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DYADIC EXPANSIONS AND THEIR APPLICATIONS

by

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Abstract

This is an expository paper devoted to surveying and explaining some of the basic concepts required and motivations for the definition and use of dyadic expansions in systems analysis and controller design. Where possible, the material is related to the inverse Nyquist array and characteristic locus design methods.

1. Introduction

The purpose of this paper is to provide an introduction to the basic concepts of, and motivations for the representation of the input/output (I/O) behaviour of an m-input/m-output proper invertible system $S(A,B,C,D)$ by dyadic expansion of its mxm transfer function matrix (TFM)

$$(m\times m) \quad G(s) \triangleq C(sI_n - A)^{-1}B + D \tag{1}$$

Attention will be focussed(1-5) on the development of theoretical tools applicable to the design of the unity-negative feedback system shown in Fig. 1 for the control of the strictly proper, invertible plant with mxm TFM $G(s)$. The TFM $K(s)$ of the forward path controller is assumed to be proper and invertible.

2. Basic Concepts

2.1 A Motivation:

The stability of the feedback system of Fig. 1 is described by the fundamental relationship between the closed-loop and open-loop characteristic polynomials $\rho_c(s)$ and $\rho_o(s)$ respectively,

$$\frac{\rho_c(s)}{\rho_o(s)} \equiv |I_m + G(s)\,K(s)| \tag{2}$$

Despite its apparent simplicity, the application of this relation to design is no trivial matter. In essence, the difficulties can be attributed to possible dynamic complexity and structural complexity of the plant $G(s)$. Dynamic complexity is due to high state dimensions $n \gg m$ (as reflected by high order transfer function elements of $G(s)$) and the presence of right-half-plane poles and zeros. These are problems common to both classical and multivariable design. In contrast, structural complexity (as reflected by the presence of complicated interconnections between subsystems of $G(s)$) seems to be a purely multivariable phenomenum. Systems may possess either, neither or both properties e.g.

the plant

$$G(s) \triangleq \text{diag.}\ \{g_1(s), g_2(s), \ldots, g_m(s)\} \tag{3}$$

may be dynamically complex but it has a particularly simple pseudo-classical/noninteracting structure.

The concept of a dyadic transfer function matrix (DTFM) has its origins in the design of nuclear reactor spatial control systems[2,3]. In the following development, however, it will be regarded as the natural first step in the analysis of structural complexity and its implications for frequency domain design techniques based on reduction of the multivariable design problem to an effective sequence of scalar classical designs.

2.2 <u>Permissible Equivalence Transformations and Transformation of the Control Problem;</u>

The natural step in the analysis and simplification of system structure is to consider simple structural transformations to the TFM $G(s)$. The simplest transformation is the map $G \rightarrow H$ specified by

$$G(s) = P_1\ H(s)\ P_2 \tag{4}$$

where P_1 and P_2 are mxm <u>constant</u>, nonsingular (real or complex) matrices. The practical advantages of such transformations are their algebraic simplicity and the fact that they introduce no new dynamics into the transformed system $H(s)$. Also, if (say) P_1 and P_2 are real, the transformation has a natural physical intepretation in terms of change of input and output variables.

The need for a physical interpretation of equation (4) leads to constraints[4] on the form of P_1 and P_2. These are defined below and justified in section 3.

Definition 1 (Physical interpretation)

The mxm nonsingular complex matrices

$$P_1 = [\alpha_1, \alpha_2, \dots, \alpha_m] \quad , \quad P_2 = \begin{bmatrix} \beta_1^T \\ \cdot \\ \cdot \\ \cdot \\ \beta_m^T \end{bmatrix} \tag{5}$$

are said to be permissible if the columns (resp. rows) of P_1(resp. P_2) are real or exist in complex conjugate pairs and if $\bar{\beta}_j = \beta_{\ell(j)}$ whenever $\bar{\alpha}_j = \alpha_{\ell(j)}$, $1 \leqslant j \leqslant m$.

It follows immediately that $P_1 P_2$ is real and nonsingular if the pair (P_1, P_2) are permissible. The identity $G(\bar{s}) \equiv \overline{G(s)}$ holds if, and only if,

$$H_{jk}(s) \equiv H_{\ell(j)\ell(k)}(s) \quad , \quad 1 \leqslant j,k \leqslant m \tag{6}$$

Also, if P_1, P_2 are permissible and complex, $H(s)$ has elements with complex coefficients. It is, of course, tempting to eliminate this non-classical description by requiring that both P_1 and P_2 are real. This would be a severe restriction on the theoretical development analogous to requiring that the eigenvectors of A are all real (this would, of course, eliminate the description of complex conjugate pair poles and oscillatory systems!).

