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STRONG UNIQUENESS FOR SDES IN HILBERT SPACES WITH NONREGULAR DRIFT

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We prove pathwise uniqueness for a class of stochastic differential equations (SDE) on a Hilbert space with cylindrical Wiener noise, whose nonlinear drift parts are sums of the sub-differential of a convex function and a bounded part. This generalizes a classical result by one of the authors to infinite dimensions. Our results also generalize and improve recent results by N. Champagnat and P. E. Jabin, proved in finite dimensions, in the case where their diffusion matrix is constant and nondegenerate and their weakly differentiable drift is the (weak) gradient of a convex function. We also prove weak existence, hence obtain unique strong solutions by the Yamada–Watanabe theorem. The proofs are based in part on a recent maximal regularity result in infinite dimensions, the theory of quasi-regular Dirichlet forms and an infinite dimensional version of a Zvonkin-type transformation. As a main application, we show pathwise uniqueness for stochastic reaction diffusion equations perturbed by a Borel measurable bounded drift. Hence, such SDE have a unique strong solution.

1. Introduction. In a separable Hilbert space H, with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, we consider the SDE

$$
\begin{aligned}
& dX_t = \big(AX_t - \nabla V(X_t) + B(X_t)\big)\,dt + dW_t, \\
& X_0 = z,
\end{aligned}
\tag{1.1}
$$

where we assume:

(H1) $A : D(A) \subset H \to H$ is a self-adjoint and strictly negative definite operator (i.e., $A \leq -\omega I$ for some $\omega > 0$), with A^{-1} of trace class.

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(H2) $V : H \to (-\infty, +\infty]$ is a *convex, proper, lower-semicontinuous, lower bounded* function; denote by D_V the set of all $x \in \{V < \infty\}$ such that V is Gâteaux differentiable at x.

(H3) For the Gateaux derivative ∇V we have for some $\varepsilon > 0$

$$
\begin{aligned}
&\gamma(D_V) = 1,\\
&\int_H \big(|V(x)|^{2+\varepsilon} + |\nabla V(x)|^2\big)\gamma(dx) < \infty, \qquad (1.2)\\
&\int_H \|D^2 V(x)\|_{\mathcal{L}(H)} \nu(dx) < \infty,
\end{aligned}
$$

where γ is the centered Gaussian measure in H with covariance $Q = -\frac{1}{2}A^{-1}$ and ν is the probability measure on H defined as

$$
\nu(dx) = \frac{1}{Z} e^{-V(x)} \gamma(dx), \qquad Z = \int_H e^{-V(x)} \gamma(dx).
$$

Clearly, γ and ν have the same zero sets. Here, the second assumption in (1.2) means that there exists $u_n \in \mathcal{F}C_b^2(H), n \in \mathbb{N}$, such that $V = \lim_{n\to\infty} u_n$ in $L^2(H, \nu)$ and $D^2 V := \lim_{n\to\infty} D^2 u_n$ in $L^2(H, \nu; L(H))$, where $\mathcal{F}C_b^2(H)$ denotes the set of all C_b^2-cylindric functions on H (see below for the precise definition) and $L(H)$ the set of all bounded linear operators from H to H.

(H4) $B : H \to H$ is Borel measurable and bounded.

(H5) W is an $(\mathcal{F}_t)$-cylindrical Brownian motion in H, on some pobability space $(\Omega, \mathcal{F}, P)$ with normal filtration $(\mathcal{F}_t), t \geq 0$.

Formally, W is a process of the form $W_t = \sum_{i=1}^{\infty} W_t^i e_i$ where W_t^i are independent real valued Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\{e_i\}_{i\in\mathbb{N}}$ is a complete orthonormal system in H; for every $h \in H$, the series $\langle W_t, h\rangle = \sum_{i=1}^{\infty} W_t^i \langle e_i, h\rangle$ converges in $L^2(\Omega)$.

REMARK 1.1. Since A is strictly negative definite, we may assume $V(x) \geq \varepsilon |x|^2, x \in H$, for some $\varepsilon > 0$ and all $x \in H$. Otherwise, replace A by $A + \frac{\omega}{2} I$ and V by $V + \frac{\omega}{2}|x|^2 + |\inf_{x\in H} V(x)|$. In particular, without loss of generality we have that $|x|^p e^{-V(x)}$ is bounded in $x \in H$ for all $p \in (0, \infty)$.

REMARK 1.2. (i) We note that if $x \in D_V$ by definition

$$
\lim_{s\to 0} \frac{1}{s}\big(V(x + sh) - V(x)\big) = \langle \nabla V(x), h\rangle
$$

for all $h \in H$ where a priori the limit is taken in the Alexandrov topology on $(-\infty, +\infty]$, since $V(x + sh)$ could be $+\infty$ for some s. On the other hand, the limit $\langle \nabla V(x), h\rangle \in \mathbb{R}$, so $V(x + sh) \in \mathbb{R}$ for $s \leq s_0$ for some small enough $s_0 > 0$.

(ii) If $\{V < \infty\}$ is open, then $\gamma(D_V) = 1$. Indeed, if $\{V < \infty\}$ is open, then V is continuous on $\{V < \infty\}$; see, for example, [20], Proposition 3.3. Since furthermore, V is then locally Lipschitz on $\{V < \infty\}$ (see, e.g., [20], Proposition 1.6), it follows by the fundamental result in [4, 19]; see also [7], Section 10.6, that $\gamma(\{V < \infty\} \setminus D_V) = 0$. But $\gamma(\{V < \infty\}) = 1$, since $V \in L^2(H, \gamma)$.

It turns out that the condition on the second (weak) derivative in (1.2) in Hypothesis (H3) is too strong for some applications (see Section 7 below). Therefore, we shall also consider the following modified version of (H3):

(H3)′ V and ∇V satisfy (H3) with the condition on the second derivative of V replaced by the following: there exists a separable Banach space $E \subset H$, continuously and densely embedded, such that $E \subset D(V)$, $\gamma(E) = 1$ and on E the function V is twice Gâteaux-differentiable such that for all $x \in E$ its second Gâteaux-derivative $V_E''(x) \in L(E, E')$ (with E' being the dual of E) extends by continuity to an element in $L(H, E')$ such that

$$\|V_E''(x)\|_{L(H,E')} \le \Psi(|x|_E)$$

for some convex function $\Psi : [0, \infty) \to [0, \infty)$. Furthermore, for γ-a.e. initial condition $z \in E$ there exists a (probabilistically) weak solution $X^V = X^V(t), t \in [0, T]$, to SDE (1.1) with $B = 0$ so that

$$\mathbb{E} \int_0^T \Psi(|X^V(s)|_E)\, ds < \infty. \tag{1.3}$$

Though (H3)′ is quite complicated to formulate, it is exactly what is fulfilled if ∇V is a polynomial. We refer to Section 7.1 below.

REMARK 1.3. We would like to stress at this point that the conditions on the second derivative of V both in (H3) and in (H3)′ are only used to be able to apply the mean value theorem in the proof of Lemma 5.2 below. For the rest of this paper, we assume that (H1), (H2), (H4), (H5) and (H3) or (H3)′ are in force.

DEFINITION 1.4. A solution of the SDE (1.1) in H is a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ on H, an H-cylindrical $(\mathcal{F}_t)$-Brownian motion $(W_t)_{t\ge 0}$ w.r.t. this space, a continuous $(\mathcal{F}_t)$-adapted process $(X_t)_{t\ge 0}$ on this space such that:

(i) $X_s \in D_V$ for $dt \otimes P$ a.e. (s, ω) and $\int_0^T |\langle \nabla V(X_s), h\rangle|\, ds < \infty$ with probability one, for every $T > 0$ and $h \in D(A)$;

(ii) for every $h \in D(A)$ and $t \ge 0$, one has

$$\langle X_t, h\rangle = \langle z, h\rangle + \int_0^t \big(\langle X_s, Ah\rangle + \langle B(X_s) - \nabla V(X_s), h\rangle\big)\, ds + \langle W_t, h\rangle$$

with probability one.

If X is $\mathcal{F}^W$-adapted, where $\mathcal{F}^W = (\mathcal{F}_t^W)_{t\ge 0}$ is the normal filtration generated by W, we say that X is a strong solution.

The Gaussian measure γ is invariant for the linear equation

$$dZ_t = AZ_t\,dt + dW_t$$

while ν is invariant for the nonlinear equation

$$dX_t = \bigl(AX_t - \nabla V(X_t)\bigr)\,dt + dW_t.$$

They are equivalent, since $V < \infty$ (hence $e^{-V} > 0$) at least on D_V and $\gamma(D_V) = 1$. Hence, the full measure sets in H are the same with respect to γ or ν. Our main uniqueness result is the following.

THEOREM 1.5. *There is a Borel set $\Xi \subset H$ with $\gamma(\Xi) = 1$ having the following property. If $z \in \Xi$ and X, Y are two solutions with initial condition x* (*in the sense of Definition 1.4*), *defined on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and w.r.t. the same cylindrical Brownian motion W, then X and Y are indistinguishable processes. Hence, by the Yamada–Watanabe theorem they are* (*probabilistically*) *strong solutions and have the same law.*

The proof is given in Section 5. This result was first proved in [11] in the case $V = 0$ (see also the more recent [12], where also the case $V = 0$, but with B only bounded on balls was treated) with a rather complex proof based on the very nontrivial maximal regularity results in $L^p(H, \gamma)$ for the Kolmogorov equation

$$(\lambda - \mathcal{L}_{A,B})u = f$$

associated to the SDE, where $\mathcal{L}_{A,B}$ is the operator formally defined as

$$\mathcal{L}_{A,B}u(x) = \tfrac{1}{2}\operatorname{Tr}\bigl(D^2u(x)\bigr) + \langle Ax + B(x), Du\rangle$$

on suitable functions u, for $x \in D(A)$. Here, we present a much simpler proof which covers also the case $V \neq 0$, based on several new ingredients.

First, in order to perform a suitable change of coordinates (analogous to [11] and [12]), we use the family of Kolmogorov equations

$$(\lambda + \lambda_i - \mathcal{L}_{A,B,V})u = f$$

or in vector form

$$(\lambda - A - \mathcal{L}_{A,B,V})U = F, \tag{1.4}$$

where $\mathcal{L}_{A,B,V}$ is the operator formally defined as

$$\mathcal{L}_{A,B,V}u(x) = \tfrac{1}{2}\operatorname{Tr}\bigl(D^2u(x)\bigr) + \langle Ax - \nabla V(x) + B(x), Du\rangle$$

on suitable functions u. The presence of the term $\lambda_i u$ in the equation adds the advantages of the resolvent of A [given by $(\lambda - A)^{-1}$] to those of the elliptic regularity theory (given by $\mathcal{L}_{A,B}$). Moreover, we use the recent maximal regularity results in $L^2(H, \nu)$ for the Kolmogorov equation

$$(\lambda - \mathcal{L}_{A,B,V})u = f$$

proved in [13].

Second, thanks to the previous new Kolmogorov equation, we may apply a trick based on Itô's formula and the multiplication by the factor e^{-A_t} (see below the definition of A_t) which greatly simplifies the proof.

Third, we use Girsanov's theorem in a better form in the proof of the main Lemma 5.2. The new proof of the lemma along with the previous two innovations allow us to use only the L^2 theory of the Kolmogorov equation, which is much simpler.

Fourth, we heavily use the theory of classical (gradient type) Dirichlet forms on infinite dimensional state spaces.

For more background literature in the finite dimensional case following the initiating work [23], we refer to [11, 12]. We only mention here the recent work [9], where SDEs with weakly differentiable drifts are studied. In the case when in [9] the diffusion matrix is constant and nondegenerate and if the weakly differentiable drift is the (weak) gradient of a convex function, our results generalize those in [9] from $\mathbb{R}^d$ to a separable Hilbert space as state space, and to the case when a bounded merely measurable drift part is added. Finally, we mention the paper [8] which concerns pathwise uniqueness for some Hölder perturbation of reaction-diffusions equations studied in spaces of continuous functions instead of square integrable function.

The organization of the paper is as follows: Section 2 is devoted to existence of solutions and Section 3 to the regularity theory of the Kolmogorov operator (1.4) above. The mentioned change of coordinates is performed in Section 4. Sections 5 and 6 contain the proof of our main Theorem 1.5. In Section 7, we present applications.

We end this section by giving the definition of Sobolev spaces and some notation. We consider an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of H which diagonalizes Q and set $Qe_k = \lambda_k e_k$ and $x_k = \langle x, e_k\rangle$ for each $x \in H$, $k \in \mathbb{N}$. We denote by P_n the orthogonal projection on the linear span of $e_1, \dots, e_n$. For each $k \in \mathbb{N} \cup \{+\infty\}$, we denote by $\mathcal{F}\mathcal{C}_b^k(H)$ the set of the cylindrical functions $\varphi(x) = \phi(x_1, \dots, x_n)$ for some $n \in \mathbb{N}$, with $\phi \in C_b^k(\mathbb{R}^n)$.

For $\mu = \gamma$ or $\mu = \nu$, the Sobolev spaces $W^{1,2}(H, \mu)$ is the completion of $\mathcal{F}\mathcal{C}_b^1(H)$ in the norm

$$\|\varphi\|^2_{W^{1,2}(H,\mu)} := \int_H \left(|\varphi|^2 + \|D\varphi\|^2\right) d\mu = \int_H \left(|\varphi|^2 + \sum_{k=1}^{\infty} (D_k\varphi)^2\right) d\mu.$$

The Sobolev spaces $W^{2,2}(H, \mu)$ is the completion of $\mathcal{F}\mathcal{C}_b^2(H)$ in the norm

$$\begin{aligned}\|u\|^2_{W^{2,2}(H,\mu)} &= \|u\|^2_{W^{1,2}(H,\mu)} + \int_H \operatorname{Tr}\left(\left[D^2 u\right]^2\right) d\mu \\ &= \|u\|^2_{W^{1,2}(H,\mu)} + \sum_{h,k\in\mathbb{N}} (D_{hk}u)^2\, d\mu.\end{aligned}$$

We denote the Borel σ-algebra on H by $\mathcal{B}(H)$ and by $B_b(H)$ the set of all bounded $\mathcal{B}(H)$-measurable functions $\varphi : H \to \mathbb{R}$. We set for a function $\varphi : H \to \mathbb{R}$

$$\|\varphi\|_\infty := \sup_{x \in H} |\varphi(x)|.$$

$I : H \to H$ denotes the identity operator on H. For $k \in \mathbb{N}$, $C_b^k(H)$ denotes the set of all $\varphi : H \to \mathbb{R}$ of class C^k, which together with all their derivatives up to order k are bounded and uniformly continuous. Furthermore, we reserve the symbol D for the closure of the derivative for $u \in \mathcal{F}C_b^1$ in $L^2(H, \mu; H)$ for $\mu = \gamma$ or $\mu = \nu$. For the Gâteaux derivative, we use the symbol ∇. Since they coincide on convex and Lipschitz functions u, in the sense that ∇u is a γ- or ν-version of Du, we shall write ∇u, whenever we want to stress that we consider that special version.

