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Congestion Behavior and Tolls in a Bottleneck Model with Stochastic Capacity

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Abstract
In this paper we investigate a bottleneck model in which the capacity of the bottleneck is assumed stochastic and follows a uniform distribution. The commuters' departure time choice is assumed to follow the user equilibrium principle according to mean trip cost. The analytical solution of the proposed model is derived. Both the analytical and numerical results show that the capacity variability would indeed change the commuters' travel behavior by increasing the mean trip cost and lengthening the peak period. We then design congestion pricing schemes within the framework of the new stochastic bottleneck model, for both a time-varying toll and a single-step coarse toll, and prove that the proposed piecewise time-varying toll can effectively cut down, and even eliminate, the queues behind the bottleneck. We also find that the single-step coarse toll could either advance or postpone the earliest departure time. Furthermore, the numerical results show that the proposed pricing schemes can indeed improve the efficiency of the stochastic bottleneck through decreasing the system's total travel cost.

Key words: bottleneck model; stochastic capacity; departure time choice; congestion pricing

1. Introduction

The well known bottleneck model was originally developed by Vickrey (1969). This model formulates the commuters' trip schedule during a morning rush hour. In the model, it is assumed that the commuters' travel cost consists of two components: the cost of travel time (including free flow travel time and queuing time) and the cost of schedule delay from early or late arrival at the workplace. The departure time choice follows the user equilibrium (UE) principle, i.e., all commuters experience the same travel cost no matter when they leave home.

The simplicity of this single bottleneck model provides a theoretical base to gain qualitative insights into alternative policy measures such as congestion pricing and metering, and several studies have since extended the basic single bottleneck model (see comprehensive reviews in Arnott et al. 1990a, 1998; de Palma et al. 2011). Smith (1984) and Daganzo (1985) proved the existence and uniqueness of the bottleneck equilibrium. Arnott et al. (1993) and Braid (1989) extended the basic bottleneck model to consider elastic demand, whilst Lindsey (2004) and Ramadurai et al (2010) developed a single bottleneck model with heterogeneous commuters.

Most of the existing literature, however, is based on deterministic settings, with either a fixed capacity and demand (Vickrey 1969; Arnott et al. 1990b; Lindsey 2004; Huang and Lam 2002), or a pre-defined elastic demand function (Arnott et al. 1993; Yang and Huang 1997). In reality, not only does travel time increase with traffic volume, but there is also a wide range of randomness in the micro behavior of traffic and traffic conditions. Variations in the behavior of individual drivers, in the performance of vehicles, in weather and lighting on driving conditions, etc, all contribute to the unpredictability or the unreliability of travel time. Variation in road capacity may also occur for physical and operational reasons, such as road works, accidents, vehicle breakdown.. It is intuitive to represent the above-mentioned variations and their impacts on network performance using probability distributions (Chen et al. 2002). Arnott et al. (1999) considered the case where the ratio of demand to capacity is stochastic and examined the effect of information on total social cost. Fosgerau (2008) derived a closed form expression for the expected cost in a bottleneck model with stochastic capacity and demand, assuming linear scheduling costs. Siu and Lo (2009) investigated the random travel delay in a single bottleneck with a heterogeneous population and arrival probability constraint. Li et al. (2008) developed numerical methods to solve a bottleneck where daily capacity is distributed uniformly between an upper and a lower bound. Arnott et al (1999) proved the existence of equilibrium for a general distribution of capacity. However Fosgerau and Jensen (2008) subsequently proved that, in a bottleneck with stochastic demand and supply and assuming the last user always departs after the preferred time, the equilibrium condition may not exist. Lindsey (2009) studied the cost recovery problem from congestion tolls when the bottleneck capacity is random. Fosgerau (2010) investigated the distribution of delays in a congested facility with random capacity and demand. Lindsey (1994, 1999) examined the properties of no-toll equilibrium and system optimum in a bottleneck model with a given joint probability distribution of capacity and demand. Lindsey's analysis focused on a general distribution of road capacity. In contrast, Peer et al. (2010) investigated the capacity changes within peak periods using the bottleneck model.

It is widely recognized that congestion pricing is an effective method to reduce traffic congestion. Vickrey (1969) proposed an optimal, continuous time-varying toll scheme which was shown to have eliminated the queuing delay at the bottleneck. Arnott et al. (1990b) developed an optimal time-varying toll and a one-step coarse toll for a deterministic bottleneck such that the average cost excluding the toll is minimal under time-variable toll. . Laih (1994, 2004) developed a multi-step toll scheme for a single bottleneck, and analyzed the amount of queue reduction and the effects of the toll scheme on equilibrium commuting behavior. Knockaert et al. (2010) studied a single-step coarse toll with inelastic demand. Recently, Lindsey et al. (2012) proposed a braking model in which drivers approaching a tolling point wait until a toll is lowered from one time-step to another. They showed that such braking model would lengthen the peak period with earlier departure and later arrival.

Comparatively, congestion pricing under uncertainty has received relatively little attention. Lindsey (1994, 1999) investigated optimal pricing and information provision under stochastic bottleneck capacity conditions. He found that, if a time-varying toll is implemented, the optimal departure is a non-decreasing function with time over the peak period and the toll charged is a concave function with time. Under the assumption of a two-point distribution of capacity, the peak starts earlier. Daniel (1995) used a discrete Markov Chain to model the expected optimal congestion toll with arrival time uncertainty. Following the perspective of Henderson (1974) to efficiently manage urban traffic congestion through pricing, Lam (2000) developed a model with congestion uncertainty in a network with parallel bottlenecks. Yao et al. (2010) considered travel cost uncertainty and stochastic tolling as a foundation for introducing congestion derivatives.

In this paper, we focus on the morning commuting problem along a single highway with stochastic capacity. We limit our analysis to day-to-day fluctuations in capacity, and assume that the capacity within the day is constant and that the bottleneck is severely congested over the peak period. The capacity fluctuation leads to variability in queue length behind the bottleneck and to variability of travel time and trip cost, which in turn directly influences the commuters' departure time choice behavior. This problem is formulated as a bottleneck model with stochastic capacity. We derive the model's analytical solution and investigate the properties of the model. More specifically, we conduct detailed analysis of the equilibrium cost patterns. A similar stochastic bottleneck model has been developed by Li et al (2008), who analyzed two equilibrium cost patterns for expected early arrivals and expected late arrivals. In this paper, we consider four possible departure-time intervals in a very congested bottleneck when users always arrive early, can arrive early or late, always arrive late and incur a queuing delay, or always arrive late and may not incur a queuing delay.

The solution of the proposed model shows that the capacity variability of the bottleneck leads to significant changes in departure time patterns, which are different to those derived under deterministic conditions. In a deterministic bottleneck model, an individual can choose either to depart in the tails of the rush hour when travel time is low and pay the penalty of arriving at work early or late, or to depart during the peak when travel time is high but schedule delay cost are low. In other words, under the deterministic equilibrium, schedule delay early and schedule delay late cannot occur simultaneously for a given departure time (Arnott et al. 1990b). We demonstrate that with stochastic capacity, commuters departing at the same time during the peak can experience early or late arrival depending on the capacity on the day.

