Seven

Let k be the smallest integer $j \geqslant 1$ such that $A^{j-1}R(B) \not\subset N(c)$. Then the sum

$$V_q \triangleq \sum_{i=o}^{q-1} A^i R(B) \quad , \qquad 1 \leq q \leq k \qquad \ldots(44)$$

is a direct sum, and

$$V_q \cap \omega = \{0\} \quad , \qquad 1 \leq q \leq k \qquad \ldots(45)$$

Moreover

$$\dim A^i R(B) = \ell \quad , \qquad o \leq i \leq k-1 \qquad \ldots(46)$$

and

$$n_z \leq (n-m) - (k-1)\ell - \dim\{A^{k-1}R(B)\} \cap N(c) \qquad \ldots(47)$$

Sufficient conditions for equality to hold in (47) is $m=\ell$ and

$$\{A^{k-1}R(B)\} \cap N(c) = \{0\} \qquad \ldots(48)$$

Proof

If $k = 1$, the result is proved immediately by application of Result 4. Therefore take $k>1$ and hence $R(B) \subset N(c)$. Note that

$$V_q \subset N(c) \quad , \qquad 1 \leq q < k \qquad \ldots(49)$$

$$V_1 = R(B) \qquad \ldots(50)$$

$$V_q \subset V_{q+1} \quad , \qquad 1 \leq q < k \qquad \ldots(51)$$

$$V_{q+1} = A^q R(B) + V_q = AV_q + R(B) \quad , \qquad 1 \leq q < k \qquad \ldots(52)$$

By result 4, $V_1 \cap \omega = \{0\}$, so assume that $V_i \cap \omega = \{0\}$, $1 \leq i \leq j$, for some $j<k$ but that $V_{j+1} \cap \omega \neq \{0\}$. Select a non-zero vector $x \in V_{j+1} \cap \omega$ and write, using equation (52), $x = Av_j + b_j$ where $v_j \in V_j$ and $b_j \in V_1$. After rearrangement, $Av_j = x - b_j \in \omega \oplus R(B)$ but, by induction, $v_j \notin \omega$ so that, if $\hat{\omega} \triangleq \omega \oplus \{v_j\}$, ω is a proper subset of $\hat{\omega}$, $\hat{\omega} \subset N(c)$ and

$$A\hat{\omega} \subset \omega + R(B) \subset \hat{\omega} + R(B)$$

contradicting the maximality of ω ie $V_{j+1} \cap \omega = \{0\}$. Hence

$$V_j \cap \omega = \{0\} \quad , \qquad 1 \leq j \leq k \qquad \ldots(53)$$

which proves (45).

Next note that $V_2 = AR(B) \oplus V_1$, for, if $\{AR(B)\} \cap R(B) \neq \{0\}$, there exists a non-zero $x \in R(B)$ such that $Ax \in R(B)$ ie $(sI_n - A)x \in R(B)$ for all complex numbers s. By assumption $R(B) \subset N(c)$ so that (Result 1) $x \in \omega_1(s)$. A contradiction is obtained by taking $s \notin Z_o(A,B,C)$ and applying Result 1 to prove that $x = 0$. Relation (44) is now proved if $k = 2$, so take $k>2$ and assume that for some $j<k-1$

$$V_{i+1} = A^i R(B) \oplus V_i \quad , \qquad 1 \leq i \leq j$$

and take $x \in \{A^{j+1}R(B)\} \cap V_{j+1}$. Write $x = A^{j+1}b$ (for some $b \in R(B)$) and $v_j \triangleq A^j b \in V_{j+1}$. By (52) and the assumption that $x \in V_{j+1}$, then $x = Av_{j-1} + \hat{b}$ for some $v_{j-1} \in V_j$ and $\hat{b} \in R(B)$. Hence $A(v_j - v_{j-1}) \in R(B)$ and noting that $j<k-1$ implies $V_j \subset N(c)$ then $v_j - v_{j-1} \in \omega$. Noting (by (51)) that $v_j - v_{j-1} \in V_{j+1}$ and $V_{j+1} \cap \omega = \{0\}$ (from (53)) then $v_j = v_{j-1} \in V_j$. But $v_j = A^j b \in A^j R(B)$ and, by the inductive assumption $\{A^j R(B)\} \cap V_j = \{0\}$, so that $A^j b = 0$ and $x = A(A^j b) = 0$, whence

$$\{A^{j+1}R(B)\} \cap V_{j+1} = \{0\} \qquad \ldots(54)$$

and

$$V_{j+2} = A^{j+1}R(B) \oplus V_{j+1} \qquad \ldots(55)$$

so that

$$V_{j+1} = A^j R(B) \oplus V_j \quad , \quad 1 \leq j < k \qquad \ldots(56)$$

and hence

$$V_k = \bigoplus_{j=o}^{k-1} A^j R(B) \qquad \ldots(57)$$

This relation also proves that k exists and is finite.

