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New Bound Characteristics of NARX Model in the Frequency Domain

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Abstract: New results about the bound characteristics of both the generalized frequency response functions (GFRFs) and the output frequency response for the NARX (Nonlinear AutoRegressive model with eXogenous input) model are established. It is shown that the magnitudes of the GFRFs and the system output spectrum can all be bounded by a polynomial function of the magnitude bound of the first order GFRF, and the coefficients of the polynomial are functions of the NARX model parameters. These new bound characteristics of the NARX model provide an important insight into the relationship between the model parameters and the magnitudes of the system frequency response functions, reveal the effect of the model parameters on the stability of the NARX model to a certain extent, and provide a useful technique to evaluate the truncation error in a Volterra series expression of nonlinear systems, and the highest order needed in the Volterra series approximation. A numerical example is given to demonstrate the effectiveness of the theoretical results.

Keywords: Bound characteristics, Frequency domain, Nonlinear systems, NARX

1 Introduction

The Nonlinear AutoRegressive model with eXogenous input (NARX) represents a wide class of nonlinear systems, and many well-known nonlinear input-output models are specific cases of this model (Chen and Billings 1989). Many research studies have been carried out for the modelling and analysis of nonlinear systems described by the NARX model (Chen, Billings, Cowan and Grant 1990, Dzielinski 1999). Based on the Volterra series approach of nonlinear systems, the NARX model has been analysed in the frequency domain, and significant results have been achieved. In George (1959), the concept of generalized frequency response functions (GFRFs) was proposed, which extended the well-known linear frequency response function to the nonlinear case. Since then considerable efforts had been focused on the GFRFs based nonlinear system frequency domain studies (Rugh 1981, George 1959, Bendat 1990, Chua and Ng 1979). In Peyton-Jones and Billings (1989), a recursive algorithm to compute the GFRFs of the NARX model was derived. Billings and Peyton-Jones (1990) extended the result to nonlinear systems described by nonlinear integro-differential equations. Swain and

Billings (2001) extended thereafter these results to the case of MIMO nonlinear systems. The output frequency characteristics of nonlinear systems were studied in Lang and Billings (1996) and Lang and Billings (1997). All these results form an important basis for further study of the analysis of nonlinear systems in the frequency domain.

In this paper, the bound characteristics of the GFRFs and the output spectrum of the NARX model are studied based on the original work in Zhang and Billings (1996) and Billings and Lang (1996). The magnitude bounds of the system frequency response functions are shown to be a polynomial function of the magnitude bound of the first order GFRF of the NARX model, and the coefficients of the polynomial are the functions of the model parameters. This provides a significant insight into the relationship between the model parameters and the system frequency response functions, though it can be regarded as an important extension of the previous work in Zhang and Billings (1996) and Billings and Lang (1996). Based on these new results, the truncation error associated with the Volterra series expression of nonlinear systems can be studied. Sufficient conditions for the BIBO stability of the NARX model can also be established. A numerical example is given to demonstrate the results.

2 The frequency response functions of nonlinear systems and the NARX model

Nonlinear systems with stable zero equilibrium point can be approximated in the neighbourhood of the equilibrium by the Volterra series

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \tag{1}$$

where $h_n(\tau_1,\cdots,\tau_n)$ is called the nth order Volterra kernel, which is a real valued function of $\tau_1,\cdots,\tau_n$, N is the maximum order of the system nonlinearity, which may need to be large enough to guarantee required accuracy of approximation. The output frequency response of the system can be described as (Lang and Billings 1996)

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_{\omega n} \tag{2}$$

where $\sigma_{\omega n}$ denotes a small unite in the n dimensional hyperplane $\omega_1+\cdots+\omega_n=\omega$, and

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \tag{3}$$

is the nth order GFRF of system (1). When the system is subject to a multi-tone input described by

$$u(t)=\sum_{i=1}^{K}|F_i|\cos(\omega_i t+\angle F_i) \tag{4}$$

the system output spectrum can be written as (Lang and Billings 1996):

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{5}$$

where,

$$F(\omega)=\begin{cases}|F_i|e^{j\angle F_i} & \text{if } \omega\in\{\omega_k, k=\pm1,\cdots,\pm K\}\\ 0 & \text{else}\end{cases} \tag{6}$$

The NARX model of nonlinear systems is given by

$$y(t)=\sum_{m=1}^{M} y_m(t) \tag{7a}$$

$$y_m(t)=\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=1}^{K} c_{p,q}(k_1,\cdots,k_{p+q})\prod_{i=1}^{p} y(t-k_i)\prod_{i=p+1}^{p+q} u(t-k_i) \tag{7b}$$

where $y_m(t)$ is the mth-order output of the system, and $p+q=m$, k_i=1,..., K, $\sum_{k_1,k_{p+q}=1}^{K}(\cdot)=\sum_{k_1=1}^{K}(\cdot)\cdots\sum_{k_{p+q}=1}^{K}(\cdot)$. The GFRFs for a specific nonlinear system can be derived by using the probing method in Rugh (1981). A recursive algorithm in (Peyton-Jones and Billings 1989) can be used to compute the GFRFs of the NARX model as follows:

$$\begin{aligned}&L_n(\omega)\cdot H_n(j\omega_1,\cdots,j\omega_n)\\ &=\sum_{k_1,k_n=1}^{K} c_{0,n}(k_1,\cdots,k_n)\exp(-j(\omega_1k_1+\cdots+\omega_nk_n))\\ &+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=1}^{K} c_{p,q}(k_1,\cdots,k_{p+q})\exp(-j(\omega_{n-q+1}k_{n-q+1}+\cdots+\omega_{p+q}k_{p+q}))H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})\\ &+\sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K} c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n)\end{aligned} \tag{8}$$

$$H_{n,p}(\cdot)=\sum_{i=1}^{n-p+1} H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)\exp(-j(\omega_1+\cdots+\omega_i)k_p) \tag{9}$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n)=H_n(j\omega_1,\cdots,j\omega_n)\exp(-j(\omega_1+\cdots+\omega_n)k_1) \tag{10}$$

where $L_n(\omega)=1-\sum_{k_1=1}^{K} c_{1,0}(k_1)\exp(-j\omega k_1)$ and $\omega=\omega_1+\cdots+\omega_n$. Moreover, $H_{n,p}(j\omega_1,\cdots,j\omega_n)$ in (9) can also be written as

$$H_{n,p}(j\omega_1,\cdots,j\omega_n)=\sum_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\prod_{i=1}^{p} H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})\exp(-j(\omega_{r_{X+1}}+\cdots+j\omega_{r_{X+r_i}})k_i),\ \text{where } X=\sum_{x=1}^{i-1} r_x \tag{11}$$

Based on equations (8)-(11), the GFRFs of the NARX model (7) of any order can be obtained. The objective of this study is to investigate the bound characteristics of the GFRFs and the output spectrum of nonlinear systems described by the NARX model to provide an important insight into the effects of the model parameters on these system frequency response functions. Note that the bounded-input bounded-output (BIBO) stability can be guaranteed by the frequency domain property of bounded-input and bounded-output spectrum. The bound characteristics of the NARX model are also significant for the system BIBO stability. Sufficient bounded stability criteria of the NARX model can be derived from the bound characteristics of the system output spectrum.