Furthermore, we investigate a time-varying toll and a single-step coarse toll within the framework of this stochastic bottleneck model. The time-varying toll is shown to effectively reduce, and in certain conditions even eliminate, the queues behind the bottleneck without changing the commuters' mean trip cost. However, time-varying pricing scheme is perceived as unpredictable by the general public and has practical difficulties to implement as it requires continuously changeable charges. A main contribution of this paper is the development of a single-step coarse toll which levies a positive, constant charge during a pre-determined time period, and we analyze its impact on the commuters' departure-time choice. The results suggest that the single-step coarse toll may either advance or postpone the earliest departure time. Both pricing schemes are shown to improve the efficiency of the stochastic bottleneck through decreasing the system's total travel cost.

The rest of this paper is organized as follows. In the next section, we provide an overview of the deterministic bottleneck model. The commuters' travel costs and departure time choice in a single bottleneck with stochastic capacity are formulated in Section 3. We also examine the model's properties in this section. In Sections 4 and 5, we develop a time-varying toll and a single-step coarse toll. Numerical examples are presented in Section 6 to illustrate the properties of the proposed model. Finally, section 7 concludes the paper.

2. Overview of the Bottleneck Model with Deterministic Capacity

Let us take a highway with a single bottleneck connecting a residential district with a central business district (CBD). Let s be the capacity of the bottleneck, t_{free} the free flow travel time of the highway, and N the travel demand from the residential district to the CBD. Throughout this paper, we set $t_{free} = 0$ for simplicity andthis will not change any properties of the bottleneck model.

By definition, the cumulative departures $R(t)$ can be formulated as follows:

$$R(t) = \int_{t_0}^{t} r(x)dx, \tag{1}$$

where $r(x)$ is the departure rate at time instant x, and t_0 the earliest time with positive departure rate.

We consider that the highway is congested during the rush hour and that the capacity of the bottleneck will have been fully utilized from time t_0. The length of the queue behind the bottleneck is therefore:

$$Q(t) = \max\left\{R(t) - s\left(t - t_0\right), 0\right\}. \tag{2}$$

The travel time for travelers departing at time t equals the queuing time and can be formulated as follows:

$$T(t)=\frac{Q(t)}{s}. \tag{3}$$

The cost for commuters traveling from home to the CBD consists of two components: the cost of travel time and the cost of schedule delay early or late. The total cost can be formulated as follows:

$$C(t)=\alpha T(t)+\begin{cases}\beta\left(t^{*}-t-T(t)\right), & \text{if } t^{*}\geq t+T(t)\\ \gamma\left(t+T(t)-t^{*}\right), & \text{if } t^{*}<t+T(t)\end{cases} \tag{4}$$

where t^{*} is the preferred arrival time (i.e. the official work start time), α , β and γ denote the values of travel time, schedule delay early (SDE) and schedule delay late (SDL) respectively. In accordance with the empirical findings in Small (1982), the following relationship holds:

$$\gamma>\alpha>\beta>0. \tag{5}$$

To ensure the existence of a deterministic equilibrium, it is necessary to assume $\beta<\alpha$; the opposite case of $\alpha\leq\beta$ is discussed in Appendix 1 of Arnott et al. (1985).

The equilibrium condition for commuters' departure time choice in a single bottleneck is defined as: no commuter can reduce his/her travel cost by unilaterally altering his/her departure time. This condition implies that all commuters incur the same cost, and therefore travel cost is constant at all times while commuters are departing, i.e., $dC(t)/dt=0$ if $r(t)>0$. Using this condition, we can obtain the commuters' departure rate during the rush hour as follows (see Arnott et al. 1990b for further details):

$$r(t)=\begin{cases}\alpha s/(\alpha-\beta), & \text{if } t_0\leq t\leq t_t\\ \alpha s/(\alpha+\gamma), & \text{if } t_t\leq t\leq t_e\end{cases} \tag{6}$$

where t_0 and t_e are, respectively, the earliest and the latest times with positive departure rate, and t_t is the watershed time for departure rate changing (i.e. the departure time at which an individual arrives at the CBD on time t^{*}). The reasoning underlying this result is that the departure rate function must be such that the marginal benefit from postponing the departure by a unit of time equals the marginal cost. In the case of departure prior to t_t, the marginal benefit from postponing the departure is the reduction in early arrival cost $\beta\left(1+T'(t)\right)$ and the marginal cost is the increased travel time cost $\alpha T'(t)$, where $T'(t)$ denotes a time derivative. Application of (3) then gives $r\left(t\right)$ for $t\in\left[t_0,t_t\right)$. The reasoning for departing after t_t is analogous. The arrival rate at work, meanwhile, is constant at s over the rush hour. Thus, a queue builds up linearly from t_0 to t_t and then dissipates linearly until it disappears at t_e. As derived in Arnott et al. (1990b), we have $t_0=t^*-\gamma N\big/\left(\left(\beta+\gamma\right)s\right)$, $t_e=t^*+\beta N\big/\left(\left(\beta+\gamma\right)s\right)$ and $t_t=t^*-\beta\gamma N\big/\left(\left(\beta+\gamma\right)\alpha s\right)$.

3. Bottleneck Model with Stochastic Capacity

3.1. Assumptions

Throughout this paper, the following four assumptions are used:

(A1) Commuters are homogeneous with the same value of time and the same values of schedule delays.

(A2) The capacity of the bottleneck is constant within a day but fluctuates from day to day. The variability of capacity is exogenous and independent of departures.

(A3) The capacity is a non-negative stochastic variable changing around a certain mean capacity. Following Kuang et al. (2007) and Li et al. (2008), we assume that stochastic capacity follows a uniform distribution within interval $[\theta\bar{s},\bar{s}]$, where $\bar{s}$ is the design capacity and $\theta(\leq 1)$ is a positive variable which denotes the lowest rate of available capacity.

(A4) Commuters are aware of the capacity degeneration probability and their departure time choice follows the UE principle in terms of mean trip cost.

Unlike the Vickrey (1969) model, we assume the capacity of the single bottleneck is stochastic, although the commuters' departure time choice is made deterministically based on mean trip cost. The constant within-day capacity assumption (A2) implies that the current model accounts for incidents happened before the peak started (when the first of the N commuters departed), but not for incidents occurring during the peak period. The later was investigated by Fosgerau (2010) and Peer et al. (2010). We consider that the commuters' travel time and their schedule delays are both stochastic due to capacity fluctuations. We assume further that the commuters learn the incident probability from their day-to-day travel (A4), and adjust their departure time in order to minimize their expected travel costs.