To prove equation (46), note that $\dim R(B) = \ell$. Suppose that $\dim A^i R(B) = \ell$ for $1 \leq i \leq j$ and $j<k-1$ and note that $\dim A^{j+1}R(B) \leq \ell$. If $\dim A^{j+1}R(B) < \ell$ then there is a non-zero $b \in R(B)$ such that $A^{j+1}b = 0$ ie $A\{A^j b\} = 0$.

By the inductive assumption, $A^j b \neq 0$ and $A^j b \in N(c)$. If $x = A^j b$ then $Ax = 0 \in R(B)$ so that $x \in \omega \cap V_k = \{0\}$ from equation (53), which produces a contradiction.

Note that

$$\omega \oplus \{\{A^{k-1}R(B)\} \cap N(c)\} \oplus \bigoplus_{j=o}^{k-2} A^j R(B) \subset N(c) \qquad \ldots(58)$$

so that

$$n_z + (k-1)\ell + \dim \{A^{k-1}R(B)\} \cap N(c) \leqslant n-m \qquad \ldots(59)$$

which proves equation (47).

To prove the final part of the result it is sufficient to prove that, $0 \leqslant j \leqslant k-1$, the maximal subspace (which exists) $\hat{\omega}_j \subset N(c)$ satisfying

$$A\hat{\omega}_j \subset \hat{\omega}_j + A^j R(B) \qquad \ldots(60)$$

is

$$\hat{\omega}_j = \omega \oplus s_j \qquad \ldots(61)$$

where

$$s_o = \{0\} \qquad \ldots(62)$$

$$s_j = \bigoplus_{i=o}^{j-1} A^i R(B) \quad , \quad j>0 \qquad \ldots(63)$$

for, if $m = \ell$ and $\{A^{k-1}R(B)\} \cap N(c) = \{0\}$, then, using (46),

$$C^n = A^{k-1}R(B) \oplus N(c) \qquad \ldots(64)$$

so that $\hat{\omega}_{k-1} = N(c)$ and, from (61),(63),

$$n-m = \dim N(c) = \sum_{j=o}^{k-2} \dim A^j R(B) + \dim \omega \qquad \ldots(65)$$

which proves the result, by equation (46).

By Result 4, $\hat{\omega}_o = \omega$, so assume, for some $0 \leqslant j < k-1$, that equation (61) holds. Now it is easily shown that $\omega \oplus s_{j+1} \subset \hat{\omega}_{j+1}$ but assume that there exists $x \notin \omega \oplus s_{j+1}$, such that $x \in \hat{\omega}_{j+1}$ ie $x \in N(c)$ and, by suitable choice of x,

$$Ax = \lambda x + y + \sum_{i=o}^{j+1} A^i b_i$$

for some complex number $\lambda, y \in \omega$ and $b_i \in R(B)$, $0 \leq i \leq j+1$. After some manipulation

$$A(x - \sum_{i=o}^{j} A^i b_{i+1}) = \lambda(x - \sum_{i=o}^{j} A^i b_{i+1}) + y + b_o + \lambda \sum_{i=o}^{j} A^i b_{i+1}$$

so that $x - \sum_{i=o}^{j} A^i b_{i+1} \in \hat{\omega}_j = \omega \oplus s_j$ and hence $x \in \omega \oplus s_{j+1}$ contrary to assumption. The result is now proved.

QED

Equation (47) has a direct interpretation in terms of the rank of the matrix $CA^{k-1}B$, for if

$$r_k = \text{rank } CA^{k-1}B \qquad \ldots(66)$$

then

$$\dim\{A^{k-1}R(B)\} \cap N(c) = \ell - r_k \qquad \ldots(67)$$

and (47) states that the number of open-loop transmission zeros

$$\begin{aligned} n_z &\leq (n-m) - (k-1)\ell - (\ell - r_k) \\ &= (n-m) - k\ell + r_k \end{aligned} \qquad \ldots(68)$$

equality holding if $m=\ell$ and $r_k=\ell$.