2. Existence. In this section, we shall prove that under conditions (H1)–(H4) from the Introduction, which will be in force in all of this paper, that the SDE (1.1) has a solution in the sense of Definition 1.4. We start with the following proposition showing that the gradient DV in $L^2(H, \gamma; H)$ and the Gâteaux derivative ∇V coincide γ-a.e.

PROPOSITION 2.1. *We have $V \in W^{1,2}(H, \gamma)$ and*

$$DV = \nabla V, \qquad \gamma\text{-}a.e.$$

The proof of Proposition 2.1 requires a numbers of lemmas.

LEMMA 2.2. *Let $k \in Q^{1/2}H$. Then*

$$\lim_{s \to 0} \frac{V(\cdot + sk) - V(\cdot)}{s} = \langle \nabla V, k \rangle \qquad in\ L^2(H, \gamma).$$

PROOF. Let $x \in \{V < \infty\}$. Then by convexity for $s \in (0, 1)$

$$V(x + sk) \le sV(x + k) + (1 - s)V(x),$$

hence

$$\frac{V(x + sk) - V(x)}{s} \le V(x + k) - V(x). \tag{2.1}$$

Since $k \in Q^{1/2}H$, by the Cameron–Martin theorem (see, e.g., [10], Section 1.2.3) the function on the right as a function of x is in $L^2(H, \gamma)$, since by assumption (H3) $V \in L^{2+\varepsilon}(H, \gamma)$.

Furthermore for $x \in D_V$ taking the limit $s \to 0$ in (2.1) we find that

$$\langle \nabla V(x), k \rangle \le V(x + k) - V(x).$$

Replacing k by sk which is also in $Q^{1/2}H$, and dividing by s, we obtain

$$\langle \nabla V(x), k\rangle \le \frac{V(x+sk)-V(x)}{s}. \tag{2.2}$$

But the left-hand side as a function of x is in $L^{2+\varepsilon}(H,\gamma)$ by assumption (H3). Hence, (2.1) and (2.2) imply the assertion of the lemma by Lebesgue's dominated convergence theorem, since $\gamma(D_V)=1$. $\square$

Before we proceed to Lemma 2.3 we need to introduce the following space:

$$\begin{aligned}\mathcal{D}_0 := \Big\{&u \in L^2(H,\gamma) : \exists F_u \in L^2(H,\gamma;H) \text{ such that}\\ &\lim_{s\to 0}\frac{1}{s}[u(\cdot+se_i)-u(\cdot)] = \langle \nabla F_u, e_i\rangle \text{ in } L^2(H,\gamma), \forall i\in\mathbb{N}\Big\}.\end{aligned} \tag{2.3}$$

Set $\widetilde{D}u := F_u$ for $u \in \mathcal{D}_0$. Then obviously $\mathcal{F}C_b^2 \subset \mathcal{D}_0$ and $D\varphi = \widetilde{D}\varphi$ for all $\varphi \in \mathcal{F}C_b^2$.

LEMMA 2.3. (i) *Let $u \in \mathcal{D}_0$, $\varphi \in \mathcal{F}C_b^2$ and $i\in\mathbb{N}$. Then*

$$\begin{aligned}\int_H \langle \widetilde{D}u(x), e_i\rangle \varphi(x)\gamma(dx) = &-\int_H u(x)\langle D\varphi(x), e_i\rangle\gamma(dx)\\ &+ 2\lambda_i\int_H u(x)\langle e_i, x\rangle\varphi(x)\gamma(dx).\end{aligned}$$

(ii) *The operator $\widetilde{D}:\mathcal{D}_0 \subset L^2(H,\gamma)\to L^2(H,\gamma;H)$ is closable.*

PROOF. (i) We have

$$\begin{aligned}&\int_H \langle \widetilde{D}u(x), e_i\rangle\varphi(x)\gamma(dx)\\ &\quad = \lim_{s\to 0}\frac{1}{s}\Big[\int_H u(x+se_i)\varphi(x)\gamma(dx) - \int_H u(x)\varphi(x)\gamma(dx)\Big].\end{aligned} \tag{2.4}$$

But by the Cameron–Martin theorem the image measure $T_{se_i}(\gamma)$ of γ under the translation $x\mapsto x+se_i$ is absolutely continuous with respect to γ with density (cf. [10], Section 1.2.3)

$$a_{se_i}(x) = e^{2s\lambda_i\langle e_i,x\rangle - s^2\lambda_i}.$$

Hence, the difference of the two integrals on the right-hand side of (2.4) can be written as

$$\int_H u(x)[\varphi(x-\lambda_i e_i)-\varphi(x)]a_{se_i}(x)\gamma(dx) + \int_H u(x)\varphi(x)\big(a_{se_i}(x)-1\big)\gamma(dx).$$

Hence, letting $s\to 0$ in (2.4) assertion (i) follows.

(ii) Suppose $u_n \in \mathcal{D}_0, n \in \mathbb{N}$, such that $u_n \to 0$ in $L^2(H,\gamma)$ and $\widetilde{D}u_n \to F$ in $L^2(H,\gamma;H)$. Then for all $\varphi \in \mathcal{F}C_b^2, i \in \mathbb{N}$, by (i)

$$\int_H \langle F(x), e_i\rangle \varphi(x)\gamma(dx) = \lim_{n\to\infty}\int_H \langle \widetilde{D}u_n, e_i\rangle \varphi(x)\gamma(dx) = 0.$$

Hence, $F = 0$. $\square$

Let us denote the closure of $(\widetilde{D}, \mathcal{D}_0)$ again by $\widetilde{D}$ and its domain by $\widetilde{W}^{1,2}(H,\gamma)$. Clearly, since $\mathcal{F}C_b^2 \subset \mathcal{D}_0$ with $D\varphi = \widetilde{D}\varphi$ for all $\varphi \in \mathcal{F}C_b^2$, it follows that $W^{1,2}(H,\gamma)$ is a closed subspace of $\widetilde{W}^{1,2}(H,\gamma)$. But in fact, they coincide.

LEMMA 2.4. *$\mathcal{F}C_b^2$ is dense in $\widetilde{W}^{1,2}(H,\gamma)$, hence*

$$W^{1,2}(H,\gamma) = \widetilde{W}^{1,2}(H,\gamma)$$

and thus $Du = \widetilde{D}u$ for all $u \in W^{1,2}(H,\gamma)$.

PROOF. Let $u \in \widetilde{W}^{1,2}(H,\gamma)$ such that

$$\int_H \langle \widetilde{D}\varphi, \widetilde{D}u\rangle\, d\gamma + \int_H \varphi u\, d\gamma = 0 \qquad \forall \varphi \in \mathcal{F}C_b^2. \tag{2.5}$$

Since $\varphi(x) = \Phi(x_1,\dots,x_N)$ for some $\Phi \in C_b^2(\mathbb{R}^N)$ and $x_i := \langle x, e_i\rangle$, $1 \le i \le N$, we have that

$$\langle \widetilde{D}\varphi, \widetilde{D}u\rangle = \sum_{i=1}^N \langle D\varphi, e_i\rangle \langle \widetilde{D}u, e_i\rangle$$

and that $\langle D\varphi, e_i\rangle \in \mathcal{F}C_b^2$. Hence, by Lemma 2.3, (2.5) is equivalent to

$$-\int_H (2\mathcal{L}^{\mathrm{OU}} - 1)\varphi u\, d\gamma = 0 \qquad \forall \varphi \in \mathcal{F}C_b^2, \tag{2.6}$$

where

$$\mathcal{L}^{\mathrm{OU}}\varphi(x) = \tfrac{1}{2}\operatorname{Tr} D^2\varphi(x) + \langle x, ADx\rangle.$$

But (2.6) implies that $u = 0$, since it is well known that $\lambda - \mathcal{L}^{\mathrm{OU}}$ has dense range in $L^2(H,\gamma)$ for $\lambda > 0$. For the convenience of the reader we recall the argument: The C_0-semigroup generated by the Friedrichs extension of the symmetric operator $(\mathcal{L}^{\mathrm{OU}}, \mathcal{F}C_b^2)$ on $L^2(H,\gamma)$ is easily seen to be given by the following Mehler formula on bounded, Borel functions $f : H \to \mathbb{R}$

$$P_t f(x) = \int_H f(e^{tA}x + y) N_{Q_t}(dy), \qquad t > 0, \tag{2.7}$$

where N_{Q_t} is the centred Gaussian measure on H with covariance operator

$$Q_t := \int_0^t e^{2sA}\, ds, \qquad t > 0.$$

Obviously, $P_t(\mathcal{F}C_b^2) \subset \mathcal{F}C_b^2$, and also

$$\left(\int_0^\infty e^{-\lambda t} P_t\, dt\right)(\mathcal{F}C_b^2) \subset \mathcal{F}C_b^2.$$

But

$$(\lambda - \mathcal{L}^{\mathrm{OU}})^{-1} = \int_0^\infty e^{-\lambda t} P_t\, dt$$

as operators on $L^2(H, \gamma)$. Hence,

$$(\lambda - \mathcal{L}^{\mathrm{OU}})^{-1}(\mathcal{F}C_b^2) \subset \mathcal{F}C_b^2$$

and so

$$\mathcal{F}C_b^2 \subset (1 - \mathcal{L}^{\mathrm{OU}})(\mathcal{F}C_b^2).$$

But $\mathcal{F}C_b^2$ is dense in $L^2(H, \gamma)$. $\square$

Now we turn back to SDE (1.1).

PROOF OF PROPOSITION 2.1. By (H2) and Lemma 2.2, we have that $V \in \widetilde{W}^{1,2}(H, \gamma)$ with $\nabla V = \widetilde{D} V$, γ-a.e. Hence, Lemma 2.4 implies the assertion. $\square$

Let us consider the case when in SDE (1.1) we have that $B = 0$, that is,

$$\begin{aligned} dX_t &= \big(AX_t - \nabla V(X_t)\big)\, dt + dW(t), \\ X_0 &= z, \end{aligned} \tag{2.8}$$

where for convenience we extend $\nabla V : D_V \to H$ by zero to the whole space D_V. The case for general B then follows easily from Girsanov's theorem.

To solve (2.8) in the (probabilistically) weak sense, we shall use [3], that is, the theory of Dirichlet forms, more precisely the so-called "classical (gradient type)" Dirichlet forms, which for the measure ν from the Introduction is just

$$\mathcal{E}_\nu(u, v) := \int_H \langle Du(x), Dv(x)\rangle \nu(dx), \qquad u, v \in D(\mathcal{E}_\nu) := W^{1,2}(H, \nu).$$

But the whole theory has been developed for arbitrary finite measures m on $(H, \mathcal{B}(H))$ which satisfy an integration by parts formula (see [3, 17] and the references therein) or even more generally for finite measures m for which $D : \mathcal{F}C_b^\infty \subset L^2(H, m) \to L^2(H, m; H)$ is closable (see [1, 2, 17]). In particular, we can also take $m := \gamma$. Let us recall the following result which is crucial for the theory of classical Dirichlet forms which we shall formulate for ν, but holds for every m as above. For its formulation, we need the notion of an "$\mathcal{E}_\nu$-nest": Let $F_n \subset H$, $n \in \mathbb{N}$, be an increasing sequence of closed sets and define for $n \in \mathbb{N}$

$$D(\mathcal{E}_\nu)|_{F_n} := \{u \in D(\mathcal{E}_\nu) : u = 0, \nu\text{-a.e. on } H \setminus F_n\}.$$

Then $(F_n)_{n\in\mathbb{N}}$ is called an $\mathcal{E}_\nu$-*nest* if

$$\bigcup_{k=1}^{\infty} D(\mathcal{E}_\nu)|_{F_n} \qquad \text{is dense in } D(\mathcal{E}_\nu),$$

with respect to the norm

$$\mathcal{E}_{\nu,1}(u,u)^{1/2} := \big(\mathcal{E}_\nu(u,u) + |u|^2_{L^2(H,\nu)}\big)^{1/2}, \qquad u \in D(\mathcal{E}_\nu),$$

that is, with respect to the norm in $W^{1,2}(H,\nu)$.

Then the crucial result already mentioned is the following.

THEOREM 2.5. *There exists an $\mathcal{E}_\nu$-nest consisting of compact sets.*

PROOF. See [21] and [17], Chapter IV, Proposition 4.2. □

Let us denote $(K_n)_{n\in\mathbb{N}}$ this $\mathcal{E}_\nu$-nest consisting of compacts. This theorem says that $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is completely determined in a K_σ set of H. Then it follows from the general theory that $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is *quasi-regular*, hence has an associate Markov process which solves (SDE) (2.8) and this Markov process also lives on this K_σ set $\bigcup_{n=1}^{\infty} K_n$, that is, the first hitting times $\sigma_{H\setminus K_n}$ of $H \setminus K_n$ converge to infinity as $n \to \infty$.

The precise formulations of these facts is the contents of Theorems 2.6 and 2.8 below. We need one more notion: A set $N \subset H$ is called $\mathcal{E}_\nu$-*exceptional*, if it is contained in the complement of an $\mathcal{E}_\nu$-nest. Clearly, this complement has ν-measure zero, hence $\nu(N) = 0$ if $N \in \mathcal{B}(H)$.

THEOREM 2.6. *There exists $S \in \mathcal{B}(H)$ such that $H \setminus S$ is $\mathcal{E}_\nu$-exceptional* [*hence $\nu(H \setminus S) = 0$*] *and for every $z \in S$ there exists a probability space $(\Omega, \mathcal{F}, P_z)$ equipped with a normal filtration $(\mathcal{F}_t)_{t\ge 0}$, independent real valued Brownian motions $W_t^k, t \ge 0, k \in \mathbb{N}$, on $(\Omega, \mathcal{F}, P_z)$ and a continuous H-valued $(\mathcal{F}_t)$-adapted process $X_t, t \ge 0$, such that P_z-a.s.*:

(i) $X_t \in S \ \forall t \ge 0$,

(ii) $\int_H \mathbb{E}_{P_z}[\int_0^t |\nabla V(X_s)|^2\,ds]\nu(dz) < \infty$ *and* $\mathbb{E}_{P_z}[\int_0^t 1_{H\setminus D_V}(X_s)\,ds] = 0$ $\forall t \ge 0$,

(iii) $\langle e_k, X_t\rangle = \langle e_k, z\rangle + \int_0^t (\langle Ae_k, X_s\rangle + \langle e_k, \nabla V(X_s)\rangle)\,ds + W_t^k, t \ge 0, k \in \mathbb{N}$.