3.2. Stochastic Trip Cost Formulation

Under the stochastic condition, definitions in (1)-(3) are still valid, and (4) can still be used to calculate the trip cost of commuters departing at time instant t. However, the trip cost is now not deterministic but stochastic. For simplicity, we set the preferred arrival time as zero, i.e., $t^* = 0$. The mean trip cost with respect to departure time t under stochastic condition can be formulated as follows:

$$\begin{aligned} E\left[C(t)\right] &= E\left[\alpha T(t) + \beta \overline{SDE}(t) + \gamma \overline{SDL}(t)\right] \\ &= \alpha E\left[T(t)\right] + \beta E\left[\overline{SDE}(t)\right] + \gamma E\left[\overline{SDL}(t)\right], \end{aligned} \tag{7}$$

where $\overline{SDE}(t)$ and $\overline{SDL}(t)$ are the schedule delay early and late for commuters departing at time t, respectively, and can be expressed as follows:

$$\overline{SDE}(t) = \max\{0, -T(t) - t\} \text{ and } \overline{SDL}(t) = \max\{0, T(t) + t\}. \tag{8}$$

3.3. Stochastic Bottleneck Model

The equilibrium condition for commuters' departure time choice in a single bottleneck with stochastic capacity is as follows: no commuter can reduce his/her mean trip cost by unilaterally altering his/her departure time. This condition implies that the commuters' mean trip cost is constant with respect to time instant if the departure rate is positive, i.e.,

$$dE[C(t)]/dt = 0, \text{ if } r(t) > 0. \tag{9}$$

The calculation of the mean trip cost relies on the calculations of the mean travel time, the mean schedule delay early and late. Because of the day-to-day stochastic capacity of the bottleneck, commuters departing at the same time may endure schedule delay early or late in different days, and may or may not encounter queuing delays. In this paper, we study a very congested bottleneck, and there are in total four situations to be considered, each corresponding to a time interval within which (I) users always arrive early, (II) depending on capacity users can arrive early or late, (III) users always arrive late and incur a queuing delay, and (IV) users always arrive late and – depending on capacity – may or may not incur a queuing delay. The four situations occur consecutively, and we use t_1, t_2, t_3 to denote the watershed lines which separate the four situations. The detailed derivation of departure rates in these four situations can be found in Appendix A; we summarize the results below.

Situation I. Users always arrive early in $[t_0, t_1]$

In this situation, no commuters experience schedule delay late subject to all possible values of the capacity of the bottleneck. The departure rate in this interval is

$$r(t) = \frac{\alpha}{\alpha - \beta} \frac{\bar{s}(1-\theta)}{\ln \theta^{-1}}, \quad t_0 \le t \le t_1, \tag{10}$$

where t_0 is the earliest time with positive departure rate and also the time with zero queue length. The boundary condition for this situation is $\overline{SDE}(t_1) = 0$ when $s = \theta\bar{s}$, and we thus have $R(t_1) = -t_0\theta\bar{s}$.

Situation II. Depending on capacity users can arrive early or late in $(t_1, t_2]$

In this situation, both schedule delay early and late may occur. If the capacity of the bottleneck is large enough, only schedule delay early will occur. On the other hand, schedule delay late occurs when the capacity is small. The watershed capacity has to generate $T(t) + t = 0$, i.e., $s = -R(t)/t_0$. The departure rate in this interval is

$$r(t) = \frac{\alpha}{A + B(\ln R(t) + 1)}, \quad t_1 < t \le t_2, \tag{11}$$

where $A = -\left(\alpha \ln\theta + \beta \ln(-t_0\bar{s}) + \gamma \ln(-t_0\theta\bar{s}) + (\beta+\gamma)\right) / (\bar{s} - \bar{s}\theta)$, $B = (\beta+\gamma)/(\bar{s} - \bar{s}\theta)$. The boundary condition for this situation is $\overline{SDE}(t_2) = \overline{SDL}(t_2) = 0$ when $s = \bar{s}$, and so we have $R(t_2) = -t_0\bar{s}$.

Situation III. Users always arrive late and incur a queuing delay in $(t_2, t_3]$

Similar to Situation I, in this situation all commuters experience schedule delay late despite maximum bottleneck capacity. The departure rate in this interval is

$$r(t)=\frac{\alpha}{\alpha+\gamma}\frac{\bar{s}(1-\theta)}{\ln\theta^{-1}},\quad t_2<t\le t_3. \tag{12}$$

The boundary condition for this situation is $R(t_3)=\bar{s}(t_3-t_0)$, i.e., the queuing length at time t_3 equals to zero when $s=\bar{s}$.

Situation IV. Users always arrive late and – depending on capacity – may or may not incur a queuing delay in $(t_3,t_e]$

Similar to Situation II, there is a watershed capacity of the bottleneck such that the queuing length falls to zero. The departure rate in this interval is

$$r(t)=\frac{(\alpha+\gamma)R(t)/(t-t_0)-(\alpha\theta+\gamma)\bar{s}}{(\alpha+\gamma)\left(\ln R(t)-\ln\left(\theta\bar{s}(t-t_0)\right)\right)},\ \ t_3<t\le t_e. \tag{13}$$

The boundary condition for this situation is $r(t_e)=0$. Equally, we have $R(t_e)=\hat{s}(t_e-t_0)$, where $\hat{s}=\bar{s}(\alpha\theta+\gamma)/(\alpha+\gamma)$.

PROPOSITION 1. The following inequality holds,

$$\hat{s}(t-t_0)\le R(t)\le\bar{s}(t-t_0),\ \ t\in[t_3,t_e]. \tag{14}$$

PROOF. Since the queue may exist behind the bottleneck for Situation IV, we have $R(t)>\theta\bar{s}(t-t_0)$, and the numerator of (13) is thus positive. By definition, the departure rate is non-negative, and the denominator of (13) must be non-negative. Additionally, we have $\hat{s}(t-t_0)\le R(t)$. Since $R(t_3)=\bar{s}(t_3-t_0)$ and the queue may not exist behind the bottleneck for Situation IV, we have $R(t)\le\bar{s}(t-t_0)$. This completes the proof. □

Since the departure rate $r(t)=0$ if $t>t_e$, the cumulative departure flow at time t_e equals the traffic demand, i.e., $R(t_e)=N=\hat{s}(t_e-t_0)$. Therefore, we have $t_e=t_0+N/\hat{s}$. Moreover, the equilibrium condition of the stochastic bottleneck model implies that $E[C(t_0)]=E[C(t_e)]$ $=E[C(t_0+N/\hat{s})]=-t_0\beta$. Thus, we have

$$t_0=\frac{N}{\hat{s}}\frac{1}{k_0-1}\ \text{and}\ t_e=\frac{N}{\hat{s}}\frac{k_0}{k_0-1}, \tag{15}$$

where $\hat{s}=\overline{s}\left(\alpha\theta+\gamma\right)/\left(\alpha+\gamma\right)$ and

$$k_0=1-\frac{\left(1-\theta\right)\left(\beta+\gamma\right)\overline{s}}{\left(\alpha+\gamma\right)\hat{s}\left(\ln\hat{s}-\ln\theta\overline{s}\right)}. \tag{16}$$

Using the boundary conditions of Situations I, II, and III, we obtain the watershed lines as follows:

$$t_1=\frac{N}{\hat{s}}\frac{k_1}{k_0-1},\ t_2=\frac{N}{\hat{s}}\frac{k_2}{k_0-1},\text{ and } t_3=\frac{N}{\hat{s}}\frac{k_3}{k_0-1}, \tag{17}$$

where $k_1=1-\frac{\alpha-\beta}{\alpha}\frac{\theta\ln\theta^{-1}}{1-\theta}$, $k_2=\frac{\alpha+\beta+\gamma}{\alpha}+\frac{\left(\alpha+\gamma\right)\ln\theta}{\alpha\left(1-\theta\right)}$ and $k_3=1+\frac{\left(1-\theta\right)\left(\gamma+\beta\right)}{\alpha\left(1-\theta\right)+\left(\alpha+\gamma\right)\ln\theta}$.