It is interesting to note that k and r_k can be obtained directly from the system transfer function matrix $G(s) = C(sI-A)^{-1}B$ as k is the uniquely defined integer such that

$$G_\infty^{(k)} \triangleq \lim_{s\to\infty} s^k G(s) \qquad \ldots(69)$$

exists and is non-zero, and

$$r_k = \text{rank } G_\infty^{(k)} \qquad \ldots(70)$$

Note also that, by equations (59) and (46), and the assumption that $n_z > 0$ implies

$$n-m > (k-1)\ell \qquad \ldots(71)$$

which provides a useful bound on the possible values of k e.g. If $m = 3$, $n = 4$, $\ell = 2$, equation (71) indicates that $k = 1$ and hence $G_\infty^{(1)}$ is non-zero and finite.

4. Multiplicity of Closed-loop Zeros

As pointed out in section 1, all open-loop zeros produce closed-loop zeros, but, in general,

$$Z_o(A,B,C) \neq Z_c(A,B,C) \qquad \ldots(72)$$

as can be seen by the example,

$$C = I_3 \quad , \qquad A = \mathrm{diag}\{1,2,3\} \quad , \qquad B = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^T$$

where $Z_o(A,B,C) = \phi$ but $3 \in Z_c(A,B,C)$ as this pole is uncontrollable. This section is devoted to the analysis of the closed-loop zeros of the system, and in particular the multiplicity of such zeros.

Definitions (Closed-loop zero multiplicity)

Let $Z_o{}^m(A,B,C)$ be the set of zeros of $|sI_{n_z} - A_{22}|$ counted with appropriate multiplicity (see Results 5,6), and $\hat{Z}(A,B,KC)$ be the set of zeros (counted with appropriate multiplicity) of the polynomial

$$P_K(s) \triangleq \begin{vmatrix} sI_n - A & -B \\ KC & 0 \end{vmatrix} \qquad \ldots(73)$$

where K is an $\ell \times m$ constant matrix, then

$$Z_c{}^m(A,B,C) \triangleq \bigcap_K \hat{Z}(A,B,KC) \qquad \ldots(74)$$

where the intersection runs over all $\ell \times m$ constant matrices K. Moreover λ is said to be a closed-loop zero of $S(A,B,C)$ with multiplicity $\hat{n}_\lambda$ if $\lambda \in Z_c{}^m(A,B,C)$ and λ occurs $\hat{n}_\lambda$ times in $Z_c{}^m(A,B,C)$.

Using the canonical form of Result 5, the polynomial of equation (73) becomes

$$P_K(s) = \begin{vmatrix} sI_n - E^{-1}AE & -E^{-1}B \\ KCE & 0 \end{vmatrix}$$

$$= (-1)^q |sI_{n_z} - A_{22}| \cdot \begin{vmatrix} sI_{n-n_z} - A_{11} & \hat{B} \\ K\hat{C} & 0 \end{vmatrix} \qquad \ldots(75)$$

for some integer q. This immediately yields the relation

$$Z_o^m(A,B,C) \subset Z_c^m(A,B,C) \qquad \ldots(76)$$

so that every open-loop zero λ with open-loop multiplicity n_λ becomes a closed-loop zero with multiplicity $\hat{n}_\lambda \geqslant n_\lambda$.

If $m=\ell$, then

$$P_K(s) = (-1)^q |sI_{n_z} - A_{22}| \cdot |K| \cdot \begin{vmatrix} sI_{n-n_z} - A_{11} & \hat{B} \\ \hat{C} & 0 \end{vmatrix}$$

$$= \alpha |K| \cdot |sI_{n_z} - A_{22}| \qquad \ldots(77)$$

where, by equation (33), α is a constant and non-zero. Hence, the following result is proved,

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<u>Result Eight</u>

If $m=\ell$, then $\qquad Z_o^m(A,B,C) = Z_c^m(A,B,C)$

That is, every open-loop zero $\lambda \in Z_o(A,B,C)$ with multiplicity n_λ is a closed-loop zero with multiplicity $\hat{n}_\lambda = n_\lambda$ and vice versa. The following result yields a construction for $Z_c^m(A,B,C)$ when $m>\ell$. If S_1 and S_2 are finite sequences of complex numbers, of length k_1 and k_2 respectively, then S_1+S_2 will denote the finite sequence of length k_1+k_2 consisting of the elements of S_1 followed by the elements of S_2.

