IDENTIFICATION OF NONLINEAR S_m SYSTEMS

by

S. A. BILLINGS, B.Eng.,Ph.D.,M.Inst.M.C.,AMIEE

S. Y. Fakhouri, B.Sc.,M.Sc.,AMIEE

Department of Control Engineering
University of Sheffield
Mappin Street
Sheffield S1 3JD

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Abstract

An identification algorithm for systems which can be represented by a nonlinear S_m model is presented. Cross-correlation techniques are employed to provide estimates of the individual linear subsystems and nonlinear coefficients from measurements of the input and noise corrupted output.

1. INTRODUCTION

Identification of nonlinear systems which can be represented by the S_m model illustrated in Fig.1 is considered, where m denotes the highest integer power nonlinearity present. This class of systems was originally studied by Baumgartner and Rugh[1] who developed identification algorithms based on steady-state sinusoidal measurements. The algorithms were extended by Wysocki and Rugh[2] who reduced the number of measurements and inputs required for identification. Sandor and Williamson[3] later achieved the same results in a form which could be extended to a more general class of systems.

In the present study a correlation algorithm which has been developed for the identification of a general class of nonlinear systems[4,5,6] is extended to provide complete identification of the S_m model. The algorithm is relatively simple to implement and provides estimates of the individual linear elements and nonlinear coefficients from input-output correlation functions computed when the input has the properties of a white Gaussian process.

2. Identification of the Linear Subsystems

Inspection of Fig.1 shows that for an input $u_2(t)$ the measured system output $z(t)$ can be expressed as

$$z(t) = \sum_{i=1}^{m} w_i(t)+v(t) \qquad (1)$$

where $w_i(t)$ is the contribution of the i'th branch or kernel of the S_m model to the output and is defined as

$$w_i(t) = \int d\tau_1 \ldots \int d\tau_i \int d\theta h_{i2}(\theta) h_{i1}(\tau_1) \ldots h_{i1}(\tau_i)$$

$$u_2(t-\tau_1-\theta) \ldots u_2(t-\tau_i-\theta) \qquad (2)$$

Define the first order output cross-correlation function

$$\phi_{u_1 z'}(\sigma) = \overline{u_1(t-\sigma) z'(t)} = \overline{u_1(t-\sigma)(z(t)-\overline{z(t)})}$$

$$= \sum_{i=1}^{m} \phi_{u_1 w_i'}(\sigma) + \phi_{u_1 v'}(\sigma) \qquad (3)$$

where the superscript ' is used throughout to denote a zero mean process.

Consideration of eqn's (1), (2) and (3) shows that for a given functional form of the input $u_2(t)$, the form of the term $\phi_{u_1 w_i'}(\sigma)$ is fixed but its amplitude is proportional to the i'th power of $u_2(t)$. Thus for a series of experiments with inputs $\alpha_j u_2(t)$ where $\alpha_j \neq \alpha_\ell \ \forall \ j \neq \ell$ the output correlation function $\phi_{u_1 z'_{\alpha_j}}(\sigma)$ is given by

$$\phi_{u_1 z'_{\alpha_j}}(\sigma) = \sum_{i=1}^{m} \alpha_j^{\,i} \phi_{u_1 w_i'}(\sigma) \qquad \text{for } j = 1,2,\ldots m \qquad (4)$$

assuming that $u_1(t)$ and $v'(t)$ the measurement noise are statistically independent where z_{α_j} is the response of the system to the input $\alpha_j u_2(t)$.

Alternatively, a series of experiments with inputs $\{\alpha_j u_2(t)\}$ and $\{-\alpha_j u_2(t)\}$ yield the correlation functions[7]

$$\phi_{u_1 o_{\alpha_j}}(\sigma) = \tfrac{1}{2}(\phi_{u_1 z'_{\alpha_j}}(\sigma) - \phi_{u_1 z'_{-\alpha_j}}(\sigma))$$

$$= \sum_{i=1}^{k} \alpha_j^{\,2i-1} \phi_{u_1 w'_{(2i-1)}}(\sigma) \quad ; \quad j = 1,2\ldots k \qquad (5)$$

$$\phi_{u_1 e_{\alpha_j}}(\sigma) = \tfrac{1}{2}(\phi_{u_1 z'_{\alpha_j}}(\sigma) + \phi_{u_1 z'_{-\alpha_j}}(\sigma))$$

