INVARIANT ZEROS OF MULTIVARIABLE SYSTEMS: A GEOMETRIC ANALYSIS

by

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Abstract

The invariant zeros of a linear multivariable system S(A,B,C) are defined geometrically. A canonical form is derived which illustrates the physical source of zeros in terms of state feedback and observability. Upper bounds on the number of zeros are derived and related to the structure of the system transfer function matrix.

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1. Introduction

This paper is concerned with a linear time-invariant system S(A,B,C) of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(t) \in R^n$$
$$y(t) = C\,x(t) \quad , \quad y(t) \in R^m \quad , \quad u(t) \in R^{\ell} \qquad \ldots(1)$$

and the geometric structure of S(A,B,C) giving rise to the presence of invariant zeros as defined by MacFarlane and Karcanius (1975) and Shaked and Karcanius (1976) or, equivalently, the transmission zeros as defined by Davison and Wang (1974).

The invariant zeros of S(A,B,C) are defined to be the set $Z_o(A,B,C)$ of complex numbers λ such that, if the system matrix

$$P(s) \triangleq \begin{pmatrix} sI_n - A & -B \\ C & 0 \end{pmatrix} \qquad \ldots(2)$$

then

$$\text{rank } P(\lambda) < n+\min(m,\ell) \qquad \ldots(3)$$

Note that $Z_o(A,B,C)$ may be empty, contain a finite number of complex numbers or include the whole complex plane. Throughout the paper it is assumed that

$$m \geqslant \ell \qquad \ldots(4)$$

$$\text{rank } B = \ell \qquad \ldots(5)$$

$$\text{rank } C = m \qquad \ldots(6)$$

$Z_o(A,B,C)$ is finite or empty ...(7)

Condition (4) is not restrictive as the case of $m<\ell$ can be analysed by applying the results to the system $S(A^T,C^T,B^T)$. Conditions (5),(6) are valid in almost all applications and (7) is crucial in the proof of the results and, apart from its general validity, is related to the desirable system characteristic of functional controllability (Rosenbrock 1970).

In feedback design applications it is necessary to consider the possibility of the introduction of zeros due to squaring down. For example, if $m = n$, then (2),(3),(5),(6) imply that $Z_o(A,B,C) = \phi$ (the empty set). If, however, $\ell<m$ and K is an ℓxm constant matrix, then, in general, $Z_o(A,B,KC) \neq \phi$. In general, for an ℓxm constant matrix K and $\ell\leq m$,

$$Z_o(A,B,C) \subseteq Z_o(A,B,KC) \qquad \text{...(8)}$$

The set of system zeros of S(A,B,C) can be defined by the relation

$$Z_c(A,B,C) \triangleq \bigcap_K Z_o(A,B,KC) \qquad \text{...(9)}$$

where the intersection runs over all ℓxm constant matrices K. Using (8), it follows that

$$Z_o(A,B,C) \subseteq Z_c(A,B,C) \qquad \text{...(10)}$$

so that every invariant zero of S(A,B,C) is also a system zero (MacFarlane and Karcanius, 1975).

2. A Geometric Definition of Invariant Zeros

By assumption $\ell\leq m$ so that, if $\lambda \in Z_o(A,B,C)$, there exists $x \in R^n$, $y \in R^\ell$ such that $||x|| + ||y|| > 0$ and

$$\begin{aligned}(\lambda I_n - A)x &= By\\ Cx &= 0\end{aligned} \qquad \text{...(11)}$$

As rank $B = \ell$ then $x \neq 0$ so, if N(C) denotes the null space of C and R(B) denotes the range space of B, the following result is obtained. This result will form the geometric characterization of invariant zeros used in this paper, and is a direct consequence of (2),(3).

Theorem 1 (A geometric definition of invariant zeros)

$$\lambda \in Z_o(A,B,C) \quad \text{iff} \quad \omega_1(\lambda) \triangleq \{(\lambda I_n - A)^{-1} R(B)\} \cap N(C) \neq \{0\} \qquad \ldots(12)$$

Proof

If $\lambda \in Z_o(A,B,C)$ then, from the previous discussion, there exists a non-zero solution vector x to (11) ie $x \in N(C)$ and $(\lambda I_n - A)x \in R(B)$ so that $x \in \omega_1(\lambda)$ and $\omega_1(\lambda) \neq \{0\}$. Conversely, if $\omega_1(\lambda) \neq \{0\}$, there exists a non-zero vector $x \in N(C)$ satisfying $(\lambda I_n - A)x \in R(B)$ ie $\lambda \in Z_o(A,B,C)$ and the result is proved. Q.E.D.

