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Identification of nonlinear heat transfer laws from boundary observations

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Abstract

We consider the problem of identifying a nonlinear heat transfer law at the boundary, or of the temperature-dependent heat transfer coefficient in a parabolic equation from boundary observations. As a practical example, this model applies to the heat transfer coefficient that describes the intensity of heat exchange between a hot wire and the cooling water in which it is placed. We reformulate the inverse problem as a variational one which aims to minimize a misfit functional and prove that it has a solution. We provide a gradient formula for the misfit functional and then use some iterative methods for solving the variational problem. Thorough investigations are made with respect to several initial guesses and amounts of noise in the input data. Numerical results show that the methods are robust, stable and accurate.

Keywords: Inverse problem; Nonlinear boundary condition; Heat transfer law.

1 Introduction

There are many physical phenomena occurring at high temperatures/high pressures or, in hostile environments, e.g. in combustion chambers, gas turbines, cooling steel or hot glass processes, gas-quenching in furnaces, etc. in which either the actual method of heat and mass transfer is not known, or it cannot be assumed that the governing boundary law has a simple form, e.g. linear Newton's law of cooling or, fourth-order power Stefan-Boltzmann's black-body radiation law. In such situations, we model these as an inverse problem of identifying a nonlinear heat transfer law at the boundary, or of the temperature-dependent heat transfer coefficient. In other fields of application, this formulation may also be considered as a model for the concentration of gaseous diffusion with an unknown chemical reaction at surface or, for the population density with an unspecified migration law at the boundary, [23].

In [11], Pilant and Rundell considered the problem of determining the heat transfer law function $g(\cdot)$ and the temperature $u(x,t)$ in the initial boundary value problem

$$u_t - u_{xx} = \gamma(x,t), \quad 0 < x < 1, 0 < t < T, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{1.2}$$

$$u_x(0,t) = g(u(0,t)), \quad 0 \le t \le T, \tag{1.3}$$

$$-u_x(1,t) = g(u(1,t)), \quad 0 \le t \le T \tag{1.4}$$

from the additional condition

$$u(0,t) = h(t), \tag{1.5}$$

where the functions γ, u_0 and h representing a heat source, an initial temperature and a boundary temperature, respectively, are given. Note that from (1.1) and (1.5) we obtain $u_x(0,t) = g(h(t))$ for $t \in [0,T]$. Under certain conditions, the authors of [11] proved that there exists a unique pair (u,g) to (1.1)–(1.5) over the interval $0 \le t \le t^*$, for some $t^* \in (0,T]$. They also proposed an iterative method for this inverse problem and tested it briefly on computer. Later on, Rundell and Yin [21] studied a similar problem, but in multidimensions. Namely, for $T > 0$ and $Q = \Omega \times (0,T]$ with Ω being a bounded domain in $\mathbb{R}^m$, they considered the problem of finding a pair of functions $u(x,t)$ and $g(s)$ defined on $\overline{Q}$ and $[A,B]$, respectively, which satisfies the equations

$$u_t - \Delta u = \gamma(x,t) \quad \text{in } Q, \tag{1.6}$$

$$u(x,0) = u_0(x) \quad \text{on } \overline{\Omega}, \tag{1.7}$$

$$\frac{\partial u}{\partial n} = g(u) + \varphi \quad \text{on } S := \partial\Omega \times [0,T], \tag{1.8}$$

and the additional condition

$$u(\xi_0, t) = h(t), \quad t \in [0,T], \tag{1.9}$$

where the functions γ, u_0, φ and h are given, ξ_0 is a fixed point of $\partial\Omega$, n is the outer normal to $\partial\Omega$, $A = \min_{\overline{Q}} u(x,t)$ and $B = \max_{\overline{Q}} u(x,t)$. Under some conditions, the authors of [21] established a stability estimate for g and from that they obtained the uniqueness of a solution to (1.6)–(1.9). It is clear that the function g can be determined only in the interval $[A,B]$, but not on the whole real axis $\mathbb{R}$. However, in [3], Choulli raised the question: how many measurements do we need to recover $g(s)$ for $s \in \mathbb{R}$? Choulli proved that: (i) if all lateral boundary measurements are available and g' is bounded, then we have uniqueness; (ii) if lateral boundary measurements are generated by a one-dimensional vector space, then we also have uniqueness, provided that $g = g_0 + g_1$, where g_0 is known and g_1 is unknown with no accumulation point of zeros. In the above context, it is also worth citing the natural linearization numerical algorithm of [4] for the identification of the nonlinear heat transfer law $g(u)$ in (1.8) when, instead of the single measurement (1.9), one has available the overdetermined measurement of the temperature u on the whole boundary S.

Finally, note that the identification of the heat transfer law $g(u)$ in (1.8) is one-dimensional although the underlying temperature state $u(x,t)$ may depend on the time t and on $x := (x_1, \ldots, x_m)$.

Similar problems have been investigated in a series of papers by Tröltzsch and Rösch [8], [20], [14]–[19]. Namely, these authors considered the problem of determining the heat transfer coefficient $\sigma(u)$ in the initial boundary value problem

$$u_t - \Delta u = 0 \quad \text{in } Q, \tag{1.10}$$

$$u(x,0) = u_0(x) \quad \text{on } \overline{\Omega}, \tag{1.11}$$

$$\frac{\partial u}{\partial n} = \sigma(u(\xi,t))(u_\infty - u(\xi,t)) \quad \text{on } S = \partial\Omega \times [0,T], \tag{1.12}$$

where u_∞ is the ambient temperature which is assumed a given constant, from various additional conditions: $u(x,t)$ is given in the whole domain Q, or $u(x,t_i)$ are given at fixed time points $t_i, i = 1, \ldots, L$, [20], [14], or u is given on the whole boundary S, [17]. They reformulated the inverse problem as an optimal control problem and proved the Fréchet differentiability of the functional to be minimized. They also solved the problem numerically by iterative methods. We also note that in some continuous casting of steel processes, the heat transfer coefficient σ in (1.12) may depend on both temperature u and time t, [5], but the investigation of this more complex inverse problem is deferred to a future work.

Later on, Lesnic and co-authors [9], [10], Janicki and Kindermann [7] also attempted to solve the inverse problems (1.1)–(1.5) and (1.10)–(1.12) numerically. For more physical meaning of these inverse problems in heat transfer, we refer the reader to the aforementioned references.