$$= \sum_{i=1}^{N} \alpha_j^{2i} \phi_{u_1 w'_{(2i)}}(\sigma) \quad ; \quad j = 1,2\ldots N \qquad (6)$$

where O_{α_j} and e_{α_j} represent the response of the odd and even order kernels respectively, and

$$k = \begin{cases} \frac{m}{2} & \text{for m even} \\ \frac{m+1}{2} & \text{for m odd} \end{cases} \quad ; \quad N = \begin{cases} \frac{m}{2} & \text{for m even} \\ \frac{m-1}{2} & \text{for m odd} \end{cases} \qquad (7)$$

Thus for any value of σ eqn (4) or eqn's (5) and (6) have a unique solution for $\phi_{u_1 w'_i}(\sigma)$ $i = 1,2\ldots m$. Whilst the procedure for eqn (4) is perfectly acceptable in many cases the latter procedure defined by eqn's (5), (6) and (7) provides more accurate estimates in the presence of noise.

Notice that although multilevel inputs must be employed only $\phi_{u_1 w'_i}(\sigma)$ not the individual outputs $w_i(t)$ must be computed. This simplifies the procedure because for a stable subsystem $\phi_{u_1 w'_i}(\sigma)$ will tend to steady-state after a small number of values typically 30-40 sample points.

The k'th branch of the S_m system illustrated in Fig.1 has the structure of the general model where $F[\cdot] = \gamma_k(\cdot)^k$. Thus setting $u_1(t) = u(t)$, $u_2(t) = u(t)+b$, where $u(t)$ is a zero mean white gaussian process, b is a non-zero mean level and employing previous results derived for the general model[4,5], the first order correlation function of the k'th branch can be expressed as

$$\phi_{uw'_k}(\sigma) = C_{Fk} \int h_{k1}(\tau_1) h_{k2}(\sigma-\tau_1) d\tau_1 \qquad (8)$$

$$C_{Fk} = \gamma_k \sum_{r=o}^{\frac{q-1}{2}} \binom{k}{2r+k-q} \mu_x^{(2r+k-q)} \frac{(2p)!}{2^p p!} (\lambda \int h_{k1}^2(\theta) d\theta)^{p-1} \qquad (9)$$

$$\text{for } k = 2,3,4\ldots m$$

where

$$\lambda = \int_{-\infty}^{\infty} \phi_{uu}(t) dt$$

$$q = \begin{cases} k & \text{for } k \text{ odd} \\ k-1 & \text{for } k \text{ even} \end{cases}$$

$$\mu_x = b\int h_{k1}(\theta)\, d\theta \quad , \quad p = \frac{q-2+1}{2}$$

and $\phi_{u_1 w_1'}(\sigma) = \lambda h_{11}(\sigma)$ for $k = 1$ (10)

Define the second order output correlation function

$$\phi_{u_1^2 z'}(\sigma) = \overline{u_1^2(t-\sigma)z'(t)}$$

$$= \sum_{i=1}^{m} \phi_{u_1^2 w_i'}(\sigma) + \phi_{u_1^2 v'}(\sigma) \qquad (11)$$

Providing $u_1(t)$ and $v(t)$ are statistically independent $\phi_{u_1^2 v'}(\sigma) = 0\ \forall\ \sigma$ and eqn (11) reduces to

$$\phi_{u_1^2 z'}(\sigma) = \sum_{i=1}^{m} \phi_{u_1^2 w_i'}(\sigma) \qquad (12)$$

Following the procedure outlined above for the evaluation of $\phi_{uw_i'}(\sigma)$ the second order correlation function of each branch of the S_m model can be isolated to yield

$$\phi_{u^2 w_k'}(\sigma) = 2C_{FFk} \int h_{k2}(\theta) h_{k1}^2(\sigma-\theta)\, d\theta \qquad (13)$$

$$C_{FFk} = \lambda^2 \gamma_k \sum_{r=0}^{\frac{s-2}{2}} \binom{k}{2r+k-s} \mu_x^{(2r+k-s)} \left(\lambda \int h_{k1}^2(\theta)\, d\theta\right)^{t-1} \frac{(2t)!}{2^{(t-1)}(t-1)!} \qquad (14)$$

for $k = 2,3...m$

where

$$s = \begin{cases} k & \text{for } k \text{ even} \\ k-1 & \text{for } k \text{ odd} \end{cases}$$

$$t = \frac{s-2r}{2}$$

and $\phi_{u^2 w_1'}(\sigma) = 0\ \forall\ \sigma$ for $k = 1$.