Result Nine

Using the notation of Result Five, then

$$Z_c^m(A,B,C) = Z_o^m(A,B,C) + Z_c^m(A_{11},\hat{B},\hat{C}) \qquad \ldots(78)$$

and, if X_λ is the linear subspace of C^{n-n_z},

$$X_\lambda \triangleq \{x \; ; \; x\in C^{n-n_z} \text{ and } (\lambda I_{n-n_z} - A_{11})x\in R(\hat{B})\} \quad , \qquad \ldots(79)$$

then $Z_c^m(A_{11},\hat{B},\hat{C})$ consists of these eigenvalues λ of A_{11} such that $\dim X_\lambda>\ell$, and λ has multiplicity greater than or equal to $\dim X_\lambda-\ell$ in $Z_c^m(A_{11},\hat{B},\hat{C})$.

Finally, sufficient conditions for $Z_c^m(A_{11},\hat{B},\hat{C}) = \phi$ is either $m=\ell$ or $S(A,B,C)$ is completely state controllable.

Proof

Relation (78) is a direct consequence of the definitions and equation (75).

By Result 5, $S(A_{11},\hat{B},\hat{C})$ has no zeros so that, using Result 1, for any complex number λ,

$$X_\lambda \cap N(\hat{c}) = \{0\} \qquad \ldots(80)$$

It follows directly that $\dim \hat{C}X_\lambda = \dim X_\lambda$ and there exists an ℓxm matrix K_λ such that $\dim K_\lambda\hat{C}X_\lambda = \min\{\ell, \dim X_\lambda\}$. If $\dim X_\lambda\leq\ell$, then

$$X_\lambda \cap N(K_\lambda\hat{C}) = \{0\}$$

so that, by Result 1, $\lambda\notin Z_o(A_{11},\hat{B},K_\lambda\hat{C})$ and hence $\lambda\notin Z_c^m(A,B,C)$ as λ is not a zero of $P_{K_\lambda}(s)$.

If $\dim X_\lambda > \ell$ then, for every K_λ, an $\dim X_\lambda - \ell$ subspace of X_λ lies in the null-space of $K_\lambda \hat{C}$. Hence $\lambda \in Z_c^m(A_{11}, \hat{B}, \hat{C})$, and by (17) and (22), λ has multiplicity greater than or equal to $\dim X_\lambda - \ell$ in $Z_c^m(A_{11}, \hat{B}, \hat{C})$. The fact that only eigenvalues of A_{11} occur in $Z_c^m(A,B,C)$ is proved by noting that, if λ is not an eigenvalue of A_{11}, $(\lambda I_{n-n_z} - A)$ is invertible and hence $\dim X_\lambda = \ell$.

Finally, if $m=\ell$, then Result 8 indicates that $Z_c^m(A_{11}, \hat{B}, \hat{C}) = \phi$. If $S(A,B,C)$ is controllable, then, writing (eqn(32))

$$E^{-1}AE = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} + E^{-1}B\begin{bmatrix} 0 & K \end{bmatrix}$$

and noting that controllability is invariant under state feedback, then $S(A_{11}, \hat{B}, \hat{C})$ is controllable. Let x_j, $1 \leqslant j \leqslant \dim X_\lambda$, be a basis for X_λ and λ an eigenvalue of A_{11}. Then,

$$A_{11}x_j = \lambda x_j + b_j \quad , \quad b_j \in R(\hat{B}) \quad , \quad 1 \leqslant j \leqslant \dim X_\lambda$$

Let P be the linear mapping represented by $x_j \to Px_j = b_j$, then, by extending P to the whole state space, write $P = \hat{B}K_\lambda$ for some matrix K_λ. That is

$$(A_{11} + \hat{B}K_\lambda)x_j = \lambda x_j \quad , \quad 1 \leqslant j \leqslant \dim X_\lambda$$

and X_λ is a subspace of the eigenspace of $A_{11} + \hat{B}K_\lambda$ corresponding to the eigenvalue λ. It follows directly from the observation that $S(A_{11}, \hat{B}, \hat{C})$ is controllable implies $S(A_{11} + \hat{B}K_\lambda, \hat{B}, \hat{C})$ is controllable that, necessarily,

$$\dim X_\lambda \leqslant \ell$$

which proves the result.