In this paper, we consider the inverse problem of determining the function $g(\cdot,\cdot)$ in the initial boundary value problem, [22],

$$u_t - \Delta u = 0 \quad \text{in } Q, \tag{1.13}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{1.14}$$

$$\frac{\partial u}{\partial n} = g(u,f) \quad \text{on } S \tag{1.15}$$

from the additional condition (1.9). Here,

> $g : I \times I \to \mathbb{R}$ (with I a subinterval of $\mathbb{R}$) is assumed to be locally Lipschitz continuous, monotone decreasing in u and increasing in f and to satisfy $g(u,u) = 0$, u_0 and f are given functions with range in I belonging, respectively, to $L^2(\Omega)$ and $L^2(S)$.

Throughout, we assume that g satisfies this condition, and write that as $g \in \mathcal{A}$. Usually, the heat transfer coefficient is identified as a function of time or space, [6], but in this paper we refer to applications where it depends on the boundary temperature.

The model (1.13)–(1.15) describes many practical situations, [1, 22]. It includes the linear boundary condition $g(u,f) = c(f-u)$ with c a positive constant. It includes nonlinear conditions of the form $g(u,f) = \phi(f) - \phi(u)$, with ϕ Lipschitz and monotone increasing on I; these include the Stefan-Boltzmann radiation condition for which $\phi(w) = w^4$ and $I = [0,\infty)$, the Michaelis-Menten law of enzyme diffusion for which $\phi(u) = cu/(u+k)$, where c and k are positive constants. It also covers the case $g(u,f) = \psi(f-u)$, where ψ is Lipschitz and monotone increasing on the "difference interval" $I - I$; in particular one can take $\psi(w) = w^{5/4}$ for $w > 0$, and $= 0$ for $w < 0$, which relates to natural convection.

As the additional condition (1.9) is pointwise, it cannot be defined if the solution is understood in the weak sense, as we intend to use in this paper. Therefore, we consider the following alternative conditions.

1) Observations on a part of the boundary:

$$u|_\Sigma = h(\xi,t), \quad (\xi,t) \in \Sigma, \tag{1.16}$$

where $\Sigma = \Gamma \times (0,T]$, Γ is a non-zero measure part of $\partial\Omega$;

2) Boundary integral observations:

$$lu := \int_{\partial\Omega} \omega(\xi) u(\xi,t) d\xi = h(t), \quad t \in (0,T], \tag{1.17}$$

where ω is a non-negative function defined on $\partial\Omega$, $\omega \in L^1(\partial\Omega)$ and $\int_{\partial\Omega} \omega(\xi)d\xi > 0$. We note that if we take ω as approximations to the Dirac δ-function, then the observations of this kind can be considered as an averaged version of (1.9). Such integral observations are alternatives to model pointwise measurements (thermocouples have non-zero width) and they will make variational methods for the inverse problem much easier.

This paper is organized as follows. In the next section we will outline some well-known results on the direct problem (1.13)–(1.15). Section 3 is devoted to the variational method for solving the inverse problem (1.13)–(1.15), (1.17), and (1.13)–(1.15), (1.16), where we formulate the method and prove the existence result for it, as well as deliver the formula for the gradient of the functional to be minimized. As a by-product, we derive also the variational method for solving the inverse problem (1.10)–(1.12), (1.17). Section 4 is devoted to presenting and discussing thoroughly the numerical results, whilst Section 5 presents the conclusions of this study.

2 Direct problem

In this section, we outline the results on the direct problem (1.13)–(1.15), [22]. We use the standard Sobolev spaces $H^1(\Omega)$, $H^{1,0}(Q)$ and $H^{1,1}(Q)$ (see e.g., [24, p. 111]).

For a Banach space B, we define

$$L^2(0,T;B) = \{u : u(t) \in B \ \text{ a.e. } t \in (0,T) \text{ and } \|u\|_{L^2(0,T;B)} < \infty\},$$

with the norm

$$\|u\|^2_{L^2(0,T;B)} = \int_0^T \|u(t)\|^2_B dt.$$

In the sequel, we shall use the space $W(0,T)$ defined as

$$W(0,T) = \{u : u \in L^2(0,T;H^1(\Omega)), u_t \in L^2(0,T;(H^1(\Omega))')\},$$

equipped with the norm

$$\|u\|^2_{W(0,T)} = \|u\|^2_{L^2(0,T;H^1(\Omega))} + \|u_t\|^2_{L^2(0,T;(H^1(\Omega))')}.$$

We take the convention of notation in [22]: letting J be a subinterval of I we shall use J as a subscript on function spaces to denote the subset of functions having essential range in J.

Definition 2.1. *Let $u_0 \in L^2_I(\Omega)$ and $f \in L^2_I(S)$. Then $u \in H^{1,0}_I(Q)$ is said to be a weak solution of* (1.13)–(1.15) *if $g(u,f) \in L^2(S)$ and for all $\eta \in H^{1,1}(Q)$ satisfying $\eta(\cdot,T) = 0$,*

$$\int_Q \Big(-u(x,t)\eta_t(x,t) + \nabla u(x,t)\cdot\nabla\eta(x,t)\Big)dxdt = \int_\Omega u_0(x)\eta(x,0)dx + \int_S g(u(\xi,t),f(\xi,t))\eta(\xi,t)d\xi dt. \tag{2.1}$$

In [22] the following results have been proved.

Theorem 2.2. *Let J be a subinterval of I such that $g(u,f)$ is uniformly Lipschitz continuous on $J \times J$. Then, for every u_0 in $L^2_J(\Omega)$ and f in $L^2_J(S)$, the problem* (1.13)–(1.15) *has a unique weak solution.*

Theorem 2.3. *Let u be a weak solution of* (1.13)–(1.15). *If u_0 and f are bounded below by m (or above by M) almost everywhere, the same is true for u.*

We note that in [22] the strict monotonicity of g is assumed. However, it is in fact not needed.

We have also stronger results.