The potential of the transformations can be illustrated by the simple example(1).

$$G(s) = \begin{bmatrix} G_1(s) & G_2(s) \\ G_2(s) & G_1(s) \end{bmatrix}$$

$$\equiv \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} G_1(s) + G_2(s) & 0 \\ 0 & G_1(s) - G_2(s) \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{7}$$

with the natural identification of P_1, $H(s)$ and P_2. The pair (P_1, P_2) is permissible and $H(s)$ has a diagonal structure independent of the dynamics $G_1(s)$ and $G_2(s)$. The description is hence a real representation of the structural complexity of $G(s)$.

An alternative example with permissible complex transformations is the structure

$$G(s) = \begin{pmatrix} G_1(s) & -G_2(s) \\ G_2(s) & G_1(s) \end{pmatrix}$$

$$\equiv \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} G_1(s) + iG_2(s) & 0 \\ 0 & G_1(s) - iG_2(s) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ \frac{1}{2}i & \frac{1}{2} \end{pmatrix} \tag{8}$$

with the natural identification of P_1, $H(s)$ and P_2.

Consider now the feedback system of Fig. 1 and the identity

$$|I_m + G(s)\ K(s)| \equiv |I_m + H(s)\ P_2\ K(s)\ P_1| \tag{9}$$

In particular, note that the design of K for G can (from the point of view of stability analysis) be replaced by the design of the transformed controller $P_2\ K(s)\ P_1$ for the transformed plant $H(s)$. The relationship between closed-loop dynamics takes the form

$$H_c(s) \triangleq \{I_m + GK\}^{-1}\ GK \equiv P_1\{\{I + HP_2\ KP_1\}^{-1}\ H\ P_2\ K\ P_1\}P_1^{-1} \tag{10}$$

i.e. the closed-loop TFM's are related by the similarity transformation P_1.

This seems to be as far as a general theory can take us without the detailed insight available from the study of special cases that occur in practice and the introduction of approximation procedures and useful, applicable stability theorems. These are described in the remainder of the paper and indicate that the advantages to be gained in the stability analysis by suitable choice of permissible P_1 and P_2 can far outweigh the increased complexity in the interpretation of closed-loop transient performance using equation (10).

3. Dyadic Transfer Function Matrices[1,3]

The equivalence transformations introduced above are an algebraic construction. Some physical insight can be obtained by consideration of dyadic transfer function matrices (DTFM's) where it is possible to ensure that $H(s)$ is diagonal (see examples (7) and (8)).

Definition[1] 2. (Structural definition of DTFM's)

An mxm invertible system is said to be dyadic if there exists constant mxm matrices P_1 and P_2 and scalar transfer functions $g_1(s), \ldots, g_m(s)$ such that the system TFM takes the form

$$G(s) \equiv P_1 \operatorname{diag} \{g_1(s), g_2(s), \ldots, g_m(s)\} P_2 \tag{11}$$

The TFM $G(s)$ is then said to be a DTFM.

It is easily verified that we can always assume that (P_1, P_2) are permissible and take $H(s) = \operatorname{diag} \{g_j(s)\}_{1 \leqslant j \leqslant m}$.

3.1 Interpretations and Alternative Definitions:

The simplest interpretation of a dyadic system is illustrated in Fig. 2., i.e. as a non-interacting system 'sandwiched' between input and output transformations P_2 and P_1 respectively. It is a natural first generalization of the pseudo-classical structure of equation (3). The interaction in the system is due solely to the presence of P_1 and P_2 i.e. the practical decision to measure $y_1, \ldots, y_m$ rather than the 'sub-system' outputs $\hat{y}_1, \ldots, \hat{y}_m$ and to control these outputs by direct manipulation of $u_1, \ldots, u_m$ rather than the 'subsystem' inputs $\hat{u}_1, \ldots, \hat{u}_m$. In some applications[1,2] these subsystems can be given a modal interpretation and the transformation of the control problem is both physically meaningful and a great advantage in stability analysis.

Perhaps the most natural definition of a DTFM is, however, modal in nature! Consider the mxm strictly proper, invertible system S(A,B,C) (not necessarily dyadic) and suppose, for simplicity, that A has distinct eigenvalues i.e. the system TFM

$$G(s) \equiv \sum_{j=1}^{n} \frac{1}{s - \lambda_j} \alpha_j \beta_j^T \qquad (12)$$

take the form of a linear combination of constant <u>dyads</u> $\{\alpha_j\beta_j^T\}_{1 \leqslant j \leqslant n}$ with first-order transfer function coefficients. It is important to note that the unordered set $\{\alpha_j\beta_j^T\}_{1 \leqslant j \leqslant n}$ may contain complex elements but is invariant under complex conjugation. The vectors α_j and β_j describe the way in which the pole at $s = \lambda_j$ affects output response and how the input excites that pole respectively.

