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SOME UNIFYING CONCEPTS IN MULTIVARIABLE FEEDBACK DESIGN

BY


Department of Control Engineering,
Mappin Street,
Sheffield S1 3JD.

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SOME UNITING CONCEPTS IN MULTIVARIABLE FEEDBACK DESIGN

D. H. Owens
Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield S1 3JD, England

Abstract. Frequency domain techniques for computer-aided design of multivariable feedback systems are now well established in the form of several, apparently distinct, design techniques. It is shown that a unified design structure can be developed based on the theoretical concepts of precompensation, eigenvalue approximation and permissible, constant, input/output transformations.

Keywords. Computer-aided system design, multivariable control systems, control system analysis, feedback, linear systems, multivariable systems.

INTRODUCTION

The use of frequency domain ideas in the analysis and design of output feedback configurations for the control of a linear time-invariant plant described by the m-input transfer function matrix (TFM) G(s) is now well established (Owens, 1978; Rosenbrock, 1974). A number of apparently distinct design techniques have been introduced based on reduction of the multivariable problem to a sequence of essentially independent scalar designs. Particular examples of such techniques are the inverse Nyquist array (INA) (Rosenbrock, 1974, Munro, 1979), the method of dyadic expansion (MDE) (Owens 1975, 1978, 1979) and the use of characteristic loci (CL) and approximately commutative control (ACC) (MacFarlane and Kouvaritakis 1977, Postlethwaite 1979, Kouvaritakis 1979).

Although the techniques have a fundamentally different conceptual basis it is clear that the design operations have an algebraic and numerical similarity. This paper describes a unified structure encompassing the apparently distinct design techniques based on the ideas of precompensation, eigenvalue approximation and the use of a class of permissible input/output transformations. Not only do these concepts provide a theoretical unity, they can enable the construction of general design programmes of great flexibility containing the above named design techniques simply as special cases.

GENERAL DESIGN PHILOSOPHY

For simplicity, consider the unity negative feedback system of Fig. 1(a). The special case of the diagonal plant G(s) = diag\( G_j(s) \), \( 1 \leq j \leq m \) represents \( m \) non-interacting systems. Each of these systems can be separately controlled by the diagonal controller \( K(s) = \text{diag}(k_1(s), k_2(s), \ldots, k_m(s)) \) and

\[
|Y(s)| = \prod_{j=1}^{m} (1+G_j(s)k_j(s))
\]

\[
H_c(s) = \text{diag}\left( \frac{G_j(s)k_j(s)}{1+G_j(s)k_j(s)} \right), 1 \leq j \leq m
\]

(1)

where \( Y(s) \triangleq I + G(s)K(s) \) is the matrix return-difference and \( H_c(s) \) is the closed-loop TFM (CLTFM). In effect the closed-loop system is noninteracting and its stability and transient performance is only 'as good as' the responses of the scalar systems

\[
G_j(s)k_j(s)
\]

\[
H_j(s) = \frac{G_j(s)k_j(s)}{1+G_j(s)k_j(s)}, 1 \leq j \leq m
\]

(2)

Although trivial, this problem does underline the observation that the presence of off-diagonal/interaction terms in the plant are a minor source of design problems. It is therefore assumed that design procedures will, in practice, attempt to reduce or eliminate (in some sense) the need to consider interaction effects in the design process and hence reduce the design to a sequence of independent scalar designs. The theoretical tools developed will, of course, reflect this general design philosophy.

TRANSFORMATION OPERATIONS

The natural approach to the diagonalization of a square matrix is the use of eigenvector methods as in the CL design method. In Fig. 1(a) let the forward path TFM \( Q \triangleq Gk \) have eigenvector matrix \( W(s) \) with inverse \( V(s) \) and corresponding eigenvalues \( \lambda_j(s) \),
In effect eigenvector transformations formally reduce the design problem to the analysis and design of the 'characteristic transfer functions' \( q_j(s) \), \( 1 \leq j \leq m \), (subjected to unity negative feedback) and the eigenvector matrix \( W(s) \).

The pseudo-classical nature of this analysis is deceptive in its simplicity as it provides few explicit design algorithms. It does however establish transformation techniques as an important component in the general design philosophy.