Theorem 2.4. *([2, 12, 13]) If $u_0 \in C(\overline{\Omega})$ and $f \in L^\infty(S)$, then there exists a unique solution of (1.13)–(1.15) in $W(0,T) \cap L^\infty(Q)$. This solution is continuous in $\overline{Q}$ and there exists a positive constant c independent of u_0, f such that*

$$\|u\|_{W(0,T)} + \|u\|_{C(\overline{Q})} \le c\Big(\|u_0\|_{C(\overline{\Omega})} + \|g\|_{L^\infty(I\times I)}\Big). \tag{2.2}$$

From now on, to emphasize the dependence of the solution u on the coefficient g, we write $u(g)$ or $u(x,t;g)$ instead of u. We shall prove that the mapping $u(g)$ is Fréchet differentiable with respect to g. In doing so, first we prove that this mapping is Lipschitz continuous. To this purpose, we assume that

$g(u,f)$ is continuously differentiable with respect to u in I and denote that by $g \in \mathcal{A}_1$.

Furthermore, since f is fixed, we shall write $g(u)$ instead of $g(u,f)$, but we always keep in mind that g depends on the both variables, and f has the same essential range in I as u_0 does. Also, as we consider g as a function of one variable, we write $\dot{g}(u)$ instead of $dg(u)/du$.

Lemma 2.5. *Let $g^1, g^2 \in \mathcal{A}_1$ such that $g^1 - g^2 \in \mathcal{A}$. Denote the solutions of (1.13)–(1.15) corresponding to g^1 and g^2 by u^1 and u^2, respectively. Further, suppose that $u_0 \in L^2_I(\Omega)$ and $f \in L^\infty_I(S)$. Then there exists a constant c such that*

$$\|u^1 - u^2\|_{W(0,T)} + \|u^1 - u^2\|_{C(\overline{Q})} \le c\|g^1 - g^2\|_{L^\infty(I\times I)}. \tag{2.3}$$

Proof. Denote $v = u^1 - u^2$. Then, v satisfies the problem

$$v_t - \Delta v = 0 \quad \text{in } Q, \tag{2.4}$$

$$v(x,0) = 0 \quad \text{in } \Omega, \tag{2.5}$$

$$\frac{\partial v}{\partial n} = g^1(u^1) - g^2(u^2) \quad \text{on } S. \tag{2.6}$$

Since $v(x,0) = 0$,

$$\begin{aligned} g^1(u^1) - g^2(u^2) &= (g^1(u^1) - g^1(u^2)) + (g^1(u^2) - g^2(u^2)) \\ &= \int_{u_2}^{u_1} \dot{g}^1(\theta)d\theta + g^1(u^2) - g^2(u^2), \end{aligned}$$

$g^1(u^2) - g^2(u^2) \in L^\infty(S)$ and $\dot{g}^1 < 0$, from Theorem 2.4 (see also [13, Proposition 3.3]) applied to the problem (2.4)-(2.6), we have $v \in W(0,T) \cap C(\overline{Q})$ and the estimate (2.3). □

Now we prove that $u(g)$ is Fréchet differentiable with respect to g. In doing so, we introduce the sensitivity problem:

$$\eta_t - \Delta\eta = 0 \quad \text{in } Q, \tag{2.7}$$

$$\eta(x,0) = 0 \quad \text{in } \Omega, \tag{2.8}$$

$$\frac{\partial \eta}{\partial n} = \dot{g}(u(g)) + z(u(g)) \quad \text{on } S. \tag{2.9}$$

Here, $z \in \mathcal{A}_1$ and $g \in \mathcal{A}_1$. Since $\eta(x,0) = 0$, there exists a unique solution of (2.7)–(2.9) in $W(0,T) \cap L^\infty(Q)$ which belongs to $C(\overline{Q})$. From the proof of Lemma 2.5 we see that η is a bounded linear operator mapping $z \in \mathcal{A}_1$ into $W(0,T)$.

We have the following result.

Theorem 2.6. *Let $u_0 \in L^2_I(\Omega)$, $f \in L^\infty_I(S)$ and $g \in \mathcal{A}_1$. Then the mapping $g \mapsto u(g)$ is Fréchet differentiable in the sense that for any $g, g+z \in \mathcal{A}_1$ there holds*

$$\lim_{\|z\|_{L^\infty(I\times I)}\to 0} \frac{\|u(g+z) - u(g) - \eta\|_{W(0,T)}}{\|z\|_{C^1(I)}} = 0. \tag{2.10}$$

Proof. Set $w = u(g+w) - u(g) - \eta$, where η is the solution of problem (2.7)–(2.9). We see that w is the solution of the problem

$$w_t - \Delta w = 0 \quad \text{in } Q, \tag{2.11}$$
$$w(x,0) = 0 \quad \text{in } \Omega, \tag{2.12}$$
$$\frac{\partial w}{\partial n} = g(u(g+z)) + z(u(g+z)) - g(u(g)) - \dot{g}(u(g))\eta - z(u(g)) \quad \text{on } S. \tag{2.13}$$

We have

$$\begin{aligned} &g(u(g+z)) + z(u(g+z)) - g(u(g)) - \dot{g}(u(g))\eta - z(u(g)) \\ &= \dot{g}(u(g))(u(g+z) - u(g) - \eta) \\ &\quad + g(u(g+z)) - g(u(g)) - \dot{g}(u(g))(u(g+z) - u(g)) \\ &\quad + z(u(g+z)) - z(u(g)). \end{aligned}$$

Thus, w is the solution of the problem

$$w_t - \Delta w = 0 \quad \text{in } Q, \tag{2.14}$$
$$w(x,0) = 0 \quad \text{in } \Omega, \tag{2.15}$$
$$\begin{aligned} \frac{\partial w}{\partial n} - \dot{g}(u(g))w &= g(u(g+z)) - g(u(g)) \\ &\quad - \dot{g}(u(g))\big(u(g+z) - u(g)\big) + z(u(g+z)) - z(u(g)) \quad \text{on } S. \end{aligned} \tag{2.16}$$

Since g is continuously differentiable, we have

$$\begin{aligned} \|g(u(g+z)) - g(u(g)) - \dot{g}(u(g))(u(g+z) - u(g))\|_{L^\infty(S)} &= o(\|u(g+z)|_S - u(g)|_S\|_{L^\infty(S)} \\ &= o(\|z\|_{L^\infty(I)}), \end{aligned}$$

due to Theorem 2.4. Furthermore,

$$\begin{aligned} \|z(u(g+z)) - z(u(g))\|_{L^\infty(S)} &= \Big\| \int_{u(g)}^{u(g+z)} \dot{z}(\theta)d\theta \Big\|_{L^\infty(S)} \\ &\le c\|\dot{z}\|_{L^\infty(I)}\|z\|_{L^\infty(I)} = o(\|z\|_{C^1(I)}). \end{aligned} \tag{2.17}$$

From the estimates (2.17) and the estimates in Theorem 2.4 we arrive at (2.10). Since η is a bounded linear operator mapping $z \in \mathcal{A}_1$ into $W(0,T)$, the theorem is proved. ∎

3 Variational method

3.1 Inverse problem (1.13)–(1.15) and (1.17) over $\mathcal{A}_1$

In this subsection we study a variational method for the inverse problem (1.13)–(1.15) and (1.17). We minimize the functional

$$J(g) = \frac{1}{2}\|lu(g) - h\|^2_{L^2(0,T)} \tag{3.1}$$

over $\mathcal{A}_1$. First, we prove that this functional is Fréchet differentiable and derive a formula for the gradient. Second, under some stronger conditions on g we shall prove that there exists a solution of the variational problem.