3 Bound characteristics of NARX model in the frequency domain

In this section, some notations and useful operators are introduced first. Then bound characteristics of the GFRFs of the NARX model are derived using these notations and operators. Finally, the bound characteristics of the system output spectrum are developed.

3.1 Notations and operators

Let $\underline{L} = \inf_{\omega \in I_\omega} \{|L_n(\omega)|\}$, where I_ω is the non-negative frequency region of the output spectrum of a NARX model. In what follows, let

$$C(p,q) = \begin{cases} \sum_{k_1,k_{p+q}=1}^{K} |c_{p,q}(k_1,\cdots,k_{p+q})|, & 1 \le q \le n-1, 1 \le p \le n-q \\ \sum_{k_1,k_n=1}^{K} |c_{0,n}(k_1,\cdots,k_n)|, & q=n, p=0 \\ \sum_{k_1,k_p=1}^{K} |c_{p,0}(k_1,\cdots,k_p)|, & q=0, 2 \le p \le n \\ 0, & else \end{cases} \tag{12}$$

Obviously, $C(p,q)$ is a nonnegative function of the coefficients $c_{pq}(.)$ defined on all $0 \le p,q \le n$. Moreover, let

$$\begin{cases} \overline{H}_{n,p} = \sup_{\omega_1 \cdots \omega_n \in R_\omega} (|H_{n,p}(\cdot)|), H_{0,0}(\cdot) = 1 \\ H_{n,0}(\cdot) = 0 \text{ for } n > 0 \\ H_{n,p}(\cdot) = 0 \text{ for } n < p \\ \overline{H}_n = \sup_{\omega_1 \cdots \omega_n \in R_\omega} (|H_n(\cdot)|) \end{cases} \tag{13}$$

where R_ω is the input frequency range of a NARX model.

In order to develop the bounded characteristics of the GFRFs of the NARX model, define two operators as follows. Consider two polynomials of degree n and m respectively,

$$f_a = a_0 + a_1 h + \cdots + a_n h^n = a \cdot \hbar_n^T, \text{ and } f_b = b_0 + b_1 h + \cdots + b_n h^m = b \cdot \hbar_m^T$$

where the coefficients $a_0, a_1, \ldots, a_n$; $b_0, b_1, \ldots, b_m$ are all real numbers, h stands for a real or complex valued function, a=[a_0, a_1, …, a_n], b=[b_0, b_1,…, b_m], and $\hbar_i$=[1, h, …, h^i].

Define a multiplication operator "$\otimes$" as

$a \otimes b = c$, where c is an n+m+1-dimension vector, $c(k)$= $\sum_{\substack{i+j=k \\ 0 \le i \le n, 0 \le j \le n}} a_i b_j$ for $0 \le k \le m+n$.

Denote $(a \otimes b)(k) = \sum_{\substack{i+j=k \\ 0 \le i \le n, 0 \le j \le m}} a_i b_j$. From this operator it follows that, for example, $f_a \cdot f_b = a \otimes b \cdot \hbar_{n+m}^T$. Similarly, define an addition operator "$\oplus$" as

$a \oplus b = c$, where c is an x-dimension vector, x=max{m,n}, c(k)=a(k)+b(k) for $0 \le k \le x$.
If k>n or m, then $a(k)$=0 or $b(k)$=0, accordingly.
From the operator"$\oplus$" it follows that, for example, $f_a + f_b = a \oplus b \cdot \hbar_{\max(n,m)}^T$. Moreover, let $\underset{(*)}{\otimes}(\cdot)$ and $\underset{(*)}{\oplus}(\cdot)$ denote the multiplication and addition in terms of the operator"$\otimes$" and "$\oplus$"for the series (.) satisfying (*), respectively.

3.2 Bound characteristics of the GFRFs

The bound characteristics of the GFRFs are derived in this section. A preliminary result is given in Lemma 1, which shows that the magnitude bound of the nth order GFRF can be recursively determined from the magnitude bounds of the lower order GFRFs. Then based on Lemma 1, Theorem 1 is established which describes the magnitude bound of the GFRFs as a polynomial function of the magnitude bound of the first order GFRF $H_1(j\omega)$.

Lemma 1. $$\overline{H}_n \le \frac{1}{\underline{L}} \sum_{m=2}^{n} \sum_{\substack{p+q=m \\ 0\le p,q\le m}} C(p,q)\overline{H}_{n-q,p}$$

$$\overline{H}_{n-q,p} \le \sup \left| \sum_{\substack{r_1\cdots r_p=1 \\ \sum r_i=n-q}}^{n-q-p+1} \prod_{i=1}^{p} H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}}) \right| = \sum_{\substack{r_1\cdots r_p=1 \\ \sum r_i=n-q}}^{n-q-p+1} \prod_{i=1}^{p} \overline{H}_{r_i}$$

where, $n>1$, $X=\sum_{x=1}^{i-1} r_x$, $\sum_{\substack{p+q=m \\ 0\le p,q\le m}}(\cdot)$ or $\sum_{\substack{p,q=0 \\ p+q=m}}^{m}(\cdot)$ denotes the sum of the corresponding terms with respect to all the combinations of (p,q) satisfying $p+q=m$ and $0\le p,q\le m$. ■

Note that $0\le p,q\le m$ denotes that $0\le p\le m$ and $0\le q\le m$, and $r_1\cdots r_p=1$ means that $r_1=1,\cdots,r_p=1$.