If equations (8) and (13) are evaluated in discrete time, estimates of the parameters in the pulse transfer functions

$$Z\{\phi_{uw_k}{}'(\sigma)\} = Z\{C_{Fk}h_{k1}(t)*h_{k2}(t)\} = \frac{B_k(z^{-1})}{A_k(z^{-1})} \quad (15)$$

$$Z\{\phi_{u^2w_k}{}'(\sigma)\} = Z\{C_{FFk}h_{k1}^2(t)*h_{k2}(t)\} = \frac{F_k(z^{-1})}{E_k(z^{-1})} \quad (16)$$

can be readily obtained using a simple least squares algorithm. Estimates of the pulse transfer functions $Z\{\mu_{k1}h_{k1}(t)\}$ and $Z\{\mu_{k2}h_{k2}(t)\}$. $k = 2,\ldots m$ can then be computed to within constant scale factors μ_{k1},μ_{k2} by decomposing the results of eqn's (15) and (16) using a multistage least squares algorithm[4].

3. Identification of the Nonlinear Coefficients

The error between the sampled process output $z(i)$ and the predicted output $\hat{z}(i)$ can be defined as

$$\begin{aligned} e(i) &= z(i)-\hat{z}(i) \\ &= z(i)-\gamma_1{}'\hat{q}_{e1}(i)-\gamma_2\sum_{j=o}^{p}\mu_{22}\hat{h}_{22}(j)\hat{q}_{e2}^{\,2}(i-j) \\ &\quad -\ldots\gamma_m{}'\sum_{j=o}^{p}\mu_{m2}\hat{h}_{m2}(j)\hat{q}_{em}^{\,m}(i-j) \end{aligned} \quad (17)$$

where $\hat{q}_{ek}(i)$ the scaled estimate of $q_k(i)$ is given by

$$\hat{q}_{ek}(i) = \sum_{j=o}^{\ell\ell}\mu_{k1}\hat{h}_{k1}(j)u_2(i-j) = \mu_{k1}q_k(i) \quad (18)$$

$$\gamma_1 = \gamma_1{}'/\lambda \quad \text{and} \quad \gamma_k = \mu_{k1}^{\,k}\,\mu_{k2}\gamma_k{}' \quad , \quad k = 2,3\ldots m \quad (19)$$

If NN measurements of the sampled process input and output are available the matrix equation

$$\begin{bmatrix} z(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z(NN) \end{bmatrix} = \begin{bmatrix} \hat{q}_{k1}(1);\ \sum_{j=o}^{p}\mu_{22}\hat{h}_{22}(j)\hat{q}_e^{\,2}(1-j);\ldots\sum_{j=o}^{p}\mu_{m2}\hat{h}_{m2}(j)\hat{q}_{em}^{\,m}(1-j) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hat{q}_1(NN);\ \sum_{j=o}^{p}\mu_{22}\hat{h}_{22}(j)\hat{q}_{e2}^{\,2}(NN-j);\ldots\sum_{j=o}^{p}\mu_{m2}\hat{h}_{m2}(j)\hat{q}_{em}^{\,m}(NN-j) \end{bmatrix}$$

$$\cdot \begin{pmatrix} \gamma_1' \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \gamma_m' \end{pmatrix} + \begin{pmatrix} e(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ e(NN) \end{pmatrix}$$

$$\underline{Z} = \underline{\phi}\ \underline{\theta} + \underline{E} \qquad (20)$$

can be formulated and the least squares estimate of the coefficients $\gamma_1' \ldots \gamma_m'$ can be computed

$$\underline{\hat{\theta}} = (\underline{\phi}^T \underline{\phi})^{-1} \underline{\phi}^T \underline{Z} \qquad (21)$$

and the identification is complete.

4. Simulation Results

The identification procedure outlined above was used to identify the parameters in a third order S_m system defined in Table 1. The system was simulated on an ICL 1906S digital computer and 30,000 points were generated by recording the response to a four level input signal $\alpha_i u_2(t)$, $i = 1,\ldots 4$ where $u_2(t)$ is a white gaussian sequence N(0.15,0.3333) and $\alpha_1 = 1.0$, $\alpha_2 = -1.0$, $\alpha_3 = 0.9$, $\alpha_4 = -0.9$. The procedure defined by eqns (5) and (6) was used to isolate the branch correlation functions.