This definition can be used to obtain an equivalent algebraic definition (Kouvaritakis and MacFarlane, 1976). Let N,M be full rank $(n-\ell)\times n$ and $n\times(n-m)$ constant matrices respectively satisfying

$$NB = 0 \quad , \quad CM = 0 \qquad \ldots(13)$$

Theorem 2 (The NAM algorithm)

$$\lambda \in Z_o(A,B,C) \quad \text{iff} \quad \text{rank } N(\lambda I_n - A)M < n-m \qquad \ldots(14)$$

Proof

If rank $N(\lambda I_n - A)M < n-m$, there exists $z \neq 0$ such that $N(\lambda I_n - A)Mz = 0$. Let $x = Mz$ then $x \neq 0$, $x \in N(C)$ and $N(\lambda I_n - A)x = 0$ ie $(\lambda I_n - A)x \in R(B)$ so that $x \in \omega_1(\lambda) \neq \{0\}$ and, by Theorem 1, $\lambda \in Z_o(A,B,C)$. Conversely, if $\lambda \in Z_o(A,B,C)$, there exists a non-zero $x \in N(C)$ such that $(\lambda I_n - A)x \in R(B)$. Equivalently, $N(\lambda I_n - A)x = 0$ and the result follows by writing $x = Mz$ and noting that $z \neq 0$. Q.E.D.

Using recent terminology (Shaked and Karcanius, 1976), the subspace $\omega_1(\lambda)$ can be identified as that subspace of N(C) spanned by the state zero directions (MacFarlane and Karcanius, 1975). It can also be shown (Shaked and Karcanius, 1976) that $\omega_1(\lambda)$ is invariant under state feedback,

$$\omega_1(\lambda) \cap R(B) = \{0\} \qquad \ldots(15)$$

and (quite obviously) $\omega_1(\lambda)$ is an (A,B)-invariant subspace of N(C).

The geometric multiplicity (MacFarlane and Karcanius, 1975) of $\lambda \in Z_o(A,B,C)$ is defined by the relation

$$d_\lambda \triangleq \text{rank defect of } P(\lambda) \qquad \ldots(16)$$

An equivalent definition is

$$d_\lambda \triangleq \dim \omega_1(\lambda) \qquad \ldots(17)$$

for, if x_j $(1 \leq j \leq \dim \omega_1(\lambda))$ is a basis for $\omega_1(\lambda)$, then $Ax_j = \lambda x_j - By_j$ $(1 \leq j \leq \dim\omega_1(\lambda))$. Equivalently, $P(\lambda)\begin{pmatrix} x_j \\ y_j \end{pmatrix} = 0$ $(1 \leq j \leq \dim\omega_1(\lambda))$ implying that $\dim \omega_1(\lambda) \leq d_\lambda$. To prove that $d_\lambda \leq \omega_1(\lambda)$ suppose that $\begin{pmatrix} x_j \\ y_j \end{pmatrix}$ $(1 \leq j \leq d_\lambda)$ are linearly independent vectors in the null space of $P(\lambda)$ and that $\sum_{j=1}^{d_\lambda} \alpha_j x_j = 0$. The relations $(\lambda I_n - A)x_j = By_j$, $1 \leq j \leq d_\lambda$, imply that $B \sum_{j=1}^{d_\lambda} \alpha_j y_j = 0$ or, by (5), $\sum_{j=1}^{d_\lambda} \alpha_j y_j = 0$, contradicting the assumption that $\begin{pmatrix} x_j \\ y_j \end{pmatrix}$, $1 \leq j \leq d_\lambda$, are linearly independent. Hence x_j, $1 \leq j \leq d_\lambda$, are linearly independent state zero directions corresponding to $\lambda \in Z_o(A,B,C)$ ie $d_\lambda \leq \dim\omega_1(\lambda)$, which proves (17).