With the above derived boundary conditions of Situations I-IV, the cumulative departure flows of a stochastic bottleneck are given in Figure 1. The earliest time instant for commuters leaving home is t_0. It can be seen in Figure 1 that, at the beginning, commuters depart from home at a constant departure rate until the watershed time instant t_1. If the capacity of the bottleneck equals $\theta\overline{s}$, commuters departing at this time instant will arrive at their workplaces on time. Afterwards, the departure rate will gradually decrease until the watershed time instant t_2. If the capacity of the bottleneck equals $\overline{s}$, commuters departing at time instant t_2 will arrive at their workplaces on time. Thereafter, the departure rate will remain constant until the watershed time instant t_3 and the queuing length at this time instant will be zero. After t_3, the departure rate continues to decrease with time till zero at time instant t_e.

The above analysis is based on the assumption that the bottleneck is severely congested due to heavy traffic demand relative to capacity. This assumption can be relaxed. We present the analytical results for lighter traffic demand in Appendix B and show that the second time period with constant departure rate will disappear if the bottleneck is not very congested.

So far, we have taken the stochastic capacity as the unique source of travel time variability. In practice, travel demand may also be variable. Including an explicit representation of stochastic demand to the existing model with stochastic capacity will introduce added complexity for the theoretical analysis. Thus in this paper, we consider only the impact of the demand elasticity. The analytical solutions of a stochastic bottleneck with demand elasticity are presented in Appendix C. The results show that the length of peak period with elastic demand could either decrease or increase with increasing θ value, depending on a demand sensitivity to cost.

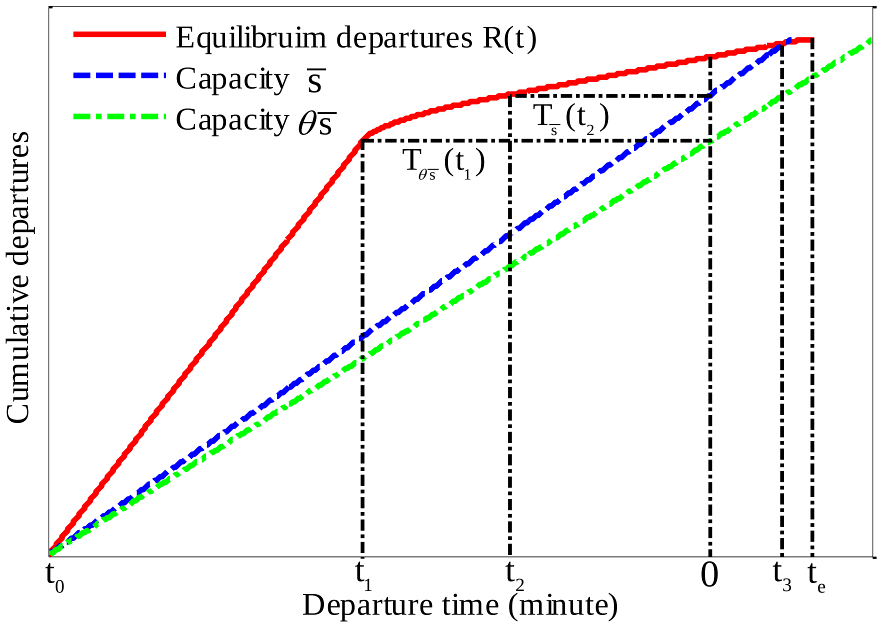

Figure 1 Equilibrium departures with stochastic capacity in a single bottleneck

3.4. Properties of the Stochastic Bottleneck Model

We present the following theorems and propositions to reveal some interesting properties of the equilibrium solution of the proposed bottleneck model.

THEOREM 1. At equilibrium state, the expected trip cost for every commuter is a strictly monotonically increasing function of the traffic demand, i.e., $\partial E[C(t_0)]/\partial N > 0$.

PROOF. Since $E[C(t_0)] = -t_0\beta$ and $t_0 = \dfrac{N}{\hat{s}(k_0 - 1)}$, then

$$E[C(t_0)] = \frac{-\beta N}{\hat{s}(k_0 - 1)}, \tag{18}$$

$$\partial E[C(t_0)]/\partial N = \frac{\beta}{\hat{s}(1 - k_0)}. \tag{19}$$

To prove $\partial E[C(t_0)]/\partial N > 0$, we only need to prove $1-k_0 > 0$. Since $0<\theta<1$ and $\hat{s} = \bar{s}\left(\alpha\theta+\gamma\right)/\left(\alpha+\gamma\right) > \bar{s}\left(\alpha\theta+\gamma\theta\right)/\left(\alpha+\gamma\right) = \theta\bar{s}$, it can be shown that $\ln\hat{s} - \ln\theta\bar{s} > 0$ holds. Thus, both the numerator and denominator of the second term on the right hand side of (16) are positive. Therefore the right hand side of (16) is less than 1. So, $1-k_0 > 0$ holds. This completes the proof. □

This theorem partly coincides with a property obtained by Fosgerau (2010) who was concerned with the marginal external social costs of capacity and demand with a general distribution.

THEOREM 2. At equilibrium state, the departure rate is a monotonically decreasing function of the departure time t, $t \in [t_0, t_e]$.

PROOF. See Appendix D. This theorem coincides with Proposition 3 in Lindsey (1994).

PROPOSITION 2. When the value of the parameter θ approaches to one, the stochastic bottleneck model follows the deterministic model.

PROOF. According to the L'Hôspital's rule, we have $\lim_{\theta\to 1} \ln\theta^{-1} = 0$ and $\lim_{\theta\to 1}\left(1-\theta\right)/\ln\theta^{-1} = 1$. Therefore,

$$\lim_{\theta\to 1}\hat{s} = \bar{s},\ \lim_{\theta\to 1}k_0 = \lim_{\theta\to 1}k_3 = -\frac{\beta}{\gamma},\ \lim_{\theta\to 1}k_1 = \lim_{\theta\to 1}k_2 = \frac{\beta}{\alpha}, \tag{20}$$

and

$$\lim_{\theta\to 1} r(t) = \begin{cases} \alpha\bar{s}/\left(\alpha-\beta\right), & \text{if } t_0 \le t \le t_1 \\ \alpha\bar{s}/\left(\alpha+\gamma\right), & \text{if } t_2 < t \le t_3 \end{cases} \tag{21}$$

Substituting (20) into (17), then the watershed times become:

$$t_0 = -\frac{\gamma}{\beta+\gamma}\frac{N}{\bar{s}},\ t_1 = t_2 = -\frac{\beta\gamma}{\beta+\gamma}\frac{N}{\alpha\bar{s}},\ t_3 = t_e = \frac{\beta}{\beta+\gamma}\frac{N}{\bar{s}}. \tag{22}$$

Substituting (22) into (21), we obtain the same traffic flow pattern as for the deterministic bottleneck model (6). This completes the proof. □

PROPOSITION 3. With a fixed number of commuters, enlarging the value of the parameter θ will result in a decrease in the length of peak period.

PROOF. The definition of $\hat{s}$ yields: $d\hat{s}/d\theta = \alpha\bar{s}/(\alpha+\gamma) > 0$. This implies that $\hat{s}$ is a monotonically increasing function of θ. From (15), the length of peak period is as follows:

$$t_e - t_0 = \frac{N}{\hat{s}}\frac{k_0}{k_0-1} - \frac{N}{\hat{s}}\frac{1}{k_0-1} = \frac{N}{\hat{s}}. \tag{23}$$

Since N is a positive constant, then $t_e - t_0$ is monotonically decreasing with respect to θ. This completes the proof. □

Under Assumption A3, the stochastic capacity follows a uniform distribution. Let e be the mean capacity and v a parameter such that $s \in [e-v, e+v]$. By this definition, we obtain $\bar{s} = e+v$, and $\theta = (e-v)/(e+v)$. With this new formulation of capacity distribution, we obtain the following results.