Least squares estimates of the parameters in the linear pulse transfer function models and the coefficients of the integer power nonlinearities are summarised in Table 1. A comparison of the estimated pulse responses and the theoretical weighting sequences $h_{11}(t)$;$h_{21}(t)$,$h_{22}(t)$;$h_{31}(t)$,$h_{32}(t)$ are illustrated in Fig.2(a), (b) and (c) respectively.

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	$H_{11}(z^{-1})$				$H_{21}(z^{-1})$		$H_{22}(z^{-1})$		$H_{31}(z^{-1})$		$H_{32}(z^{-1})$				Nonlinear coefficients		
Parameter	n_1	n_2	d_1	d_2	n_1	d_1	n_1	d_1	n_1	d_1	n_1	n_2	d_1	d_2	γ_1	γ_2	γ_3
Theoretical values	0.2	0.0	-1.5	0.62	0.6	-0.8	15.0	-0.4	0.8	-0.7	2.5	0.0	-1.68	0.8	1.0	1.0	1.0
Estimates	0.202	-0.003	-1.496	0.614	0.593	-0.8	15.01	-0.398	0.829	-0.672	2.412	-0.07	-1.704	0.83	0.784	1.168	1.02

TABLE 1 A Summary of the Identification Results

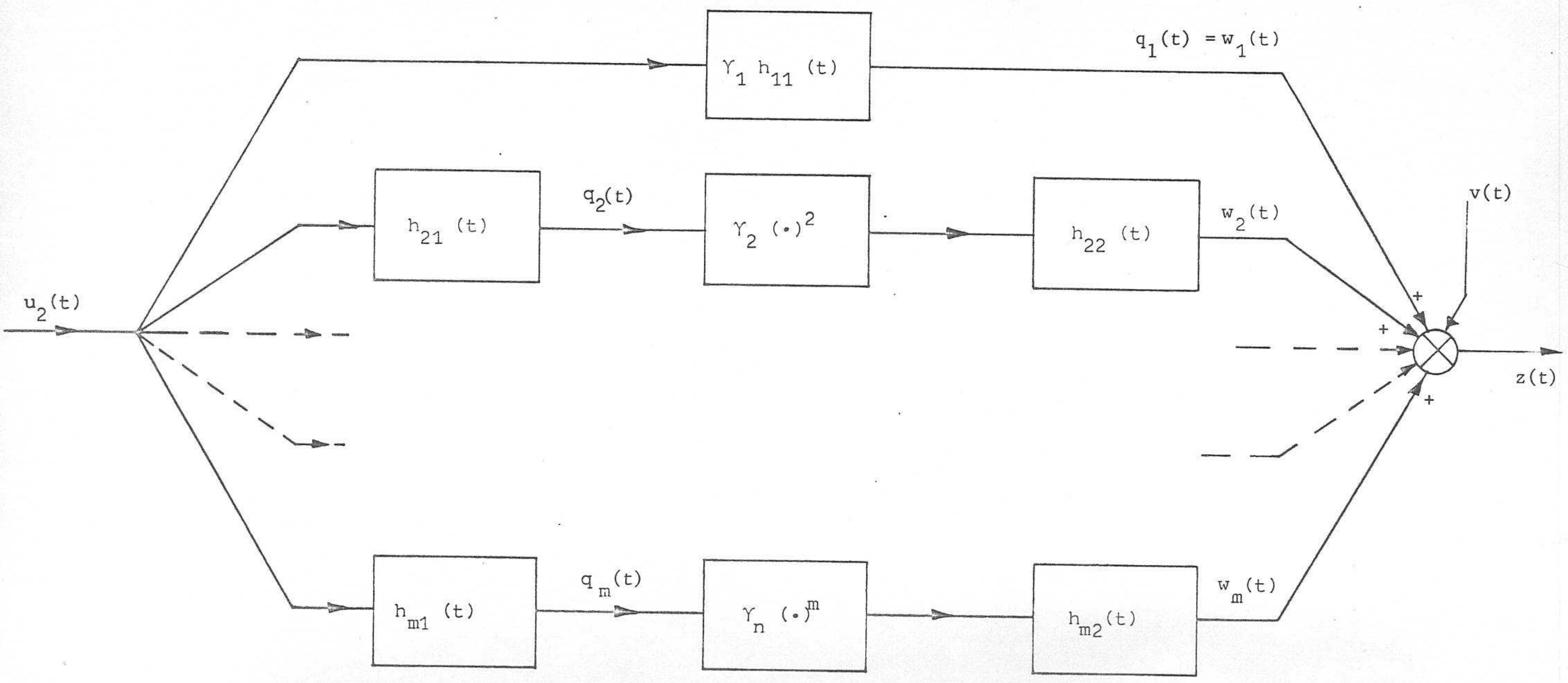
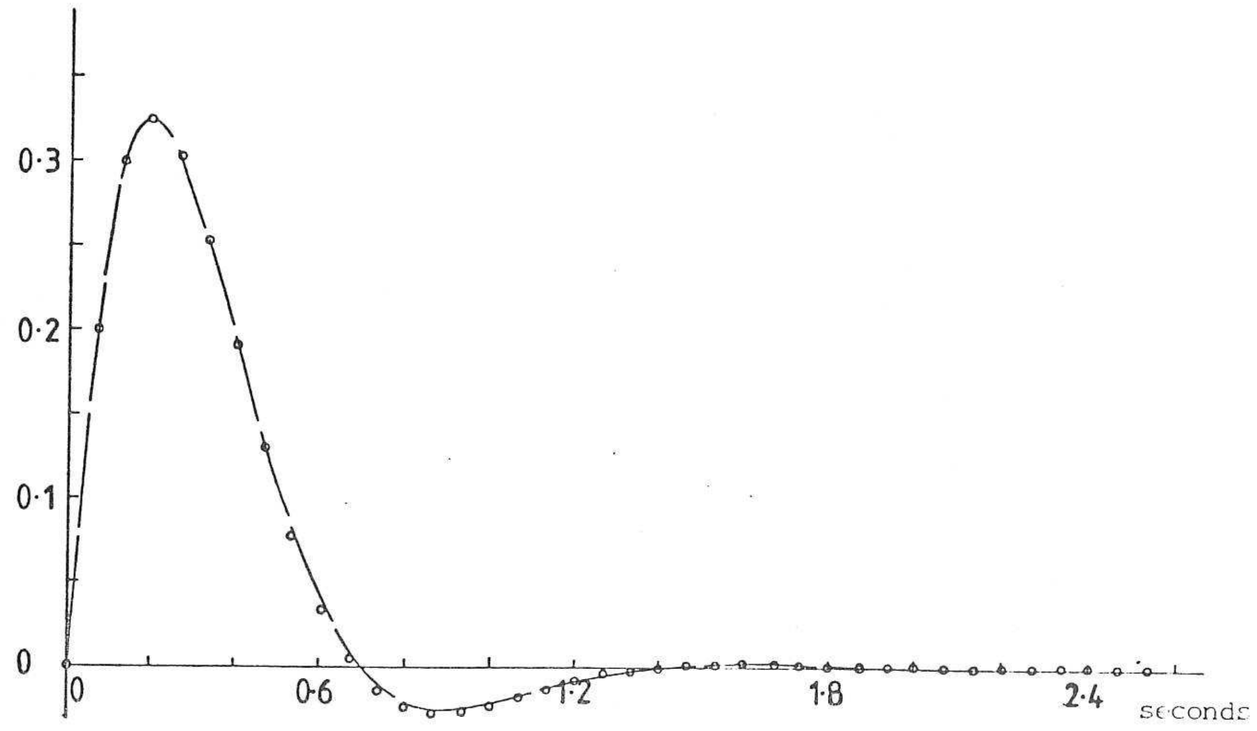