Finally, an immediate consequence of (15) and the relation $\omega_1(\lambda) \subset N(C)$ is that, defining the nxn matrix

$$E_\lambda = [B \; P \; Q] \qquad \ldots(18)$$

where the columns of the $n \times d_\lambda$ matrix Q span $\omega_1(\lambda)$, it is always possible to choose P such that E_λ is invertible, and

$$CE_\lambda = [C_\lambda \quad 0_{m,d_\lambda}] \qquad \ldots(19)$$

$$E_\lambda^{-1} B = \begin{pmatrix} I_\ell \\ 0_{n-\ell,\ell} \end{pmatrix} \triangleq \begin{pmatrix} B_\lambda \\ 0_{d_\lambda,\ell} \end{pmatrix} \qquad \ldots(20)$$

$$E_\lambda^{-1} A E_\lambda = \left(\begin{array}{c|c} A_{11} & K_\lambda \\ \cline{2-2} & 0 \\ \hline A_{21} & \lambda I_{d_\lambda} \end{array}\right) \qquad \ldots(21)$$

where K_λ is an $\ell \times d_\lambda$ matrix.

3. Algebraic Multiplicity and Decomposition of the State Space

It has been noted (section 2) that $\omega_1(\lambda)$ $(\lambda \in Z_o(A,B,C))$ is an (A,B)-invariant subspace of N(c). Let ω be the maximal (A,B)-invariant subspace of N(c), then (Bengtsson, 1973)

$$\omega \cap R(B) = \{0\} \qquad \ldots(22)$$

and ω is invariant under state feedback. Equation (22) has a direct interpretation in terms of a state space cononical form defined by the following result.

Theorem 3 (A Zero Canonical Form)

There exists a similarity transformation E such that

$$CE = \begin{bmatrix} \hat{C} & 0_{m,n_z} \end{bmatrix} \qquad \ldots(23)$$

$$E^{-1}B = \begin{bmatrix} I_\ell \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{B} \\ 0_{n_z,\ell} \end{bmatrix} \qquad \ldots(24)$$

$$E^{-1}AE = \left[\begin{array}{c|c} A_{11} & \begin{array}{c} K \\ \hline 0 \end{array} \\ \hline A_{21} & A_{22} \end{array}\right] \qquad \ldots(25)$$

where

$$n_z \triangleq \dim \omega \quad , \qquad \ldots(26)$$

and K, A_{22} are $\ell x n_z$ and $n_z x n_z$ matrices respectively. Moreover, the system $S(A_{11},\hat{B},\hat{C})$ is observable and

$$Z_o(A_{11},\hat{B},\hat{C}) = \phi \qquad \ldots(27)$$

Proof

Write $E = [B,P,Q]$ where the columns of the nxn_z matrix Q span ω. Relation (22), with suitable choice of P, ensures that E is invertible (23)-(25) follow directly, noting that $\omega \subset N(c)$ implies $CQ = 0$.

To prove the second part of the result note that $Z_o(A_{11},\hat{B},\hat{C})$ is finite or empty (as $Z_o(A,B,C)$ is finite or empty) and assume there exists $\lambda \in Z_o(A_{11},B,C)$. Applying the construction defined by (18)-(21) there is a transformation T such that

$$T^{-1}\hat{B} = \hat{B} \qquad \hat{C}T = [\hat{C}_1 \quad 0_{m,1}]$$

$$T^{-1}A_{11}T = \begin{pmatrix} \hat{A}_{11} & \begin{matrix}\hat{k}\\ \hline 0\end{matrix} \\ \hline \hat{A}_{21} & \lambda \end{pmatrix}$$

for some ℓx1 vector $\hat{k}$. Applying the transformation $\hat{T} \triangleq \text{block diag}(T,I_{n_z})$ to the canonical form of equations (23)-(25), it follows that the last n_z+1 columns of $E\hat{T}$ span an n_z+1 dimensional (A,B)-invariant subspace of N(c) contradicting the definition of ω. Hence $Z_o(A_{11},\hat{B},\hat{C}) = \phi$.

Finally, if $S(A_{11},\hat{B},\hat{C})$ is not observable, there exists a scalar λ and a non-zero vector $x \in N(c)$ such that $(\lambda I-A)x = 0$ ie $\omega_1(\lambda) \neq \{0\}$ which, by theorem 1, contradicts (27). Q.E.D.

The following result identifies the properties of A_{22} which lead to a natural definition of the algebraic multiplicity of any $\lambda \in Z_o(A,B,C)$.