Hence (*by density*), *we have a solution of* (2.8) *in the sense of Definition* 1.4. *Furthermore*, *up to completing $\mathcal{F}_t$ w.r.t. P_z*, $(\Omega, \mathcal{F})$, $X_t, t \ge 0$, *and* $(\mathcal{F}_t)$ *can be taken canonical*, *independent of $z \in S$ and then*

$$\mathbf{M} := \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, (X_t)_{t\ge 0}, (P_z)_{z\in S}\big),$$

forms a conservative Markov process, *with invariant measure ν*.

PROOF. The assertion follows from [3], Theorem 5.7. □

For later use, we define the Borel set

$$H_V := \left\{ z \in H : \mathbb{E}_{P_z}\left[\int_0^T \left|\nabla V(X_s)\right|^2 ds \right] < \infty \right\} \tag{2.9}$$

and note that by Theorem 2.6(ii) we have $\nu(H_V) = \gamma(H_V) = 1$.

In fact, by the convexity of V we also have uniqueness for the solutions to (2.8). We recall that the sub-differential ∂V of V is monotone (which is trivial to prove; see, e.g., [20], Example 2.2a) and that for $x \in D_V$, $\partial V(x) = \nabla V(x)$; see, for example, [5], page 8. Hence, we have

$$\langle \nabla V(x) - \nabla V(y), x - y \rangle \geq 0, \qquad x, y \in D_V. \tag{2.10}$$

THEOREM 2.7. *Let S be as in Theorem 2.6 and $z \in S$. Then pathwise uniqueness holds for all solutions in the sense of Definition 1.4 for SDE (2.8). In particular, uniqueness in law holds for these solutions.*

PROOF. The first assertion is an immediate consequence of the monotonicity (2.10), since a part of our Definition 1.4 requires that the solutions are in D_V $dt \otimes P$-a.e.; see, for example, [15], proof of the claim page 1008/1009 or [18], Section 4, for details. The second assertion then follows by the Yamada–Watanabe theorem (see, e.g., [22] which easily can be adapted to apply to our case here). □

THEOREM 2.8. *Let $\mathbf{M}$ be as in Theorem 2.6 and let $(F_n)_{n\in\mathbb{N}}$ be an $\mathcal{E}_\nu$-nest. Then*

$$P_z\Big[\lim_{n\to\infty} \sigma_{H\setminus F_n} = \infty\Big] = 1,$$

for all $z \in S \setminus N$, for some $\mathcal{E}_\nu$-exceptional set N, where for a closed set $F \subset H$

$$\sigma_{H\setminus F} := \inf\{t > 0 : X_t \in H \setminus F\}$$

is the first hitting time of $H \setminus F$.

PROOF. Since $\mathbf{M}$ is conservative its lifetime ζ is infinity. So, the assertion follows from [17], Chapter V, Proposition 5.30. □

Below we shall use the following simple lemma.

LEMMA 2.9. *Let $(E, \|\cdot\|)$ be a Banach space and $V : E \to (-\infty, \infty]$ a convex function.*

(i) *Let $K \subset E$ be convex and compact such that $V(K)$ is a bounded subset of $\mathbb{R}$. Then the restriction of V to K is Lipschitz.*

(ii) *Assume that V is lower semi-continuous and $K \subset E$ compact such that $V(K)$ is an upper bounded subset of $\mathbb{R}$. Then the restriction of V to K is Lipschitz.*

PROOF. (i) The proof is a simple modification of the classical proof that a continuous convex function on an open subset of E is locally Lipschitz (see [20], Proposition 1.6). For the convenience of the reader, we give the argument:

Define

$$M = \sup_{x \in K} |V(x)|$$

and

$$d := \operatorname{diam}(K) \qquad \big(:= \sup\{\|x - y\| : x, y \in K\}\big).$$

Let $x, y \in K$. Set $\alpha := \|x - y\|$ and

$$z := y + \frac{d}{\alpha}(y - x).$$

Then $\|x - y\| \le d$, hence $z \in K$ since K is convex. Furthermore,

$$y = \frac{\alpha}{\alpha + d} z + \frac{d}{\alpha + d} x,$$

hence

$$f(y) \le \frac{\alpha}{\alpha + d} f(z) + \frac{d}{\alpha + d} f(x),$$

so,

$$f(y) - f(x) \le \frac{\alpha}{\alpha + d}\big(f(z) - f(x)\big) \le \frac{2M}{d} \|x - y\|.$$

Interchanging x and y in this argument, implies the assertion.

(ii) This is an easy consequence of (i). Let K_1 be the closed convex hull of K. Then by Mazur's theorem K_1 is still compact and by convexity $V(K_1)$ is an upper bounded subset of $\mathbb{R}$. But $V(K_1)$ is also lower bounded, since V is lower semi-continuous. Hence, by (i) V is Lipschitz on K_1, hence on K. □

Now let us come back to our convex function $V : H \to (-\infty, \infty]$ satisfying (H2) and (H3). We know by Proposition 2.1 that $V \in W^{1,2}(H, \nu) = D(\mathcal{E}_\nu)$. Since $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is quasi-regular, it follows by [17], Chapter IV, Proposition 3.3, that there exists an $\mathcal{E}_\nu$-nest $(F_n)_{n \in \mathbb{N}}$ and a $\mathcal{B}(H)$-measurable function $\widetilde{V} : H \to \mathbb{R}$ such that

$$\widetilde{V} = V\, \nu\text{-a.e. and } \widetilde{V}|_{F_n} \text{ is continuous for every } n \in \mathbb{N}, \tag{2.11}$$

where $\widetilde{V}|_{F_n}$ denotes the restriction of $\widetilde{V}$ to F_n. By [17], Chapter III, Theorem 2.11, $(F_n \cap K_n)_{n \in \mathbb{N}}$ is again an $\mathcal{E}_\nu$-nest, where $(K_n)_{n \in \mathbb{N}}$ is the $\mathcal{E}_\nu$-nest of compacts from

Theorem 2.5. Since $\nu(U) > 0$ for every nonempty open set $U \subset H$, by [17], Chapter III, Proposition 3.8, we can find an $\mathcal{E}_\nu$-nest $(\widetilde{F}_n)_{n\in\mathbb{N}}$ such that $\widetilde{F}_n \subset F_n \cap K_n$ and the restriction of ν to $\widetilde{F}_n$ has full topological support on $\widetilde{F}_n$ for every $n \in \mathbb{N}$, that is, $\nu(U \cap \widetilde{F}_n) > 0$ for all open $U \subset H$ with $U \cap \widetilde{F}_n \neq \varnothing$. (Such an $\mathcal{E}_\nu$ set is called *regular*.) Since we want to fix this special regular $\mathcal{E}_\nu$-nest of compacts depending on V below, we assign to it a special notation and set

$$K_n^V := \widetilde{F}_n, \qquad n \in \mathbb{N}. \tag{2.12}$$

Now we can prove the following result which will be crucially used in Section 6.

PROPOSITION 2.10. (i) *Let $n \in \mathbb{N}$ and K_n^V as in* (2.12). *Then $V|_{K_n^V}$ is real valued, continuous and bounded. Furthermore, $V(x) = \widetilde{V}(x)$ for every $x \in \bigcup_{n=1}^\infty K_n^V$.*

(ii) *There exists $S_V \in \mathcal{B}(H)$ such that $H \setminus S_V$ is $\mathcal{E}_\nu$-exceptional, Theorem* 2.6 *holds with S_V replacing S and for every $z \in S_V$*

$$P_z\Big[\lim_{n\to\infty} \sigma_{H\setminus K_n^V} = \infty\Big] = 1. \tag{2.13}$$

PROOF. (i) Since $K_n^V \subset F_n$, we have for $\widetilde{V}$ from (2.11), that $V|_{K_n^V} - \widetilde{V}|_{K_n^V}$ is lower semi-continuous on K_n^V with respect to the metric on K_n^V induced by $|\cdot|$. Hence, $\{V|_{K_n^V} - \widetilde{V}|_{K_n^V} > 0\} = K_n^V \cap U$ for some open subset $U \subset H$. Since $V|_{K_n^V} = \widetilde{V}|_{K_n^V}$ ν-a.s., it follows, since $(K_n^V)_{n\in\mathbb{N}}$ is a regular $\mathcal{E}_\nu$-nest that

$$V(x) \le \widetilde{V}(x) \qquad \text{for every } x \in K_n^V.$$

But $\widetilde{V}|_{K_n^V}$ is continuous, hence bounded, because K_n^V is compact, so $V(K_n^V)$ is an upper bounded subset of $\mathbb{R}$, so by Lemma 2.9(ii) $V|_{K_n^V}$ is Lipschitz. But then $\{V|_{K_n^V} \neq \widetilde{V}|_{K_n^V}\} = K_n^V \cap U$ for some open subset $U \subset H$. Since $(K_n^V)_{n\in\mathbb{N}}$ is a regular $\mathcal{E}_\nu$-nest, we conclude that

$$V(x) = \widetilde{V}(x) \qquad \text{for every } x \in K_n^V.$$

Hence, assertion (i) is proved.

(ii) By Theorem 2.8, we know that there exists an $\mathcal{E}_\nu$-nest $(F_n)_{n\in\mathbb{N}}$ such that

$$P_z\Big[\lim_{n\to\infty} \sigma_{H\setminus K_n^V} = \infty\Big] = 1 \qquad \forall z \in \bigcup_{n=1}^\infty F_n.$$

Then by a standard procedure (see, e.g., [17], page 114) one can construct the desired set $S_V \in \mathcal{B}(H)$. $\square$

For the rest of this section, we fix S_V as in Proposition 2.10.

COROLLARY 2.11. *Let $z \in S_V$ and $(X_t)_{t\ge 0}$ a solution to* (2.8) *on some probability space $(\Omega, \mathcal{F}, P)$ with normal filtration and cylindrical $(\mathcal{F}_t)$-Brownian motion $W = W_t, t \ge 0$. Then* (2.13) *holds with P replacing P_z.*

PROOF. This follows from the last part of Theorem 2.7. □

It is now easy to prove existence of (probabilistic) weak solutions to SDE (1.1) and uniqueness in law

THEOREM 2.12. *For every $z \in S_V$, there exists a solution $Y = Y_t, t \in [0, T]$, to SDE* (1.1) *on some probability space* $(\Omega', \mathcal{F}', P')$ *in the sense of Definition* 1.4 *and this solution is unique in law. Furthermore*, (2.13) *holds with P' replacing P_z and Y replacing X, where $X = X_t, t \geq 0$, is the process from Theorem* 2.6 *and if $z \in S_V \cap H_V$* [*with H_V as in* (2.9)], *then*

$$\int_0^T |\nabla V(Y_s)|^2 \, ds < \infty \qquad P'\text{-a.s.} \tag{2.14}$$

PROOF. This is now an easy consequence of Theorems 2.6, 2.7 and Girsanov's theorem (see, e.g., [11], Appendix A1) which easily extends to the present case since uniqueness in law holds for SDE (2.8). To prove the last part, we note that by Girsanov's theorem there exists a probability density $\rho : \Omega \to (0, \infty)$ such that

$$(\rho \cdot P') \circ Y^{-1} = P_z \circ X^{-1}.$$

Hence, $P_z \circ X^{-1} = \rho_0 P' \circ Y^{-1}$, where ρ_0 is the $P' \circ Y^{-1}$-a.s. unique function such that $\rho_0(Y) = \mathbb{E}_{P'}[\rho|\sigma(Y)]$ P-a.s. and $\sigma(Y)$ denotes the σ-algebra generated by $Y_t, t \in [0, T]$. So, (2.13) and (2.14) follow, if $\rho_0 > 0 P' \circ Y^{-1}$-a.e. To show the latter, we first note that

$$P' \circ Y^{-1}(\{\rho_0 = 0\}) = P'(\{\mathbb{E}_{P'}[\rho|\sigma(Y)] = 0\}).$$

But since

$$\rho = e^{-\int_0^T \langle B(Y_s), dW_s\rangle - (1/2)\int_0^T |B(Y_s)|^2 \, ds}$$

and W is $\sigma(Y)$-measurable by SDE (1.1), it follows that $\mathbb{E}_{P'}[\rho|\sigma(Y)] = \rho$. But $\rho > 0$. □

3. Regularity theory for the corresponding Kolmogorov operator.

3.1. *Uniform estimates on Lipschitz norms.* First, we are concerned with the scalar equation

$$\lambda u - \mathcal{L}u - \langle B(x), Du\rangle = f, \tag{3.1}$$

where $\lambda > 0$, $f \in B_b(H)$ and $\mathcal{L}$ is the Kolmogorov operator

$$\mathcal{L}u(x) = \tfrac{1}{2} \operatorname{Tr}[D^2 u(x)] + \langle Ax - DV(x), Du(x)\rangle, \qquad x \in H. \tag{3.2}$$

Since the corresponding Dirichlet form

$$\mathcal{E}_B(v, w) := \frac{1}{2}\int_H \langle Dv, Dw\rangle \, d\nu - \int_H \langle B, Dv\rangle w \, d\nu + \lambda \int_H vw \, d\nu,$$

$v, w \in W^{1,2}(H, \nu)$, is weakly sectorial for λ big enough, it follows by [17], Chapter 1 and Section 3e in Chapter II, that (3.1) has a unique solution $u \in L^2(H, \nu)$ such that $u \in D(\mathcal{L})$. We need, however, Lipschitz regularity for u and an estimate for its ν-a.e. defined Gâteaux derivative in terms of $\|u\|_\infty$. To prove this, we also need the Kolmogorov operator associated to the linear equation that one obtains, when $B = V = 0$, in SDE (1.1), that is, the Ornstein–Uhlenbeck operator

$$\mathcal{L}^{\mathrm{OU}} u(x) = \tfrac{1}{2} \operatorname{Tr}\big[D^2 u(x)\big] + \big\langle Ax, Du(x)\big\rangle, \qquad x \in H. \tag{3.3}$$

As initial domains of $\mathcal{L}$, $\mathcal{L}^{\mathrm{OU}}$ and $\mathcal{L} + \langle B, D\rangle$, we take the set $\mathcal{E}_A(H)$ defined to consist of the linear span of all real parts of functions $\varphi : H \to \mathbb{R}$ of the form $\varphi(x) = e^{i\langle h,x\rangle}$, $x \in H$, with $h \in D(A)$. It is easy to check that $\mathcal{E}_A(H) \subset W^{1,2}(H, \gamma)$ densely and $\mathcal{E}_A(H) \subset W^{1,2}(H, \nu)$ densely. Then rewriting the last term in the above expression as $\langle Ax, Du(x)\rangle$, the above operators are well defined for $u \in \mathcal{E}_A(H)$. Below we are going to use results from [14] in a substantial way with $F := \partial V$, the sub-differential of V, which is maximal monotone (see, e.g., [5]) and which is in general multi-valued, but single-valued on $D_V \subset D(F)$ because $\partial V(x) = \nabla V(x)$ for $x \in D_V$.