Proof of Lemma 1. From (8)(12)(13), and noting $\underline{L}$ is the lower bound of $L_n(\omega)$, it follows

$$\begin{aligned}
|H_n(j\omega_1,\cdots,j\omega_n)| &\le \frac{1}{\underline{L}} \sum_{k_1,k_n=1}^{K} |c_{0,n}(k_1,\cdots,k_n)||H_{0,0}(j\omega_1,\cdots,j\omega_n)| \\
&+\frac{1}{\underline{L}} \sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_n=1}^{K} |c_{p,q}(k_1,\cdots,k_{p+q})||H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})| + \frac{1}{\underline{L}}\sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K} |c_{p,0}(k_1,\cdots,k_p)||H_{n,p}(j\omega_1,\cdots,j\omega_n)| \\
&\le \frac{1}{\underline{L}} C(0,n)\overline{H}_{0,0} + \frac{1}{\underline{L}}\sum_{q=1}^{n-1}\sum_{p=1}^{n-q} C(p,q)\overline{H}_{n-q,p} + \frac{1}{\underline{L}}\sum_{p=2}^{n} C(p,0)\overline{H}_{n,p} \\
&= \frac{1}{\underline{L}}\sum_{q=0}^{n}\sum_{p=0}^{n-q} C(p,q)\overline{H}_{n-q,p}
\end{aligned} \tag{14}$$

It can be easily seen that $\sum_{q=0}^{n}\sum_{p=0}^{n-q} C(p,q)\overline{H}_{n-q,p}$ includes all the permutations of (p,q) satisfying $p+q=m$, $0\le p,q\le m$, and m=2,…,n. Hence, it follows $\sum_{q=0}^{n}\sum_{p=0}^{n-q} C(p,q)\overline{H}_{n-q,p} = \sum_{m=2}^{n}\sum_{\substack{p+q=m \\ 0\le p,q\le m}} C(p,q)\overline{H}_{n-q,p}$

From (11), it can be derived that

$$\overline{H_{np}} = \sup|H_{np}(j\omega_1,\cdots,j\omega_n)| = \sup\left| \sum_{\substack{r_1\cdots r_p=1 \\ \sum r_i=n}}^{n-p+1} \prod_{i=1}^{p} H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})\exp\left(-j(\omega_{r_{X+1}}+\cdots+\omega_{r_{X+r_i}})k_i\right) \right|$$

$$\le \sup \sum_{\substack{r_1 \cdots r_p = 1 \\ \sum r_i = n}}^{n-p+1} \prod_{i=1}^{p} \left| H_{r_i}(j\omega_{r_{X+1}}, \cdots, j\omega_{r_{X+r_i}}) \right| = \sum_{\substack{r_1 \cdots r_p = 1 \\ \sum r_i = n}}^{n-p+1} \prod_{i=1}^{p} \overline{H}_{r_i}$$

Therefore $$\overline{H}_{n-q,p} \le \sup \sum_{\substack{r_1 \cdots r_p = 1 \\ \sum r_i = n-q}}^{n-q-p+1} \prod_{i=1}^{p} \left| H_{r_i}(j\omega_{r_{X+1}}, \cdots, j\omega_{r_{X+r_i}}) \right| = \sum_{\substack{r_1 \cdots r_p = 1 \\ \sum r_i = n-q}}^{n-q-p+1} \prod_{i=1}^{p} \overline{H}_{r_i} .$$

This completes the proof. ■

Although Lemma 1 shows essentially the same result as those obtained in Zhang and Billings (1996) and Billings and Lang (1996), Lemma 1 provides a general expression for the magnitude bound of the nth-order GFRF in terms of the model parameters and $\overline{H}_1 \cdots \overline{H}_{n-1}$, and compared with the result in Billings and Lang (1996), it is much simpler in form and derived in a more systematic approach. Based on Lemma 1 and by using the new operators defined in section 3.1, a more comprehensive result about the bound of the GFRFs of the NARX model can be obtained.

Theorem 1. Consider the nth-order GFRF for the NARX model (7). There exists a series of scalar positive real numbers $b_{n0}, b_{n1}, \cdots, b_{nn}$, such that

$$\left| H_n(j\omega_1, \cdots, j\omega_n) \right| \le b_{n0} + b_{n1}\overline{H}_1 + b_{n2}\overline{H}_1{}^2 + \cdots + b_{nn}\overline{H}_1{}^n \tag{15a}$$

where the coefficients $b_{n0}, b_{n1}, \ldots, b_{nn}$ can be recursively determined as follows (denote $b_n = [b_{n0} \quad b_{n1} \quad \cdots \quad b_{nn}]$):

$$b_{nk} = \frac{1}{L}C(k, n-k) + \frac{1}{L}\left(\mathop{\oplus}_{m=2}^{n-1} \mathop{\oplus}_{\substack{p+q=m \\ 0 \le p,q \le m}} \left(C(p,q) \cdot \mathop{\oplus}_{\substack{\sum r_i = n-q \\ 1 \le r_1 \cdots r_p \le n-m+1}} \left(\mathop{\otimes}_{i=1}^{p} b_{r_i} \right) \right) \right)(k) \text{ for } 0 \le k \le n \tag{15b}$$

$$b_2 = [b_{20}, b_{21}, b_{22}] = \left[\frac{1}{L}C(0,2), \frac{1}{L}C(1,1), \frac{1}{L}C(2,0) \right] \tag{15c}$$

$$b_1 = [b_{10}, b_{11}] = [0,1] \tag{15e}$$

Moreover, $\mathop{\otimes}_{i=1}^{p} b_{r_i} = 0$ if $p<1$, and $\mathop{\oplus}_{m=2}^{n}(\cdot) = 0$ if $n<2$.