Let $\epsilon > 0$ and z be in $\mathcal{A}_1$ such that $g + \epsilon z \in \mathcal{A}_1$ for $0 \le \epsilon \le \epsilon_0$, ϵ_0 is given and sufficiently small. Denoting u^ϵ the solution of (1.13)–(1.15) with g replaced by $g + \epsilon z$, we have

$$\begin{aligned} J(g+\epsilon z) - J(g) &= \frac{1}{2}\|lu(g+\epsilon z) - h\|^2_{L^2(0,T)} - \frac{1}{2}\|lu(g) - h\|^2_{L^2(0,T)} \\ &= \frac{1}{2}\|l(u^\epsilon - u(g))\|^2_{L^2(0,T)} + < l(u^\epsilon - u(g)), lu(g) - h >_{L^2(0,T)} . \end{aligned}$$

Letting $\epsilon \to 0$, in virtue of Lemma 2.5, we have $\|l(u^\epsilon - u(g))\|^2_{L^2(0,T)} = o(\|z\|_{L^\infty(I)})$. On the other hand, since $u(g)$ is Fréchet differentiable, $J(g)$ is also Fréchet differentiable and its gradient has the form

$$\begin{aligned} J'(g)z &= < l(\dot{u}(g)z), lu(g) - h >_{L^2(0,T)} \\ &= \int_0^T \Big(\int_{\partial\Omega} \omega(\xi)\eta(\xi,t)d\xi \Big) \Big(\int_{\partial\Omega} \omega(\xi) u(g)|_S d\xi - h(t) \Big) dt, \end{aligned} \tag{3.2}$$

where η is the solution of the sensitivity problem (2.7)–(2.9).

Introduce the adjoint problem

$$-\varphi_t - \Delta\varphi = 0 \quad \text{in } Q, \tag{3.3}$$

$$\varphi(x,T) = 0 \quad \text{in } \Omega, \tag{3.4}$$

$$\frac{\partial\varphi}{\partial n} = \dot{g}(u(g))\varphi + \omega(\xi)\Big(\int_{\partial\Omega} \omega(\xi)u(g)|_S d\xi - h(t) \Big) \quad \text{on } S. \tag{3.5}$$

Since $\dot{g}(u(g)) < 0$ and $\omega(\xi)\Big(\int_{\partial\Omega} \omega(\xi)u(g)|_S d\xi - h(t) \Big) \in L^2(S)$, this problem has a unique weak solution in $W(0,T)$ and due to Green's formula [24, Theorem 3.18], we have

$$\int_0^T \Big(\int_{\partial\Omega} \omega(\xi)\eta(\xi,t)d\xi \Big) \Big(\int_{\partial\Omega} \omega(\xi)u(g)|_S d\xi - h(t) \Big) dt = \int_S z(u(g))\varphi(\xi,t) d\xi dt.$$

Thus,

$$J'(g)z = \int_S z(u(g))\varphi(\xi,t)d\xi dt. \tag{3.6}$$

We summarize this result as follows.

Theorem 3.1. *The functional $J(g)$ is Fréchet differentiable in $\mathcal{A}_1$ and its gradient has the form* (3.6).

From this statement, we can derive the necessary first order optimality condition.

Theorem 3.2. *Let $g^* \in \mathcal{A}_1$ be a minimizer of the function (3.1) over $\mathcal{A}_1$. Then for any $z = g - g^* \in \mathcal{A}_1$,*

$$J'(g^*)z = \int_S z(u(g^*))\varphi(\xi, t; g^*)d\xi dt \geq 0, \tag{3.7}$$

where $\varphi(x, t; g^*)$ is the solution of the adjoint system (3.3)–(3.5) with $g = g^*$.

Now we prove the existence of a minimizer of the function (3.1) over an admissible set. We introduce the set $\mathcal{A}_2$ as follows ([20]),

$$\begin{aligned}\mathcal{A}_2 := \Big\{ & g \in C^{1,\nu}[I], m_1 \leq g(u) \leq M_1, M_2 \leq \dot{g}(u) \leq 0, \forall u \in I, \\ & \sup_{u_1, u_2 \in I} \frac{|\dot{g}(u_1) - \dot{g}(u_2)|}{|u_1 - u_2|^\nu} \leq C \Big\}.\end{aligned} \tag{3.8}$$

Here, ν, m_1, M_1, M_2 and C are given.

Suppose that $u_0 \in C^\beta(\overline{\Omega})$ for some constant $\beta \in (0, 1]$. Then, due to [12, Corollary 3.2], $u \in C^{\gamma,\gamma/2}(\overline{Q})$ for some $\gamma \in (0, 1)$.

Following [20], set

$$T_{ad} := \Big\{ (g, u(g)) : g \in \mathcal{A}_2; u \in C^{\gamma,\gamma/2}(\overline{Q}) \Big\}.$$

Lemma 3.3. *The set T_{ad} is precompact in $C^1[I] \times C(\overline{Q})$.*

Proof. The set $\mathcal{A}_2$ is compact in $C^1[I]$ (see [20]). Due to [12, Corollary 3.2], when $g \in \mathcal{A}_2$ the solution $u(g)$ is bounded in $C^{\gamma,\gamma/2}(\overline{Q})$, which is compactly imbedded in $C(\overline{Q})$. Hence, T_{ad} is precompact. □

Theorem 3.4. *The set T_{ad} is closed in $C^1[I] \times C(\overline{Q})$.*

Proof. Let (g_n, u_n) be a convergent sequence in T_{ad} with the limit (g, u). We shall prove that $u = u(g)$. Since $g_n \in \mathcal{A}_2$, following [20], g also belongs to $\mathcal{A}_2$. Besides, since u_0 and f have the essential range in I, so do the functions u_n and u. It follows that

$$\begin{aligned}|g(u) - g_n(u_n)| &\leq |g(u) - g(u_n)| + |g(u_n) - g_n(u_n)| \\ &\leq M_1 \|u - u_n\|_{C(\overline{Q})} + \|g - g_n\|_{L^\infty(I)}.\end{aligned}$$

Hence, $g_n(x, t) = g_n(u_n)$ converges uniformly to $g(x, t) = g(u)$. From Definition 2.1 we see that $u = u(g)$. □

Since $J(g)$ is continuous, from the last theorem we have the following result.