QED

Result 9 gives a complete characterization of $Z_c^m(A,B,C)$. From the practical viewpoint however, the important conclusion is that, if either $S(A,B,C)$ is controllable or $m=\ell$, then

$$Z_c{}^m(A,B,C) = Z_o{}^m(A,B,C) \qquad \ldots(81)$$

That is, under almost all practical conditions, the open-loop zero set characterizes completely the closed-loop zero set of the system. If $m=\ell$, then, independent of the proportional controller used, the only zeros generated are the open-loop zeros (with appropriate multiplicity). Equivalently, the only finite poles at infinitely high gain are $\lambda \in Z_o(A,B,C)$ with multiplicity n_λ. If $m>\ell$, then the choice of a particular controller will generate zeros $\lambda \notin Z_o(A,B,C)$ but these are controller dependent and are easily eliminated by variations in controller parameters.

5. Inductive Construction of ω

From the analysis of the previous sections the maximal subspace ω is the fundamental concept used in the proof of all results, and, in particular, the construction of the canonical form. The following result indicates a sequential technique for the construction of ω. The technique is used in the examples of section 7.

Result Ten

If $\lambda \in Z_o(A,B,C)$ and

$$V_o(\lambda) = \{0\} \qquad \ldots(82)$$

$$V_{j+1}(\lambda) \triangleq \{(\lambda I_n - A)^{-1}(V_j(\lambda)+R(B))\} \cap N(c) \quad , \quad j>0 \qquad \ldots(83)$$

then there exists an integer $k(\lambda) \geq 1$ such that $V_{j+1}(\lambda) = V_j(\lambda)$ for all $j \geq k(\lambda)$ and

$$\omega = \bigoplus_{\lambda \in Z_o(A,B,C)} V_{k(\lambda)}(\lambda) \qquad \ldots(84)$$

Proof

Note that $V_j(\lambda) \subset N(c)$, $j \geq 0$, and $V_j(\lambda) \subset V_{j+1}(\lambda)$, $j \geq 0$, for $V_o(\lambda) \subset V_1(\lambda) = \omega_1(\lambda)$

so assume that $V_{j-1}(\lambda)\subset V_j(\lambda)$ for some $j\geq 1$. Take $x\in V_j(\lambda)$, then $(\lambda I_n - A)x\in V_{j-1}(\lambda)+R(B)\subset V_j(\lambda)+R(B)$ ie $x\in V_{j+1}(\lambda)$ and hence $V_j(\lambda)\subset V_{j+1}(\lambda)$. As $N(c)$ is finite dimensional, then $k(\lambda)$ exists and $k(\lambda)\geq 1$ as $\omega_1(\lambda) \neq \{0\}$. Moreover $AV_j(\lambda)\subset V_j(\lambda)+V_{j-1}(\lambda)+R(B) = V_j(\lambda)+R(B)$ so that $V_j(\lambda)\in\omega_B$, from which

$$V \triangleq \sum_{\lambda\in Z_o(A,B,C)} V_{k(\lambda)} \subset \omega \qquad \text{...(85)}$$

A glance at the canonical form with A_{22} in Jordan form indicates that $\omega\subset V$ ie $V = \omega$. In a similar manner, the canonical form indicates that the sum of (85) is direct, which proves the result.

QED

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6. System Zeros and State Feedback

The canonical form of Result 5 can be used to gain some insight into the physical structure of the system and the physical source of open-loop transmission zeros. Write

$$E^{-1}x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} , \quad x_1\in R^{n-n_z} , \quad x_2\in R^{n_z} \qquad \text{...(86)}$$

then, from equations (30)-(32)

$$\dot{x}_1 = A_{11}x_1 + \hat{B}\{u + Kx_2\} \qquad \text{...(87)}$$

$$\dot{x}_2 = A_{22}x_2 + A_{21}x_1 \qquad \text{...(88)}$$

$$y = \hat{C}x_1 \qquad \text{...(89)}$$

A schematic representation of this system is shown in Fig.1, which demonstrates that $S(A,B,C)$ consists of a forward path system $S(A_{11},\hat{B},\hat{C})$ which (Result 5) has no open-loop transmission zeros and a dynamic state feedback $S(A_{22},A_{21},K)$ of state dimension n_z with poles $\lambda\in Z_o(A,B,C)$ with multiplicity n_λ.