The dyads $\{\alpha_j\beta_j^T\}_{1 \leqslant j \leqslant n}$ can be regarded as generating a subspace of the vector space of all mxm complex matrices of dimension equal to $\ell \leqslant n$ (and normally $\ell \ll n$). Without loss of generality, suppose that $\alpha_1\beta_1^T,\ldots,\alpha_\ell\beta_\ell^T$ span this subspace. It follows that

$$\begin{aligned} G(s) &\equiv \sum_{j=1}^{\ell} g_j(s)\, \alpha_j\beta_j^T \\ &\equiv [\alpha_1,\alpha_2,\ldots,\alpha_\ell]\ \mathrm{diag}\{g_j(s)\}_{1 \leqslant j \leqslant \ell} \begin{pmatrix} \beta_1^T \\ \vdots \\ \beta_\ell^T \end{pmatrix} \end{aligned} \qquad (13)$$

where $g_j(s)$ are scalar transfer functions. In effect, system dynamics can be regarded as being generated by ℓ (possibly overlapping) groups of modes characterized by the transfer functions $g_j(s)$, $1 \leqslant j \leqslant \ell$. Each group is excited only by the projected input $\beta_j^T u(s)$ and contributes to the output in a vector form described by α_j as can be seen by writing the relation $y = Gu$ in the form,

$$y(s) = \sum_{j=1}^{\ell} \alpha_j \{g_j(s) \{\beta_j^T u(s)\}\} \qquad (14)$$

The system is dyadic if, and only if, $\ell = m$ when equation (13) reduces to equation (11). In particular, the invariance of the unordered set $\{\alpha_j\beta_j^T\}$ under complex conjugation and the possibility of complex conjugate pairs due to complex conjugate pair poles explains the need to introduce complex permissible transformations into definition 1.

The modal interpretation is perhaps most succinctly stated by the following equivalent definition (simply set $K_D = (P_1P_2)^{-1}$)

Definition 3 (Modal definition of DTFM's)

An mxm, invertible system is dyadic if, and only if, there exists a real, constant, nonsingular mxm matrix K_D such that $G(s)\ K_D$ has m linearly independent, constant eigenvectors $\alpha_1, \alpha_2, \ldots, \alpha_m$

$$G(s)\ K_D\ \alpha_j = g_j(s)\ \alpha_j \tag{15}$$

where $g_j(s)$, $1 \leqslant j \leqslant m$, are scalar transfer functions.

The eigenvector interpretation of equation (15) is given the natural modal interpretation which can have great significance in applications.

Finally, definition 3 can be restated by noting that GK_D commutes with any other matrix having eigenvector matrix P_1.

Definition 4 (Invariance definition of DTFM's)

An mxm, invertible system is dyadic if, and only if, there exists real, nonsingular, constant mxm matrices K_D and P_o such that P_o has distinct eigenvalues and

$$G(s)K_D\ P_o \equiv P_o\ G(s)\ K_D \tag{16}$$

Equation (16) is an invariance relation for G which, in general terms, states that, if the input u produces the output y, then the input $K_D\ P_o\ K_D^{-1}$ u will produce the output P_oy. In specific applications the invariance is a reflection of system structural properties such as spatial symmetry[1,2]. For example, the highly symmetric structure of equation (7) satisfies the invariance relation

$$G(s)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \equiv \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} G(s) \qquad (17)$$

reflecting the indistinguishability of the system dynamics under simultaneous interchange of input and output subscripts. The matrix P_o in this case is the generator for a two dimensional representation of the reflection group[2].

3.2 <u>Unity Feedback Control</u>[1]

The interpretation given in 3.1 suggests that the transformation $G(s) \rightarrow H(s) \triangleq \mathrm{diag}\{g_j(s)\}_{1\leq j\leq m}$ is a natural, physically meaningful transformation. It also suggests the choice of a diagonal controller $P_2K(s)P_1$ ie

$$K(s) \triangleq P_2^{-1}\ \mathrm{diag}\ \{k_j(s)\}_{1\leq j\leq m} P_1^{-1} \qquad (18)$$

which is a DTFM with real coefficients if $k_j(\bar{s}) \equiv \overline{k_{\ell(j)}(s)}$, $1\leq j\leq m$. The resulting closed-loop stability is easily assessed using the identity

$$\frac{\rho_c(s)}{\rho_o(s)} = |I+GK| = |I+HP_2KP_1| \equiv \prod_{j=1}^{m} (1+g_j(s)k_j(s)) \qquad (19)$$

In fact, the stability analysis reduces to the design of compensation elements $k_j(s)$ for each subsystem transfer function $g_j(s)$, $1\leq j\leq m$. This analysis is independent of P_1 and P_2 and has an essentially classical flavour. Closed-loop transient performance is, however, dependent on P_1 via the closed-loop TFM

$$H_c(s) \equiv P_1 \operatorname{diag}\left\{ \frac{g_j(s)k_j(s)}{1 + g_j(s)k_j(s)} \right\}_{1 \le j \le m} P_1^{-1} \tag{20}$$

The analysis of this relationship and its implications for the choice of $\{k_j(s)\}$ to achieve the design objectives is a new and essentially multivariable problem. Examples indicate[1] that the problems are surmountable and can be based on a few simple empirical guidelines.