PROPOSITION 4. When the value of the parameter v approaches to zero, the stochastic bottleneck model follows the deterministic model.

PROOF. Since $\theta = (e-v)/(e+v)$, when the value of the parameter v approaches to zero, the parameter θ approaches to one. The rest of the proof is the same as that of Proposition 2. □

So far, we have provided two formulations of the uniform capacity distribution, i.e. $[\theta\bar{s}, \bar{s}]$ and $[e-v, e+v]$. It can be seen that, both the expectation and the variance of the bottleneck capacity will change with θ, whilst only the variance of the bottleneck capacity will change with v. Thus, Proposition 4 is slightly different from Proposition 2.

PROPOSITION 5. With a fixed number of commuters, enlarging the value of the parameter v will result in a decrease in the length of peak period.

PROOF. Substituting $\bar{s} = e+v$ and $\theta = (e-v)/(e+v)$ into $\hat{s} = \bar{s}(\alpha\theta+\gamma)/(\alpha+\gamma)$ leads to $d\hat{s}/dv = (\gamma-\alpha)/(\gamma+\alpha) > 0$. This implies that $\hat{s}$ is monotonically increasing with respect to v. According to (23), the length of peak period $t_e - t_0$ is monotonically decreasing with respect to $\hat{s}$. Therefore, $t_e - t_0$ is also monotonically decreasing with respect to v. This completes the proof. □

4. Time-varying Toll

Arnott et al. (1990b) designed an optimal time-varying toll under a deterministic bottleneck capacity. They showed that the toll doesn't change the schedule delay costs, but can completely eliminate the waiting time caused by queuing, and therefore the waiting time cost could be replaced by toll charge. Using control theory, Lindsey (1994, 1999) derived the optimal pricing with stochastic bottleneck capacity condition. In this section, we investigate a time-varying toll in the case of a stochastic bottleneck capacity and a constant departure rate over a fixed peak period (which is set to be the same as the peak period without toll). We show that under this tolling scheme, the queues can be reduced significantly but not eliminated completely unless the capacity is constant.

4.1. A Time-varying Toll Scheme

Let commuters depart at a constant rate $r = \hat{s} = \bar{s}\left(\alpha\theta+\gamma\right)/\left(\alpha+\gamma\right)$. (7) gives the expected travel cost function at departure time t in the absence of toll. Hence, the time-dependent toll can be formulated as

$$\rho(t) = E\left[C(t)\right] - E\left[c(t)\right], \tag{24}$$

where the lowercase 'c' stands for the expected travel cost exclusive of the toll. Because a constant departure rate is used and the mean trip cost is kept unchanged, this scheme may or may not be the first-best toll; therefore the optimal control theory method used by Lindsey (1994, 1999) to solve the time-varying toll problem is not required here.

Similar to the analysis for the no-toll equilibrium in Section 3, the morning commuting problem with time-dependent toll can be analytically investigated for the following three situations, each corresponding to a time interval: (I) users always arrive early, (II) depending on capacity users can arrive early or late, and (III) users always arrive late and incur a queuing delay. In all three situations, queue may exist depending on the capacity value. Let t_0 and t_e have the same definitions as in the previous section, t_1 and t_2 now define the watershed time points between the above three situations. With a constant departure rate, i.e., $r(t)=\hat{s}$, $t \in [t_0, t_e]$, the cumulative departures at time t can be easily computed as,

$$R(t)=\hat{s}\left(t-t_0\right). \tag{25}$$

Substituting (25) into (2), the queue length becomes,

$$Q(t)=\max\left\{\left(\hat{s}-s\right)\left(t-t_0\right),0\right\},\ \ s\in[\theta\bar{s},\bar{s}]. \tag{26}$$

Substituting (3), (25) and (26) into (4), we can obtain the expected trip cost in the absence of toll, for each of the three time intervals separately.

Let $E\left[C(t)\right]=-t_0\beta$, the time-varying toll can be formulated as a piecewise function, corresponding to the three time intervals $[t_0,t_1]$, $[t_1,t_2]$ and $[t_2,t_e]$ as,

$$\rho(t)=\begin{cases}\beta\left(t-t_0\right)-\left(\alpha-\beta\right)\left(t-t_0\right)\xi_1, & t\in[t_0,t_1]\\ -\alpha\left(t-t_0\right)\xi_1+\beta\left(t-t_0\right)\xi_2+\left(\beta+\gamma\right)t_0\xi_3-\gamma\left(t-t_0\right)\xi_4, & t\in[t_1,t_2]\\ -\left(\beta+\gamma\right)t_0-\left(\alpha\theta+\gamma\right)\left(t-t_0\right)\xi_5, & t\in[t_2,t_e]\end{cases} \tag{27}$$

where $\xi_1=\left(\hat{s}\left(\ln\hat{s}-\ln(\theta\bar{s})\right)-\hat{s}+\theta\bar{s}\right)/\left(\bar{s}(1-\theta)\right)$, $\xi_2=\left(\hat{s}\left(\ln t_0-\ln\left(t_0-t\right)\right)+\bar{s}\right)/\left(\bar{s}(1-\theta)\right)$, $\xi_3=\bar{s}\theta/\left(\bar{s}(1-\theta)\right)$, $\xi_4=\hat{s}\left(\ln\left(\hat{s}(t_0-t)\right)-\ln\left(t_0\theta\bar{s}\right)-1\right)/\left(\bar{s}(1-\theta)\right)$, $\xi_5=\left(\ln\hat{s}-\ln(\theta\bar{s})\right)/\left(1-\theta\right)$.

The detailed derivation of the time-varying toll is given in Appendix E. It can be proved that as the value of θ approaches one, $\rho(t)$ becomes the one with deterministic capacity as derived by Arnott et al (1990b).

Lindsey (1994, 1999) proved that a socially optimal departure rate can be decentralized using a time-varying toll. He showed that, in a special case of a two-point distribution of capacity, when a time-varying toll is used to support the social optimum, the expected individual trip cost is greater than or equal to that in the absence of toll. We herein follow the first-best system optimum strategy designed for the deterministic bottleneck (Arnott et al., 1990b) and develop a time-varying toll for the constant departure rate over the peak period which maintains the same mean individual trip cost. Thus, the proposed time-varying toll is in fact a second-best system optimum strategy for the studied stochastic bottleneck. However, such a tolling scheme may be more acceptable since the concept of the mean value is closer to reality.

4.2. Properties of the Time-varying Toll

This section presents the properties of the equilibrium solution under the time-varying toll. We use subscript ρ to denote parameters or variables associated with this pricing scheme.

PROPOSITION 6. Under the time-varying pricing scheme, the equilibrium solution results in shorter queue length and shorter travel time than no-toll scheme, i.e., $Q_\rho(t) \le Q(t)$ and $T_\rho(t) \le T(t)$ and the toll is non-negative, i.e., toll $\rho(t) \ge 0$, $\forall t \in [t_0, t_e]$.