Noting that $S(A_{22},A_{21},K)$ is not directly affected by the control input $u(t)$ and makes no direct contribution to the output, then, intuitively, it is seen that the existence of transmission zeros in $S(A,B,C)$ is related to the existence of inherent dynamic feedback within the system structure. To illustrate this point, consider the dynamics of a point thermal reactor. In the absence of delayed neutrons, the dynamics of a small perturbation $n(t)$ in neutron population about a steady state n_o is described by the linear differential equation

$$\ell^* \dot{n}(t) = n_o \, \delta k(t) \qquad \ldots(90)$$

where ℓ^* is the neutron mean life-time, and $k(t)$ is the change in system reactivity. System reactivity is affected by many system parameters including $n(t)$, control input $u(t)$ and xenon-135 fission product $x(t)$,

$$\delta k(t) = \alpha_n n(t) + u(t) + \alpha_x x(t) \qquad \ldots(91)$$

and xenon is generated by the dynamic chain,

$$\dot{i}(t) = -\lambda_1 i(t) + \beta_1 n(t) \qquad \ldots(92)$$

$$\dot{x}(t) = -\lambda_2 x(t) + \lambda_1 i(t) + \beta_2 n(t) \qquad \ldots(93)$$

where $i(t)$ is the iodine-135 concentration in the reactor curve. The system output is $y(t) = n(t)$ and it is easily seen that the states $i(t),x(t)$ represent inherent dynamic feedback effects as they make no direct contribution to the output and are not affected directly by the control input $u(t)$. From the above comments we expect the zeros of the system to be $-\lambda_1$ and $-\lambda_2$, which is easily verified from the state space form,

$$\frac{d}{dt}\begin{bmatrix} n \\ i \\ x \end{bmatrix} = \begin{bmatrix} n_o\alpha_n/\ell^* & 0 & n_o\alpha_x/\ell^* \\ \beta_1 & -\lambda_1 & 0 \\ \beta_2 & \lambda_1 & -\lambda_2 \end{bmatrix}\begin{bmatrix} n \\ i \\ x \end{bmatrix} + \frac{n_o}{\ell^*}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \quad \ldots(94)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} n & i & x \end{bmatrix}^T \quad \ldots(95)$$

which (Result 5) is in Zero canonical form with $n_z = 2$,

$$A_{22} = \begin{bmatrix} -\lambda_1 & 0 \\ \lambda_1 & -\lambda_2 \end{bmatrix} \quad \ldots(96)$$

so that $Z_o^m(A,B,C) = \{-\lambda_1, -\lambda_2\}$ (Result 6). This result is easily checked from the system transfer function

$$\frac{n(s)}{u(s)} = \frac{n_o\alpha_n}{\ell^*} / \{s + \frac{n_o}{\ell^*}\alpha_n + \frac{\alpha_x}{(s+\lambda_2)}\{\frac{\lambda_1\beta_1}{s+\lambda_1} + \beta_2\}\} \quad \ldots(97)$$

7. Illustrative Examples

Example 1

Consider the example of section 2,

$$g(s) = (s+1)^2/s^3$$

with state space representation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

so that $Z_o(A,B,C) = \{-1\}$, rank $P(-1) = 3$ and, by Result 3, $\omega_1(-1)$ is a one dimensional subspace of $N(c)$ spanned by the solution of the equation

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} x_1 = - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

ie $$x_1 = [0 \quad +1 \quad +1]^T$$

Using result 10, $V_1(-1) = \omega_1(-1)$, and to construct $V_2(-1)$, we solve the equation (α_1, α_2 arbitrary)

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} x = \alpha_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} , \quad x \in N(c)$$

which has only two linearly independent solutions $x = x_1$ and

$$x = x_2 = [0 \quad 0 \quad 1]^T$$

As span $\{x_1, x_2\} = N(c)$ then $\omega = V_2(-1)$, and, by Result 5, the similarity transformation required to transform the system to zero canonical form is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

i.e. $$CE = [1 \quad 0 \quad 0]$$

$$E^{-1}B = [1 \quad 0 \quad 0]^T$$

$$E^{-1}AE = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ -3 & 1 & -1 \end{bmatrix}$$

so that

$$A_{22} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

and -1 is a zero of the system with multiplicity 2, in agreement with the transfer function representation.