Proof. Use the induction method. For the second and third order GFRFs, it is easy to obtain from Lemma 1 that

$$\left| H_2(j\omega_1, j\omega_2) \right| \le \frac{1}{L} \sum_{m=2}^{2} \sum_{\substack{p+q=m \\ 0 \le p,q \le m}} C(p,q)\overline{H}_{2-q,p} = \frac{1}{L}\left(C(0,2) + C(1,1)\overline{H}_{1,1} + C(2,0)\overline{H}_{2,2} \right)$$

$$= \frac{1}{L}\left(C(0,2) + C(1,1)\overline{H}_1 + C(2,0)\overline{H}_1{}^2 \right) = b_2 \cdot h_2^T$$

$$\left| H_3(j\omega_1, j\omega_2, j\omega_3) \right| \le \frac{1}{L} \sum_{m=2}^{3} \sum_{\substack{p+q=m \\ 0 \le p,q \le m}} C(p,q)\overline{H}_{3-q,p}$$

$$= \frac{1}{L}\left(\begin{aligned} &\left(C(0,3) + \frac{1}{L}C(1,1)C(0,2) \right) + \left(C(1,2) + \frac{1}{L}C(1,1)^2 + \frac{2}{L}C(2,0)C(0,2) \right)\overline{H}_1 \\ &+ \left(C(2,1) + \frac{1}{L}C(1,1)C(2,0) + \frac{2}{L}C(2,0)C(1,1) \right)\overline{H}_1{}^2 + \left(C(3,0) + \frac{2}{L}C(2,0)^2 \right)\overline{H}_1{}^3 \end{aligned} \right)$$

$$= b_3 \cdot h_3^T$$

Hence, the theorem holds for n=2 and 3. Consider the nth order GFRF under the assumption that the theorem holds for all the GFRFs of orders less than n. From Lemma 1,

$$|H_n(j\omega_1,\cdots,j\omega_n)| \le \frac{1}{\underline{L}}\sum_{m=2}^{n}\sum_{\substack{p+q=m \\ 0\le p,q\le m}} C(p,q)\sup\left(\sum_{\substack{1\le r_1\cdots r_p\le n-m+1 \\ \sum r_i=n-q}}\prod_{i=1}^{p}|H_{r_i}(.)|\right) \tag{16}$$

Note $1\le n-m+1\le n-1$ and $0\le\sum r_i = n-q\le n$, each $|H_{r_i}(.)|$ is bounded by a polynomial of the form of (15a) with degree $r_i(\le n-1)$, and $\prod_{i=1}^{p}|H_{r_i}(.)|$ is therefore bounded by a polynomial of the form (15a) with degree n-q ($\le n$). It follows from inequality (16) that $|H_n(j\omega_1,\cdots,j\omega_n)|$ must be bounded by a polynomial of the form (15a) with degree n.

The explicit expression for the coefficients in (15a) is derived as follows. It follows from (16) that

$$|H_n(j\omega_1,\cdots,j\omega_n)| \le \frac{1}{\underline{L}}\sum_{\substack{p+q=n \\ 0\le p,q\le n}} C(p,q)\left(\sup\sum_{\substack{1\le r_1\cdots r_p\le 1 \\ \sum r_i=n-q}}\prod_{i=1}^{p}|H_{r_i}(.)|\right)+\frac{1}{\underline{L}}\sum_{m=2}^{n-1}\sum_{\substack{p+q=m \\ 0\le p,q\le m}} C(p,q)\left(\sup\sum_{\substack{1\le r_1\cdots r_p\le n-m+1 \\ \sum r_i=n-q}}\prod_{i=1}^{p}|H_{r_i}(.)|\right)$$

$$= \frac{1}{\underline{L}}\left(C(0,n)+C(1,n-1)\overline{H}_1+\cdots+C(n,0)\overline{H}_1^{\,n}\right)+\frac{1}{\underline{L}}\sup\sum_{m=2}^{n-1}\sum_{\substack{p+q=m \\ 0\le p,q\le m}} C(p,q)\left(\sum_{\substack{1\le r_1\cdots r_p\le n-m+1 \\ \sum r_i=n-q}}\prod_{i=1}^{p}|H_{r_i}(.)|\right) \tag{17}$$

Because

$$|H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})| \le b_{r_i,0}+b_{r_i,1}\overline{H}_1+\cdots+b_{r_i,r_i}\overline{H}_1^{\,r_i} = b_{r_i}\cdot h_{r_i}^T \quad \text{for } 1\le r_i\le n-m+1$$

where $b_{r_i}=[b_{r_i,0}\ \ b_{r_i,1}\ \ \cdots\ \ b_{r_i,r_i}]$ and $h_{r_i}=[1\ \ \overline{H}_1\ \ \cdots\ \ \overline{H}_1^{\,r_i}]$, it can be derived by using the operators "$\otimes$" and "$\oplus$" that

$$\sum_{\substack{1\le r_1\cdots r_p\le n-m+1 \\ \sum r_i=n-q}}\prod_{i=1}^{p}|H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})| = \sum_{\substack{1\le r_1\cdots r_p\le n-m+1 \\ \sum r_i=n-q}}\bigotimes_{i=1}^{p} b_{r_i}\cdot h_{n-q} = \left(\bigoplus_{\substack{\sum r_i=n-q \\ 1\le r_1\cdots r_p\le n-m+1}}\left(\bigotimes_{i=1}^{p} b_{r_i}\right)\right)\cdot h_{n-q}$$

Therefore,

$$\sum_{m=2}^{n-1}\sum_{\substack{p+q=m \\ 0\le p,q\le m}} C(p,q)\left(\sum_{\substack{1\le r_1\cdots r_p\le n-m+1 \\ \sum r_i=n-q}}\prod_{i=1}^{p}|H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})|\right) = \left(\bigoplus_{m=2}^{n-1}\bigoplus_{\substack{p+q=m \\ 0\le p,q\le m}}\left(C(p,q)\cdot\bigoplus_{\substack{\sum r_i=n-q \\ 1\le r_1\cdots r_p\le n-m+1}}\left(\bigotimes_{i=1}^{p} b_{r_i}\right)\right)\right)\cdot h_n$$

and (17) can be written as

$$|H_n(j\omega_1,\cdots,j\omega_n)| \le \frac{1}{\underline{L}}\left(C(0,n)+C(1,n-1)\overline{H}_1+\cdots+C(n,0)\overline{H}_1^{\,n}\right)+\frac{1}{\underline{L}}\left(\bigoplus_{m=2}^{n-1}\bigoplus_{\substack{p+q=m \\ 0\le p,q\le m}}\left(C(p,q)\cdot\bigoplus_{\substack{\sum r_i=n-q \\ 1\le r_1\cdots r_p\le n-m+1}}\left(\bigotimes_{i=1}^{p} b_{r_i}\right)\right)\right)\cdot h_n$$

This proves equation (15b). (15c) follows from the first two steps of the recursive computation. The proof of Theorem 1 is thus completed. ■

Theorem 1 throws that the magnitude of the nth-order GFRF can be bounded by a polynomial function of the magnitude bound of the first order GFRF $H_1(j\omega_1)$ of degree n, and the coefficients of the polynomial are the functions of the model parameters. This reveals an explicit relationship between the NARX model parameters and the magnitude bound of the nth-order GFRF, and is therefore important for the system analysis. From Theorem 1, the magnitude bounds of any order GFRFs for the NARX model can readily be computed from the model parameters and the first order GFRF.