Theorem 3.5. *The problem of minimizing $J(g)$ over $\mathcal{A}_2$ admits at least one solution.*

3.2 Inverse problem (1.13)–(1.15) and (1.16) over $\mathcal{A}_1$

To solve the inverse problem (1.13)–(1.15) and (1.16), we approach it similarly: minimize the functional

$$J(g) = \frac{1}{2} \|u(g) - h(\cdot, \cdot)\|^2_{L^2(\Sigma)} \tag{3.9}$$

over $\mathcal{A}_1$. All the above results are valid for this functional, except for the formula of the gradient of J. To obtain it, we introduce the adjoint problem

$$-\varphi_t - \Delta\varphi = 0 \quad \text{in } Q, \tag{3.10}$$

$$\varphi(x,T) = 0 \quad \text{in } \Omega, \tag{3.11}$$

$$\frac{\partial \varphi}{\partial n} = \dot{g}(u(g))\varphi + \Big(u(\xi,t) - h(\xi,t)\Big)\chi_\Sigma(\xi,t) \quad \text{on } S. \tag{3.12}$$

Here, χ_Σ is the characteristic function of Σ: $\chi_\Sigma(\xi,t) = 1$ if $(\xi,t) \in \Sigma$, $= 0$ otherwise. Taking z as in the previous subsection, we obtain that the gradient of $J(g)$ has the same form of (3.6).

3.3 Inverse problem (1.10)–(1.12) and (1.17) over $\mathcal{A}_2$

As a by-product, we consider now the variational method for (1.10)–(1.12) and (1.17) under the condition that $\sigma \in \mathcal{A}_2$. We denote the solution of (1.10)–(1.12) by $u(\sigma)$: A function $u \in H^{1,0}(Q)$ is said to be a weak solution of (1.10)–(1.12) if for all $\eta \in H^{1,1}(Q)$ satisfying $\eta(\cdot,T) = 0$,

$$\int_Q \Big(- u(x,t)\eta_t(x,t) + \nabla u(x,t) \cdot \nabla \eta(x,t)\Big) dxdt = \int_\Omega u_0(x)\eta(x,0)dx + \int_S \sigma(u(\xi,t))(u_\infty - u(\xi,t))\eta(\xi,t)d\xi dt. \tag{3.13}$$

If we suppose that $u_0 \in C(\overline{\Omega})$, due to [2, 12, 13], there exists a weak solution of (1.10)–(1.12) in $W(0,T) \cap L^\infty(Q)$ which belongs to $C(\overline{Q})$ and if $u_0 \in C^\beta(\overline{\Omega})$ for some constant $\beta \in (0,1]$, then $u \in C^{\gamma,\gamma/2}(\overline{Q})$ for some $\gamma \in (0,1)$. Furthermore, as noted in [20], due the maximum principle, $\min\{u_\infty, \inf_{x\in\Omega} u_0(x)\} \le u(x,t) \le \max\{u_\infty, \sup_{x\in\Omega} u_0(x)\}$.

Now we consider the problem of minimizing the functional

$$J(\sigma) = \frac{1}{2}\|lu(\sigma) - h\|^2_{L^2(0,T)} \tag{3.14}$$

over $\mathcal{A}_2$. It can be proved that there exists a solution of this minimization problem.

It is proved in [15] that the mapping from $\sigma \in C^1(I)$ to $u(\sigma) \in C(Q)$ is Fréchet differentiable. Here, $I := \Big[\min\{u_\infty, \inf_{x\in\Omega} u_0(x)\}, \max\{u_\infty, \sup_{x\in\Omega} u_0(x)\}\Big]$. We note however that this result can be proved also by the same way as above. If we take the variation z as in §3.1, then the Fréchet derivative $\eta = \dot{u}(\sigma)z$ satisfies the sensitivity problem ([15])

$$\eta_t - \Delta\eta = 0 \quad \text{in } Q, \tag{3.15}$$

$$\eta(x,0) = 0 \quad \text{in } \Omega, \tag{3.16}$$

$$\frac{\partial \eta}{\partial n} = \Big(\dot{\sigma}(u(\sigma))(u_\infty - u(\sigma)) - \sigma(u)\Big)\eta + z(u(\sigma))\Big(u_\infty - u(\sigma)\Big) \quad \text{on } S. \tag{3.17}$$

Due to [2, 12, 13], there exists a unique weak solution of (3.15)–(3.17) in $W(0,T) \cap L^\infty(Q)$ which belongs to $C(\overline{Q})$.

Now we derive a formula for the gradient of J. In doing so, we introduce the adjoint problem:

$$-\varphi_t - \Delta\varphi = 0 \quad \text{in } Q, \tag{3.18}$$

$$\varphi(x,T) = 0 \quad \text{in } \Omega, \tag{3.19}$$

$$\frac{\partial \varphi}{\partial n} = \Big(\dot{\sigma}(u(\sigma))(u_\infty - u(\sigma)) - \sigma(u)\Big)\varphi + \omega(\xi)\Big(\int_{\partial\Omega} \omega(\xi)u(\xi,t;\sigma)d\xi - h(t)\Big) \quad \text{on } S. \tag{3.20}$$

Similarly to the above, we can prove that

$$J'(\sigma)z = \int_S z(u(\sigma))\Big(u_\infty - u(\sigma)\Big)\varphi(\xi,t)d\xi dt. \tag{3.21}$$

4 Numerical results

We tested our algorithms for the two-dimensional domain $\Omega = (0,1)\times(0,1)$ and $T = 1$. For the temperature we take the exact solution to be given by,

$$u_{exact}(x,t) = \frac{100}{4\pi t}\exp\left(-\frac{|x-x_0|^2}{4t}\right), \tag{4.1}$$

where $x_0 = (-2;-2)$. This gives the intial condition (1.14) given by $u(x,0) = u_0(x) = 0$. From (4.1) it is easy to check that the minimum of u_{exact} occurs at $t = 0$ giving $A = 0$, whilst the maximum of u_{exact} occurs at $t = T = 1$ and $x = (0;0)$ giving $B = \frac{100}{4\pi}e^{-2}$.