Finally, it is interesting to note that the number of zeros can be easily deduced from Result 7. Noting that $CB = 1 \neq 0$, then $k = 1$ and $R(B) \cap N(c) = \{0\}$ so that, as $m=\ell=1$, $n_z = n-m = 2$. Also, by Result 8, $Z_c^{\ m} = Z_o^{\ m}$ and the only finite poles at infinitely high gain proportional feedback control are -1 with multiplicity 2.

Example 2

Consider the non-square system,

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = x \qquad (\text{ie} \quad C = I_3)$$

then rank $P(s) = 4$ for all s and the system has no open-loop transmission zeros. Let

$$K = [k_1 \quad k_2 \quad k_3]$$

be a squaring down matrix, then

$$\begin{vmatrix} s-1 & 0 & 0 & -1 \\ 0 & s-2 & 0 & -1 \\ 0 & 0 & s-3 & 0 \\ k_1 & k_2 & k_3 & 0 \end{vmatrix}$$

$$= (s-3) \begin{vmatrix} s-1 & 0 & -1 \\ 0 & s-2 & -1 \\ k_1 & k_2 & 0 \end{vmatrix}$$

$$= (s-3)\{s(k_2+k_1) - (k_2+2k_1)\}$$

so that $Z_c^{\ m} = \{-3\}$. Note that $Z_c^{\ m}$ is non-empty as the system is uncontrollable (Result 9), and that the gain term k_3 has no effect on the closed-loop zero positions.

8. Summary and Conclusions

Using a definition proposed by Davison and Wang (1974), a geometric definition of system transmission zeros has been proposed for the case of $m \geq \ell$. A distinction has been noted between open and closed-loop zeros of a system and it has been demonstrated that the two classes of zeros coincide if either $m=\ell$ or the system is controllable.

The basic tool in the characterization and analysis in the maximal subspace $\omega \subset N(c)$ satisfying

$$A\omega \subset \omega + R(B)$$

when, provided the system has only a finite number of transmission zeros, the fact that $\omega \cap R(B) = \{0\}$ leads directly to a canonical decomposition of (A,B,C) which illustrates the physical source of zeros as due to inherent dynamic state feedback within the system structure.

The total number of open-loop transmission zeros has been shown to be $n_z = \dim \omega$ and the multiplicity of each zero has been identified using a Jordan-type decomposition of the restriction of A to ω. Useful bounds have been obtained for n_z in Results 4 and 7 using parameters easily calculated from the system transfer function matrix $G(s)$.

Finally, the geometric techniques used in decoupling theory and pole allocation (Morse and Wonham, 1971) have been shown to be a powerful tool in the characterization and analysis of system zeros. Future work should make possible deeper results and may provide insight into feedback design and compensation.

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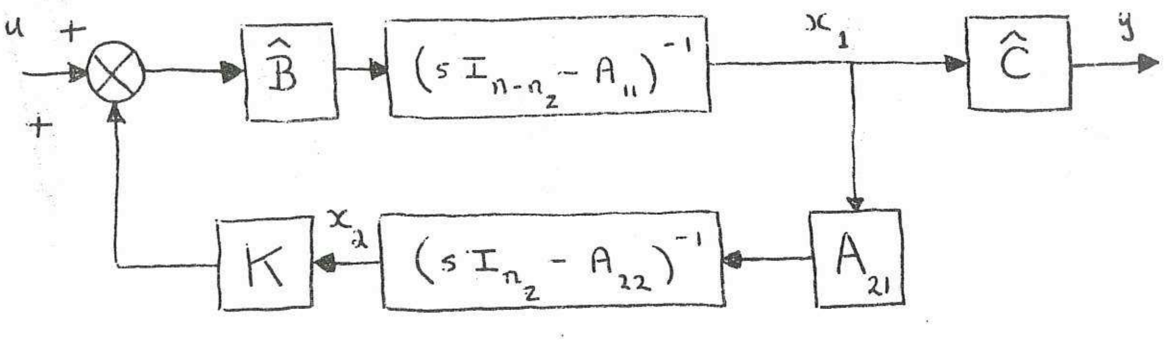

Fig. 1. State Feedback Representation of the Canonical Form.