3.3 Bound Characteristics of the output spectrum

Based on Theorem 1, a bound function in polynomial form can be derived for the system output spectrum in terms of the magnitude bound of $H_1(j\omega_1)$, and a sufficient condition for the convergence of the bound function can be obtained in terms of the system model parameters which can guarantee the BIBO stability of the NARX model. The results for the boundedness of the output spectrum of the NARX model (7) when subject to a general input are given in the following theorem.

Theorem 2. Assume the input of the NARX model (7) is a general input with spectrum $U(j\omega)$ defined by $U(j\omega)=\begin{cases}U(j\omega) & \omega\in R_\omega \\ 0 & \text{otherwise}\end{cases}$. Then the output spectrum of the NARX model is bounded by

$$|Y(j\omega)|\le \mathop{\oplus}_{n=1}^{N}\frac{1}{(2\pi)^{n-1}}\cdot b_n\cdot h_n^T\cdot \underbrace{|U|*\cdots*|U(j\omega)|}=\left(\mathop{\oplus}_{n=1}^{N}\alpha_n b_n\right)\cdot h_N^T \tag{18a}$$

and the series on the right side of (18a) is convergent if the model parameters satisfy

$$\lim_{\substack{N\to\infty\\ k\to\infty}}\sqrt[k]{\left(\mathop{\oplus}_{n=1}^{N}\alpha_n b_n\right)(k)}<\frac{1}{\overline{H}_1} \tag{18b}$$

In (18a,b), $h_N=[1\ \ \overline{H}_1\ \ \cdots\ \ \overline{H}_1{}^N]$, $b_n=[b_{n0}\ \ b_{n1}\ \ \cdots\ \ b_{nn}]$, $\alpha_n=(2\pi)^{1-n}\underbrace{|U|*\cdots*|U(j\omega)|}_{n}$, and

$\underbrace{|U|*\cdots*|U(j\omega)|}_{n}=\frac{1}{\sqrt{n}}\int_{\omega_1+\cdots+\omega_n=\omega}\prod_{i=1}^{n}|U(j\omega_i)|d\sigma_{\omega n}$.

Proof. It can be derived from equation (2) that

$$|Y(j\omega)|\le\sum_{n=1}^{N}\left|\frac{|H_n(j\omega_1^*,\cdots,j\omega_n^*)|}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega\right|\le\sum_{n=1}^{N}\frac{|H_n(j\omega_1^*,\cdots,j\omega_n^*)|}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}\prod_{i=1}^{n}|U(j\omega_i)|d\sigma_{\omega n}$$
$$=\sum_{n=1}^{N}\frac{1}{(2\pi)^{n-1}}|H_n(j\omega_1^*,\cdots,j\omega_n^*)|\underbrace{|U|*\cdots*|U(j\omega)|}_{n} \tag{19}$$

where $(j\omega_1^*,\cdots,j\omega_n^*)$ is a point on the hyper-plane $\omega_1+\cdots+\omega_n=\omega$. According to Theorem 1,

$$|H_n(.)|\le b_n\cdot h_n^T=b_{n0}+b_{n1}\overline{H}_1+b_{n2}\overline{H}_1{}^2+\cdots+b_{nn}\overline{H}_1{}^n$$

Thus using the operator "$\oplus$", inequality (19) yields

$$|Y(j\omega)| \le \overset{N}{\underset{n=1}{\oplus}} \frac{1}{(2\pi)^{n-1}} \cdot b_n \cdot h_n^T \cdot |U| * \cdots * |U(j\omega)| = \left(\overset{N}{\underset{n=1}{\oplus}} \alpha_n b_n \right) \cdot h_N^T$$

which can be rewritten as

$$|Y(j\omega)| \le \overline{Y} = \left(\overset{N}{\underset{n=1}{\oplus}} \alpha_n b_n \right)(0) + \left(\overset{N}{\underset{n=1}{\oplus}} \alpha_n b_n \right)(1)\overline{H}_1 + \left(\overset{N}{\underset{n=1}{\oplus}} \alpha_n b_n \right)(2)\overline{H}_1^2 + \cdots + \left(\overset{N}{\underset{n=1}{\oplus}} \alpha_n b_n \right)(k)\overline{H}_1^k + \cdots \quad (20)$$

The bound of the output spectrum is in general an infinite series as given by (20). The convergence of the series guarantees the stability of the NARX model. According to Cauchy's criterion (Weisstein 1999) for convergence, a sufficient condition for the convergence of the series in (20) is $\lim_{\substack{N\to\infty \\ k\to\infty}} \sqrt[k]{\left(\overset{N}{\underset{n=1}{\oplus}} \alpha_n b_n \right)(k)\overline{H}_1^k} = \overline{H}_1 \lim_{k\to\infty} \sqrt[k]{\left(\overset{\infty}{\underset{n=1}{\oplus}} \alpha_n b_n \right)(k)} < 1$. This completes the proof. ■

Note that in Theorem 2, $b_n = [b_{n0} \quad b_{n1} \quad \cdots \quad b_{nn}]$ can be determined according to Theorem 1, and $\underbrace{|U| * \cdots * |U(j\omega)|}_{n} = \frac{1}{\sqrt{n}} \int_{\omega_1 + \cdots + \omega_n = \omega} \prod_{i=1}^{n} |U(j\omega_i)| d\sigma_{\omega n}$ can be calculated by an algorithm given in Billings and Lang (1996). Similarly, the following result can be obtained for the output spectrum of the NARX model (7) when the input is a multi-tone signal.

Theorem 3. Assume the input of the NARX model (7) is the multi-tone signal (4). Then the output spectrum of the NARX model is bounded by

$$|Y(j\omega)| \le \overset{N}{\underset{n=1}{\oplus}} \left(2^{-n} \cdot b_n \cdot h_n^T \cdot \sum_{\omega_{k_1} + \cdots + \omega_{k_n} = \omega} |F(\omega_{k_1}) \cdots F(\omega_{k_n})| \right) = \left(\overset{N}{\underset{n=1}{\oplus}} \beta_n b_n \right) \cdot h_N^T \quad (21a)$$

and the series on the right side of (21a) is convergent if the system model parameters satisfy

$$\lim_{\substack{N\to\infty \\ k\to\infty}} \sqrt[k]{\left(\overset{N}{\underset{n=1}{\oplus}} \beta_n b_n \right)(k)} < \frac{1}{\overline{H}_1} \quad (21b)$$

In (21a,b) $h_N = [1 \quad \overline{H}_1 \quad \cdots \quad \overline{H}_1^{\,N}]$, $b_n = [b_{n0} \quad b_{n1} \quad \cdots \quad b_{nn}]$ which can be determined according to Theorem 1, $\beta_n = 2^{-n} \sum_{\omega_{k_1} + \cdots + \omega_{k_n} = \omega} |F(\omega_{k_1}) \cdots F(\omega_{k_n})|$.