The analysis can be extended to cope systematically with component failures[1-3] and, as such, forms a fairly complete design theory for dyadic systems. The resulting controller can be implemented in the normal manner or, if both P_1 and P_2 are real for example, as three cascaded blocks P_2^{-1}, $\operatorname{diag}\{k_j\}$ and P_1^{-1}.

Finally, the controller TFM is a DTFM with its own modal interpretation. A particularly neat modal interpretation of the control system is obtained by noting that

$$G(s)P_2^{-1}P_1^{-1}P_1P_2K(s) \equiv P_1P_2K(s)G(s)P_2^{-1}P_1^{-1} \tag{21}$$

and hence that P_1P_2K is a commutative controller[6] for $GP_2^{-1}P_1^{-1}$. In terms of definition 3 it is straightforward to verify that equations (19) and (20) remain unchanged if we set $K(s) = K_DK_1(s)$ with $K_1(s) = P_1\operatorname{diag}\{k_j(s)\}P_1^{-1}$ and that the use of the constant precompensator K_D enables the design of the commutative controller K_1 for GK_D.

4. Dyadic Approximations[1,3-5]

It is tempting to attempt to bridge the gap between the general theory of section 2 and the conceptual, analytical and design simplicity of dyadic systems by approximation of any mxm invertible plant by a dyadic system for the purpose of design studies. The success of the attempt[1] will, of course, depend on careful choice of definitions and parameters and the development of applicable stability theorems. A number of techniques have been suggested[1,3,7] and, in principle, there are an infinite number of possibilities. The approaches outlined below are perhaps the most general constructions that need be considered for applications.

4.1 Dyadic and Inverse Dyadic Approximations

With (P_1,P_2) permissible, assuming that $g_j(s) \triangleq H_{jj}(s) \not\equiv 0$, $1 \leq j \leq m$, then the DTFM

$$G_A(s) \triangleq P_1 \text{ diag } \{g_j(s)\}_{1\leq j\leq m} P_2 \tag{22}$$

obtained by neglecting the off-diagonal terms of H is the natural dyadic approximation[1] (DA) to G using (P_1,P_2). An alternative approach using the inverse system is to write

$$\hat{G}(s) = P_2^{-1} \hat{H}(s) P_1^{-1} \tag{23}$$

(where $\hat{L}(s)$ will denote the inverse of the TFM L(s)), when, if $g_j^{-1}(s) \triangleq \hat{H}_{jj}(s) \not\equiv 0$ $(1\leq j\leq m)$, equation (22) is the natural inverse dyadic approximation[1] (IDA) to G.

In both cases, the approximation is physically realizable in the sense that $G_A(\bar{s}) \equiv \overline{G_A(s)}$ by (6) and definition 1.

The remaining problems are, firstly, to consider the construction of useful criteria for the structure required for H and, secondly, to investigate the existence of permissible (P_1, P_2) achieving such structures. These problems are considered in the next two subsections.

4.2 Approximation and Diagonal Dominance:

In general terms the validity of the approximation can be checked by simulation or by abstract stability criteria[8,9]. The natural intuitive approach is to look for conditions on the off-diagonal terms of H (or $\hat{H}$) such that they are 'small enough' to allow the DA or IDA to be confidently used in the feedback stability analysis. The dynamics of G could then be regarded as being dominated by the m independent modal groupings represented by the transfer functions $g_j(s)$, $1 \leq j \leq m$.

The criterion for smallness is simply diagonal dominance[1,10,11]. A number of useful results can be derived[1] from the basic encirclement theorems underlying the inverse Nyquist array (INA) design method[1,10,11]. For example, the identity

$$|I_m + GK| \equiv |I_m + HP_2KP_1| \equiv \frac{|I_m + P_1^{-1}\hat{K}P_2^{-1}\hat{H}|}{|P_1^{-1}\hat{K}P_2^{-1}\hat{H}|} \qquad (24)$$

leads to the following stability theorem:

Theorem 1

Let D be the usual Nyquist contour in the complex plane and let n_c (resp. n_o) be the number of closed (resp. open)-loop poles in the interior of D. Suppose that both $P_1^{-1}\hat{K}(s)P_2^{-1}\hat{H}(s)$ and $I_m + P_1^{-1}\hat{K}(s)P_2^{-1}\hat{H}(s)$ are diagonally (row) dominant at each point s on D. Let the jth diagonal term of $P_1^{-1}\hat{K}(s)P_2^{-1}\hat{H}(s)$ map D onto the closed contour Γ_j encircling the (-1,0) point and origin of the complex plant $\hat{n}_j$ and $\tilde{n}_j$ times in a clockwise manner, $1 \le j \le m$. Then

$$n_c - n_o = \sum_{j=1}^{m} (\hat{n}_j - \tilde{n}_j) \tag{25}$$

Taking the case of $\hat{H}(s)$ diagonally (row) dominant on D and the controller (equation (18)) suggested by the IDA, the diagonal terms of $P_1^{-1}\hat{K}P_2^{-1}\hat{H}$ are simply $k_j^{-1}(s)g_j^{-1}(s)$, $1 \le j \le m$. Theorem 1 can now be interpreted as stating that the IDA can be confidently used in stability analysis if the dominance conditions are satisfied.