We consider the physical examples of retrieving a linear Newton's law and a nonlinear radiative fourth-power law in the boundary condition (1.15) which is written in the slightly modified notation form

$$\frac{\partial u}{\partial n} = g(u) - g_{exact}(f), \quad \text{on } S,$$

where the input function f is given by

$$f = \frac{\partial u_{exact}}{\partial n} + u_{exact}, \quad \text{on } S$$

in the linear case $g_{exact}(f) = -f$, and

$$f = \left(\frac{\partial u_{exact}}{\partial n} + u^4_{exact}\right)^{1/4}, \quad \text{on } S$$

in the nonlinear case $g_{exact}(f) = -f^4$. One can calculate the extremum points of the function f on S in the above expressions and obtain that $[m := min_S f, M := max_S f] \supset [A,B] = [0, \frac{100}{4\pi}e^{-2}]$. From the max-min principle Theorem 2.3 we know that $m \le u \le M$, and we have available these upper, M, and lower, m, bounds because the functions u_0 and f are given input data. However, in our preliminary numerical investigation reported in this section, we have taken that the full information about the end points A and B is available, although from Theorem 2.3 we only know that $[A,B]$ is a subset of the known interval $[m,M]$. In the absence of such information being *a priori* available, one could run the inverse problem on the wider interval $[m,M]$ and retain *a posteriori* the function g only on the reliably obtained range of the function u.

We investigate two weight functions in the boundary integral observations (1.17), namely,

$$\omega(\xi) = \begin{cases} \frac{1}{\varepsilon} & \text{for } \xi \in [(0;0),(\varepsilon;0)], \\ 0 & \text{otherwise,} \end{cases} \qquad \varepsilon = 10^{-5}, \tag{4.2}$$

and

$$\omega(\xi) = \xi_1^2 + \xi_2^2 + 1, \tag{4.3}$$

where $\xi = (\xi_1;\xi_2)$. Note that the weight function (4.2) with ε vanishingly small is supposed to mimic the case of a pointwise measurement (1.9) at the origin $\xi_0 = (0;0)$.

We employ the Gauss-Newton method for minimizing the cost functional (3.1), namely,

$$J(g) = \frac{1}{2}\|lu(g) - h\|^2_{L^2(0,T)} =: \frac{1}{2}\|\Phi(g)\|^2_{L^2(0,T)}, \tag{4.4}$$

as follows. For a given g_n, we consider the sub-problem to minimize (with respect to $z \in L^2(I)$)

$$\frac{1}{2}\|\Phi(g_n) + \Phi'(g_n)z\|^2_{L^2(0,T)} + \frac{\alpha_n}{2}\|z\|^2_{L^2(I)}, \quad \text{Method 1 (M1)}, \tag{4.5}$$

or

$$\frac{1}{2}\|\Phi(g_n) + \Phi'(g_n)z\|^2_{L^2(0,T)} + \frac{\alpha_n}{2}\|z - g_n + g_0\|^2_{L^2(I)}, \quad \text{Method 2 (M2)}. \tag{4.6}$$

Then we update the new iteration as

$$g_{n+1} = g_n + 0.5z. \tag{4.7}$$

Here we choose the regularization parameters

$$\alpha_n = \frac{0.001}{n+1}. \tag{4.8}$$

The direct and adjoint problems are solved using the boundary element method (BEM) with 128 boundary elements and 32 time steps. We also use a partition of the interval $[A, B]$ into 32 sub-intervals.

In what follows, we present the numerical results for both cases of linear and nonlinear unknown functions $g(u)$ using methods M1 and M2 for several choices of initial guess g_0 and noisy data $||h^\delta - h||_{L^2(0,T)} \leq \delta$.

4.1 The linear case

In this case, we wish to retrieve the linear function $g(u) = -u$. Consider three sufficiently different initial guesses

$$g_0(u) \in \Big\{0, -\frac{1}{2}u, -\frac{1}{B}u^2\Big\}. \tag{4.9}$$

For the weight function (4.2), Figures 1 and 2 show the numerical solutions obtained using methods (M1) and (M2), for various initial guesses (4.9), and amounts of noise $\delta = 0.001$ and $\delta = 0.01$, respectively. Figures 3 and 4 presents the same results as Figures 1 and 2, respectively, but for the weight function (4.3). From Figures 1–4 it can be seen that both methods (M1) and (M2) perform similarly well and show independence of the initial guesses (4.9). Except for some isolated large jumps occurring at $u = B$, the numerical results are accurate, stable and robust, i.e. independent of the initial guesses. By comparing Figures 1 and 2 with Figures 3 and 4 it seems that the choice of weight function (4.2) or (4.3) slightly influence the behaviour of the numerical results. In particular, Figures 3 and 4 show a staircase behaviour of the numerical results for $g(u)$, as a function of u, and there is also some slight dependence on the initial guess (4.9) which is more pronounced for method (M2) and for the larger noise $\delta = 0.01$. The numerical solution for the function g is not so smooth because we approximate it by piecewise constant functions.

4.2 The nonlinear case

In this case, we wish to retrieve the nonlinear function $g(u) = -u^4$. Consider three sufficiently different initial guesses