Theorem 4 (Invariant zeros and the eigenvalues of A_{22})

λ is an eigenvalue of A_{22} iff $\lambda \in Z_o(A,B,C)$

Proof

By inspection of the canonical form, if λ is an eigenvalue of A_{22}, there exists a non-zero vector $x \in \omega$ such that $Ax = \lambda x + b$ ($b \in R(B)$). Noting that $\omega \subset N(c)$ then (Theorem 1) $\omega_1(\lambda) \neq \{0\}$ and $\lambda \in Z_o(A,B,C)$. Conversely, if $\lambda \in Z_o(A,B,C)$, then (by the maximality of ω) $\omega_1(\lambda) \subset \omega$ ie λ is an eigenvalue of A_{22}. Q.E.D.

Write, using theorem 4,

$$|sI_{n_z} - A_{22}| = \prod_{\lambda \in Z_o(A,B,C)} (s-\lambda)^{n_\lambda} \qquad \ldots(28)$$

where n_λ is the multiplicity of λ as an eigenvalue of A_{22}, and noting that

$$n_z = \sum_{\lambda \in Z_o(A,B,C)} n_\lambda \qquad \ldots(29)$$

it is natural to define

$$n_\lambda \triangleq \text{algebraic multiplicity of } \lambda \in Z_o(A,B,C) \qquad \ldots(30)$$

$$n_z \triangleq \text{total number of invariant zeros} \qquad \ldots(31)$$

These definitions are consistent with those used by MacFarlane and Karcanius (1975).

In geometric terms, the geometric multiplicity of $\lambda \in Z_o(A,B,C)$ is simply the dimension of $\omega_1(\lambda)$ (the subspace generated by the state zero directions corresponding to the invariant zero λ). The geometric characterization of n_λ is obtained by choosing a direct sum decomposition of ω of the form

$$\omega = \bigoplus_{j=1}^{p} I(\lambda_j) \qquad \ldots(32)$$

where, if $Z_o(A,B,C) = \{\lambda_j\}_{1 \leq j \leq p}$, then A_{22} has a Jordan form

$$A_{22} = \text{block diag } (J(\lambda_1),\ldots,J(\lambda_p)) \qquad \ldots(33)$$

where $J(\lambda_j)$ is the Jordan block corresponding to the zero λ_j ie

$$n_\lambda = \dim I(\lambda) \quad , \quad \lambda \in Z_o(A,B,C) \qquad \ldots(34)$$

4. Upper Bounds for n_z

This section presents new results describing upper bounds for n_z. To initiate the analysis, note that (22) implies that the sum $\omega + \{R(B) \cap N(c)\}$ is direct. Moreover $\omega + \{R(B) \cap N(c)\} \subset N(c)$ or

$$n_z \leq n - m - \dim R(B) \cap N(c) \qquad \ldots(35)$$

If $m = \ell$ and $R(B) \cap N(c) = \{0\}$, then $C^n = R(B) \oplus N(c)$ which implies that $\omega = N(c)$ and hence equality holds in (35) ie $n_z = n-m$.

A better estimate is obtained using the following result.

Theorem 5

Let k be the smallest integer $j \geq 1$, such that $A^{j-1}R(B) \not\subset N(c)$. Then the sum

$$V_q \triangleq \sum_{i=o}^{q-1} A^i R(B) \quad , \quad 1 \leq q \leq k \qquad \ldots(36)$$

is a direct sum, and

$$V_q \cap \omega = \{0\} \quad , \quad 1 \leq q \leq k \qquad \ldots(37)$$

Moreover

$$\dim A^i R(B) = \ell \quad , \quad 0 \leq i \leq k-1 \qquad \ldots(38)$$

and

$$n_z \leq (n-m) - (k-1)\ell - \dim\{A^{k-1}R(B)\} \cap N(c) \qquad \ldots(39)$$

Sufficient conditions for equality to hold in (39) is $m = \ell$ and

$$\{A^{k-1}R(B)\} \cap N(c) = \{0\} \qquad \ldots(40)$$

Proof

If $k = 1$, the result is proved using the discussion preceding this theorem. Therefore take $k>1$ and hence $R(B) \subset N(c)$. Note that

$$V_q \subset N(c) \qquad 1 \leq q < k \qquad \ldots(41)$$

$$V_1 = R(B) \qquad \ldots(42)$$

$$V_q \subset V_{q+1} \qquad 1 \leq q < k \qquad \ldots(43)$$

$$V_{q+1} = A^q R(B) + V_q = AV_q + R(B) \quad , \quad 1 \leq q < k \qquad \ldots(44)$$