Let us first check that assumptions (H1) and (H2) in there are satisfied.

First, Hypothesis 1.1 in [14] is satisfied since we are in the special case $A = A^*$ and $C = I$. Hypothesis 1.2(ii) is satisfied for $\mathcal{L}$ defined above, replacing N_0 in [14] with $F_0 := \nabla V$, since by integrating by parts we have

$$\int_H \mathcal{L}\varphi\psi \, d\nu = -\frac{1}{2} \int_H \langle D\varphi, D\psi\rangle \, d\nu \qquad \forall \varphi, \psi \in \mathcal{E}_A(H)$$

and thus, taking $\psi = 1$,

$$\int_H \mathcal{L}\varphi \, d\nu = 0 \qquad \forall \varphi, \psi \in \mathcal{E}_A(H). \tag{3.4}$$

Here, F_0 is the minimal section of F in [14], and hence $\nabla V = F_0$ on $D(V) \subset D(F)$, so Hypothesis 1.2(iii) holds. Hypothesis 1.2(i) follows from Remark 1.1.

The first result we now deduce from [14] is the following.

PROPOSITION 3.1. *$(\mathcal{L}, \mathcal{E}_A(H))$ is closable on $L^2(H, \nu)$ and its closure $(\mathcal{L}, D(\mathcal{L}))$ is m-dissipative on $L^2(H, \nu)$.*

PROOF. This is a special case of [14], Theorem 2.3. □

For later use, we need to replace $\mathcal{E}_A(H)$ in Proposition 3.1 above by $\mathcal{F}C_b^2$ (defined in the Introduction of this paper). We need the following easy lemma.

LEMMA 3.2. *Let $\varphi \in C_b^2(\mathbb{R}^d)$. Then there exists a sequence φ_n, $n \in \mathbb{N}$, each φ_n consisting of linear combinations of functions of type $x \to \cos\langle a, x\rangle_{\mathbb{R}^d}$, $a \in \mathbb{R}^d$, such that $\sup_{n\in\mathbb{N}}\{\|\varphi_n\|_\infty + \|D\varphi_n\|_\infty + \|D^2\varphi_n\|_\infty\} < \infty$ and*

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \qquad \lim_{n\to\infty} D\varphi_n(x) = D\varphi(x), \qquad \lim_{n\to\infty} D^2\varphi_n(x) = D^2\varphi(x),$$

for all $x \in \mathbb{R}^d$.

PROOF. First assume that $\varphi \in C_b^\infty(\mathbb{R}^d)$ with compact support. Then we have

$$\varphi(x) = \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle_{\mathbb{R}^d}} \widehat{\varphi}(\xi)\, d\xi, \qquad x \in \mathbb{R}^d,$$

where $\widehat{\varphi}$ is in the Schwartz test function space, with the corresponding integral representations for $D\varphi$ and $D^2\varphi$.

Discretizing the integrals immediately implies the assertion since $x \mapsto (1 + |x|^2)\widehat{\varphi}$ is Lebesgue integrable. Replacing φ by $\chi_n\varphi$ where $\chi_n, n \in \mathbb{N}$, is a suitable sequence of localizing functions (*bump functions*), the result follows for all $\varphi \in C_b^\infty(\mathbb{R}^d)$ by regularization through convolution with a Dirac sequence. $\square$

As an immediate consequence of Proposition 3.1 and Lemma 3.2, we get the following.

PROPOSITION 3.3. *$(\mathcal{L}, \mathcal{F}C_b^2)$ is closable on $L^2(H,\nu)$ and the closure $(\mathcal{L}, D(\mathcal{L}))$ is the same as that in Proposition 3.1, hence it is m-dissipative on $L^2(H,\nu)$. Furthermore,*

$$\mathcal{L}u = \mathcal{L}^{\mathrm{OU}}u - \langle \nabla V, Du\rangle \qquad \forall u \in \mathcal{F}C_b^\infty.$$

Since $(\mathcal{L}, D(\mathcal{L}))$ is an m-dissipative operator on $L^2(H,\nu)$ by Proposition 3.1, every $\lambda > 0$ is in its resolvent set, hence $(\lambda - \mathcal{L})^{-1}$ exists as a bounded operator on $L^2(H,\nu)$. The following is one of the main results in [14].

THEOREM 3.4. *Let $\lambda > 0$ and $f \in B_b(H)$. Then there exists a ν-version of $(\lambda - \mathcal{L})^{-1} f$ denoted by $R_\lambda f$, which is Lipschitz on H, more precisely*

$$\big|R_\lambda f(x) - R_\lambda f(y)\big| \le \sqrt{\frac{\pi}{\lambda}}\|f\|_\infty |x-y| \qquad \forall x, y \in H. \tag{3.5}$$

PROOF. We first notice that H_0, defined in [14] to be the topological support of ν, in our case is equal to H, since ν has the same zero sets as the (nondegenerate) Gaussian measure γ on H. Hence, the assertion follows from the last sentence of [14], Proposition 5.2. $\square$

REMARK 3.5. In fact, each R_λ is a kernel of total mass λ^{-1}, absolutely continuous with respect to ν and $(R_\lambda)_{\lambda>0}$ forms a resolvent of kernels on $(H, \mathcal{B}(H))$. We refer to [14], Section 5, for details.

Now we are going to solve (3.1) for each $f \in B_b(H)$ if λ is large enough, and show that the solution $u \in L^2(H,\nu)$ has a ν-version which is Lipschitz continuous, with Lipschitz constant dominated up to a constant by $\|f\|_\infty$.

First, we need the following.

LEMMA 3.6. *Let $g : H \to \mathbb{R}$ be Lipschitz. Then $g \in W^{1,2}(H,\gamma)$, hence also in $W^{1,2}(H,\nu)$ and $\|Dg\|_\infty \le \|g\|_{\mathrm{Lip}}$ (= Lipschitz norm of g). Furthermore, $Dg = \nabla g$, γ-a.e. where ∇g is the Gâteaux derivative of g which exists γ-a.e.*

PROOF. By the fundamental result in [4, 19] the set D_g of all $x \in H$ where g is Gâteaux-(even Fréchet-) differentiable has γ measure one. Let ∇g denote its Gâteaux derivative. Since $|\nabla g| \in L^\infty(H,\mu)$, it follows trivially that $g \in \mathcal{D}_0$ defined in (2.3). Hence, by Lemma 2.4 the assertion follows. □

LEMMA 3.7. *Consider the operator $T_\lambda : L^\infty(H,\nu) \to L^\infty(H,\nu)$ defined by*

$$T_\lambda\varphi = \langle B, \nabla R_\lambda\varphi\rangle, \qquad \varphi \in L^\infty(H,\nu).$$

Then for $\lambda \ge 4\pi\|B\|_\infty^2$

$$\|T_\lambda\varphi\|_{L^\infty(H,\nu)} \le \tfrac{1}{2}\|\varphi\|_{L^\infty(H,\nu)} \qquad \forall \varphi \in L^\infty(H,\nu).$$

PROOF. We have by (3.5) and Lemma 3.6 that for $\varphi \in L^\infty(H,\mu)$

$$\|T_\lambda\varphi\|_{L^\infty(H,\nu)} \le \|B\|_\infty\sqrt{\frac{\pi}{\lambda}}\|\varphi\|_{L^\infty(H,\nu)},$$

and the assertion follows. □

PROPOSITION 3.8. *Let $f \in B_b(H)$ and $\lambda \ge 4\pi\|B\|_\infty^2$. Then (3.1) has a unique solution given by the Lipschitz function*

$$u := R_\lambda\big((I - T_\lambda)^{-1} f\big).$$

This solution is Lipschitz on H with Lipschitz norm

$$\|u\|_{\mathrm{Lip}} \le 2\sqrt{\frac{\pi}{\lambda}}\|f\|_\infty.$$

PROOF. Since the operator norm of T_λ is less than $\frac{1}{2}$, the operator $(I - T_\lambda)^{-1}$ exists as a continuous operator on $L^\infty(H,\nu)$ with operator norm less than 2. Furthermore, by Theorem 3.4 and Lemma 3.6

$$\begin{aligned}(\lambda - \mathcal{L})R_\lambda\big((I - T_\lambda)^{-1} f\big) &- \big\langle B, DR_\lambda\big((I - T_\lambda)^{-1} f\big)\big\rangle\\ &= (I - T_\lambda)^{-1} f - T_\lambda\big((I - T_\lambda)^{-1} f\big) = f.\end{aligned}$$

The final part follows from (3.5) □

Having established the result for the scalar equation (3.1) for $\lambda \ge 4\pi\|B\|_\infty^2$, we may prove it for the vector equation (1.4), whose solution U has components u^i satisfying the equation

$$(\lambda + \lambda_i)u^i - \mathcal{L}u^i - \big\langle B(x), Du^i\big\rangle = f^i, \tag{3.6}$$

where f^i are the components of the vector function $F:H \to H$. $[U(x) = \sum_{i=1}^\infty u^i(x)e_i,\ F(x) = \sum_{i=1}^\infty f^i(x)e_i]$.

We have by Proposition 3.8

$$\left|u^i(x) - u^i(y)\right|^2 \le \frac{4\pi}{\lambda + \lambda_i}\|f^i\|_\infty^2|x-y|^2 \le \frac{4\pi}{\lambda+\lambda_i}\|F\|_\infty^2|x-y|^2,$$

hence

$$\sum_{i=1}^\infty\left|u^i(x) - u^i(y)\right|^2 \le c(\lambda)^2\|F\|_\infty^2|x-y|^2,$$

where $c(\lambda) := \sum_{i=1}^\infty \frac{4\pi}{\lambda+\lambda_i}$. This series converges and $\lim_{\lambda\to\infty} c(\lambda) = 0$. Moreover, $|U(x) - U(y)|^2 = \sum_{i=1}^\infty |u^i(x) - u^i(y)|^2$, hence we have proved the following.

LEMMA 3.9. *$U(= U_\lambda)$ defined above satisfies*

$$\left|U(x) - U(y)\right| \le c(\lambda)\|F\|_\infty|x-y|, \qquad x, y \in H$$

with $\lim_{\lambda\to\infty} c(\lambda) = 0$.

3.2. *Itô formula for Lipschitz functions.* Below we want to apply Itô's formula to $u(X_t), t \ge 0$, where u is as in Proposition 3.8 and $(X_t)_{t\ge 0}$ are the paths of the Markov process $\mathbf{M}$ from Theorem 2.6. Since u is only Lipschitz and we are on the infinite dimensional state space H, this is a delicate issue. To give a technically clean proof, we need a specific approximation of the solution u in Proposition 3.8 by functions $u_n \in \mathcal{F}C_b^2, n \in \mathbb{N}$. More precisely, we shall prove the following result.

PROPOSITION 3.10. *Let $\lambda > 0$ and $g \in B_b(H) \cap D(\mathcal{L}^{\mathrm{OU}}, C_{b,2}^1(H))$ (for the definition of the latter see below). Set*

$$w := R_\lambda g.$$

Then there exists a sequence $u_n \in \mathcal{F}C_b^2, n \in \mathbb{N}$, such that

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\left(\|u_n\|_\infty + \|\nabla u_n\|_\infty\right) \le 2\sqrt{\pi}\max\{\lambda^{-1}, \lambda^{-1/2}\}\|g\|_\infty, \\
&\lim_{n\to\infty}\int_H\left[\left|\mathcal{L}^{\mathrm{OU}}(w-u_n)\right|^2 + \left|\nabla(w-u_n)\right|^2 + (w-u_n)^2\right]d\nu = 0.
\end{aligned} \tag{3.7}$$

In particular, $u_n \to w$ as $n \to \infty$ in $\mathcal{L}$-graph norm $[$on $L^2(H,\nu)]$ and

$$\mathcal{L} = \mathcal{L}^{\mathrm{OU}}w - \langle\nabla V, \nabla w\rangle.$$

For the proof, we need some more details from [14].

Define for $\lambda > 0$ and $\varphi \in B_b(H)$

$$R\left(\lambda, \mathcal{L}^{\mathrm{OU}}\right)\varphi(x) = \int_0^\infty e^{-\lambda t}P_t\varphi(x)\,dt, \tag{3.8}$$

where P_t is defined as in (2.7). Then

$$R\big(\lambda, \mathcal{L}^{\mathrm{OU}}\big)\big(C^1_{b,2}(H)\big) \subset C^1_{b,2}(H),$$

where $C^1_{b,2}(H)$ denotes the set of all $\varphi \in C^1_b(H)$ such that

$$\sup_{x\in H} \frac{|\varphi(x)|}{1+|x|^2} < \infty \quad \text{and} \quad \sup_{x\in H} \frac{|D\varphi(x)|}{1+|x|^2} < \infty.$$

As in [14], we set

$$D\big(\mathcal{L}^{\mathrm{OU}}, C^1_{b,2}(H)\big) := R\big(\lambda, \mathcal{L}^{\mathrm{OU}}\big)\big(C^1_{b,2}(H)\big),$$

which by this resolvent equation is independent of $\lambda > 0$ and is a natural domain for the operator $\mathcal{L}^{\mathrm{OU}}$.

PROPOSITION 3.11. *Let $u \in D(\mathcal{L}^{\mathrm{OU}}, C^1_{b,2}(H))$. Then there exists $\varphi_n \in \mathcal{E}_A(H)$, $n \in \mathbb{N}$, such that $\varphi_n \to u$ in ν-measure and for some $C \in (0,\infty)$*

$$\big|\varphi_n(x)\big| + \big|D\varphi_n(x)\big| + \big|\mathcal{L}^{\mathrm{OU}}\varphi_n(x)\big| \le C\big(1+|x|^2\big) \qquad \forall x \in H, n \in \mathbb{N}.$$

In particular, $u \in D(\mathcal{L})$ and $\varphi_n \to u$ in the graph norm of $\mathcal{L}$ on $L^2(H,\nu)$ and

$$\mathcal{L}u = \mathcal{L}^{\mathrm{OU}}u - \langle \nabla V, Du\rangle.$$

PROOF. Since convergence in measure comes from a metrizable topology, this follows from [14], Lemma 2.2, Lebesgue's dominatd convergence theorem, Remark 0 and the fact that $(\mathcal{L}, \mathcal{E}_A(H))$ is closable on $L^2(H,\nu)$. $\square$

Now let us recall the approximation procedure for ∂V, more precisely for its sub-differential $F := -\partial V$ with domain $D(F)$, performed in [14]. [We recall that ∇V is maximal monotone (see, e.g., [5]), hence we can consider its Yosida approximations.] For $\alpha \in (0,\infty)$, we set

$$F_\alpha(x) := \frac{1}{\alpha}\big(J_\alpha(x) - x\big), \qquad x \in H,$$

where

$$J_\alpha(x) := (I - \alpha F)^{-1}(x), \qquad x \in H.$$

It is well known (see, e.g., [5]) that

$$\begin{aligned} &\lim_{\alpha\to 0} F_\alpha(x) = F_0(x) \qquad \forall x \in D(F), \\ &\big|F_\alpha(x)\big| \le F_0(x) \qquad \forall x \in D(F), \end{aligned} \tag{3.9}$$

where

$$F_0(x) := \inf_{y\in F(x)} |y|.$$

[Recall that $F(x) = \partial V(x)$ is in general multi-valued unless $x \in D_V$, when $\partial V(x) = \nabla V(x)$.] We need a further standard regularization by setting

$$F_{\alpha,\beta}(x) := \int_H e^{\beta B} F_\alpha\big(e^{\beta B} + y\big) N_{(1/2)B^{-1}(e^{2\beta B}-1)}(dy), \qquad \alpha, \beta \in (0,\infty), \tag{3.10}$$

where $B : D(B) \subset H \to H$ is a self-adjoint negative definite operator such that B^{-1} is of trace class. Then $F_{\alpha,\beta}$ is dissipative, of class C^∞ has bounded derivatives of all orders and

$$F_{\alpha,\beta} \to F_\alpha \text{ pointwise as } \beta \to 0 \tag{3.11}$$

(see [16], Theorem 9.19).