$$g_0(u) \in \Big\{0, -B^3u, -\frac{1}{2}u^4\Big\}. \tag{4.10}$$

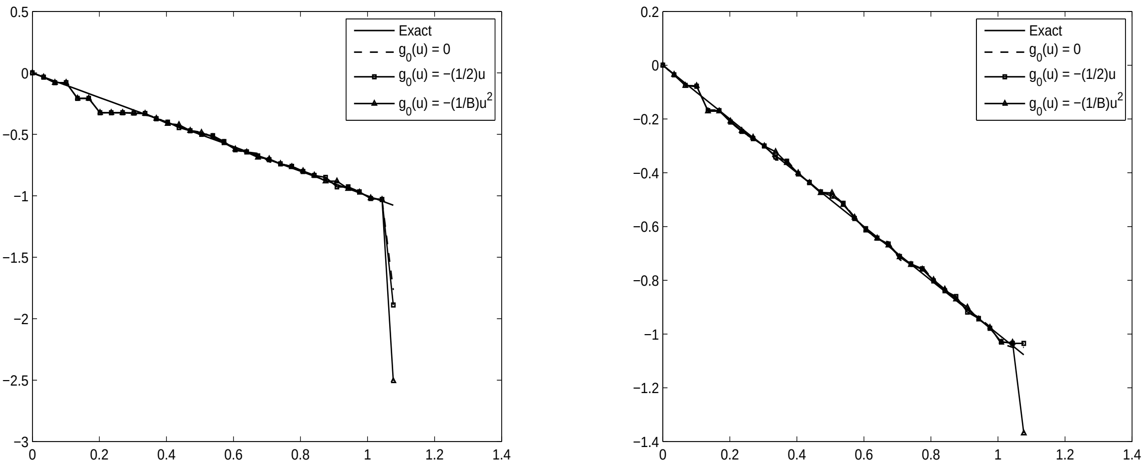

Figure 1: The exact linear function $g(u) = -u$ in comparison with the numerical solutions obtained using method (M1) (left) and method (M2) (right), for $\delta = 0.001$ noise. The weight function ω is given by (4.2).

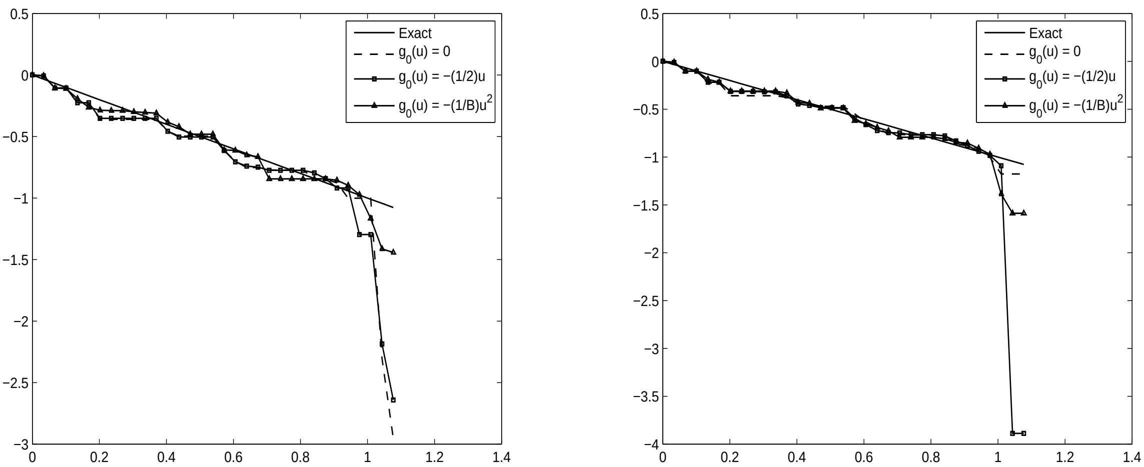

Figure 2: The exact linear function $g(u) = -u$ in comparison with the numerical solutions obtained using method (M1) (left) and method (M2) (right), for $\delta = 0.01$ noise. The weight function ω is given by (4.2).

For simplicity, we only show the numerical results obtained using the method (M2). Figures 5 and 6 show the numerical solutions obtained using method (M2), for various initial guesses (4.10) and amounts of noise $\delta \in \{0.001, 0.01\}$, for the weight functions (4.2) and (4.3), respectively. Similar conclusions to those obtained for the linear case of the previous section can be drawn from these figures.

5 Conclusions

This paper presented a novel theoretical and numerical nonlinear analysis of a multi-dimensional inverse heat conduction problem which allows the determination of the heat transfer law from bound-

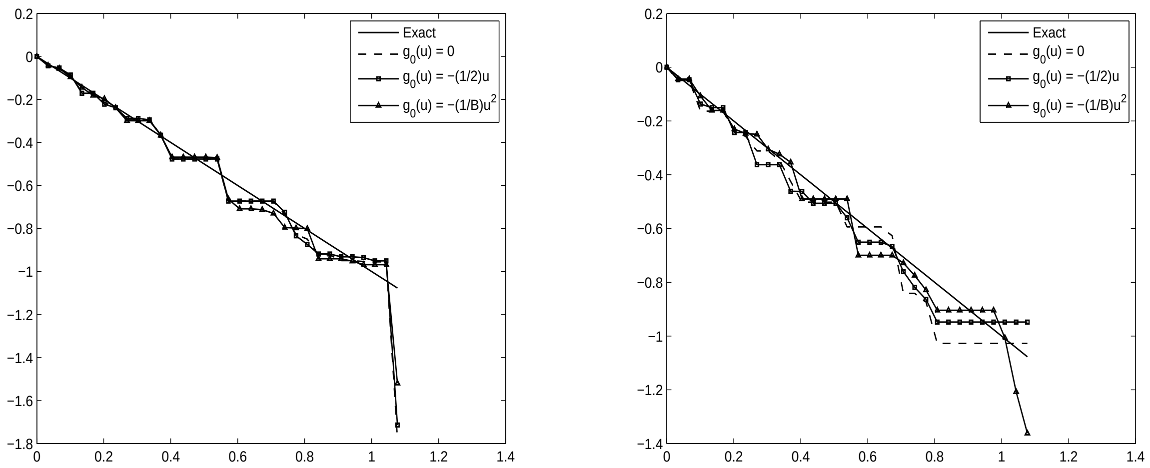

Figure 3: The exact linear function $g(u) = -u$ in comparison with the numerical solutions obtained using method (M1) (left) and method (M2) (right), for $\delta = 0.001$ noise. The weight function ω is given by (4.3).