Using (22), $V_1 \cap \omega = \{0\}$, so assume that $V_i \cap \omega = \{0\}$, $1 \leq i \leq j$, for some $j<k$ but that $V_{j+1} \cap \omega \neq \{0\}$. Select a non-zero vector $x \in V_{j+1} \cap \omega$ and write, using (44), $x = Av_j + b_j$ ($v_j \in V_j$, $b_j \in R(B)$) ie $Av_j = x - b_j \in \omega \oplus R(B)$

but, by induction, $v_j \notin \omega$ so that, if $\hat{\omega} \triangleq \omega \oplus \{v_j\}$, then $\hat{\omega}$ is an (A,B)-invariant subspace of $N(c)$ containing ω as a strictly proper subspace which is a contradiction. Hence $V_{j+1} \cap \omega = \{0\}$ and

$$V_j \cap \omega = \{0\} \qquad 1 \leq j \leq k \qquad \ldots(45)$$

which proves (37).

Next, write $V_k = AV_{k-1} + R(B)$ and note that $\{AV_{k-1}\} \cap R(B) \neq \{0\}$ implies that $V_{k-1} \cap \omega \neq \{0\}$ contradicting (45) so that $V_k = AV_{k-1} \oplus R(B)$. Using similar reasoning for V_j, $j<k$, implies that $V_{j+1} = AV_j \oplus R(B)$, $1 \leq j < k$, ie $V_k = AV_{k-1} \oplus R(B) = A(AV_{k-2} \oplus R(B)) \oplus R(B) = A^2V_{k-2} \oplus AR(B) \oplus R(B)$ where, without loss of generality, we have assumed that A is nonsingular. Continuation of the decomposition leads to the relation

$$V_k = \bigoplus_{i=1}^{k} A^{i-1}R(B) \qquad \ldots(46)$$

which proves (36) and also demonstrates that k exists and is finite.

To prove (38), note that, without loss of generality, we may assume that A is invertible so that the assumption that $\dim R(B) = \ell$ implies that $\dim A^iR(B) = \ell$, $0 \leq i < k$.

It follows directly from (38), (46) that

$$\omega \oplus \{\bigoplus_{j=o}^{k-2} A^jR(B)\} \oplus \{\{A^{k-1}R(B)\} \cap N(c)\} \subset N(c) \qquad \ldots(47)$$

so that

$$n_z + (k-1)\ell + \dim\{A^{k-1}R(B)\} \cap N(c) \leq n-m \qquad \ldots(48)$$

which proves equation (39).

To prove the final part of the result it is sufficient to prove that, $0 \leq j \leq k-1$, the maximal (A,A^jB)-invariant subspace $\hat{\omega}_j \subset N(c)$ is

$$\hat{\omega}_j = \omega \oplus s_j \qquad \ldots(49)$$

where $\qquad s_o = \{0\}$

$$s_j = \bigoplus_{i=o}^{j-1} A^i R(B) \quad , \qquad j>0 \qquad \ldots(50)$$

for, if $m = \ell$ and $\{A^{k-1}R(B)\} \cap N(c) = \{0\}$, then, using (38),

$$C^n = A^{k-1}R(B) \oplus N(c) \qquad \ldots(51)$$

so that $\hat{\omega}_{k-1} = N(c)$ and

$$n-m = \dim N(c) = \sum_{j=o}^{k-2} \dim A^j R(B) + \dim \omega \qquad \ldots(52)$$

which proves the result by (38).

Note that $\hat{\omega}_o = \omega$ and assume, for some $0 \leq j < k-1$, that (49) holds. It is easily seen that $\omega \oplus s_{j+1} \subset \hat{\omega}_{j+1}$ so assume that there exists $x \notin \omega \oplus s_{j+1}$ such that $x \in \hat{\omega}_{j+1}$ ie $x \in N(c)$ and, by suitable choice of x,

$$Ax = \lambda x + y + \sum_{i=o}^{j+1} A^i b_i \qquad \ldots(53)$$

for some $\lambda \in C$, $y \in \omega$ and $b_i \in R(B)$, $0 \leq i \leq j+1$. After some manipulation

$$A(x - \sum_{i=o}^{j} A^i b_{i+1}) = \lambda(x - \sum_{i=o}^{j} A^i b_{i+1})$$

$$+ y + b_o + \lambda \sum_{i=o}^{j} A^i b_{i+1} \qquad \ldots(54)$$

so that $x - \sum_{i=o}^{j} A^i b_{i+1} \in \hat{\omega}_j = \omega \oplus s_j$ and hence $x \in \omega \oplus s_{j+1}$ contrary to assumption. The result is now proven. Q.E.D.