PROOF. In Section 3, for time interval $[t_0, t_3]$, the stochastic bottleneck was depicted as a congested commuting system in which the inequality $R(t) \ge \bar{s}(t - t_0)$ holds. In time interval $[t_3, t_e]$, commuters may experience queue only, yet we have the inequality $R(t) \ge \hat{s}(t - t_0)$. Hence, the inequality $R(t) \ge \hat{s}(t - t_0)$ is always true during the peak period. Comparing (2) with (26), we have

$$Q(t) = \max\left\{R(t) - s(t - t_0), 0\right\} \ge \max\left\{(\hat{s} - s)(t - t_0), 0\right\} = Q_\rho(t), \quad s \in [\theta\bar{s}, \bar{s}]. \tag{28}$$

Substituting (28) into (3), we get the inequality $T(t) \ge T_\rho(t)$.

Considering the mean trip cost function, we have

$$\begin{aligned} E\left[C(t)\right] &= E\left[\alpha T(t) - \beta\left(T(t) + t\right) + \gamma\left(T(t) + t\right)\right] = E\left[(\alpha - \beta + \gamma)T(t) + (\gamma - \beta)t\right] \\ &\ge E\left[(\alpha - \beta + \gamma)T_\rho(t) + (\gamma - \beta)t\right] \\ &= E\left[\alpha T_\rho(t) - \beta\left(T_\rho(t) + t\right) + \gamma\left(T_\rho(t) + t\right)\right] = E\left[c_\rho(t)\right] \end{aligned}$$

So, $\rho(t) = E\left[C(t)\right] - E\left[c(t)\right] \ge 0$ holds. The above completes the proof. □

PROPOSITION 7. At either no-toll equilibrium or time-varying toll equilibrium, the expected queuing time is identical for the last commuter who leaves home at time t_e.

PROOF. The cumulative departures at time t_e is $R(t_e) = \hat{s}(t_e - t_0)$ at both equilibrium states, and the queuing time for commuters who leave home at time t_e equals

$\mathrm{T}(t_e)=\max\left\{\left(\hat{s}\left(t_e-t_0\right)-s\left(t_e-t_0\right)\right)/s\,,0\right\}$. Although departure rates are different in these two equilibrium conditions, their expected travel times are identical, i.e.,

$$\begin{aligned} E\left[T(t_e)\right] &= E\left[\max\left\{s^{-1}\left[R(t_e)-s(t_e-t_0)\right],0\right\}\right] \\ &= \int_{\theta\bar{s}}^{\frac{R(t_e)}{t_e-t_0}}\left(s^{-1}R(t_e)-t_e+t_0\right)f(s)ds=\int_{\theta\bar{s}}^{\hat{s}}\left(s^{-1}\hat{s}\left(t_e-t_0\right)-t_e+t_0\right)f(s)ds \\ &= E\left[\max\left\{s^{-1}\left(\hat{s}-s\right)\left(t_e-t_0\right),0\right\}\right]=E\left[T_\rho(t_e)\right]. \end{aligned}$$

This completes the proof. □

The above proof was based on a uniformly distributed capacity function. Here, we consider more general distribution functions for the bottleneck capacity. Let $f(s)$ be a general probability density function of the bottleneck capacity and $\hat{s}$ the average capacity of the bottleneck realized in no-toll equilibrium. We assume that the departure rate is a non-increasing function of time t. Under this assumption, if the capacity is larger than $\hat{s}$, then both the queuing length and the queuing time of the last commuter will be zero. Then, $E\left[T(t_e)\right]=\int_0^{\hat{s}}s^{-1}\left[R(t_e)-s(t_e-t_0)\right]f(s)ds$ and $E\left[T_\rho(t_e)\right]=\int_0^{\hat{s}}s^{-1}(\hat{s}-s)(t_e-t_0)f(s)ds$ holds. Since $R(t_e)=\hat{s}\left(t_e-t_0\right)$, then $E\left[T(t_e)\right]=E\left[T_\rho(t_e)\right]$. Therefore, Proposition 7 continues to hold when the bottleneck capacity follows other distribution functions.

In summary, under time-varying toll scheme, the peak-period does not change and the flow pattern is similar to that achieved in deterministic social optimum. However, queuing delay and capacity underutilization can occur at any time.

5. Single-step Coarse Toll

The toll scheme formulated in Section 4 varies continuously over time. Such a complex pricing structure is not very well accepted by travelers as they cannot predict the amount of charging they would have to pay in advance. This impels researchers to develop more practical tolling schemes, including one that varies in steps over time. In the context of bottleneck problems, Arnott et al. (1990b) studied a simple step toll, which has a positive and constant value during a certain interval and zero otherwise. This has been referred as a single-step coarse toll in the literature. If $\alpha \geq \gamma$, some drivers depart after the mass. This is the case considered by Laih (1994). If $\alpha < \gamma$, no drivers depart after the mass which is the case considered by Arnott et al (1990b). As explained by Arnott et al (1990b), the reason for a mass of departures immediately after the toll has been lifted is due to the fact that the last person to arrive before the toll is lifted must have the same trip cost as the first person to arrive after the toll is lifted. The latter must therefore incur an additional travel time plus schedule delay costs which are equal to toll, and as such is higher than the former. This is impossible unless there is a mass of departures just after the toll is lifted. According to empirical results (Small, 1982), the shadow value of one minute late is significantly larger than the shadow value of travel time, and hence we treat only the case $\alpha < \gamma$ here, i.e., no driver chooses to depart after the mass.

Similar to Arnott et al (1990b), we introduce here a coarse toll into the bottleneck model with stochastic capacity. It is assumed that a coarse toll equilibrium exists with capacity η, $\eta \in [\theta \bar{s}, \bar{s}]$. Commuters departing in interval $[t^+, t^-]$ will be charged by a fixed toll τ, here t^+ and t^- denote the starting and ending points of the toll, respectively. The objective, in the following subsection, is to find the optimal fee and time interval based on capacity η, $\eta \in [\theta \bar{s}, \bar{s}]$.

5.1. Equilibrium Departure Pattern with a Coarse Toll

Intuitively, when the toll is set too high or the charging time interval is too long, there could be times when no one utilizes the bottleneck. We aim to derive the optimal toll and the optimal charging interval which would minimize the total travel cost of the commuting system.

We divide all commuters into three groups: N_0 commuters who go through the bottleneck before the tolling period; N_1 commuters who have to pay a constant toll when passing through the bottleneck; and N_2 commuters after the tolling period.

Case I. Before tolling period $[t_0', t^+]$

Under equilibrium, the mean trip cost of the last commuter who does not need to pay the toll should be the same as the trip cost of the first commuter who does. For this to happen, there must be a period which has no departures between the two time instances that the above two commuters departed. This corresponds to a scenario whereby, early in the morning, commuters depart at a high rate and pay no toll. This departure rate is the same as that in the no-toll equilibrium. Then, commuters cease to depart for a while and the queue dissipates gradually as travelers are being served by the bottleneck. In equilibrium, the expected trip cost of every commuter should be the same as that experienced by the first commuter, i.e.,

$$\mathrm{E}\left[\mathrm{C}(\mathrm{t})\right] = -\mathrm{t}_0'\beta, \tag{29}$$

where t_0' denotes the departure time of the first commuter under the single-step course toll regime. The departure rate follows that given by (10) and the boundary condition for this group of commuters is:

$$\mathrm{N}_0 = \eta\left(\mathrm{t}^+ - \mathrm{t}_0'\right), \tag{30}$$

where $\eta \in [\theta\bar{s}, \bar{s}]$ is the bottleneck capacity. In equilibrium, $\mathrm{E}\left[\mathrm{C}(\mathrm{t}_0')\right] = \mathrm{E}\left[\mathrm{c}(\mathrm{t}^+, \eta)\right] + \tau$, where $\mathrm{E}\left[\mathrm{c}(\mathrm{t}^+, \eta)\right]$ is formulated in Appendix F. Therefore, the relationship between t_0', t^+ and τ can be formulated as follows:

$$\mathrm{t}_0' = \frac{\tau}{(\alpha - \beta)\psi - \beta} + \mathrm{t}^+. \tag{31}$$

where $\psi = \left(\eta\left(\ln\eta - \ln\theta\bar{s}\right) - \left(\eta - \theta\bar{s}\right)\right) / \left(\bar{s}\left(1 - \theta\right)\right)$.