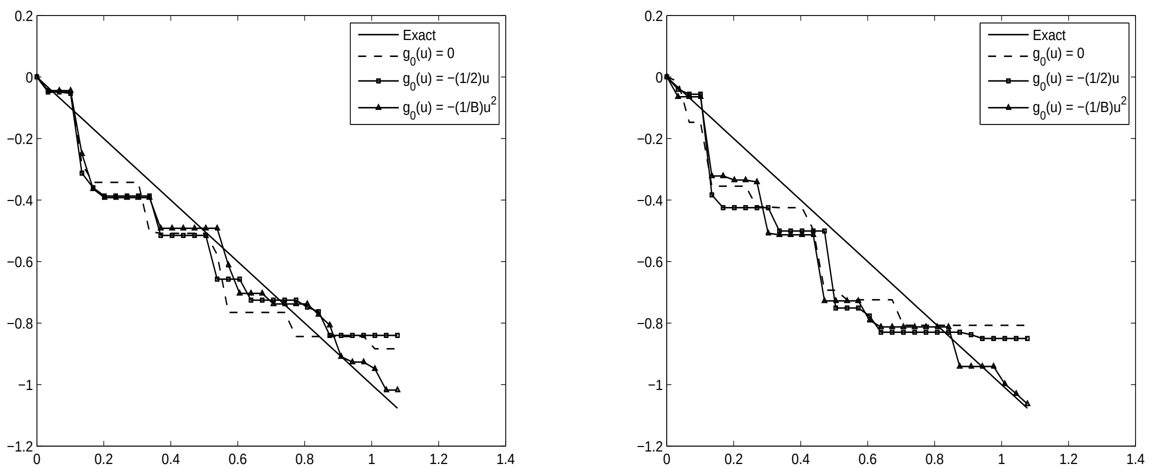

Figure 4: The exact linear function $g(u) = -u$ in comparison with the numerical solutions obtained using method (M1) (left) and method (M2) (right), for $\delta = 0.01$ noise. The weight function ω is given by (4.3).

ary temperature integral observations (1.17). A weak form variational formulation was adopted in which the least-squares functional (3.1) or (3.9) was minimized over a couple of admissible sets $\mathcal{A}_1$ (sections 3.1 and 3.2) or $\mathcal{A}_2$ (section 3.3). The Fréchet differentiability (Theorem 3.1) of the objective functional and the existence of its minimizer (Theorem 3.5), as well as explicit formulae (3.6) and (3.21) for the gradient have all been rigorously established. The numerical solution was found by employing the Gauss-Newton method for the nonlinear minimization of (4.4) based on either (4.5) (method (M1)) or, (4.6) (method (M2)). The numerically obtained results in Section 4 demonstrated that the methods proposed were able to retrieve in an accurate, stable and robust manner, the unknown both linear (section 4.1) and nonlinear (section 4.2) heat transfer laws $g(u)$, from the noisy boundary temperature integral measurements (1.17). Future work will consider a more general nonlinear identification of a heat transfer law $g(u, t)$ depending on both

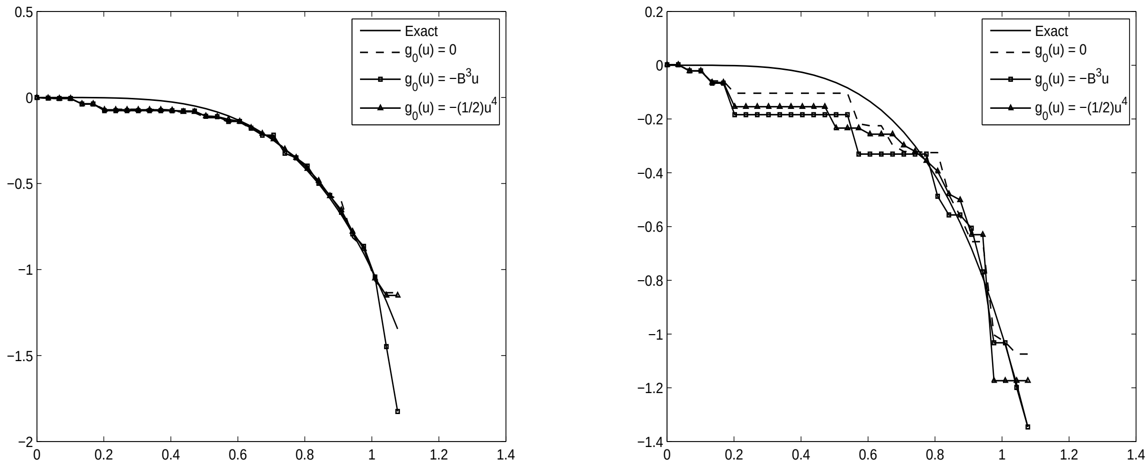

Figure 5: The exact nonlinear function $g(u) = -u^4$ in comparison with the numerical solutions obtained using method (M2): $\delta = 0.001$ noise (left) and $\delta = 0.01$ noise (right). The weight function ω is given by (4.2).

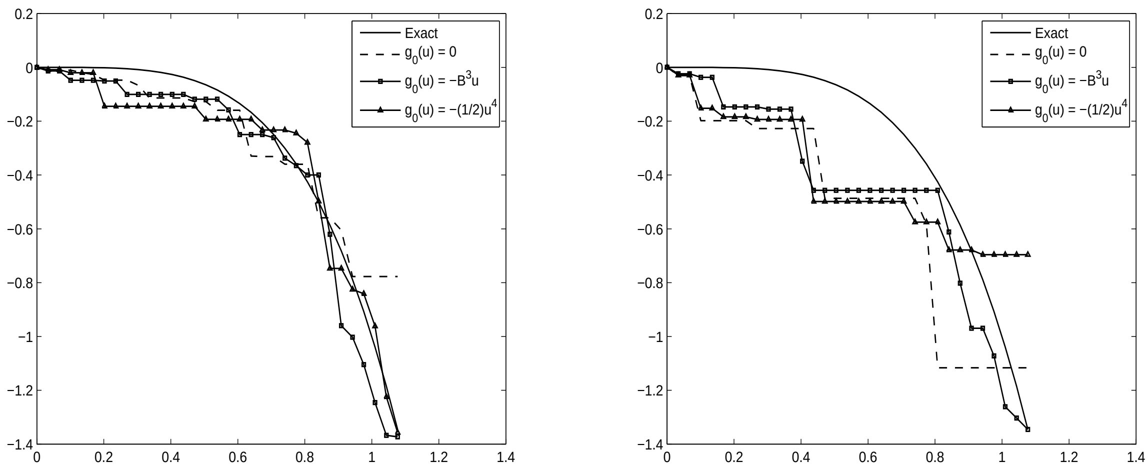

Figure 6: The exact nonlinear function $g(u) = -u^4$ in comparison with the numerical solutions obtained using method (M2): $\delta = 0.001$ noise (left) and $\delta = 0.01$ noise (right). The weight function ω is given by (4.3).

the temperature u and the time t.

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