Now let us fix $\lambda > 0$ and consider the equation for $v \in C_b^2(H)$

$$\lambda u - \mathcal{L}^{\mathrm{OU}} u - \langle F_{\alpha,\beta}, Du\rangle = v. \tag{3.12}$$

Then by [14], page 268, there exists a linear map

$$R_\lambda^{\alpha,\beta} : C_b^2(H) \to D\big(\mathcal{L}^{\mathrm{OU}}, C_{b,2}^1(H)\big) \cap C_b^2(H)$$

[in fact given by the resolvent of the SDE corresponding to the Kolmogorov operator on the left-hand side of (3.12)] such that $R_\lambda^{\alpha,\beta} v$ is a solution to (3.12) for each $v \in C_b^2(H)$. In particular,

$$\|R_\lambda^{\alpha,\beta} v\|_\infty \le \frac{1}{\lambda}\|v\|_\infty, \qquad \lambda > 0, v \in C_b^2(H). \tag{3.13}$$

We also have by [14], (4.7), that

$$\sup_{x\in H}\big|\nabla R_\lambda^{\alpha,\beta} v(x)\big| \le \sqrt{\frac{\pi}{\lambda}}\|v\|_\infty, \qquad \lambda > 0, v \in C_b^2(H). \tag{3.14}$$

Now the proof of Proposition 3.10 will be the consequence of the following two lemmas.

LEMMA 3.12. *Let $\alpha_n \in (0,\infty)$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \alpha_n = 0$. Then there exists $\beta_n \in (0,\infty)$, $n \in \mathbb{N}$, such that for all $v \in C_b^2(H)$ we have that*

$$\lim_{n\to\infty} R_\lambda^{\alpha_n,\beta_n} v = R_\lambda v,$$

in $\mathcal{L}$-graph norm [*on* $L^2(H,\nu)$].

PROOF (cf. the proof of [14], Theorem 2.3). Since $D(F) \supset D_V$, so $\nu(D(F)) \ge \nu(D_V) = 1$, it follows by (3.9) and Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \int_H |F_{\alpha_n} - \nabla V|^2 \, d\nu = 0. \tag{3.15}$$

Since by the definition of $F_{\alpha,\beta}$ we have that for each $\alpha > 0$ there exists $c_\alpha \in (0, \infty)$ such that

$$\left|F_{\alpha,\beta}(x)\right| \le c_\alpha\left(1 + |x|\right) \qquad \forall x \in H,$$

it follows by (3.11) that for $n \in \mathbb{N}$ there exists $\beta_n \in (0, \frac{1}{n})$, such that

$$\int_H \left|F_{\alpha_n,\beta_n}(x) - F_{\alpha_n}(x)\right| \nu(dx) \le \frac{1}{n}.$$

Hence, by (3.15)

$$\lim_{n\to\infty} \int_H |F_{\alpha_n,\beta_n} - \nabla V|^2 \, d\nu = 0. \tag{3.16}$$

Now let $v \in C_b^2(H)$. Then $R_\lambda^{\alpha_n,\beta_n} v \in C_b^2(H) \cap D(\mathcal{L}^{\mathrm{OU}}, C_{b,2}^1(H))$, hence by Proposition 3.11 and, because $R_\lambda^{\alpha_n,\beta_n} v$ solves (3.12), we have

$$(\lambda - \mathcal{L}) R_\lambda^{\alpha_n,\beta_n} v = v + \langle F_{\alpha_n,\beta_n} - \nabla V, \nabla R_\lambda^{\alpha_n,\beta_n} v\rangle, \tag{3.17}$$

consequently,

$$R_\lambda^{\alpha_n,\beta_n} v = (\lambda - \mathcal{L})^{-1} v + (\lambda - \mathcal{L})^{-1}\left(\langle F_{\alpha_n,\beta_n} - \nabla V, \nabla R_\lambda^{\alpha_n,\beta_n} v\rangle\right). \tag{3.18}$$

But by (3.14) and (3.16)

$$\lim_{n\to\infty} \int_H \left|\langle F_{\alpha_n,\beta_n} - \nabla V, \nabla R_\lambda^{\alpha_n,\beta_n} v\rangle\right|^2 d\nu = 0.$$

Hence, (3.17) and (3.18) imply the assertion, because $(\lambda - \mathcal{L})^{-1}$ is continuous on $L^2(H, \nu)$ ad $R_\lambda v$ is a ν-version of $(\lambda - \mathcal{L})^{-1} v$. $\square$

LEMMA 3.13. *Let λ, g and w be as in Proposition 3.10. Then there exist $u_n \in C_b^2(H) \cap D(\mathcal{L}^{\mathrm{OU}}, C_{b,2}^1(H))$, $n \in \mathbb{N}$, such that*

$$\sup_{n\in\mathbb{N}} \left(\|u_n\|_\infty + \|\nabla u_n\|_\infty\right) \le 2\sqrt{\pi} \max\{\lambda^{-1}, \lambda^{-1/2}\} \|g\|_\infty,$$

and (3.7) holds for these u_n, $n \in \mathbb{N}$.

PROOF. Since $C_b^2(H) \subset L^2(H, \nu)$ densely, we can find $v_k \in C_b^2(H)$, $k \in \mathbb{N}$, such that

$$\sup_{k\in\mathbb{N}} \|v_k\|_\infty \le 2\|g\|_\infty$$

and

$$\lim_{k\to\infty} \int_H |g - v_k|^2 \, d\nu = 0,$$

hence by the continuity of $(\lambda - \mathcal{L})^{-1}$,

$$\lim_{k\to\infty} \int_H \big|R_\lambda(g - v_k)\big|^2 \, d\nu = 0.$$

Therefore, $R_\lambda v_k \to R_\lambda g$ in $\mathcal{L}$-graph norm [on $L^2(H,\nu)$] as $k \to \infty$. Hence, by Lemma 3.12 we can choose a subsequence $(k_n)_{n\in\mathbb{N}}$ such that

$$R_\lambda^{\alpha_{n_k},\beta_{n_k}} v_k \to R_\lambda g \qquad \text{in } \mathcal{L}\text{-graph norm}\big(\text{on } L^2(H,\nu)\big) \text{ as } k \to \infty.$$

Taking $u_k := R_\lambda^{\alpha_{n_k},\beta_{n_k}} v_k$, $k \in \mathbb{N}$, the assertion follows from (3.13) and (3.14), recalling that convergence in $\mathcal{L}$-graph norm implies convergence in $W^{1,2}(H,\nu)$. □

PROOF OF PROPOSITION 3.10. Let $u \in C_b^2(H) \cap D(\mathcal{L}^{\mathrm{OU}}, C_{b,2}^1(H))$ and define $u_n := u \circ P_n \in \mathcal{F}C_b^2$, $n \in \mathbb{N}$. Then $\|u_n\|_\infty \le \|u\|_\infty$ and $\|\nabla u_n\|_\infty \le \|\nabla u\|_\infty$. Furthermore, $u_n \to u$, $\nabla u_n \to \nabla u$ and $\mathcal{L}^{\mathrm{OU}} u_n \to \mathcal{L}^{\mathrm{OU}} u$ pointwise on H as $n \to \infty$. Furthermore, $\mathcal{L}^{\mathrm{OU}} u_n \to \mathcal{L}^{\mathrm{OU}} u$ in $L^2(H,\gamma)$, hence in $L^2(H,\nu)$ as $n \to \infty$. Now the assertion follows by Lemma 3.13. □

COROLLARY 3.14. *Let $f \in B_b(H)$, $\lambda \ge 4\pi\|B\|_\infty^2$ and u as in Proposition 3.8, that is,*

$$u := R_\lambda\big((I - T_\lambda)^{-1} f\big).$$

Let $u_n \in \mathcal{F}C_b^2 \cap D(\mathcal{L}^{\mathrm{OU}}, C_{b,2}^1(H))$, $n \in \mathbb{N}$, be as in Proposition 3.10 with $g := (I - T_\lambda)^{-1} f$ [$\in B_b(H)$, with $\|g\|_\infty \le 2\|f\|_\infty$ by the proof of Proposition 3.8]. Consider the Markov process

$$\mathbf{M} := \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, (X_t)_{t\ge0}, (P_z)_{z\in S_V}\big)$$

from Theorem 2.6, with S_V defined in Proposition 2.10. Then there exists an $\mathcal{E}_\nu$-nest $(F_k^{\lambda,f})_{k\in\mathbb{N}}$ of compacts such that for every $k \in \mathbb{N}$, $F_k^{\lambda,f} \subset S_V$ and some subsequence $n_l \to \infty$:

(i) $u_{n_l}(z) \to u(z)$,
(ii) $\mathbb{E}_{P_z} \int_0^\infty e^{-\lambda s} |\nabla u - \nabla u_{n_l}|^2(X_s)\, ds = R_\lambda(|\nabla u - \nabla u_{n_l}|^2)(z) \to 0$,
(iii) $\mathbb{E}_{P_z} \int_0^\infty e^{-s} |\mathcal{L}(u - u_{n_l})(X_s)|\, ds \to 0$,

uniformly in $z \in F_k^{\lambda,f}$. In particular, for all $z \in \bigcup_{k=1}^\infty F_k^{\lambda,f} \setminus N$ with an $\mathcal{E}_\nu$-exceptional set N, we have that P_z-a.e. the following Itô formula holds:

$$u(X_t) - z - \int_0^t \mathcal{L}u(X_s)\, ds = \int_0^t \langle \nabla u(X_s), dW(s)\rangle \qquad \forall t \ge 0. \tag{3.19}$$

PROOF. Since the convergence of all three sequences in (i)–(iii) takes place in $W^{1,2}(H,\nu)$, the existence of such an $\mathcal{E}_\nu$-nest and subsequence $(n_l)_{l\in\mathbb{N}}$ follows from [17], Chapter III, Proposition 3.5, and Theorem 2.5 above. By Theorem 2.8 for $z \in \bigcup_{k=1}^\infty F_k^{\lambda,f} \setminus N$, for some $\mathcal{E}_\nu$-exceptional set N we know that

$P_z[\bigcup_{k=1}^\infty \{\tau_{H\setminus F_k^{\lambda,f}} > t\}] = 1$ for all $t \geq 0$. So, fix $z \in \bigcup_{k=1}^\infty F_k^{\lambda,f} \setminus N$. Then by the classical Itô formula on finite dimensional Euclidean space and by Theorem 2.6(iii), we have P_z-a.s.

$$\begin{aligned} & u_{n_l}(X_t) - z - \int_0^t \big(\mathcal{L}^{\mathrm{OU}} u_{n_l} - \langle \nabla V, \nabla u_{n_l}\rangle\big)(X_s)\, ds \\ & \quad = \int_0^t \big\langle \nabla u_{n_l}(X_s), dW(s)\big\rangle \qquad \forall t \geq 0. \end{aligned} \tag{3.20}$$

Fix $t > 0$. Then on $\{\tau_{H\setminus F_k^{\lambda,f}} > t\}$ we have by (ii) above that $u_{n_l}(X_t) \to u(X_t)$ as $n \to \infty$ and by the last part of Proposition 3.3 and (iii) above

$$\begin{aligned} & E_{P_z} \int_0^t \big| \big(\mathcal{L}u - \big(\mathcal{L}^{\mathrm{OU}} u_{n_l} - \langle \nabla V, \nabla u_{n_l}\rangle\big)(X_s)\big)\big|\, ds \\ & \quad \leq e^t \mathbb{E}_{P_z} \int_0^\infty e^{-s} \big|\mathcal{L}(u - u_{n_l})(X_s)\big|\, ds \to 0 \qquad \text{as } l \to \infty, \end{aligned}$$

and also that by Itô's isometry and by (ii) above

$$\begin{aligned} & \mathbb{E}_{P_z} \bigg| \int_0^t \big\langle \nabla u(X_s) - \nabla u_{n_l}(X_s), dW_s\big\rangle \bigg|^2 \\ & \quad \leq \mathbb{E}_{P_z} \int_0^t \big|\nabla u(X_s) - \nabla u_{n_l}(X_s)\big|^2\, ds \\ & \quad \leq e^{\lambda t} \int_0^\infty e^{-\lambda s} \mathbb{E}_{P_z}\big(\big|\nabla u(X_s) - \nabla u_{n_l}(X_s)\big|^2\big)\, ds \\ & \quad = e^{\lambda t} R_\lambda\big(|\nabla u - \nabla u_{n_l}|^2\big)(z) \to 0 \qquad \text{as } l \to \infty. \end{aligned}$$

Hence, on $\bigcup_{k=1}^\infty \{\tau_{H\setminus F_k^{\lambda,f}} > t\}$ we can pass to the limit in (3.20) to get (3.19). $\square$

REMARK 3.15. By the same standard procedure already mentioned at the end of the proof of Proposition 2.10, we can find $S_V^{\lambda,f}$ such that $H \setminus S_V^{\lambda,f}$ is $\mathcal{E}_\nu$-exceptional and Theorem 2.6, Proposition 2.10, Theorem 2.12 hold with $S_V^{\lambda,f}$ replacing S_V and for all $z \in S_V^{\lambda,f}$, (i)–(iii) in Corollary 3.14 hold and (3.19) holds P_z-a.s.

3.3. *Maximal regularity estimates.* Let us first consider again the solution u of the scalar equation (3.1). The following result is the main technical ingredient of this paper, on the Kolmogorov equation; see [13], Proposition 4.2.