For conceptual and numerical simplicity, the transformations suggested here (Owens 1979) (a) introduce no extra dynamics into the system and (b) have a meaningful physical interpretation. The transformations take the form \( G(s) \rightarrow \hat{G}(s) \) defined by

\[
\hat{G}(s) = P_1^{-1} G(s) P_2
\]

where \( P_1 \) and \( P_2 \) are square, nonsingular matrices.

Such transformations certainly introduce no extra dynamics and the TFH \( \hat{G}(s) \) can be interpreted as the TFH between \( u(s) = \hat{u}(s) \) and \( \hat{y}(s) = \hat{P}_1^{-1} y(s) \).

It is a natural first step to demand that the transformation pair \( (P_1^{-1}, P_2) \) be real.

In such a case the operations are a generalization of many previously defined design operations:

**Constant precompensation:** The map \( G(s) \rightarrow G(s)K \) (where \( K \) is a constant, nonsingular controller) can be regarded as the map \( G \rightarrow GK \) induced by the transformation pair \( (I, K^{-1}) \) and describes such operations as alignment of high frequency eigenvectors in the CL and ACC method and pseudodiagonalization in the direct Nyquist array (Rosenbrock 1974).

**Constant pre- and post-compensation:** Writing equation (6) in the form

\[
\hat{G}(s) = \hat{P}_2 \hat{G}(s) \hat{P}_1
\]

where \( \hat{L}(s) \) denotes the inverse of the TFH \( L(s) \) it is seen that the pair \( (P_1^{-1}, P_2) \) can be given an interpretation in terms of constant post- and pre-compensation operations as used, for example, in the INA.

There are sound theoretical and physical reasons (Owens 1979) for permitting the transformation matrices to have complex elements in a restricted form, just as it is necessary to allow complex conjugate pair eigenvectors in spectral analysis of oscillatory linear systems. This device also allows many basic operations of the MDE (Owens 1979) to be described by transformations of the form described. The class of transformations permitted are as follows:

**Definition (Owens 1979)**

The non nonsingular complex matrices

\[
P_1 = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}, \quad P_2 = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_m \end{bmatrix}
\]

are said to be permissible if the columns (resp. rows) of \( P_1 \) (resp. \( P_2 \)) are real or exist in complex conjugate pairs and if

\[\beta_j = \beta_k(j) \text{ whenever } a_j = a_k(j), 1 \leq j \leq m.\]

(Note: the map \( j \mapsto k(j) \) is a one-to-one map of the integer set \( \{1, 2, 3, \ldots, n\} \) onto itself by the structure and nonsingularity of \( P_1 \) and \( k(1) \)=j, \( 1 \leq j \leq m \). The potential benefit of introducing complex transformations can be illustrated by examples (Owens 1979) and the basic theorems of the MDE.

**Theorem 1 (Owens 1979)**

If \( |G(\omega_1)| \neq 0 \) and \( G(-\omega_1)G^{-1}(\omega_1) \) has a complete set of eigenvectors then there exists a permissible transformation \( (P_1(\omega_1), P_2(\omega_1)) \) such that \( \hat{G}(s) \) is diagonal at the point \( s = \omega_1 \) and diagonally dominant in an open frequency interval containing that point.

Equivalently, under mild (generic) conditions, permissible transformations can be used to transform a given system into a system possessing no interaction at a specified frequency point. Moreover it follows that the class of permissible transformations is the smallest class that can achieve this objective.

We now describe some properties of permissible transformations (the proofs are straightforward and omitted here):

**Theorem 2**

If the transformation pair \( (P_1, P_2) \) is permissible, then (a) \( P_0 \hat{P}_1^{-1} P_2 \) is real and nonsingular and (b) \( (P_1^{-1}, P_2^{-1}) \) are real and nonsingular and (b) \( (P_1^{-1}, P_2^{-1}) \) are real and nonsingular and (c) \( (P_1^{-1}, P_2^{-1}) \) are real and nonsingular and (d) \( (P_2^{-1}, P_1^{-1}) \) are real and nonsingular and (d) \( (P_2^{-1}, P_1^{-1}) \) are real and nonsingular and (d) \( (P_2^{-1}, P_1^{-1}) \) are real and nonsingular
TRANSFORMATION OF THE CONTROL PROBLEM