Proof. From equation (5), it follows that

$$|Y(j\omega)| \le \sum_{n=1}^{N} \frac{1}{2^n} \sum_{\omega_{k_1} + \cdots + \omega_{k_n} = \omega} |H_n(j\omega_{k_1}, \cdots, j\omega_{k_n})| |F(\omega_{k_1}) \cdots F(\omega_{k_n})|$$

$$\le \sum_{n=1}^{N} \left(|H_n(j\omega_{k_1}, \cdots, j\omega_{k_n})| \cdot 2^{-n} \sum_{\omega_{k_1} + \cdots + \omega_{k_n} = \omega} |F(\omega_{k_1}) \cdots F(\omega_{k_n})| \right)$$

According to Theorem 1, and following a similar process as the proof of Theorem 2, the conclusion of the theorem can be reached. ■

In order to illustrate the results above, consider a specific but frequently encountered case of the NARX model (7). When there are only pure output nonlinearities in (7), the NARX model is

$$y(t)=\sum_{m=p=1}^{M}\left(\sum_{k_1,k_p=1}^{K}c_{p,0}(k_1,\ldots,k_p)\prod_{i=1}^{p}y(t-k_i)+\delta(m-1)\sum_{k_1=1}^{K}c_{0,1}(k_1)u(t-k_1)\right) \tag{22}$$

where $\delta(m)=\begin{cases}1, & m=0\\ 0, & \text{else}\end{cases}$. For many engineering systems, this model can be regarded as a general linear/nonlinear state feedback system, and consequently has significance in the analysis and synthesis of feedback control systems in practical applications (Jing, Lang and Billings 2006). When the input is only a sinusoidal signal $u(t)=F_d\sin(\omega_0 t)\quad(F_d>0)$, then $F(\omega_{k_l})=-jk_lF_d$ for $k_l=\pm1,\ \omega_{k_l}=k_l\omega_0$, and $l=1,\cdots,n$ in (5). In this case, the following result can be achieved.

Corollary 1. Assume the nonlinear system described by NARX model (22) is subject to the input signal $u(t)=F_d\sin(\omega_0 t)\quad(F_d>0)$. The nth-order GFRF for this nonlinear system is bounded by

$$|H_n(j\omega_1,\cdots,j\omega_n)|\le b_{nn}\overline{H}_1^{\,n} \tag{23a}$$

and the output spectrum of the NARX model is bounded by

$$|Y(j\omega)|\le\sum_{n=0}^{\lfloor N-1/2\rfloor}C_{2n+1}^{n}\left(\frac{F_d}{2}\right)^{2n+1}b_{2n+1,2n+1}\overline{H}_1^{\,2n+1} \tag{23b}$$

which is convergent if the system model parameters satisfy

$$\lim_{n\to\infty}\sqrt[2n+1]{C_{2n+1}^{n}b_{2n+1,2n+1}}<\frac{2}{F_d\overline{H}_1} \tag{23c}$$

where $b_{nn}=\frac{1}{\underline{L}}C(n,0)+\frac{1}{\underline{L}}\sum_{m=2}^{n-1}\left(C(m,0)\sum_{\substack{\sum r_i=n\\ 1\le r_1\cdots r_p\le n-m+1}}\prod_{i=1}^{m}b_{r_ir_i}\right)$, $\lfloor\cdot\rfloor$ is to take the integer part of (.).

Proof. According to (15b) in Theorem 1,

$$b_{nk}=\frac{1}{\underline{L}}\left(\bigoplus_{m=2}^{n-1}C(m,0)\bigoplus_{\substack{\sum r_i=n\\ 1\le r_1\cdots r_p\le n-m+1}}\left(\bigotimes_{i=1}^{m}b_{r_i}\right)\right)(k)\ \text{ for } 0\le k<n \tag{24a}$$

$$b_{nn}=\frac{1}{\underline{L}}C(n,0)+\frac{1}{\underline{L}}\left(\bigoplus_{m=2}^{n-1}C(m,0)\bigoplus_{\substack{\sum r_i=n\\ 1\le r_1\cdots r_p\le n-m+1}}\left(\bigotimes_{i=1}^{m}b_{r_i}\right)\right)(n) \tag{24b}$$

Note b_1=[0,1] and $b_2=\left[0,0,\frac{1}{\underline{L_2(\omega)}}C(2,0)\right]$. It is easy to show that $b_{nk}=0$ for $0\le k<n$ in (24a). Hence (24b) can be written as

$$b_{nn}=\frac{1}{\underline{L}}C(n,0)+\frac{1}{\underline{L}}\sum_{m=2}^{n-1}\left(C(m,0)\sum_{\substack{\sum r_i=n\\ 1\le r_1\cdots r_p\le n-m+1}}\prod_{i=1}^{m}b_{r_ir_i}\right) \tag{24c}$$

Hence, from Theorem 1 $|H_n(j\omega_1,\cdots,j\omega_n)|\le b_{nn}\overline{H}_1^{\,n}$. From (21a), it follows that

$$|Y(j\omega)|\le\left(\bigoplus_{n=1}^{N}\beta_nb_n\right)\cdot h_N^T=\sum_{n=1}^{N}\beta_nb_{nn}\overline{H}_1^{\,n} \tag{25}$$

Note that, when the input is a single tone function,

$$\beta_n = 2^{-n} \sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} \left|F(\omega_{k_1})\cdots F(\omega_{k_n})\right| = \begin{cases} (\frac{F_d}{2})^n \sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} 1, & \omega \in \left\{ \omega_{k_1}+\cdots+\omega_{k_n} \middle| \begin{matrix} \omega_{k_l}=k_l\omega_0, k_l=\pm 1, \\ 1\le l\le n \end{matrix} \right\} \\ 0 & \text{else} \end{cases} \quad (26a)$$