LEMMA 3.16. *We have that $u \in W^{2,2}(H,\nu)$ and there is a constant $C > 0$ such that, for all $\lambda \geq 1$,*

$$\int_H \big|Du(x)\big|^2 \nu(dx) \leq \frac{C}{\lambda} \int_H \big|f(x)\big|^2 \nu(dx), \tag{3.21}$$

$$\int_H \|D^2u(x)\|_{HS}^2 \nu(dx) \le C \int_H |f(x)|^2 \nu(dx). \tag{3.22}$$

We then apply this result componentwise to equation (1.4).

THEOREM 3.17. *Let $U(x) = \sum_{i=1}^{\infty} u^i(x)e_i$ be the solution of equation* (1.4) *with $F(x) = \sum_{i=1}^{\infty} f^i(x)e_i$, namely $u = u^i$ satisfies equation* (3.6) *with $f = f^i$, for every $i \in \mathbb{N}$. Then*

$$\int_H \sum_{i=1}^{\infty} (\lambda_i |Du^i(x)|^2 + \|D^2u^i(x)\|_{HS}^2)\nu(dx) \le C \int_H (|F(x)|^2 + |B(x)|^2)\nu(dx).$$

PROOF. We apply the lemma and get

$$\begin{aligned}\int_H |Du^i(x)|^2 \nu(dx) &\le \frac{C}{\lambda + \lambda_i} \int_H (|f^i(x)|^2 + |\langle B(x), e_i\rangle|^2)\nu(dx) \\ &\le \frac{C}{\lambda_i} \int_H (|f^i(x)|^2 + |\langle B(x), e_i\rangle|^2)\nu(dx),\end{aligned}$$

$$\int_H \|D^2u^i(x)\|_{HS}^2 \nu(dx) \le C \int_H (|f^i(x)|^2 + |\langle B(x), e_i\rangle|^2)\nu(dx).$$

Therefore,

$$\begin{aligned}&\int_H \sum_{i=1}^{\infty} (\lambda_i |Du^i(x)|^2 + \|D^2u^i(x)\|_{HS}^2)\nu(dx) \\ &\qquad \le 2C \int_H (|F(x)|^2 + |B(x)|^2)\nu(dx) < \infty.\end{aligned}$$

The proof is complete. □

REMARK 3.18. Consider the situation of Lemma 3.16 and let $(u_l)_{l\in\mathbb{N}}$ be the sequence $(u_{n_l})_{l\in\mathbb{N}}$ from Corollary 3.14. Then it follows by Proposition 3.10 and Corollary 3.14 that as $n \to \infty$

$$f_n := (\lambda - \mathcal{L})u_n + \langle B, Du_n\rangle \to f \qquad \text{in } L^2(H, \nu).$$

Hence, by (3.22),

$$\lim_{n\to\infty} \int_H \|D^2(u - u_n)\|_{HS}^2 \, d\nu = 0.$$

This will be crucially used to justify the application of mean value theorem in the proof of Lemma 5.2 below.

4. New formulation of the SDE. In this section, we fix U, u^i as in Theorem 3.17 with $f^i := \langle B, e_i\rangle$ and $F := B$. Let $\lambda \geq 4\pi\|B\|_\infty^2$ so large that $c(\lambda) \leq \frac{1}{2}\|B\|_\infty^{-1}$ where $c(\lambda)$ is as in Lemma 3.9. Again, we write x^i for $\langle x, e_i\rangle$, $u^i(x)$ for $\langle U(x), e_i\rangle$, and so on. Below we shall apply Corollary 3.14 with f replaced by B^i and u^i replacing u, for $i \in \mathbb{N}$.

REMARK 4.1. As the corresponding sets of allowed starting points $S_V^{B^i,\lambda}$, $i \in \mathbb{N}$, are concerned, as in Remark 3.15, by a standard diagonal procedure we can find $S_V \subset \bigcap_{i\in\mathbb{N}} S_V^{B^i,\lambda}$ such that $H \setminus S_V$ is $\mathcal{E}_\nu$-exceptional and Theorem 2.6, Proposition 2.10, Theorem 2.12 hold with this (smaller) S_V and for all $z \in S_V$ (i)–(iii) in Corollary 3.14 hold and (3.19) holds P_z-a.s.

Below we fix this set $S_V (\subset H)$.

LEMMA 4.2. *Let $z \in S_V$ and set*

$$\varphi(x) = x + U(x), \qquad x \in H,$$

namely $\varphi^i(x) = x^i + u^i(x)$ and let X be a solution of the SDE (1.1). *Then for each $i \in \mathbb{N}$*

$$\begin{aligned} d\varphi^i(X_t) = {}& \big(-\lambda_i X_t^i - D_i V(X_t)\big)\,dt + (\lambda + \lambda_i)u^i(X_t)\,dt \\ & + \big\langle Du^i(X_t), dW_t\big\rangle + dW_t^i. \end{aligned} \tag{4.1}$$

PROOF. Fix $i \in \mathbb{N}$. Let us first prove the following.

Claim: We have P_z-a.e.

$$\begin{aligned} u^i(X_t) = {}& u^i(z) + \int_0^t \big(\mathcal{L}^{\mathrm{OU}} u^i(X_s) - \big\langle \nabla V(X_s) - B(X_s), Du(X_s)\big\rangle\big)\,ds \\ & + \int_0^t \big\langle Du^i(X_s), dW_s, \big\rangle\,ds, \qquad t \geq 0. \end{aligned} \tag{4.2}$$

Indeed, considering the set Ω_0 of all $\omega \in \Omega$ such that (4.2) holds, we have to prove that $P(\Omega_0) = 1$. But by Girsanov's theorem this is equivalent to (3.19) with u^i replacing u. Hence, the claim is proved.

As a consequence, we obtain that

$$\begin{aligned} du^i(X_t) &= \mathcal{L}u^i(X_t)\,dt + B(X_t)\,dt + \big\langle Du^i(X_t), dW_t\big\rangle \\ &= -B^i(X_t)\,dt + (\lambda + \lambda_i)u^i(X_t)\,dt + \big\langle Du^i(X_t), dW_t\big\rangle \end{aligned}$$

and thus

$$\begin{aligned} dX_t^i = {}& \big(-\lambda_i X_t^i - D_i V(X_t)\big)\,dt - du^i(X_t) \\ & + (\lambda + \lambda_i)u^i(X_t)\,dt + \big\langle Du^i(X_t), dW_t\big\rangle + dW_t^i. \end{aligned}$$

Then

$$d\varphi^i(X_t) = \big(-\lambda_i X_t^i - D_i V(X_t)\big)\,dt + (\lambda + \lambda_i) u^i(X_t)\,dt$$
$$+ \langle Du^i(X_t), dW_t\rangle + dW_t^i. \qquad \square$$

In vector form, we could write (4.1) as

$$dX_t = \big(AX_t - \nabla V(X_t)\big)\,dt - dU(X_t) + (\lambda - A)U(X_t)\,dt + DU(X_t)\,dW_t + dW_t$$

and

$$d\varphi(X_t) = \big(AX_t - \nabla V(X_t)\big)\,dt + (\lambda - A)U(X_t)\,dt + DU(X_t)\,dW_t + dW_t.$$

5. Proof of Theorem 1.5. Consider the situation described at the beginning of Section 4 with S_V being the set of all allowed starting points from Remark 4.1. In particular, by our choice of λ we have

$$\sup_{x\in H} \|\nabla U(x)\|_{\mathcal{L}(H)} \le \tfrac{1}{2}.$$

LEMMA 5.1. *For every $x, y \in H$, we have*

$$\tfrac{1}{2}|x-y| \le \big|\varphi(x) - \varphi(y)\big| \le \tfrac{3}{2}|x-y|.$$

In particular, φ is injective and its inverse is Lipschitz continuous.

PROOF. One has

$$|x-y| \le \big|x + U(x) - y - U(y)\big| + \big|U(x) - U(y)\big|$$
$$\le \big|\varphi(x) - \varphi(y)\big| + \tfrac{1}{2}|x-y|,$$

where we have used

$$\big|U(x) - U(y)\big| \le \sup_{x\in H} \|DU(x)\|\,|x-y| \le \tfrac{1}{2}|x-y|.$$

The claim follows. $\square$

Let X and Y be two solutions with initial condition x, defined on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ and w.r.t. the same cylindrical $(\mathcal{F}_t)$-Brownian motion W.

LEMMA 5.2. *There is a Borel set $\Xi \subset S_V$ with $\gamma(\Xi) = 1$ having the following property: If $z \in \Xi$ and X, Y are two solutions with initial condition z (in the sense of Definition 1.4), defined on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ and w.r.t. the same $(\mathcal{F}_t)$-cylindrical Brownian motion W, then*

$$A_{t,z} < \infty$$

with probability one, *for every* $t \geq 0$, *where the process* $A_{t,z}$ *is defined as*

$$
\begin{aligned}
A_{t,z} = {} & 2 \int_0^t \frac{|\nabla V(X_s) - \nabla V(Y_s)|}{|\varphi(X_s) - \varphi(Y_s)|} 1_{\varphi(X_s) \neq \varphi(Y_s)}\, ds \\
& + 2 \sum_{i=1}^{\infty} \int_0^t \frac{(u^i(X_s) - u^i(Y_s))^2}{|\varphi(X_s) - \varphi(Y_s)|^2} 1_{\varphi(X_s) \neq \varphi(Y_s)}\, ds \\
& + \sum_{i=1}^{\infty} \int_0^t \frac{|Du^i(X_s) - Du^i(Y_s)|^2}{|\varphi(X_s) - \varphi(Y_s)|^2} 1_{\varphi(X_s) \neq \varphi(Y_s)}\, ds.
\end{aligned} \tag{5.1}
$$

PROOF. Let us first treat the case when (H3) holds. By the mean value theorem and Lemma 5.1, we have for ν-a.e. $z \in S_V$

$$A_t \leq 4 N_{t,z},$$

where

$$
\begin{aligned}
N_{t,z} := {} & 2 \int_0^1 \int_0^t \| D^2 V(Z_s^\alpha) \|_{\mathcal{L}(H)}\, d\alpha\, ds \\
& + \sum_{i=1}^{\infty} \int_0^1 \int_0^t \bigl(2\lambda_i |Du^i(Z_s^\alpha)|^2 + \|D^2 u^i(Z_s^\alpha)\|_{HS}^2\bigr)\, d\alpha\, ds,
\end{aligned}
$$

where

$$Z_t^\alpha = \alpha X_t + (1-\alpha) Y_t.$$

Let us briefly show why we can indeed use the mean value theorem here. We do it separately for all three differences under the integrals in (5.1). However, we only explain it for the last difference. The other two can be treated analogously. So, fix $i \in \mathbb{N}$. We want to prove that for γ-a.e. starting point $z \in H$ we have $P \otimes dt$-a.e.

$$Du^i(X_s) - Du^i(Y_s) = \int_0^1 D^2 u^i\bigl(\alpha X_s + (1-\alpha) Y_s\bigr)(X_s - Y_s)\, d\alpha. \tag{5.2}$$

We know by Corollary 3.14 and Remark 3.18 that there exists $u_n \in \mathcal{F}C_b^2$, $n \in \mathbb{N}$, such that for $\lambda \geq 4\pi \|B\|_\infty^2$ and all $z \in S_V$

$$\lim_{n\to\infty} \mathbb{E}_{P_z}\biggl[\int_0^\infty e^{-\lambda s} |Du^i - Du_n|^2 (X_s^V)\, ds\biggr] = 0 \tag{5.3}$$

and

$$\lim_{n\to\infty} \int_H \|D^2 u^i - D^2 u_n\|_{HS}^2\, d\nu = 0. \tag{5.4}$$

Here, P_z is from the Markov process

$$\mathbf{M} := \bigl(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t^V)_{t\geq 0}, (P_z)_{z \in S_V}\bigr)$$

in Corollary 3.14 [and we changed notation and used $(X_t^V)_{t\geq 0}$ instead of $(X_t)_{t\geq 0}$ in Corollary 3.14 to avoid confusion with our fixed solution $(X_t)_{t\in[0,T]}$ above].

Recalling that by Girsanov's theorem both X and Y have laws which are equivalent to the law of $X^V := X_t^V$, $t \in [0,T]$, it follows by (5.3) that as $n \to \infty$

$$\int_0^T |Du^i(X_s) - Du_n(X_s)|^2 \, ds \to 0, \qquad \int_0^T |Du^i(Y_s) - Du_n(Y_s)|^2 \, ds \to 0,$$

in probability. If we can show that also

$$\int_0^T \int_0^1 \|D^2 u^i - D^2 u_n\|_{HS}\big(\alpha X_s + (1-\alpha)Y_s\big)|X_s - Y_s| \, d\alpha \, ds \to 0 \tag{5.5}$$

in probability as $n \to \infty$, (5.2) follows, since it trivially holds for u_n replacing u^i.

But the expression in (5.5) is bounded by

$$\sup_{s\in[0,T]} |X_s - Y_s| \int_0^T \int_0^1 \|D^2 u^i - D^2 u_n\|_{HS}\big(\alpha X_s + (1-\alpha)Y_s\big) \, d\alpha \, ds$$

and by the continuity of sample paths

$$\sup_{s\in[0,T]} |X_s - Y_s| < \infty, \qquad P\text{-a.s.}$$

Furthermore, it follows from (5.4) and the proof of Lemma 6.1 and Corollary 6.2 below that for ν-a.e. $z \in S_V$

$$\int_0^T \int_0^1 \|D^2 u^i - D^2 u_n\|_{HS}\big(\alpha X_s + (1-\alpha)Y_s\big) \, d\alpha \, ds \to 0$$

as $n \to \infty$ P-a.s. Hence, (5.5) follows.

By assumption (1.2) in (H3) we know that

$$\int_H \|D^2 V(x)\|_{\mathcal{L}(H)} \nu(dx) < \infty$$

and by Theorem 3.17 we know that

$$\int_H \sum_{i=1}^\infty \big(\lambda_i |Du^i(x)|^2 + \|D^2 u^i(x)\|_{HS}^2\big) \nu(dx) < \infty.$$

Thus, we may apply Corollary 6.2 below with

$$f(x) = \|D^2 V(x)\|_{\mathcal{L}(H)} + \sum_{i=1}^\infty \big(2\lambda_i |Du^i(x)|^2 + \|D^2 u^i(x)\|_{HS}^2\big)$$

and get that $\int_0^1 \int_0^t f(Z_s^\alpha) \, d\alpha \, ds < \infty$ with probability one, for every $t \geq 0$ and ν-a.e. $z \in S^V$, that is,

$$N_{t,z} < \infty$$

with probability one, for every $t \geq 0$, which completes the proof since $A_t \leq 4N_{t,z}$.