The previous section has shown that constant plant and controller operations can be regarded as permissible transformations and that this is true of all three design techniques considered. Another major application of the ideas is in complete transformation of the design problem. More precisely (Owens 1979), the analysis and design of the configuration of Fig. 1(a) can be replaced by that shown in Fig. 1(b), provided that

\[ K(s) = P_2^{-1}K(s)P_1^{-1} \]  

(9)

is if the forward path control systems are related by the permissible (Theorem 2(d)) transformation \((P_2^{-1}, P_1^{-1})\). In particular, the relations

\[ |L + G(s)K(s)| \equiv |L + G(s)K(s)| \]  

(10)

\[ (X + GK)_1^{-1}GK \equiv P_1^{-1}(I + G)(G)K_1^{-1} \]  

(11)

indicate that the closed-loop stability of the two systems is identical and that the CLTFN's are related by the permissible (similarity) transformation \((P_1^{-1}, P_2^{-1})\) (Theorem 2(b)).

The unifying influence of this transformation can be illustrated as follows:

**Constant precorrection.** The design of the factored controller \(K(s) = K_1(s)K_p(s)\) (where \(K_p\) is a real, nonsingular precorrector) for the plant \(G(s)\) can be regarded as the design of the controller \(K(s) = K(s)\) for the transformed plant \(G(s) = G(s)K_p\). The permissible transformation in this case is \((I_m, K_p^{-1})\).

**Constant pre- and postcompensation.** Consider Fig. 1(c) with real, nonsingular constant postcompensator \(L\) and forward path controller \(K_c(s) = K_p(s)\). The design procedure can be regarded as the design of the controller \(K(s) = K(s)\) for the transformed plant \(G(s) = GC(s)L\) generated by the permissible transformation \((L^{-1}, K_p)\). The transformation of equation (9) then generates the configuration of Fig. 1(a) with forward path controller \(K(s) = K(s)\) and \(K_p(s)\) derived from Fig. 1(c) by 'moving \(L\) around the loop'.

**Approximately commutative control.** An approximately commutative controller takes the form \(K(s) = W_i \text{diag}(k_j(s))\) for some desired frequency points \(s = \omega_i\). Comparing this with equation (10) it is seen that \(K(s)\) can be regarded as generated from \(K(s) = \text{diag}(k_j(s))\) for some induced by the permissible transformation \((P_1^{-1}, P_2^{-1}) = (V_1^{-1}, V_2^{-1})\). The corresponding transformed plant is \(G(s) = G(s)K_p\).

**Method of dyadic expansion.** This is the only method (at this stage) that explicitly makes use of complex permissible transformations. The pair \((P_1, P_2)\) are computed (Owens 1978) to generate (Theorem 1) a transformed plant \(G\) that is diagonal at a desired frequency point \(s = \omega_1\). The controller \(K(s)\) is set equal to a diagonal matrix \(\text{diag}(k_j(s))\) and the controller \(K(s)\) generated by equation (10).

**REALIZABILITY CONDITIONS**

In practice, both \(G\) and \(K\) have rational polynomial elements. For physical realizability these polynomials must have real coefficients. These conditions are satisfied (for \(G\) say) iff

\[ \tilde{G}(s) = \tilde{G}(s) \]  

(12)

**Theorem 3**

If \((P_1, P_2)\) is permissible, then eqn. (12) holds iff

\[ \tilde{G}_{jk}(s) = \tilde{G}_{jk}(s) \]  

(13)

**Proof** Substitute eqns (6) and (8) into (12)

\[ \tilde{G}_{jk}(s)\tilde{G}(s)\tilde{G}(s)^T = \tilde{X}_{jk}(s)\tilde{G}(s)\tilde{G}(s)^T \]  

(14)

and equate coefficients of \(a_j \beta_j^T \forall j, k\).

It is important to note that this theorem can also be applied to construct suitable constraints on \(K(s)\) to ensure the physical realizability of \(K(s)\). More precisely, as \((P_2^{-1}, P_1^{-1})\) is permissible (Theorem 2(d)), then:

**Corollary**

\[ K(s) = \tilde{G}(s) \]  

iff \(\tilde{K}(s) = \tilde{G}(s)\) \(\forall j, k\).