Case II. During tolling period $[t^+, t^-]$

Commuters start to leave home when the progress of queue dissipation extends to the starting point of the toll, t^+. In the deterministic bottleneck model, the optimal single step toll is timed such that the queue has just disappeared by the time the toll kicks in. While, for the stochastic model, the time when the coarse toll is lifted is based on the capacity η. The latter does not represent the maximal capacity of the bottleneck, therefore the queue may not be eliminated completely. In this case, the toll is constant and hence does not affect the departure rates. Then, when heavy congestion exists, we can derive the departure rates in four scenes as in Section 3: (i) users always arrive early, (ii) depending on capacity users can arrive early or late, (iii) users always arrive late and incur a queuing delay, and (iv) users always arrive late and – depending on capacity – may or may not incur a queuing delay. These four scenes occur consecutively, and we denote t_1', t_2', t_3' as the new watershed lines which separate the four scenes.

Here, we present the departure rate functions for the first three intervals. The methods used to obtain these functions are similar to those used in the no-toll equilibrium.

$$r(t)=\frac{\alpha}{\alpha-\beta}\frac{\bar{s}(1-\theta)}{\ln\theta^{-1}},\quad t\in[t^+,t_1'], \tag{32}$$

$$r\left(t\right)=\frac{\alpha}{A+B\left(\ln R(t)+1\right)},\quad t\in[t_1',t_2'], \tag{33}$$

where A and B are those defined in (11), and

$$r(t)=\frac{\alpha}{\alpha+\gamma}\frac{\bar{s}\left(1-\theta\right)}{\ln\theta^{-1}},\quad t\in[t_2',t_3']. \tag{34}$$

In the fourth interval $t\in[t_3',t^-]$, there is no schedule early but queue possibly exists. Considering the boundary condition $r(t^-)=0$ and the possible results of the bottleneck capacity η, we have

$$r(t)=\begin{cases}\left(\dfrac{R(t)-N_0}{\left(t-t^+\right)}-\hat{s}\right)\Bigg/\ln\dfrac{R(t)-N_0}{\left(t-t^+\right)\theta\bar{s}}, & \text{for } t\in[t_3',t^-],\ \text{if } \eta\le\hat{s}\\ \left(\dfrac{R(t)}{\left(t-t_0'\right)}-\hat{s}\right)\Bigg/\ln\dfrac{R(t)}{\left(t-t_0'\right)\theta\bar{s}}, & \text{for } t\in[t_3',t^-],\ \text{otherwise}\end{cases} \tag{35}$$

The total number of commuters in $[t^+,t^-]$ is

$$N_1=\begin{cases}\hat{s}\left(t^--t^+\right), & \text{if } \eta\le\hat{s}\\ \hat{s}\left(t^--t_0'\right)-\eta\left(t^+-t_0'\right), & \text{otherwise}\end{cases} \tag{36}$$

In order to derive the optimal charging time interval, we design a pricing scheme as follows:

$$\tau = \mathrm{E}\left[\mathrm{C(t)}\right] - \mathrm{E}\left[\mathrm{c}(\mathrm{t}^-,\eta)\right], \tag{37}$$

$$\tau = \mathrm{E}\left[\mathrm{C(t)}\right] - \mathrm{E}\left[\mathrm{c}(\mathrm{t}^+,\eta)\right]. \tag{38}$$

In equilibrium, commuters will have the same travel costs, i.e., $\mathrm{E}\left[\mathrm{c}(\mathrm{t}^-,\eta)\right] = \mathrm{E}\left[\mathrm{c}(\mathrm{t}^+,\eta)\right]$ holds for any value of η under the coarse toll scheme. Following the definitions of $\mathrm{E}[\mathrm{c}(\mathrm{t}^+,\eta)]$ and $\mathrm{E}[\mathrm{c}(\mathrm{t}^-,\eta)]$ in Appendix F, we derive the relation between t^+ and t^- as follows:

For $\eta \le \hat{\mathrm{s}}$,

$$\mathrm{t}^- = \frac{\dfrac{\tau(\beta+\gamma)\psi}{(\alpha-\beta)\psi-\beta} + \left((\alpha+\gamma)\phi-\beta\right)\mathrm{t}^+}{\gamma+(\alpha+\gamma)\phi}. \tag{39}$$

For $\eta > \hat{\mathrm{s}}$,

$$\mathrm{t}^- = \frac{\dfrac{(\alpha+\gamma)\phi-(\alpha-\beta)\psi}{(\alpha-\beta)\psi-\beta}\tau + \left((\alpha+\gamma)\phi-\beta\right)\mathrm{t}^+}{\gamma+(\alpha+\gamma)\phi}. \tag{40}$$

where $\psi = \dfrac{\eta\left(\ln\eta - \ln(\theta\bar{\mathrm{s}})\right) - (\eta-\theta\bar{\mathrm{s}})}{\bar{\mathrm{s}}(1-\theta)}$ and $\phi = \dfrac{\hat{\mathrm{s}}\left(\ln\hat{\mathrm{s}} - \ln(\theta\bar{\mathrm{s}})\right) - (\hat{\mathrm{s}}-\theta\bar{\mathrm{s}})}{\bar{\mathrm{s}}(1-\theta)}$. It can be seen that the ending time of the toll depends on the starting time of toll, the toll itself and the capacity η.

Case III. After tolling period

The toll is applied at t^+ and lifted at t^-. In the deterministic case, t^+, t^- and τ should be chosen so that the queue is zero at the moment the toll is applied and also immediately before it is lifted (Arnott et al. 1990b). To achieve equilibrium, there is a period τ/α adjacent to the instant t^+ without any departures, while a mass of individuals departs immediately after the instant t^-. Arnott et al. (1990b) showed that in the deterministic model, the relation between the toll and the size of the mass was $\tau = (\alpha+\gamma)\mathrm{N}_2/(2\bar{\mathrm{s}})$. Similarly, in the stochastic model, it can be shown that

$$\tau = (\alpha+\gamma)\int_{\theta\bar{s}}^{\bar{s}} \frac{N_2}{2s} f(s)ds, \tag{41}$$

where $f(s)$ is the probability density function of the bottleneck capacity. Then, for a uniformly distributed capacity function, the number of commuters departing after the toll becomes:

$$N_2 = \frac{2\bar{s}(1-\theta)\tau}{(\ln\bar{s} - \ln\theta\bar{s})(\alpha+\gamma)}. \tag{42}$$