4.3 Approximations Exact at a Given Frequency (1,4,5,7)

The diagonal dominance of $\hat{H}$ (or H) is an intuitively reasonable objective in the choice of permissible (P_1,P_2) (and consequent DA or IDA). It is not reasonable, however, to expect that there will always exist P_1,P_2 such that dominance is achieved on the whole of D. It is however possible (under only weak conditions) to achieve diagonal dominance over an open frequency internal about any specified frequency point $s = i\omega_1$ (a somewhat surprising result!).

Theorem 2 (1,4)

If $G(i\omega_1)$ is finite and nonsingular, then there exists a permissible transformation $(P_1(\omega_1),P_2(\omega_1))$ such that $H(s,\omega_1) \triangleq P_1^{-1}(\omega_1)$ $G(s)P_2^{-1}(\omega_1)$ is diagonal at the point $s = i\omega_1$ if, and only if, the matrix

$$M(\omega_1) \triangleq G(-i\omega_1)G^{-1}(i\omega_1) \tag{26}$$

has a complete set of eigenvectors. $P_1(\omega_1)$ (resp. $P_2^{-1}(\omega_1)$) is then an eigenvector matrix of $M(\omega_1)$ (resp. $N(\omega_1) \triangleq G^{-1}(i\omega_1)G(-i\omega_1)$).

The theorem essentially states that it is, in general, possible to choose permissible P_1, P_2 such that H(and hence $\hat{H}$) are <u>diagonal</u> at a specified frequency point and hence (by continuity) diagonally dominant in some open interval containing that point. The DA (or IDA)

$$G_A(s,\omega_1) \triangleq P_1(\omega_1)\, \mathrm{diag}\{g_j(s,\omega_1)\}_{1\leq j\leq m}\, P_2(\omega_1) \tag{27}$$

derived from $P_1(\omega_1)$ and $P_2(\omega_1)$ is hence <u>exact</u> at the specified point $s=i\omega_1$ and hence can be regarded as a good representation of plant dynamics in an open interval surrounding this point. This intuitive idea can be formulated in terms of approximation of the characteristic transfer functions(12-14) $h_j(s)$ of $H(s,\omega_1)$ by noting that the natural approximations,(1,5)

$$h_j(s) \simeq g_j(s,\omega_1) \qquad ,1\leq j\leq m \tag{28}$$

are <u>exact</u> at $s=i\omega_1$ and in error at other frequency points with bounds specified by Gershgorin's theorem. These ideas are illustrated in Fig.3 for the case of m=2.

Finally, it is easily verified(1) that, if G(s) is a DTFM, then $G_A(s,\omega_1) \equiv G(s)$ (i.e. the best DA or IDA to a DTFM is the DTFM itself) indicating that the construction is a natural generalization of the concepts of section 3. The structural, modal and invariance properties of G_A (see definitions 2-4) can hence be regarded as reflections of structural, modal and invariance properties of G at (and in the vicinity of) the point $s=i\omega_1$!

5. Applications to Feedback Design

The theoretical concepts described above do not, in themselves, form a complete design technqiue although a number of highly successful possibilities can be described and derived by suitable choice of permissible P_1, P_2. This is obvious if the plant is dyadic when the design technique of section 3 can be highly successful(1-3).

5.1 Dyadic Approximation and the INA

If the plant is not dyadic, it is a natural first step to look for the existence of permissible P_1, P_2 such that $\hat{H}$ (say) is diagonally dominant on D. The transformations may be suggested by physical insight into process dynamics or deduced from computer algorithms. As indicated in section 4.2 the design could proceed to design a diagonal 'controller' $P_2K(s)P_1$ for the transformed plant using the INA technique.

The similarity to the INA can be strengthened by suggesting that (in the INA sense) there exists constant, nonsingular pre-and post-compensators L_1, L_2 respectively and that $L_2\hat{G}L_1$ is diagonally dominant on D. It follows that the choice of $P_1=L_1, P_2=L_2$ will ensure the dominance of $\hat{H}=P_2\hat{G}P_1$. For example, the choice of $P_1=I_m$ and P_2 by the pseudo-diagonalization procedure[10] could be successful. In this sense the concepts described in this paper can be regarded as an extension of the INA procedures to include permissible equivalence transformations. This class of transformations is much richer than the class of real pre-and postcompensators as is indicated (implicitly) in theorem 2 (which is not valid if we restrict attention to real transformations).