The following result is also important:

**Theorem 4**

The set of TFN's \(G\) satisfying equation (13) is closed under multiplication and inversion.

**Proof** Suppose that \(G\) and \(K\) satisfy (13) and let \(\tilde{G} = \tilde{G}\) then

\[ \tilde{G}_{jk}(s) = \tilde{G}_{jk}(s) \]  

(13)

\[ \tilde{G}_{jk}(s) = \tilde{G}_{jk}(s) \]  

(13)
\[
\sum_{i=1}^{m} G_i(j)K(j)(i)K(i)(j) = \sum_{i=1}^{m} K_i(j)(i)K(i)(j) \tag{15}
\]

If \( G \) satisfies (13) with inverse \( K \) then
\[
\delta_{jk} = \sum_{i=1}^{m} G_{jk}(i)K(i)(j) \tag{16}
\]

\[
\delta_{jk} = \sum_{i=1}^{m} G_{jk}(i)K(i)(j) \tag{17}
\]

The result follows from the uniqueness of the inverse and \( \delta_{jk} = \delta_{k(j)(j)(k)} \) if \( j \neq k \).

**DIAGONAL DOMINANCE**

The choice of \((P_1, P_2)\) can be based on physical grounds, trial and error or theoretical synthesis procedures. In all cases a well-defined objective is required. Our attitude that permissible transformation techniques can only be of value if the transformed plant \( G \) is 'easier to control' than \( G \). Within our general design philosophy, design problems reduce as interaction effects are reduced. Therefore the attitude taken here is that permissible transformations will normally be constructed to reduce interaction effects in \( G \) over a desired frequency band.

The natural (but non-unique) measure of interaction effects in \( G \) in a given frequency interval is its degree of diagonal dominance (Rosenbrock 1974, Owens 1978, 1979). In practical terms, if we can choose permissible \((P_1, P_2)\) such that \( \hat{G} \) (or its inverse) is diagonally dominant (d.d.) on the Nyquist D-contour the basic theorems of the INA method can be used to design a diagonal \( \hat{G}(s) \).

If attention is restricted to real \((P_1, P_2)\) then the alignment and pseudodiagonalization algorithms (p.d.) can be applied, but success is not guaranteed even over a finite frequency interval. The potential benefit of accepting complex permissible transformations is illustrated by the basic theorem (Theorem 1) of the MDE technique: diagonal dominance over a frequency interval containing a specified point is (almost) always possible.

In the case of dyadic transfer function matrices (DTFM's) (Owens 1978, 1979), when \( G \) is diagonal, the design approach is particularly simple.

**PRECOMPENSATION**

Constant permissible transformation of \( G \) does not guarantee the d.d. of \( \hat{G} \) over a frequency interval large enough to have practical value. In such cases we can follow the INA by writing \( \hat{G}(s) = K(s)\hat{K}(s) \) where \( \hat{K}(s) \) is a dynamic precompensator introduced to ensure that \( G_p \) (or its inverse) is d.d. over a desired frequency interval and \( K \) is a compensator introduced to inject phase and gain compensation. We will assume that the compensator \( K \) is diagonal, which is the natural choice if \( G_p \) is approximately diagonal in the frequency interval of interest.

The only constraint on the choice of \( K_p \) and \( K_c \) is the physical realizability of \( K \).

This is ensured (Theorems 3 and 4) if \( K_p \) and \( K_c \) (or their inverses) satisfy constraints of the form of equation (13).

The choice of \( K_p \) can proceed on a trial and error basis of row or column operations or algorithmic basis as in pseudodiagonalization. The precise method is not relevant here. If, however, \( K_p \) is generated as a product of trial precompensators, Theorem 4 indicates that each factor must satisfy constraints of the form of equation (13). The resulting controller \( K \) can be implemented directly or in the form of cascaded blocks \( K = K_p K_c \) where (Theorems 2, 3) \( K(s) = P_2^{-1}K_p(s)P_1^{-1} \) and \( K_c(s) = P_1^{-1}P_2^{-1}K_c(s) \). \( P_1 \) and \( P_2 \) are physically realizable.