Now let us consider the case when (H3)$'$ holds. Clearly, we then handle the second and the third term in the right-hand side of (4.2) as above. For the first term, the treatment is different, but simpler. Indeed, we have by (H3)$'$, Lemma 4.2 and by the mean value theorem that

$$\begin{aligned}\int_0^T &\frac{|\nabla V(X_s) - \nabla V(Y_s)|}{|\varphi(X_s) - \varphi(Y_s)|} 1_{\varphi(X_s)\neq\varphi(Y_s)}\, ds \\ &\leq 2\int_0^T \int_0^1 \|V_E''(Z_s^\alpha)\|_{L(H,E')}\, d\alpha\, ds \\ &\leq 2\int_0^T \big[\Psi(|X_s|_E) + \Psi(|Y_s|_E)\big]\, ds.\end{aligned}$$

But again using Girsanov's theorem we know that the laws of X and Y are equivalent to that of X^V, hence the last expression is finite P-a.e. $\square$

We may now prove Theorem 1.5. Let $z \in \Xi$. By Lemma 4.2,

$$\begin{aligned}&d\big(\varphi^i(X_t) - \varphi^i(Y_t)\big) \\ &\quad = -\big(\lambda_i\big(X_t^i - Y_t^i\big) + D_i V(X_t) - D_i V(Y_t)\big)\, dt \\ &\qquad + (\lambda + \lambda_i)\big(u^i(X_t) - u^i(Y_t)\big)\, dt + \big\langle Du^i(X_t) - Du^i(Y_t), dW_t\big\rangle.\end{aligned}$$

Hence, by Itô's formula, we get

$$\begin{aligned}&d\big(\varphi^i(X_t) - \varphi^i(Y_t)\big)^2 \\ &\quad = -2\big(\varphi^i(X_t) - \varphi^i(Y_t)\big)\big(\lambda_i\big(X_t^i - Y_t^i\big) + D_i V(X_t) - D_i V(Y_t)\big)\, dt \\ &\qquad + 2\big(\varphi^i(X_t) - \varphi^i(Y_t)\big)(\lambda + \lambda_i)\big(u^i(X_t) - u^i(Y_t)\big)\, dt \\ &\qquad + 2\big(\varphi^i(X_t) - \varphi^i(Y_t)\big)\big\langle Du^i(X_t) - Du^i(Y_t), dW_t\big\rangle \\ &\qquad + \big|Du^i(X_t) - Du^i(Y_t)\big|^2\, dt.\end{aligned}$$

By definition of φ in Lemma 4.2, in the lines above there are the terms $-2(u^i(X_t) - u^i(Y_t))\lambda_i(X_t^i - Y_t^i)$ and $2(X_t^i - Y_t^i)\lambda_i(u^i(X_t) - u^i(Y_t))$ which cancel each other. Moreover, the term $-2(X_t^i - Y_t^i)\lambda_i(X_t^i - Y_t^i)$ is negative. Thus, we deduce

$$\begin{aligned}d\big(\varphi^i(X_t) - \varphi^i(Y_t)\big)^2 &\leq -2\big(\varphi^i(X_t) - \varphi^i(Y_t)\big)\big(D_i V(X_t) - D_i V(Y_t)\big)\, dt \\ &\quad + 2\lambda\big(\varphi^i(X_t) - \varphi^i(Y_t)\big)\big(u^i(X_t) - u^i(Y_t)\big)\, dt \\ &\quad + 2\lambda_i\big(u^i(X_t) - u^i(Y_t)\big)^2\, dt \\ &\quad + 2\big(\varphi^i(X_t) - \varphi^i(Y_t)\big)\big\langle Du^i(X_t) - Du^i(Y_t), dW_t\big\rangle \\ &\quad + \big|Du^i(X_t) - Du^i(Y_t)\big|^2\, dt.\end{aligned}$$

Let $A_t = A_{t,z}$ be the process introduced in Lemma 5.2. We have

$$\begin{aligned}
&d\big(e^{-A_t}\big(\varphi^i(X_t)-\varphi^i(Y_t)\big)^2\big)\\
&\quad\le -2e^{-A_t}\big(\varphi^i(X_t)-\varphi^i(Y_t)\big)\big(D_iV(X_t)-D_iV(Y_t)\big)\,dt\\
&\qquad+2\lambda e^{-A_t}\big(\varphi^i(X_t)-\varphi^i(Y_t)\big)\big(u^i(X_t)-u^i(Y_t)\big)\,dt\\
&\qquad+2e^{-A_t}\big(\varphi^i(X_t)-\varphi^i(Y_t)\big)\langle Du^i(X_t)-Du^i(Y_t),dW_t\rangle\\
&\qquad+2\lambda_i e^{-A_t}\big(u^i(X_t)-u^i(Y_t)\big)^2\,dt\\
&\qquad+e^{-A_t}\big|Du^i(X_t)-Du^i(Y_t)\big|^2\,dt-e^{-A_t}\big(\varphi^i(X_t)-\varphi^i(Y_t)\big)^2\,dA_t
\end{aligned}$$

and thus, for every $N>0$, summing the previous inequality for $i=1,\dots,N$, we get

$$\begin{aligned}
&d\big(e^{-A_t}\big|P_N\big(\varphi(X_t)-\varphi(Y_t)\big)\big|^2\big)\\
&\quad\le -2e^{-A_t}\langle P_N\big(\varphi(X_t)-\varphi(Y_t)\big),P_N\big(\nabla V(X_t)-\nabla V(Y_t)\big)\rangle\,dt\\
&\qquad+2\lambda e^{-A_t}\langle P_N\big(\varphi(X_t)-\varphi(Y_t)\big),U(X_t)-U(Y_t)\rangle\,dt\\
&\qquad+2e^{-A_t}\sum_{i=1}^N\big(\varphi^i(X_t)-\varphi^i(Y_t)\big)\langle Du^i(X_t)-Du^i(Y_t),dW_t\rangle\\
&\qquad+2e^{-A_t}\sum_{i=1}^N\lambda_i\big(u^i(X_t)-u^i(Y_t)\big)^2\,dt\\
&\qquad+e^{-A_t}\sum_{i=1}^N\big|Du^i(X_t)-Du^i(Y_t)\big|^2\,dt\\
&\qquad-e^{-A_t}\big|P_N\big(\varphi(X_t)-\varphi(Y_t)\big)\big|^2\,dA_t.
\end{aligned}$$

Substituting dA_t, taking expectation and using simple inequalities we get

$$\begin{aligned}
&\mathbb{E}\big[e^{-A_t}\big|P_N\big(\varphi(X_t)-\varphi(Y_t)\big)\big|^2\big]\\
&\quad\le 2\lambda\int_0^t\mathbb{E}\big[e^{-A_s}|\varphi(X_s)-\varphi(Y_s)||U(X_s)-U(Y_s)|\big]\,ds\\
&\qquad+2\int_0^t\mathbb{E}\big[e^{-A_s}|\varphi(X_s)-\varphi(Y_s)|\big|P_N\big(\nabla V(X_s)-\nabla V(Y_s)\big)\big|\big]\,ds\\
&\qquad-2\int_0^t\mathbb{E}\bigg[e^{-A_s}\big|P_N\big(\varphi(X_s)-\varphi(Y_s)\big)\big|^2\\
&\qquad\qquad\times\frac{2|\nabla V(X_s)-\nabla V(Y_s)|}{|\varphi(X_s)-\varphi(Y_s)|}1_{\varphi(X_s)\ne\varphi(Y_s)}\bigg]\,ds
\end{aligned}$$

$$+\int_0^t \mathbb{E}[e^{-A_s} g_s]\,ds$$
$$-\int_0^t E\left[e^{-A_s} g_s \frac{|P_N(\varphi(X_s)-\varphi(Y_s))|^2}{|\varphi(X_s)-\varphi(Y_s)|^2} 1_{\varphi(X_s)\neq\varphi(Y_s)}\right] ds,$$

where for shortness of notation we have written

$$g_s := 2\sum_{i=1}^{\infty} \lambda_i \big(u^i(X_s) - u^i(Y_s)\big)^2 + \sum_{i=1}^{\infty} \big|Du^i(X_s) - Du^i(Y_s)\big|^2.$$

By monotone convergence, we may take the limit as $N \to \infty$ and deduce

$$\begin{aligned}
&\mathbb{E}\big[e^{-A_t}\big|\varphi(X_t)-\varphi(Y_t)\big|^2\big] \\
&\quad\leq 2\lambda \int_0^t \mathbb{E}\big[e^{-A_s}\big|\varphi(X_s)-\varphi(Y_s)\big|\big|U(X_s)-U(Y_s)\big|\big]\,ds \\
&\qquad + 2\int_0^t \mathbb{E}\big[e^{-A_s}\big|\varphi(X_s)-\varphi(Y_s)\big|\big|\nabla V(X_s)-\nabla V(Y_s)\big|\big]\,ds \\
&\qquad - 2\int_0^t \mathbb{E}\left[e^{-A_s}\big|\varphi(X_s)-\varphi(Y_s)\big|^2 \frac{2|\nabla V(X_s)-\nabla V(Y_s)|}{|\varphi(X_s)-\varphi(Y_s)|} 1_{\varphi(X_s)\neq\varphi(Y_s)}\right] ds \\
&\qquad + \int_0^t \mathbb{E}[e^{-A_s} g_s]\,ds - \int_0^t \mathbb{E}\left[e^{-A_s} g_s \frac{|\varphi(X_s)-\varphi(Y_s)|^2}{|\varphi(X_s)-\varphi(Y_s)|^2} 1_{\varphi(X_s)\neq\varphi(Y_s)}\right] ds.
\end{aligned}$$

Notice that by Lemma 5.1, $X_s = Y_s$ if and only if $\varphi(X_s) = \varphi(Y_s)$. Hence, we may drop the indicator function $1_{\varphi(X_s)\neq\varphi(Y_s)}$ in all integrals in the above inequality.

Therefore, certain terms cancel in the previous inequality and we get

$$\mathbb{E}\big[e^{-A_t}\big|\varphi(X_t)-\varphi(Y_t)\big|^2\big] \leq 2\lambda \int_0^t \mathbb{E}\big[e^{-A_s}\big|\varphi(X_s)-\varphi(Y_s)\big|\big|U(X_s)-U(Y_s)\big|\big]\,ds.$$

Using Lemmas 3.9 and 5.1, we get

$$\mathbb{E}\big[e^{-A_t}\big|\varphi(X_t)-\varphi(Y_t)\big|^2\big] \leq 2\lambda C \int_0^t \mathbb{E}\big[e^{-A_s}\big|\varphi(X_s)-\varphi(Y_s)\big|^2\big]\,ds,$$

whence $\mathbb{E}[e^{-A_t}|\varphi(X_t)-\varphi(Y_t)|^2] = 0$ by Gronwall's lemma, and thus $\varphi(X_t) = \varphi(Y_t)$ with probability one (since $A_t < \infty$ a.s.), for all $t \geq 0$; the same is true for the identity $X_t = Y_t$ since φ is invertible and finally X and Y are also indistinguishable since they are continuous processes.

To complete the proof, we have to prove Corollary 6.2 below, which was used in the proof of Lemma 5.2.

6. Main lemmata.

Let S_V as in Remark 4.1 and H_V as in (2.9) and set

$$\Xi_V := S_V \cap H_V. \tag{6.1}$$

LEMMA 6.1. *Let $f : H \to [0, \infty)$ be a Borel measurable function such that*

$$\int_H f(x)\gamma(dx) < \infty. \tag{6.2}$$

Then there is a Borel set $\Xi \subset S_V \cap H_V$ with $\gamma(\Xi) = 1$ having the following property. Given any $z \in \Xi$ and any two solutions X, Y with initial condition z (as in the statement of Theorem 1.5) for all $T > 0$ we have

$$P\left(\int_0^1 \int_0^T f(Z_s^\alpha)\, ds\, d\alpha < \infty\right) = 1,$$

where $Z_t^\alpha = \alpha X_t + (1-\alpha)Y_t$.

PROOF. *Step* 1 (Estimates on OU process). A number $T > 0$ is fixed throughout the proof. From the assumption on f, it follows that there is a Borel set $\Xi_f \subset H$, with Ξ_f^c of γ-measure zero, such that

$$\mathbb{E}\left[\int_0^T f(Z_s^{\mathrm{OU},z})\, ds\right] = \int_0^T \left(\int_H f(x)p_{s,z}(dx)\right) ds < \infty$$

for all $z \in \Xi_f$, where $p_{s,z}(dx)$ is the law at time s of the Ornstein–Uhlenbeck process $Z_s^{\mathrm{OU},z}$, that is, the solution of the equation

$$dZ_t = AZ_t\, dt + dW_t, \qquad Z_0 = z. \tag{6.3}$$

Indeed, we have

$$\begin{aligned}
&\int_H \left(\int_0^T \left(\int_H f(x)p_{s,z}(dx)\right) ds\right)\gamma(dz) \\
&\quad = \int_0^T \left(\int_H \int_H f(x)p_{s,z}(dx)\gamma(dz)\right) ds \\
&\quad = \int_0^T \left(\int_H f(z)\gamma(dz)\right) ds = T\int_H f(z)\gamma(dz).
\end{aligned}$$

This implies $\int_0^T (\int_H f(x)p_{s,z}(dx))\, ds < \infty$ for γ-a.e. z.

Step 2 (Girsanov transform). Let Ξ_V as in (6.1) and Ξ_f be given as in step 1. Let $\Xi = \Xi_V \cap \Xi_f$, of full γ-measure. In the sequel, $z \in \Xi$ will be given, thus we avoid to index all quantities by z.

From Theorem 2.12, we have

$$\int_0^T |\nabla V(X_s)|^2\, ds + \int_0^T |\nabla V(Y_s)|^2\, ds < \infty$$

for all $T > 0$, with probability one.

Let us introduce the sequence $\{\tau^n\}$ of stopping times defined as

$$\begin{aligned}
\tau^n &= \tau_B^n \wedge \tau_{V,1}^n \wedge \tau_{V,2}^n, \\
\tau_B^n &:= \inf\left\{t \geq 0 : \left|\int_0^t B(X_s)\, dW_s\right| + \left|\int_0^t B(Y_s)\, dW_s\right| \geq n\right\} \wedge T, \\
\tau_{V,1}^n &:= \inf\left\{t \geq 0 : \left|\int_0^t \langle \nabla V(X_s), dW_s\rangle\right| + \left|\int_0^t \langle \nabla V(Y_s), dW_s\rangle\right| \geq n\right\} \wedge T, \\
\tau_{V,2}^n &:= \inf\left\{t \geq 0 : \int_0^t |\nabla V(X_s)|^2\, ds + \int_0^t |\nabla V(Y_s)|^2\, ds \geq n\right\} \wedge T
\end{aligned}$$

for $n \geq 1$ (an infimum is equal to $+\infty$ if the corresponding set is empty). All stochastic and Lebesgue integrals are well defined and continuous in t, hence we have $\tau^n = T$ eventually, with probability one. In order to prove the lemma, it is sufficient to prove that $E[\int_0^1 \int_0^{T\wedge\tau^n} f(Z_s^\alpha)\, ds\, d\alpha] < \infty$ for each n.