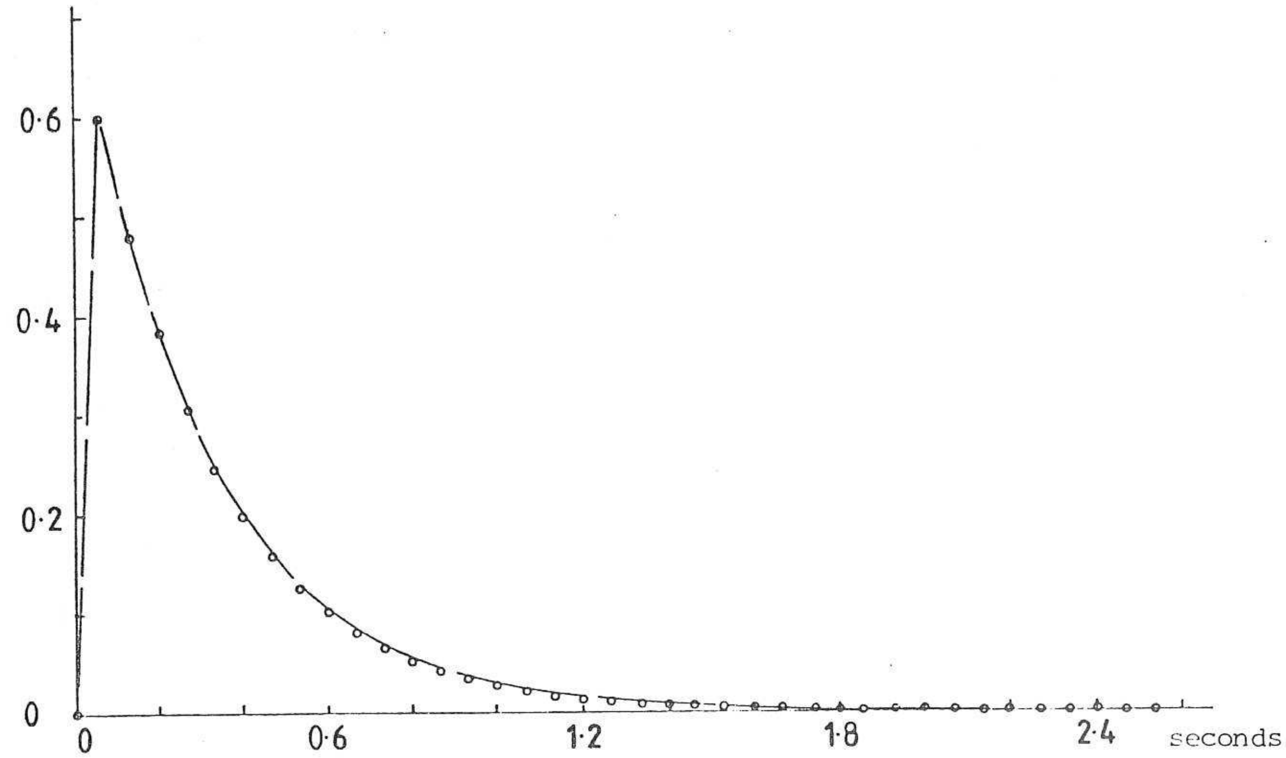
FIG 1 THE S_m MODEL



o o o Theoretical response $h_{11}(K)$

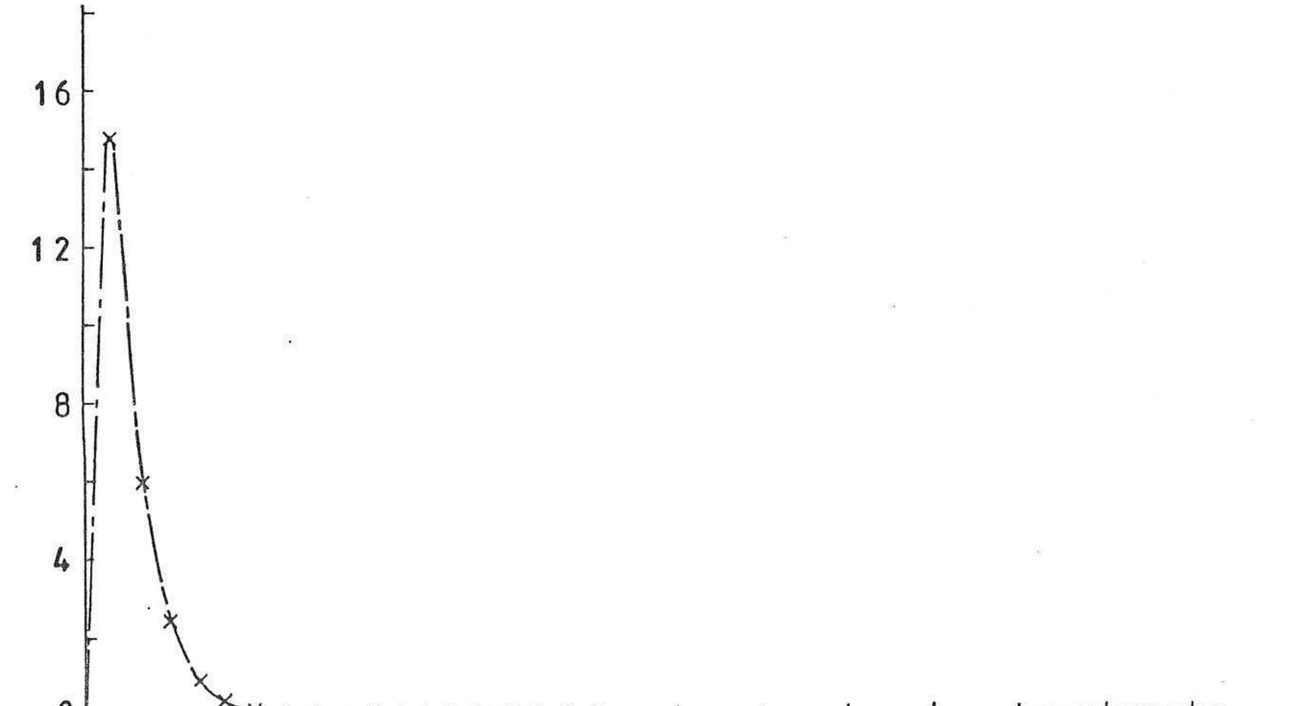
— — Estimated values $\hat{h}_{11}(K)$

FIG 2(a) A comparison of impulse responses for the first order Kernel of the S_m model