In many situations, it may be that we cannot find permissible P_1, P_2 to achieve dominance of $\hat{H}$ (or H). In such situations, origin shifts[1,15] could be involved and/or a precompensator introduced into the transformed controller $P_2K(s)P_1 \triangleq K_c(s)\ \mathrm{diag}\{\ k_j(s)\}_{1\leq j\leq m}$. Alternatively, the dyadic approximations can be used[1,5] in conjunction with characteristic locus methods.

5.2. Dyadic Approximation and Characteristic Loci[1,5]

A primary objective of characteristic locus design methods[12-14] is the systematic manipulation and compensation of the characteristic loci of the forward path TFM $Q(s)\triangleq G(s)K(s)$ by suitable choice of K. In this context, theorem 2 suggests a powerful and systematic technique[1,5].

Consider the basic problem of the choice of $K(s)$ to produce desired gain and phase characteristics of the characteristic loci in the vicinity of a specified frequency point $s=i\omega_1$. Suppose also that the conditions of theorem 2 are satisfied and that the permissible pair $(P_1(\omega_1)\ P_2(\omega_1))$ has been computed to produce the dyadic approximation $G_A(s,\omega_1)$. Consider the use of the dyadic controller (c.f. equation (18)) suggested by G_A.

$$K(s,\omega_1) \triangleq P_2^{-1}(\omega_1)\ \mathrm{diag}\{\ k_j(s,\omega_1)\}_{1\leqslant j\leqslant m}\ P_1^{-1}(\omega_1) \tag{29}$$

with $k_j(\bar{s}) \equiv \overline{k_{\ell(j)}(s)}$, $1\leqslant j\leqslant m$, to ensure physical realizability. The identity,

$$|I_m+GK| \equiv |I_m+ H(s,\omega_1)\ \mathrm{diag}\{\ k_j(s,\omega_1)\}_{1\leqslant j\leqslant m}| \tag{30}$$

and application of Gershgorins theorem indicates that the characteristic transfer functions $q_j(s)$, $1\leqslant j\leqslant m$, of Q are identical to those of $H\ \mathrm{diag}\{k_j\}$ and that the natural approximations

$$q_j(s) \simeq g_j(s,\omega_1)\ k_j(s,\omega_1) \qquad ,1\leqslant j\leqslant m \tag{31}$$

(obtained by taking the diagonal terms) are <u>exact</u> at $s=i\omega_1$ and in error at other points to an extent defined by the Gershgorin circles. Noting[1,5] that the relative magnitude of the circles are independent of the compensation elements and are also small in the vicinity of $s=i\omega_1$, it follows that equation (31) can be used with confidence to design the required gain and phase characteristics of the loci in an open frequency interval containing $s=i\omega_1$.

The above basic methodology has been used to formulate a systematic design technique[1]. The technique has the advantage of guaranteed and quantified accuracy in the analysis and design of compensation elements.

The eigenvector matrix of $Q(i\omega_1)$ is simply $P_1(\omega_1)$ amd cannot be included as a design parameter. This is in contrast to alternative technqiues[12,13] which put emphasis on eigenvector manipulation using the alignment concept. This has an intuitive relationship to high-frequency interaction but the success of the approximations used in the construction of the consequent 'approximately commutative controller' is not guaranteed. The two techniques do, however, have a well-defined relationship[16].

6. Conclusions: A Unified Approach

Although it has its origins in the modal structure of nuclear reactor spatial dynamics and the derived concept of a dyadic transfer function matrix, the concepts of dyadic approximation can be extended to provide a systematic and physically motivated framework for the description and derivation of design techniques based on reduction of the multivariable problem to a sequence of scalar design procedures.

In practice, the methodology has a close relationship to the use of real constant pre-and post-compensators in the inverse Nyquist array method. Although post-compensation is not a popular design tool, the example of dyadic systems indicates the power of the technqües amd demonstrates that it has a distinctive physical interpretation in terms of modal description of system I/O behaviour. In the authors opinion, it has great potential and should not be ignored. The techniques described here go much further by suggesting (with strong physical foundations) that complex, permissible transformations are valid physical concepts of great value in design. This is adequately demonstrated when the techniques are combined with the eigen/modal concepts of the characteristic locus method, yielding a systematic technique for manipulation and compensation of the loci in the vicinity of a specified frequency point. This approach generates a distinct design procedure that can have great success in practice[1,2,3,5,17].

The algebraic similarity of the INA and the techniques described suggest that they could usefully be unified in a single design package, based on the basic operations of permissible equivalence transformation and diagonal dominance checks. The same is also true of 'approximately commutative control'[12,13] which uses dyadic control elements of the form

$$K(s) = \hat{W}\, \mathrm{diag}\{k_j(s)\}_{1\leqslant j\leqslant m}\, \hat{V} \tag{32}$$

where $\hat{W}$ and $\hat{V}$ are real approximations to the eigenvector and inverse eigenvector matrices of G (say) at a specified frequency point. Comparing (32) with (18), the dyadic controller can be regarded as being induced by the real permissible transformation $(\hat{V}^{-1}, \hat{W}^{-1})$ and the diagonal controller diag $\{k_j\}$ as acting in the transformed plant $H = \hat{V}G\hat{W}$. In fact, the identity