Consider the frequency of interest is $\omega = \omega_0$. It is easy to verify that

$$\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega_0} 1 = \begin{cases} C_n^{\frac{n-1}{2}} & n = 2k+1, k = 0,1,2... \\ 0 & \text{else} \end{cases} \quad (26b)$$

where, $C_n^m = \frac{n\cdot(n-1)\cdot\cdots\cdot(n-m+1)}{m\cdot(m-1)\cdot\cdots\cdot 2} = \frac{n!}{m!(n-m)!}$. Note that β_n is zero if n is an even number, it is derived from (24c) and (25) that

$$|Y(j\omega)| \le \sum_{n=0}^{\lfloor \frac{N-1}{2} \rfloor} C_{2n+1}^n (\frac{F_d}{2})^{2n+1} b_{2n+1,2n+1} \overline{H}_1^{\,2n+1}$$

From Cauchy's criterion, if (23c) holds, the bound of $|Y(j\omega)|$ is convergent. This completes the proof. ■

Corollary 1 gives a very clear and simple expression for the boundedness of the frequency response of the NARX model (22) in terms of the model parameters and the bound of the 1st order GFRF. The effect of the system model parameters on the boundedness of the system output spectrum and consequently the BIBO stability of the NARX model can be analysed through checking the inequality (23c). This simple analytical bound expression for the output frequency response function also provides a very useful and simple method to evaluate the truncation error associated with the Volterra series expression of nonlinear systems and the highest order N needed in the Volterra series' approximation. Although the check of the stability for a nonlinear system theoretically involves the computation of a limitation as given in (18b) or (21b) or (23c), the result obtained for a sufficiently large N and K or n should be sufficient to provide a significant indication of the system stability.

4 Numerical examples

Consider a nonlinear system

$$y(t) = 0.15y(t-2) + 0.1u(t-1) - 0.05y(t-1)y(t-2) - 0.02y(t-1)^2 - 0.01y(t-1)^3 \quad (27)$$

which can be written into the form (22) with $c_{1,0}(2) = 0.15, c_{0,1}(1) = 0.1$, $c_{2,0}(1,2) = -0.05, c_{2,0}(1,1) = -0.02, c_{3,0}(1,1,1) = -0.01$ else $c_{p,q}(\cdot) = 0$, and K=2, M=3. There are only pure output nonlinear terms in this model.

Compute the magnitude bound of the GFRFs up to 5th order for system (27) according to Corollary 1. From equation (8), it can be obtained

$$|H_1(j\omega)| = \left| \frac{0.1\exp^{-j\omega}}{1-0.15\exp^{-j2\omega}} \right| = \frac{0.1}{L}$$

where $L = \left|1-0.15\exp^{-j2\omega}\right| = \sqrt{(1-0.15\cos 2\omega)^2 + (0.15\sin 2\omega)^2}$. It is easy to have $\underline{L} = 0.7225$ so that $\overline{H}_1$ =0.1384. According to Corollary 1, only $b_{n,n}$ is needed for evaluating the magnitude bounds of the GFRFs:

For n=1 and 2, $b_{1,1}=1, b_{2,2}=\frac{1}{\underline{L}}C(2,0)=0.07/\underline{L}=0.09689$, so that

$$\left|H_2(j\omega_1, j\omega_2)\right| \le b_2 \cdot h_2^T = \frac{0.07}{\underline{L}}\overline{H}_1^{\ 2} = 0.001856$$

For n=3,

$$b_{3,3} = \frac{1}{\underline{L}}C(3,0) + \frac{1}{\underline{L}}\sum_{m=2}^{2}\left(C(m,0) \sum_{\substack{\sum r_i = 3 \\ 1\le r_1 \cdots r_p \le 2-m+2}} \prod_{i=1}^{m} b_{r_i r_i} \right)$$

$$= \frac{0.01}{\underline{L}} + \frac{1}{\underline{L}}C(2,0)\sum_{\substack{\sum r_i = 3 \\ 1\le r_1 \cdots r_p \le 2-2+2}} \prod_{i=1}^{2} b_{r_i r_i} = \frac{0.01}{\underline{L}} + \frac{0.07}{\underline{L}}(2b_{11}b_{22}) = 0.03261$$

thus $\left|H_3(j\omega_1, \cdots, j\omega_3)\right| \le b_3 \cdot h_3^T = 0.03261\overline{H}_1^{\ 3} = 0.0000864609$

For n=4,

$$b_{44} = \frac{1}{\underline{L}}C(4,0) + \frac{1}{\underline{L}}\sum_{m=2}^{3}\left(C(m,0) \sum_{\substack{\sum r_i = 4 \\ 1\le r_1 \cdots r_p \le 4-m+1}} \prod_{i=1}^{m} b_{r_i r_i} \right)$$

$$= \frac{1}{\underline{L}}\left(C(2,0)\sum_{\substack{\sum r_i = 4 \\ 1\le r_1 \cdots r_p \le 4-2+1}} \prod_{i=1}^{2} b_{r_i r_i} + C(3,0)\sum_{\substack{\sum r_i = 4 \\ 1\le r_1 \cdots r_p \le 4-3+1}} \prod_{i=1}^{3} b_{r_i r_i} \right) = \frac{1}{\underline{L}}\left(0.07(2b_{33} + b_{22}^2) + 0.01(3b_{22})\right) = 0.01125$$

thus

$$\left|H_4(j\omega_1, \cdots, j\omega_4)\right| \le b_4 \cdot h_4^T = 0.01125\overline{H}_1^{\ 4} = 0.0000041289$$

For n=5,

$$b_{55} = \frac{1}{\underline{L}}C(5,0) + \frac{1}{\underline{L}}\sum_{m=2}^{4}\left(C(m,0) \sum_{\substack{\sum r_i = 5 \\ 1\le r_1 \cdots r_p \le 5-m+1}} \prod_{i=1}^{m} b_{r_i r_i} \right)$$

$$= \frac{1}{\underline{L}}\left(C(2,0)\sum_{\substack{\sum r_i = 5 \\ 1\le r_1 \cdots r_p \le 5-2+1}} \prod_{i=1}^{2} b_{r_i r_i} + C(3,0)\sum_{\substack{\sum r_i = 5 \\ 1\le r_1 \cdots r_p \le 5-3+1}} \prod_{i=1}^{3} b_{r_i r_i} + C(4,0)\sum_{\substack{\sum r_i = 5 \\ 1\le r_1 \cdots r_p \le 5-4+1}} \prod_{i=1}^{4} b_{r_i r_i} \right)$$

$$= \frac{1}{\underline{L}}\left(0.07(2b_{22}b_{33} + 2b_{44}) + 0.01\cdot 3(b_{33} + b_{22}^2)\right) = 0.004537$$

thus

$$\left|H_5(j\omega_1, \cdots, j\omega_5)\right| \le b_5 \cdot h_5^T = 0.004537\overline{H}_1^{\ 5} \le 0.00000023036$$