o o o Theoretical Response $h_{21}(K)$

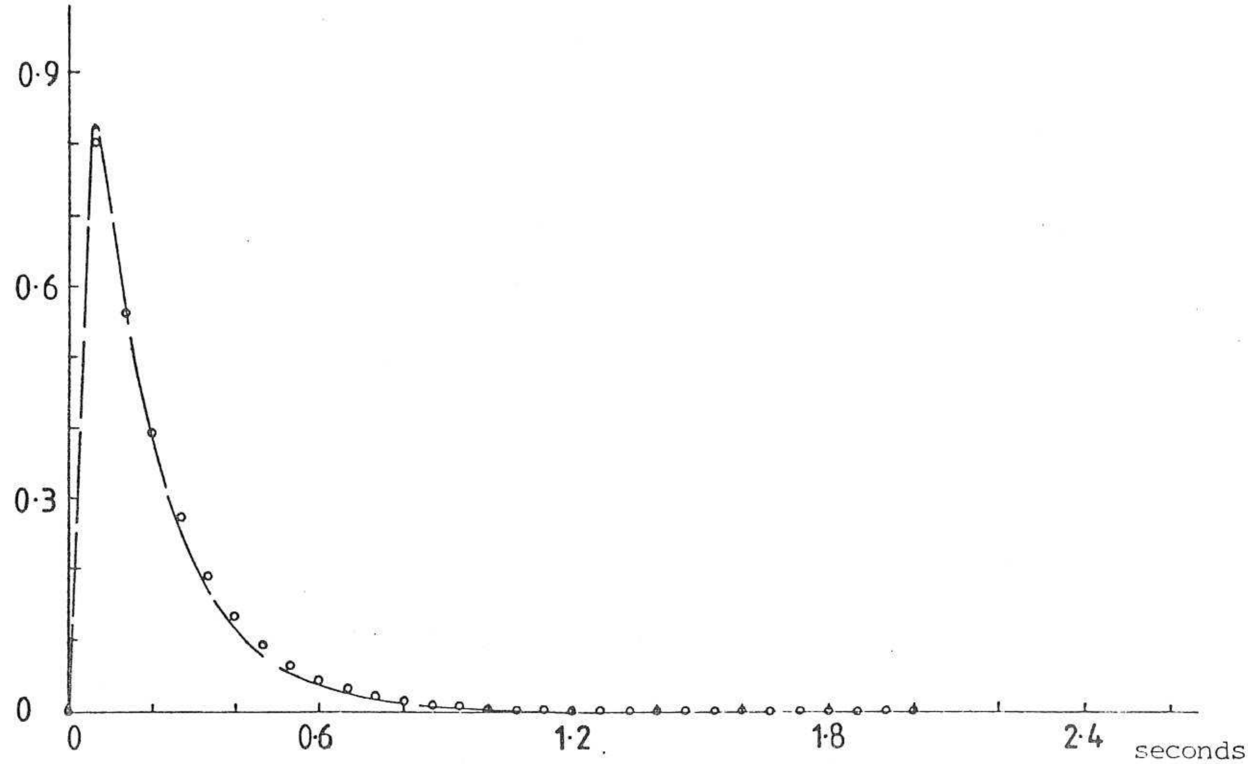
— — Estimated values $\hat{h}_{21}(K)$



X X X Theoretical Response $h_{22}(K)$

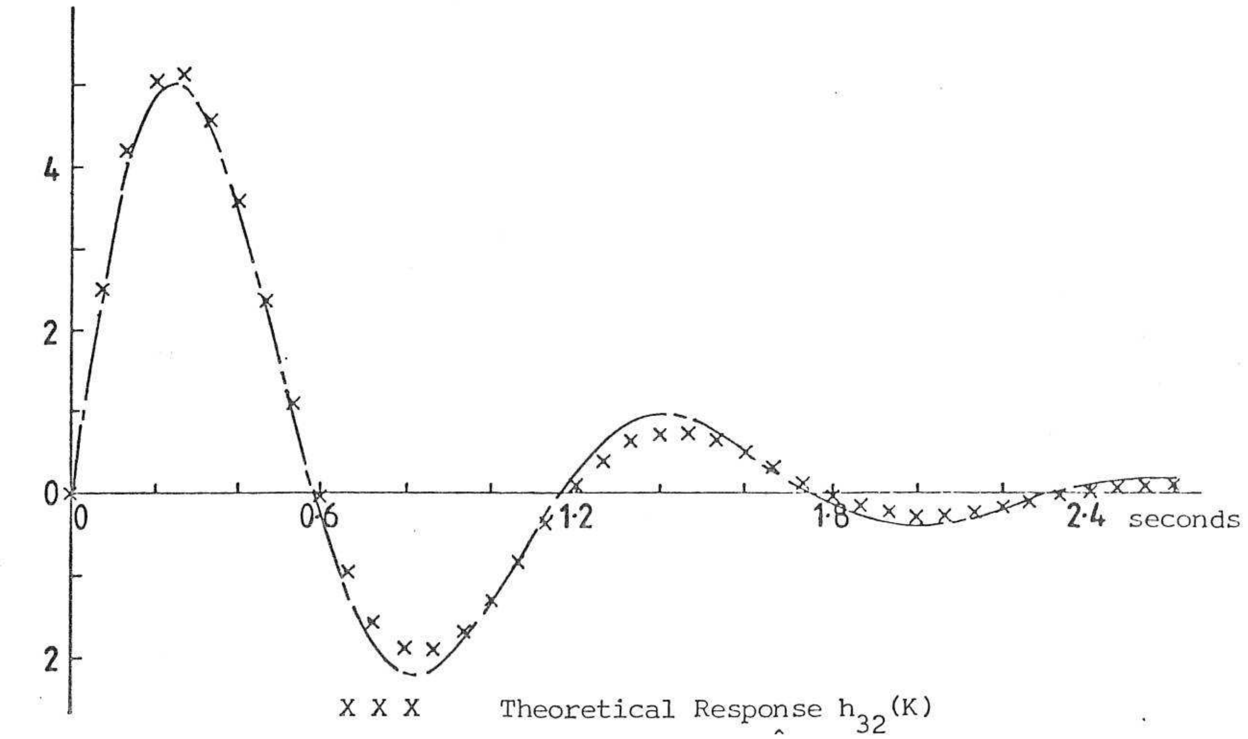
— — — Estimated values $\hat{h}_{22}(K)$

FIG 2(b) A comparison of impulse responses for the second order Kernel of the S_m model



o o o Theoretical Response $h_{31}(K)$

— — Estimated values $\hat{h}_{31}(K)$



X X X Theoretical Response $h_{32}(K)$

— — — Estimated values $\hat{h}_{32}(K)$

FIG 2(c) A comparison of impulse responses for the third order Kernel of the S_m model