Equation (39) has a direct interpretation in terms of the rank of the matrix $CA^{k-1}B$ for, if

$$r_k \triangleq \text{rank } CA^{k-1}B \qquad \ldots(55)$$

then

$$\dim\{A^{k-1}R(B)\} \cap N(c) = \ell - r_k \qquad \ldots(56)$$

and (39) states that

$$n_z \leqslant (n-m) - (k-1)\ell - (\ell - r_k)$$
$$= n-m - k\ell + r_k \qquad \ldots(57)$$

Both k and r_k can be obtained from the system transfer function matrix $G(s) = C(sI_n - A)^{-1}B$ as k is the uniquely defined integer such that

$$G_\infty^{(k)} \triangleq \lim_{s\to\infty} s^k G(s) \qquad \ldots(58)$$

exists and is non-zero, and

$$r_k \equiv \text{rank } G_\infty^{(k)} \qquad \ldots(59)$$

Note finally that the constraint $n_z \geqslant 0$ implies that

$$n-m \geqslant (k-1)\ell \qquad \ldots(60)$$

which provides a useful bound on the possible values of k.

5. System Zeros

The set of system zeros (equation (9)) and their multiplicity can be characterized by the monic greatest common divisor $\hat{z}(s)$ of all polynomials of the form

$$P_K(s) \triangleq \begin{vmatrix} sI_n - A & -B \\ KC & 0 \end{vmatrix} \qquad \ldots(61)$$

where K is any ℓxm constant matrix. Writing

$$\hat{z}(s) = \prod_{\lambda \in Z_c(A,B,C)} (s-\lambda)^{r_\lambda} \qquad \ldots(62)$$

r_λ is defined to be the multiplicity of $\lambda \in Z_c(A,B,C)$. If $m = \ell$, then, using theorem 3, (61) implies that

$$\hat{z}(s) = |sI_{n_z} - A_{22}| \qquad \ldots(63)$$

so that $Z_o(A,B,C) = Z_c(A,B,C)$ and $r_\lambda = n_\lambda$ for all λ, (MacFarlane and Karcanius, 1975). In the case of $m>\ell$, it is possible to transform $S(A_{11},\hat{B},\hat{C})$ to the form,

$$A_{11} = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \hat{B}_1 \\ 0 \end{pmatrix}, \quad \hat{C} = [\hat{C}_1, \hat{C}_2] \qquad \ldots(64)$$

where the qxq matrix M_{11} characterizes the restriction of A_{11} to the controllable subspace of $S(A_{11},\hat{B},\hat{C})$. Moreover, using theorem 3, $S(M_{11},\hat{B}_1,\hat{C}_1)$ is both controllable and observable and has no invariant zeros. Also, after some manipulation

$$P_K(s) = |sI_{n_z} - A_{22}| \cdot |sI_{n-n_z-q} - M_{22}| \cdot \begin{vmatrix} sI_q - M_{11} & -\hat{B}_1 \\ K\hat{C}_1 & 0 \end{vmatrix} \qquad \ldots(65)$$

so that

$$\hat{z}(s) = |sI_{n_z} - A_{22}| \cdot |sI_{n-n_z-q} - M_{22}| \qquad \ldots(66)$$

as the greatest common divisor of the third term in (65) is just unity. To prove this, suppose that

$$Q_K(s) = \begin{vmatrix} sI_q - M_{11} & -\hat{B}_1 \\ K\hat{C}_1 & 0 \end{vmatrix} \qquad \ldots(67)$$

has a zero λ, independent of the matrix K used. Let ν be the maximal subspace of C^q such that $(\lambda I - M_{11})\nu \subset R(\hat{B}_1)$, then, as $S(M_{11},\hat{B}_1,\hat{C}_1)$ is controllable, we have dim $\nu \leq \ell$ and the fact that $S(M_{11},\hat{B}_1,\hat{C}_1)$ has no invariant zeros implies (theorem 1) that $\nu \cap N(\hat{C}_1) = \{0\}$. It follows directly that there exists K_λ such that the map $K_\lambda \hat{C}_1 : \nu \to C^\ell$ is injective and hence λ cannot be a zero of $Q_{K_\lambda}(s)$ contradicting the assumption.