**A GENERAL DESIGN ALGORITHM**

The basic building blocks of permissible transformation, diagonal dominance checks and precompensation have been shown to provide a unified description of most of the basic operations in the INA, CL and ACC and MDE design techniques. We now show that they do, in fact, generate these algorithms and can generate new algorithms by (a) new combinations of these building blocks and (b) introducing new algorithms for constructing \((P_1, P_2)\), \( K_p \) and \( K_c \).

A common feature in the three design techniques is manipulation and compensation of plant behaviour in an open, connected subset \( S \) of the Nyquist D-contour. A general design algorithm for this purpose is proposed in Fig. 2. The INA, CL and ACC, and the MDE are simply specific cases of this algorithm generated by specific choices of \((P_1, P_2)\), \( K_p \) and \( K_c \).

INA: Take \( S = \mathbb{D} \) and the design algorithm in terms of inverse systems. Choose real pre- and post-compensators \( P_1^{-1} \) and \( P_2^{-1} \) respectively to achieve row dominance of \( \hat{G} = P_2^{-1}G P_1^{-1} \) on \( S \). Next choose an inverse
The apparently diverse nature of several frequency domain multivariable design techniques has been unified in the theoretical framework of permissible transformations, diagonal dominance checks and precompensation. A general design algorithm for system compensation in a subset of the Nyquist D-contour has been constructed from these three modules and it has been seen that the INA, CL and ACC and the MDE fit into this framework simply as special cases.

A major step is the introduction of complex permissible transformations and, hence, TFM's with rational polynomial elements with complex coefficients. This departure from normal engineering practice is meaningful and, moreover, vital if further progress in design methods is to be made at a theoretical level. Examples (Owens 1979a) certainly suggest that the benefits can outweigh any apparent difficulties and the basic theorem of the MDE (Theorem 1) also implies that, if diagonalization/diagonal dominance is taken as an important part of our design philosophy, then the acceptance of complex permissible transformations is necessary if design algorithms are to have any (generic) guarantee of success.

The structure of the general design algorithm suggests that any division of the three design techniques (in design programme terms) is purely artificial. In particular, the general design algorithm could be realized as a general design package of great flexibility, containing these techniques simply as alternative routes through the package obtained by specialized choices of transformation and the use (or otherwise) of precompensators. Such a programme would need a reconsideration of data handling to allow complex elements but the potential benefits obtained by (a) breaking down the barriers between established techniques and (b) the generation of new algorithms by, for example, the introduction of precompensation and diagonal dominance checks into the ACC and MDE, will, in the authors opinion, far outweigh the initial interpretive difficulties.

Finally, the emergence of a root-locus theory for multivariable systems (Owens 1978, 1979b, Postlethwaite 1979) has focussed attention on the configuration of Fig. 1(d). Minor loop feedback \( H \) plays an important role in compensation studies for multivariable root-loci (Owens 1979b) and can be a great help in achieving diagonal dominance as is easily seen by examining the inverse forward path system \( K(K + H) \). The introduction of permissible transformations into this picture is achieved by performing the transformations

\[
G = P \frac{G K}{P_c}, \quad K = P^{-1} K P^{-1}, \quad K_c = P K P^{-1},
\]

\[
P = \frac{1}{P_c L}, \quad K = P L P^{-1} H, \quad H = P H P^{-1}.
\]

and considering the configuration with \( G, K, K_c \) and \( H \) replaced by \( G, K, K_c \) and \( H \) respectively. The identities (cf. eqns (10), (11))
\[
|I_m + (I_m + G_K) H^{-1} G_K K_p| = |I_m + (I_m + G_K) H^{-1} G_K K_p| 
\]
\[
(18) 
\]
\[
(1_m + G_K H)^{-1} G_K K_p^{-1} (1_m + G_K H)^{-1} G_K K_p^{-1} 
\]
\[
(19) 
\]

indicate that the two configurations have identical stability characteristics with CLTPM's related by the permissible transformation \((P_1, P_2^{-1})\).

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Fig. 1. Multivariable Feedback Systems

Fig. 2. General Design Algorithm