$$|I_m+G(s)K(s)| \equiv |I_m+H(s)\,\mathrm{diag}\{\,k_j(s)\}_{1\leqslant j\leqslant m}| \tag{33}$$

suggests that the approximations inherent in approximately commutative control can be investigated via the diagonal dominance of H. More precisely if the approximation is exact, then H is diagonal at the specified frequency and, at other points, the characteristic loci of GK lie in the union of the Gershgorin circles centred on the diagonal element $H_{jj}(s)\, k_j(s)$ $,1\leqslant j\leqslant m$. In other cases, the magnitude of the circles will give an estimate of the error involved and the approximations $q_j(s)\simeq H_{jj}(s)k_j(s)$ could be used as a basis for design studies.

7. References

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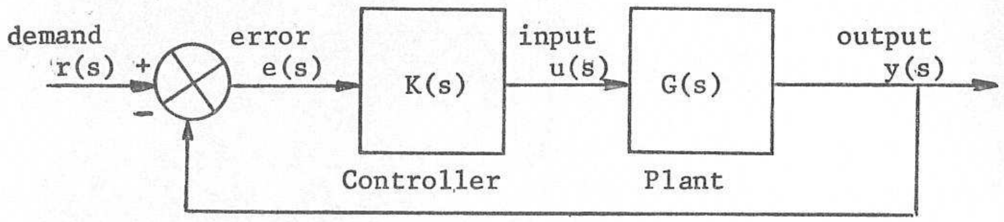

Fig. 1. Unity negative feedback system.