6. Zeros and State Feedback: A Physical Interpretation

The canonical form of theorem 3 can be used to gain some insight into the physical structure of the system and the physical source of invariant and system zeros. Write,

$$E^{-1}x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad x_1 \in R^{n-n_z}, \quad x_2 \in R^{n_z} \qquad \ldots(68)$$

then

$$\dot{x}_1 = A_{11}x_1 + \hat{B}\{u + Kx_2\}$$

$$\dot{x}_2 = A_{22}x_2 + A_{21}x_1$$

$$y = \hat{C}x_1 \qquad \ldots(69)$$

A schematic representation of this system is shown in Fig.1(a) which demonstrates that S(A,B,C) consists of an observable forward path system $S(A_{11},\hat{B},\hat{C})$ possessing no invariant zeros and a dynamic state feedback $S(A_{22},A_{21},K)$ of state dimension n_z with poles $\lambda \in Z_o(A,B,C)$ of multiplicity n_λ.

The invariant zeros are a subset of the system zeros. The remaining system zeros are generated by the uncontrollable subspace of $S(A_{11},\hat{B},\hat{C})$. Using the decomposition of (64), the system S(A,B,C) has a decomposition of the form shown in Fig.1(b) where $S(M_{11},\hat{B}_1,\hat{C}_1)$ is both controllable and observable and possesses no invariant zeros. The system zeros are the roots of $\hat{z}(s)$ (equation (66)) with appropriate multiplicity.

Noting that $S(A_{22},A_{21},K)$ is not directly affected by the control input u(t) and makes no direct contribution to the output, then, intuitively, it is seen that the existence of invariant zeros is intimately related to the existence of inherent dynamic feedback within the system structure. To illustrate this point, consider the dynamics of a point thermal nuclear reactor. In the absence of delayed neutrons, the dynamics of a small perturbation n(t) in neutron population about a steady state n_o is described by the linear differential equation,

$$\ell^* \dot{n}(t) = n_o \, \delta k(t) \qquad \ldots(70)$$

where ℓ^* is the neutron mean life-time and $\delta k(t)$ is the change in system reactivity. System reactivity is affected by many system parameters including n(t), control input u(t) and xenon-135 fission product x(t),

$$\delta k(t) = \alpha_n n(t) + \alpha_u u(t) + \alpha_x x(t) \qquad \ldots(71)$$

and xenon is generated via the dynamic chain

$$\frac{di(t)}{dt} = -\lambda_1 i(t) + \beta_1 n(t)$$

$$\frac{dx(t)}{dt} = -\lambda_2 x(t) + \lambda_1 i(t) + \beta_2 n(t) \qquad \ldots(72)$$

where i(t) is the iodine-135 concentration in the reactor core. The system output is y(t) = n(t) and it is easily seen that the states x(t),i(t) represent inherent dynamic feedback effects as they make no contribution to the output and are not affected directly by the input. From the above comments we expect the invariant zeros of the system to be $-\lambda_1$ and $-\lambda_2$ which is easily verified from the state space form

$$\frac{d}{dt}\begin{pmatrix} n \\ i \\ x \end{pmatrix} = \begin{pmatrix} n_o\alpha_n/\ell^* & 0 & n_o\alpha_x/\ell^* \\ \beta_1 & -\lambda_1 & 0 \\ \beta_2 & \lambda_1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} n \\ i \\ x \end{pmatrix} + \frac{n_o}{\ell^*}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u$$

$$y = [1 \quad 0 \quad 0] \; [n \quad i \quad x]^T \qquad \ldots(73)$$

which (Theorem 3) is in zero canonical form with $n_z = 2$

$$A_{22} = \begin{pmatrix} -\lambda_1 & 0 \\ \lambda_1 & -\lambda_2 \end{pmatrix} \qquad \ldots(74)$$

so that $Z_o(A,B,C) = \{-\lambda_1, -\lambda_2\}$ each possessing unity algebraic and geometric multiplicity.