Let us also introduce the stochastic processes

$$\begin{aligned}
b_s^\alpha &:= \alpha B(X_s) + (1-\alpha) B(Y_s), \\
v_s^\alpha &:= \alpha \nabla V(X_s) + (1-\alpha) \nabla V(Y_s)
\end{aligned}$$

and the stochastic exponentials

$$\rho_t^\alpha := \exp\left(-\int_0^t \langle b_s^\alpha - v_s^\alpha, dW_s\rangle - \frac{1}{2}\int_0^t |b_s^\alpha - v_s^\alpha|^2\, ds\right).$$

Denote

$$\rho_t^{\alpha,n} := \rho_{t\wedge\tau^n}^\alpha = \exp\left(-\int_0^t \langle 1_{s\leq\tau^n}(b_s^\alpha - v_s^\alpha), dW_s\rangle - \frac{1}{2}\int_0^t 1_{s\leq\tau^n}|b_s^\alpha - v_s^\alpha|^2\, ds\right).$$

By Novikov's criterium, this is a martingale (indeed $\int_0^T 1_{s\leq\tau^n}|b_s^\alpha - v_s^\alpha|^2\, ds$ is a bounded r.v. We may thus introduce the following new measures (and the corresponding expectations)

$$Q^{\alpha,n}(A) := \mathbb{E}[\rho_T^{\alpha,n} 1_A].$$

Girsanov's theorem implies that

$$\begin{aligned}
\widetilde{W}_t^{n,\alpha} &:= W_t + \int_0^t 1_{s\leq\tau^n}(b_s^\alpha - v_s^\alpha)\, ds \\
&= W_t + \int_0^{t\wedge\tau^n} (b_s^\alpha - v_s^\alpha)\, ds
\end{aligned}$$

is a new cylindrical Brownian motion.

Step 3 (Auxiliary process and conclusion). Recall also that Z_t^α (with the new notation) satisfies

$$dZ_t^\alpha = AZ_t^\alpha\, dt + (b_s^\alpha - v_s^\alpha)\, dt + dW_t.$$

Let us introduce the auxiliary process $Z_t^{\alpha,n}$ which solves, in the sense of Definition 1.4, the equation

$$Z_t^{\alpha,n} = z + \int_0^t AZ_s^{\alpha,n}\,ds + \int_0^t 1_{s\le\tau^n}\big(b_s^\alpha - v_s^\alpha\big)\,ds + W_t.$$

It exists, by the explicit formula

$$Z_t^{\alpha,n} = e^{tA}z + \int_0^t e^{(t-s)A}1_{s\le\tau^n}\big(b_s^\alpha - v_s^\alpha\big)\,ds + \int_0^t e^{(t-s)A}\,dW_s,$$

where e^{tA} is the analytic semigroup in H generated by A (taking inner product with the elements e_k of the basis, it is not difficult to check that this mild formula gives a solution in the weak sense of Definition 1.4). This process satisfies also

$$Z_t^{\alpha,n} = z + \int_0^t AZ_s^{\alpha,n}\,ds + \widetilde{W}_t^{n,\alpha}$$

by the definition of $\widetilde{W}_t^{n,\alpha}$, hence its law under $Q^{\alpha,n}$ is the same as the Gaussian law of Z_t^{OU} under P. Moreover,

$$Z_t^{\alpha,n} = Z_t^\alpha \qquad \text{for } t\in[0,\tau^n]$$

(indeed, by the weak formulation, the process $Y_t = Z_t^{\alpha,n} - Z_t^\alpha$ verifies, pathwise, on $[0,\tau^n]$, the equation $Y_t' = AY_t$, $Y_0 = 0$, in the weak sense of Definition 1.4 and thus, taking inner product with the elements e_k of the basis, one proves $Y = 0$).

Therefore,

$$\begin{aligned}\mathbb{E}^{Q^{\alpha,n}}\bigg[\int_0^{T\wedge\tau^n} f\big(Z_s^\alpha\big)\,ds\bigg] &= \mathbb{E}^{Q^{\alpha,n}}\bigg[\int_0^{T\wedge\tau^n} f\big(Z_s^{\alpha,n}\big)\,ds\bigg]\\ &\le \mathbb{E}^{Q^{\alpha,n}}\bigg[\int_0^{T} f\big(Z_s^{\alpha,n}\big)\,ds\bigg]\\ &= \mathbb{E}\bigg[\int_0^{T} f\big(Z_s^{\mathrm{OU}}\big)\,ds\bigg] =: C' < \infty.\end{aligned}$$

But

$$\begin{aligned}\mathbb{E}^{Q^{\alpha,n}}\bigg[\int_0^{T\wedge\tau^n} f\big(Z_s^\alpha\big)\,ds\bigg] &= \mathbb{E}\bigg[\rho_T^{\alpha,n}\int_0^{T\wedge\tau^n} f\big(Z_s^\alpha\big)\,ds\bigg]\\ &\ge C_n\mathbb{E}\bigg[\int_0^{T\wedge\tau^n} f\big(Z_s^\alpha\big)\,ds\bigg],\end{aligned}$$

where $C_n > 0$ is a constant such that $\rho_T^{\alpha,n} \ge C_n$: it exists because

$$\big(\rho_T^{\alpha,n}\big)^{-1} := \exp\bigg(\int_0^{T\wedge\tau^n}\langle b_s^\alpha - v_s^\alpha, dW_s\rangle + \frac{1}{2}\int_0^{T\wedge\tau^n}|b_s^\alpha - v_s^\alpha|^2\,ds\bigg)$$

and τ^n includes the stopping of all these integrals. Therefore,

$$\mathbb{E}\bigg[\int_0^{T\wedge\tau^n} f\big(Z_s^\alpha\big)\,ds\bigg] \le \frac{C'}{C_n}$$

and thus also

$$\mathbb{E}\left[\int_0^1 \int_0^{T\wedge\tau^n} f(Z_s^\alpha)\,ds\,d\alpha\right] \le \frac{C'}{C_n}.$$

The proof is complete. □

The next corollary extends the previous result to the case when $\int_H f(x)\nu(dx) < \infty$. Clearly,

$$\int_H f(x)\nu(dx) \le \frac{1}{Z}\int_H f(x)\gamma(dx)$$

but not conversely, without additional assumptions on V. Hence, Corollary 6.2 implies Lemma 6.1, but not conversely, in an obvious way. However, we may easily deduce Corollary 6.2 from Lemma 6.1 by assumptions (H1)–(H3).

COROLLARY 6.2. *Let $f : H \to [0, \infty)$ be a Borel measurable function such that*

$$\int_H f(x)\nu(dx) < \infty. \tag{6.4}$$

Then there is a Borel set $\Xi \subset S_V \cap H_V$ with $\nu(\Xi) = 1$ [equivalently $\gamma(\Xi) = 1$] having the property stated in Lemma 6.1.

PROOF. Since $\int_H f(x)e^{-V(x)}\gamma(dx) < \infty$, we may apply Lemma 6.1 to the function $f(x)e^{-V(x)}$ instead of $f(x)$ and get, as a result, that there is a Borel set $\Xi \subset S_V \cap H_V$ with $\gamma(\Xi) = 1$ having the following property: given any $z \in \Xi$ and any two solutions X, Y as in the statement of Theorem 1.5, for all $T > 0$ we have

$$P\left(\int_0^1 \int_0^T f(Z_s^\alpha)e^{-V(Z_s^\alpha)}\,ds\,d\alpha < \infty\right) = 1, \tag{6.5}$$

where $Z_t^\alpha = \alpha X_t + (1-\alpha)Y_t$. Take $z \in \Xi$. Since $V(Z_s^\alpha) \le V(X_t) + V(Y_t)$ (recall that $V \ge 0$ by Remark 1.1) and by Theorem 2.12,

$$P\left(\bigcup_{n=1}^\infty \{\sigma_{H\setminus K_n^V}^{X,Y} > T\}\right) = 1,$$

where

$$\sigma_{H\setminus K_n^V}^{X,Y} := \min(\sigma_{H\setminus K_n^V}^{X}, \sigma_{H\setminus K_n^V}^{Y})$$

and $\sigma_{H\setminus K_n^V}^{X}, \sigma_{H\setminus K_n^V}^{Y}$ are the first hitting times of $H \setminus K_n^V$ of X, Y, respectively, we have by (6.5)

$$\int_0^1 \int_0^T f(Z_s^\alpha)e^{-V(X_s)}e^{-V(Y_s)}\,ds\,d\alpha < \infty \qquad \text{on } \bigcup_{n=1}^\infty \{\sigma_{H\setminus K_n^V}^{X,Y} > T\},\ P\text{-a.s.}$$

But for $\omega \in \{\sigma^{X,Y}_{H\setminus K_n^V} > T\}$ and $M_n := \sup\{V(z) : z \in K_n^V\}$

$$\int_0^1 \int_0^T f\big(Z_s^\alpha(\omega)\big)\,ds\,d\alpha \le e^{2M_n} \int_0^1 \int_0^T f\big(Z_s^\alpha(\omega)\big)e^{-V(X_s)}e^{-V(Y_s)}\,ds\,d\alpha < \infty.$$

Hence,

$$P\left(\int_0^1 \int_0^T f\big(Z_s^\alpha(\omega)\big)\,ds\,d\alpha < \infty\right) = 1. \qquad \square$$

7. Applications.

7.1. *Reaction–diffusion equations.* Let $H := L^2((0,1), d\xi)$, with $d\xi =$ Lebesgue measure and $A = -\Delta$ with domain $H^2(0,1) \cap H_0^1(0,1)$, that is, A is the Dirichlet Laplacian on $(0,1)$. Then clearly (H1) holds.

Let $m \in [1, \infty)$ and

$$V(x) := \begin{cases} \displaystyle\int_0^1 \big|x(\xi)\big|^{m+1}\,d\xi, & \text{if } x \in L^{m+1}\big((0,1), d\xi\big), \\ +\infty, & \text{else.} \end{cases} \tag{7.1}$$

V obviously satisfies (H2). Now we are going to verify (H3)$'$ for this convex functional. (Of course, then according to Remark 1.1 we subsequently replace this V by $V + \frac{\omega}{2}|\cdot|_H^2$.)

For the separable Banach space E in (H3)$'$, we take

$$E := L^{2m}\big((0,1), d\xi\big) =: L^{2m}. \tag{7.2}$$

Then by elementary calculations for $x \in E$

$$V_E'(x) = (m+1)|x|^{m-1}x \in H \subset L^{2m/(2m-1)} = E', \tag{7.3}$$

$$V_E''(x)(h_1, h_2) = m(m+1)\int_0^1 \big|x(\xi)\big|^{m-1} h_1(\xi)h_2(\xi)\,d\xi, \tag{7.4}$$

for $h_1, h_2 \in E$. Obviously, the right-hand side of (7.4) is also defined for $h_1, h_2 \in H$ and by Hölder's inequality, continuous in $(h_1, h_2) \in E \times H$ with respect to the product topology. Hence, for all $x \in E$

$$V_E''(x) \subset L\big(H, E'\big)$$

and furthermore (again by Hölder's inequality)

$$\|V_E''(x)\|_{L(H,E')} \le |x|_E^{m-1}.$$

Equation (7.3) implies that $E \subset D_V$. But our Gaussian measure $\gamma = N_{-(1/2)A^{-1}}$ is known to have full mass even on $C([0,1]; \mathbb{R})$ because it is the law of the Brownian Bridge, hence $\gamma(E) = 1$ and so, $\gamma(D_V) = 1$. Furthermore, then obviously by Fernique's theorem the first inequality in (1.2) is satisfied.

It remains to verify (1.3), that is, for γ-a.e. initial condition $z \in H$

$$\mathbb{E}\int_0^T \left|X^V(s)\right|_E^{m-1} ds < \infty, \tag{7.5}$$

where $X^V(t), t \in [0, T]$, solves SDE (1.1) with $B = 0$. But the existence of such a process for γ-a.e. $z \in H$ follows from Theorem 2.5 in Section 2 above. That this process satisfies (7.5) follows from results in [6]. Indeed, it follows by [6], Theorem 3.6 and Proposition 4.1, and Fatou's lemma that even

$$\mathbb{E}\int_0^T \left|X^V(s)\right|_E^{2m} ds < \infty$$

for (γ-a.e.) $z \in E$.

Hence, (H3)$'$ is verified and our main result, Theorem 1.5, applies to this case.

7.2. *Weakly differentiable drifts*. The main motivation to also consider condition (H3), that is, to assume that the (γ-weak) second derivative D^2V of V exists and is in $L^1(H, \gamma; L(H))$, was to make a connection between our results and those in finite dimensions by [9]. As mentioned in the Introduction, our results generalize some of the results of [9] in the special case when $H = \mathbb{R}^d$. In addition, since we work with respect to a Gaussian measure (and not Lebesgue measure on $\mathbb{R}^d$) our integrability conditions are generically weaker than those in [9]. As far as the infinite dimensional case is concerned, one might ask what are examples of such functions V satisfying condition (H3). There are plenty of them and let us briefly describe a whole class of such functions.

Let $\varphi : H \to [0, \infty]$ be convex, lower semicontinuous, $\varphi \in L^{2+\delta}(H, \gamma)$ for some $\delta > 0$, and Gâteaux differentiable, γ-a.e., that is, $\gamma(D_\varphi) = 1$. Define

$$V(x) := R\big(\lambda, \mathcal{L}^{\mathrm{OU}}\big)\varphi(x), \qquad x \in H, \tag{7.6}$$

with $R(\lambda, \mathcal{L}^{\mathrm{OU}})$ defined as in (3.8), that is, it is the resolvent of the Ornstein–Uhlenbeck operator $\mathcal{L}^{\mathrm{OU}}$. Then it is elementary to check from the definition that $V : H \to [0, \infty]$ is also convex and lower semicontinuous.

Furthermore, V is in the $L^2(H, \gamma)$-domain of $\mathcal{L}^{\mathrm{OU}}$. Hence, by the maximal regularity result of [13] (already recalled in Section 3.3 above) applied to the case when $U \equiv 0$, we conclude that $V \in W^{2,2}(H, \gamma)$, in particular we have

$$\int_H \|D^2V\|_{HS}^2 \, d\gamma < \infty,$$

which is stronger than the second part of condition (1.2) in (H3).

Of course, one needs additional, but obviously quite mild bounds on $\nabla\varphi$, to ensure that $\gamma(D_V) = 1$ and $\nabla V \in L^2(H, \gamma)$. But then the class of V defined in (7.6) satisfy (H3). To be concrete in choosing φ above, consider the situation of Section 7.1. Then if we take $\varphi := V$ as defined in (7.1), the new V given by (7.6) satisfy (H3).

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