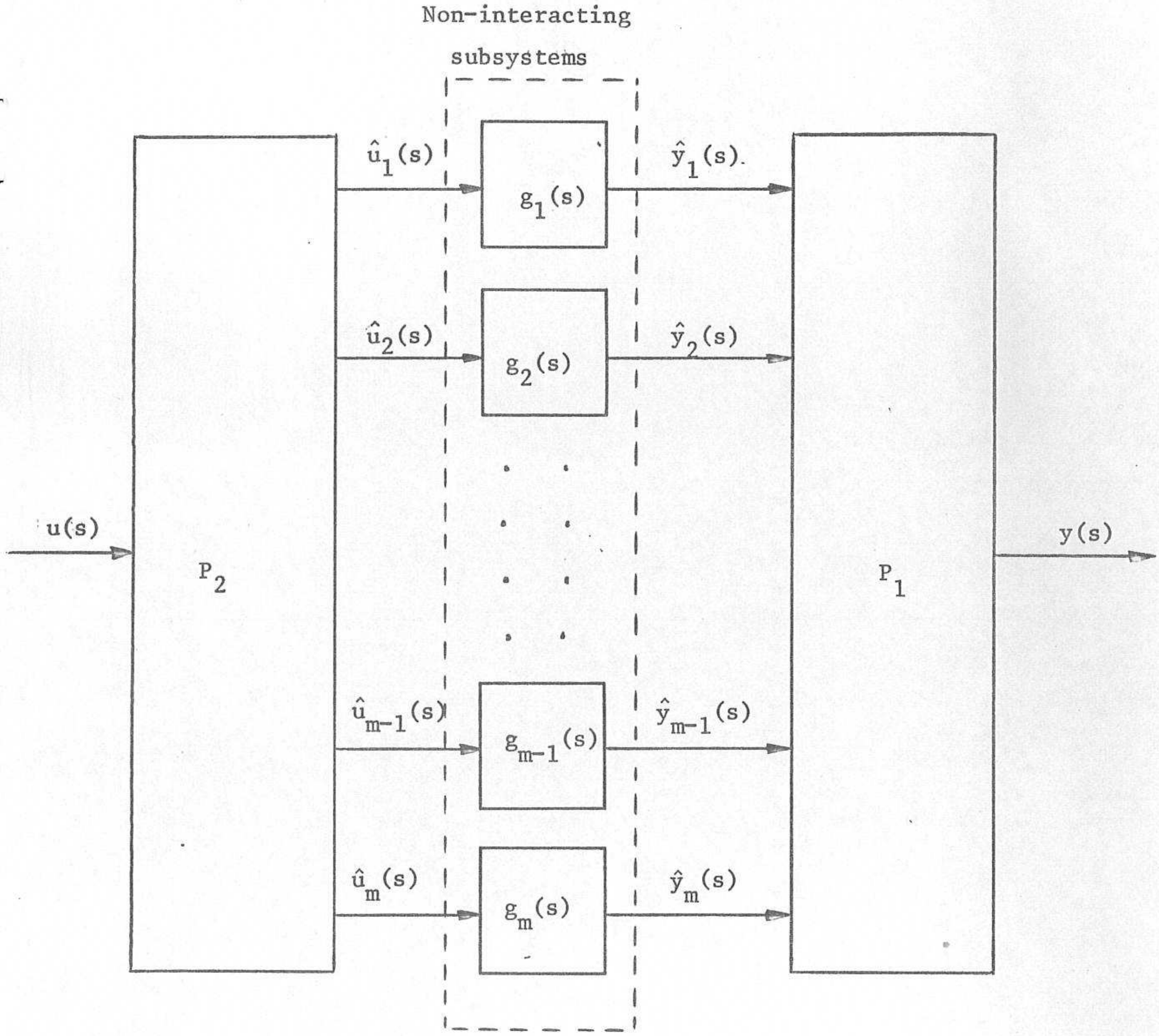

Fig. 2. Structural Decomposition of a Dyadic System

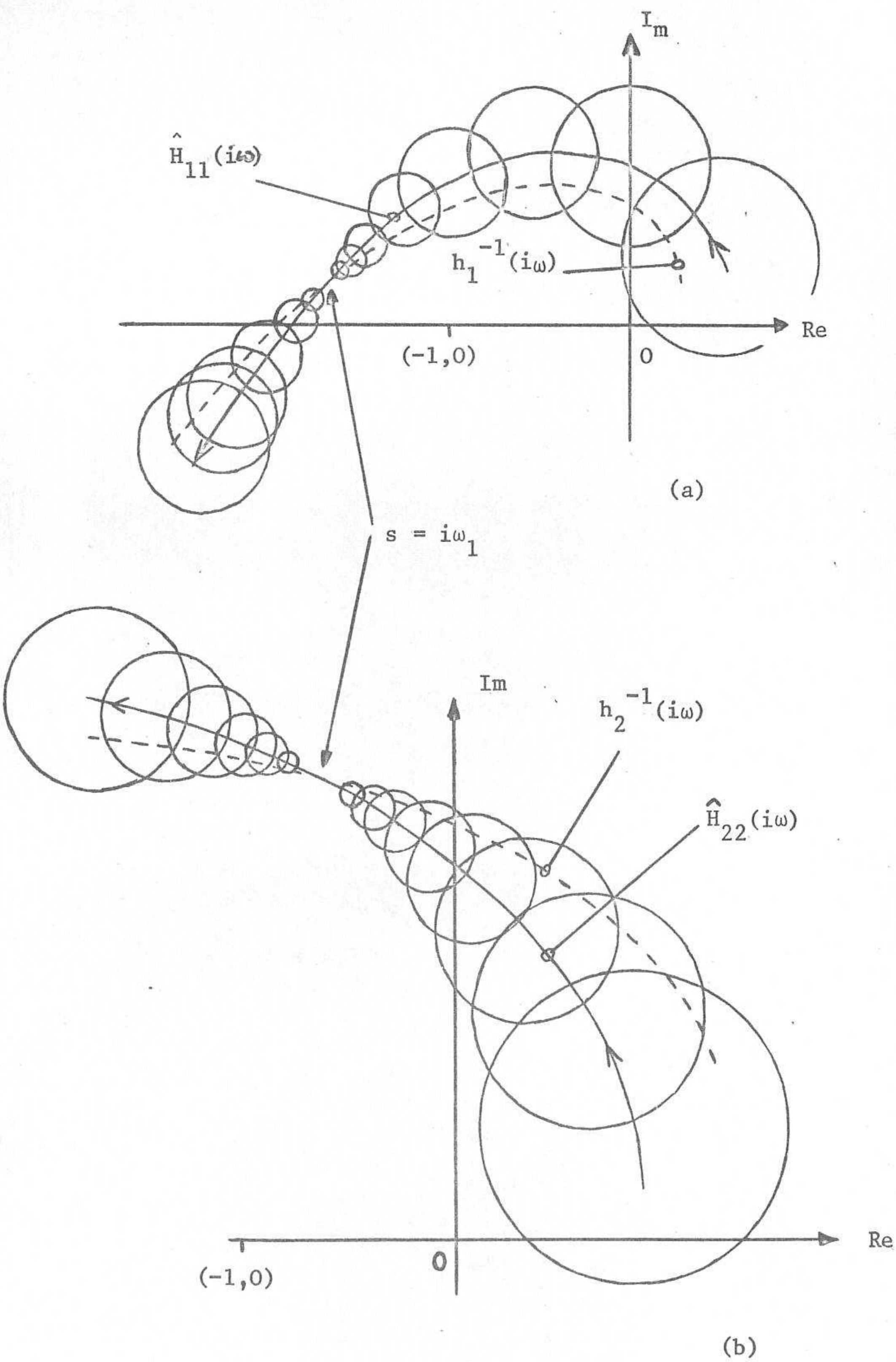

Fig. 3 Diagonal Terms of $\hat{H}(i\omega), \omega > o$, with Gershgorin Circles

(a) $\hat{H}_{11}(i\omega)$ and $h_1^{-1}(i\omega)$

(b) $\hat{H}_{22}(i\omega)$ and $h_2^{-1}(i\omega)$