7. Conclusions

Using a definition (MacFarlane and Karcanius, 1975, Shaked and Karcanius, 1976) of invariant zeros of a system S(A,B,C), a geometric definition has been proposed for the case of $m \geq \ell$. The results lead directly to a canonical decomposition of S(A,B,C) which leads to a natural definition of the algebraic multiplicity of each invariant zero and illustrates their physical source as due to inherent dynamic state feedback within the system structure. Useful new upper bounds on the total number of invariant zeros have been obtained using parameters easily calculated from the system transfer function matrix. Finally, the set of system zeros (and associated multiplicity) has been identified in terms of the invariant zeros and a canonical decomposition of the system forward path (Fig.1) into controllable and uncontrollable subsystems.

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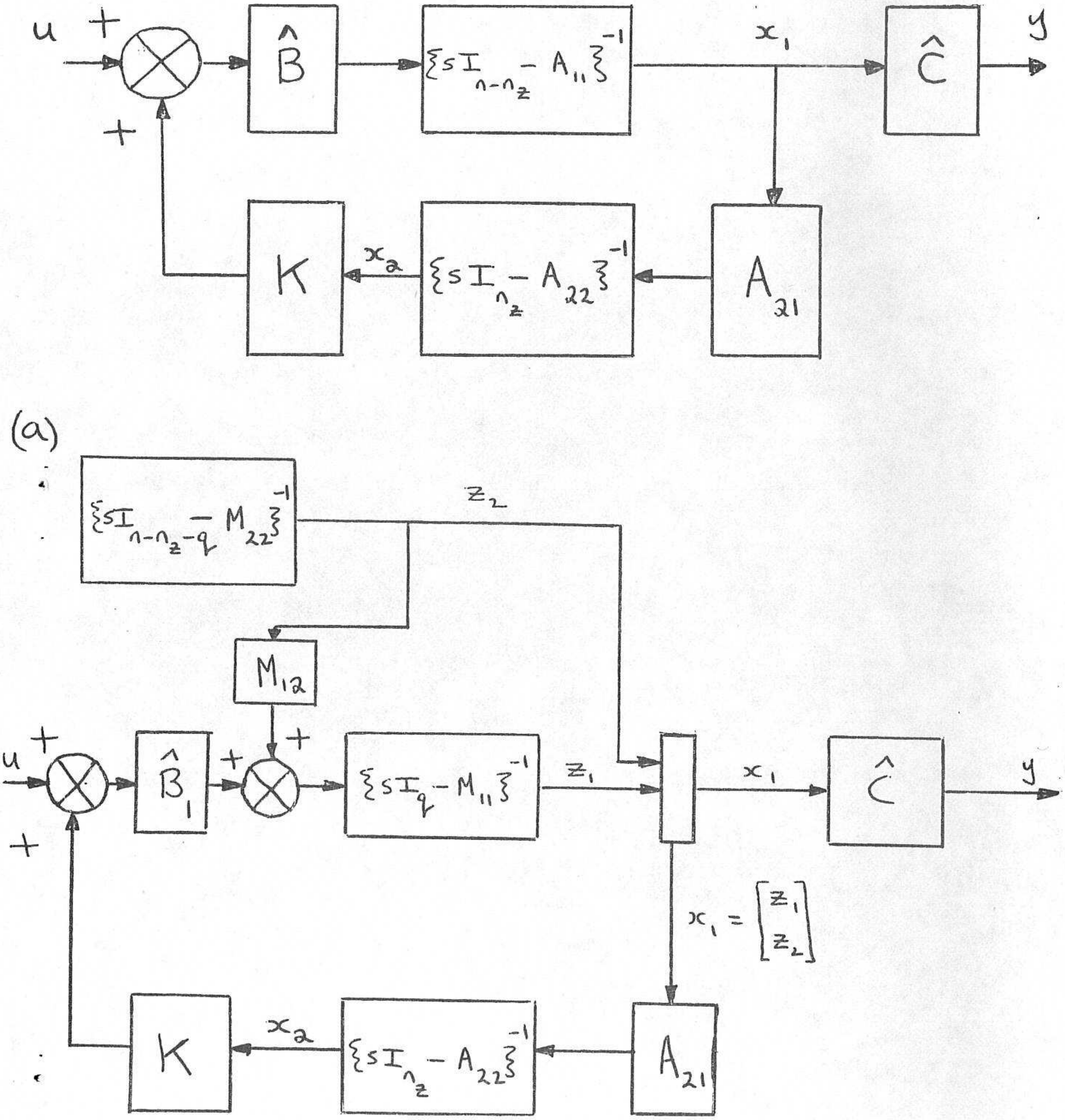

Fig. 1. Representation of $S(A, B, C)$