Carrying on with the above recursive calculation process, the magnitude bound of the GFRFs of any order can be obtained according to Corollary 1. It should be noted from the above computation that, with the order n going larger, b_{nn} is becoming smaller, and so is the magnitude bound of the nth order GFRF. These information can be used to determine the truncation error of the Volterra series expression of system (27) and to determine the largest order N in the Volterra series approximation (Billings and Lang 1997).

To demonstrate the bound characteristics of the system output spectrum of the NARX model, consider system (27) is subject to input $u(t)=10\sin(\omega_0 t)$ $(F_d>0)$. Then, according to Corollary 1,

$$
\begin{aligned}
|Y(j\omega_0)| &\le \sum_{n=0}^{\lfloor N-1/2 \rfloor} C_{2n+1}^{n}(\frac{F_d}{2})^{2n+1} b_{2n+1,2n+1}\overline{H}_1^{\,2n+1} \\
&= \frac{F_d}{2}\overline{H}_1 + \frac{3F_d^3}{8}0.03262\overline{H}_1^{\,3} + \frac{5F_d^5}{16}0.004537\overline{H}_1^{\,5} + \frac{35F_d^7}{128}0.00086719\overline{H}_1^{\,5} + \cdots + C_{2n+1}^{n}(\frac{F_d}{2})^{2n+1} b_{2n+1,2n+1}\overline{H}_1^{\,2n+1}
\end{aligned}
$$

To check the convergence of this series in the bound expression, the condition

$$\lim_{n\to\infty} \sqrt[2n+1]{C_{2n+1}^{n} b_{2n+1,2n+1}} < \frac{2}{F_d\overline{H}_1}$$

should be analysed. Note that if

$$\lim_{n\to\infty} b_{2n+1,2n+1} < \left(\frac{2}{F_d\overline{H}_1}\right)^{2n+1}\frac{1}{C_{2n+1}^{n}} = \frac{1.4451^{2n+1}}{C_{2n+1}^{n}}$$

then the convergent condition must hold. Let $b(n)=b_{2n+1,2n+1}$ and $bb(n)=\dfrac{1.4451^{2n+1}}{C_{2n+1}^{n}}$, which can be easily computed for any n by a computer program. Obviously, if $b(n)<bb(n)$, then the bound series is convergent. The result is shown in Figure 1, which indicates the convergence of the bound series where $b(n)=b_{2n+1,2n+1}$ is computed up to the 41st order. Figure 1 indicates a very quick convergent rate of the bound series in this specific case.

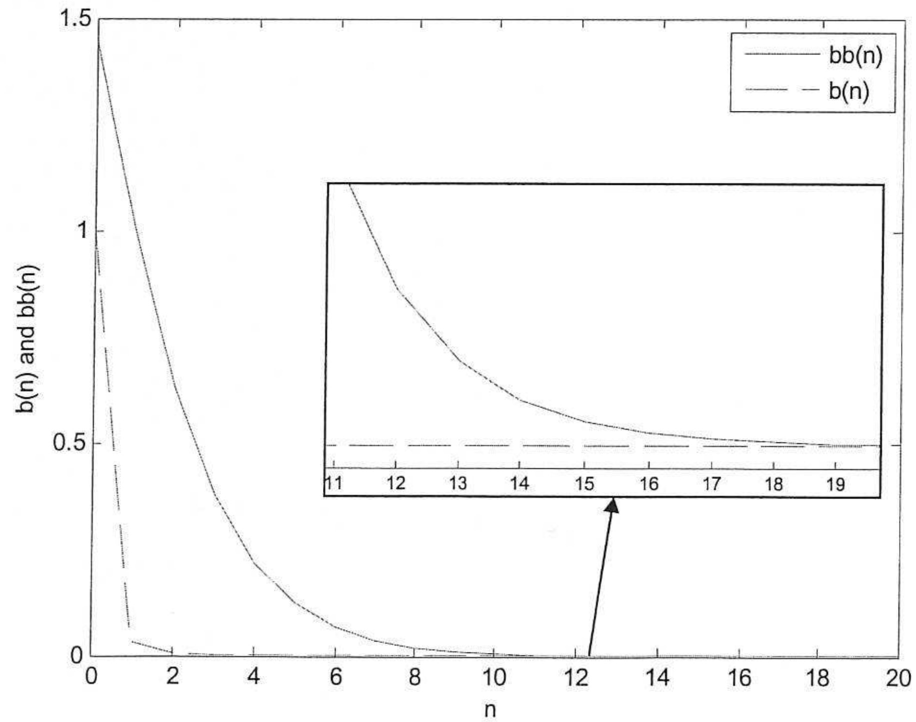

Figure 1. Boundedness of the output spectrum

Moreover, it shall be noted that through symbolic manipulations, an analytical expression for the bound expressions of both the GFRFs and the output spectrum of system (27) can be obtained in terms of model parameters $c_{p,q}(\cdot)$. Thus the magnitude of the GFRFs and output spectrum can be optimized and analysed with respect to considered model parameters. This issue will be discussed in later publications.

5 Conclusions

The bound characteristics of the frequency response functions of the NARX model including the GFRFs and the output spectrum are investigated in this paper. The magnitude bounds of the GFRFs and system output spectrum can all be expressed as a polynomial function of the magnitude bound of the first order frequency response function, and the coefficients of the polynomial are the functions of the system model parameters. These bound characteristics reveal an important relationship between the model parameters and the boundedness of the system frequency response functions, and provide a significant insight into the truncation error associated with the Volterra series' approximation of nonlinear systems. Sufficient conditions for the BIBO stability of the NARX model can also be derived from these results. Note that the boundedness results derived in this paper are based on the use of the triangular inequality. This may introduce conservatism to a certain extent. Further studies will focus on practical applications of the established theoretical results, and the development of methods to reduce possible conservatism associated with these the boundedness results.

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