Using the conservation condition $N_2 = N - N_0 - N_1$, and substituting (30) and (36) into (42), the toll start time can be found as:

$$t^+ = \frac{N - \kappa_2\tau}{\kappa_1}, \tag{43}$$

where

$$\kappa_1 = \frac{(\beta+\gamma)\hat{s}}{\gamma+(\alpha+\gamma)\phi},$$

$$\kappa_2 = \begin{cases} \dfrac{2\bar{s}(1-\theta)}{(\ln\bar{s}-\ln\theta\bar{s})(\alpha+\gamma)} + \dfrac{(\beta+\gamma)\psi\hat{s} - \eta(\alpha+\gamma)\phi - \eta\gamma}{\gamma+(\alpha+\gamma)\phi}\dfrac{1}{(\alpha-\beta)\psi-\beta}, & \text{if } \eta \le \hat{s}, \\ \dfrac{2\bar{s}(1-\theta)}{(\ln\bar{s}-\ln\theta\bar{s})(\alpha+\gamma)} + \dfrac{(\beta-\alpha)\psi\hat{s} - \hat{s}\gamma}{\gamma+(\alpha+\gamma)\phi}\dfrac{1}{(\alpha-\beta)\psi-\beta}, & \text{otherwise} \end{cases}$$

Hence, once the coarse toll and the parameter η are given, we can get t^+ by (43), t^- by (39), N_2 by (42), t_0' by (31), N_0 by (30), $N_1 = N - N_0 - N_2$, as well as departure rates in all intervals corresponding to the three cases discussed above. Clearly, to minimize the total travel cost of the system, the coarse toll and the parameter η should be optimized.

5.2. An Optimal Coarse Toll

In this subsection, we try to find an optimal coarse toll scheme under stochastic capacity in the bottleneck model. Firstly, we design a system in such a way as to minimize the expected system trip cost $G(\eta,\tau)$, excluding toll, as follows:

$$\min\ G(\eta,\tau) = E[C(t)]N - \tau N_1, \tag{44}$$

where $E[C(t)]$ refers to the mean trip cost, τ the toll and $N \geq 0, N_1 \geq 0$. Substituting (29) and (31) into (44), and observing the conservation rule $N_1 = N - N_0 - N_2$, we can rewrite the system trip cost as

$$G(\eta,\tau) = -\beta N\left(\frac{\tau}{(\alpha-\beta)\psi-\beta} + t^{+}\right) - (N - N_0 - N_2)\tau \,. \tag{45}$$

Replace N_0 and N_2 by (30) and (42) respectively, and note $t_0' = \dfrac{\tau}{(\alpha-\beta)\psi-\beta} + t^{+}$, Equation (45) becomes

$$G(\eta,\tau) = -\beta N t^{+} - \frac{(\alpha-\beta)\psi}{(\alpha-\beta)\psi-\beta} N\tau - \tau^{2}\left\{\frac{\eta}{(\alpha-\beta)\psi-\beta} - \frac{2\bar{s}(1-\theta)}{(\ln\bar{s}-\ln\theta\bar{s})(\alpha+\gamma)}\right\}. \tag{46}$$

Let the first-order partial derivative with respect to t^{+} be zero,

$$\frac{\partial G(\eta,\tau)}{\partial t^{+}} = 0 \,. \tag{47}$$

Substitute (43) into (47) and note $\partial\tau/\partial t^{+} = -\kappa_1/\kappa_2$, we obtain

$$-\kappa_2^{\,2}\beta N + \frac{\kappa_2\kappa_1(\alpha-\beta)\psi N}{(\alpha-\beta)\psi-\beta}$$
$$+2\left(N-\kappa_1 t^{+}\right)\kappa_1\left(\frac{\eta}{(\alpha-\beta)\psi-\beta} - \frac{2\bar{s}(1-\theta)}{(\ln\bar{s}-\ln\theta\bar{s})(\alpha+\gamma)}\right) = 0 \,. \tag{48}$$

From (48), the optimal t^{+} can be derived as follows:

$$t^{+} = g_1(\eta)N \,, \tag{49}$$

where $g_1(\eta) = \left(\dfrac{\kappa_2(\alpha-\beta)\psi/2\kappa_1}{(\alpha-\beta)\psi-\beta} - \dfrac{\kappa_2^{\,2}\beta}{2\kappa_1^{\,2}}\right)\Big/\left(\dfrac{\eta}{(\alpha-\beta)\psi-\beta} - \dfrac{2\bar{s}(1-\theta)}{(\ln\bar{s}-\ln\theta\bar{s})(\alpha+\gamma)}\right) + \dfrac{1}{\kappa_1}$.

Substitute (49) into (43), the optimal coarse toll is,

$$\tau = g_2(\eta)N \,, \tag{50}$$

where $g_2(\eta) = \dfrac{1-\kappa_1 g_1(\eta)}{\kappa_2}$.

Finally, substitute (30), (31), (36), (49) and (50) into (44), the expected system travel cost (excluding toll) is worked out as follows:

$$G(\eta,\tau) = g(\eta)N^2 , \tag{51}$$

where $g(\eta) = -\beta g_1(\eta) - \frac{(\alpha-\beta)\psi}{(\alpha-\beta)\psi-\beta} g_2(\eta) - \left\{ \frac{\eta}{(\alpha-\beta)\psi-\beta} - \frac{2\bar{s}(1-\theta)}{(\ln\bar{s}-\ln\theta\bar{s})(\alpha+\gamma)} \right\} g_2(\eta)^2$.

The method to study the case of $\alpha \geq \gamma$, i.e., some drivers depart after the mass, is similar to the above (see Appendix G, for details).

The efficiency of the optimal single-step toll subject to stochastic capacity η can be measured as follows:

$$\omega_\eta = \frac{TC_{NT} - TC_\eta}{TC_{NT} - TC_{TV}} , \tag{52}$$

where TC_{NT} and TC_{TV} are the system's total travel cost (excluding toll) generated under a no-toll scheme and a time-varying toll scheme respectively, and TC_η is the same as $G(\eta,\tau)$ which is defined in (44). The subscript η denotes the single-step coarse toll scheme. Also, the efficiency of the single-step coarse toll based on optimal capacity can be compared with the efficiency based on average capacity. However, it would be extremely difficult to obtain the relative cost reductions analytically for any θ-value in $\eta \in [\theta\bar{s}, \bar{s}]$. In the following section, we will numerically investigate ω_η and the relative efficiency. The results are presented in the last two columns in Table 4 and in Figure 11.

PROPOSITION 8. The optimal η determined by system optimal equilibrium is independent of N.

PROOF. From the above analysis, the function of the system travel cost excluding toll can be formulate as $G(\eta,\tau) = g(\eta)N^2$. In order to obtain the optimal η-value, we have to let the partial derivative of this function with respect to η be zero, i.e., $\partial\left(g(\eta)N^2\right)/\partial\eta = 0$. This means $\partial g(\eta)/\partial\eta = 0$. Since $g(\eta)$ is independent of N, so is the optimal value of η. This completes the proof. □

PROPOSITION 9. In equilibrium, the earliest departure time t_0' under the single-step coarse toll scheme is larger than that without toll, when parameter θ approaches to one.

PROOF. The earliest departure time without toll when parameter θ approaches to one has been shown in (22), i.e., $t_0 = -\frac{\gamma}{\beta+\gamma}\frac{N}{\bar{s}}$. In the stochastic bottleneck model with single-step toll, letting θ approach to one, we have $\lim_{\theta\to 1}\psi = \lim_{\theta\to 1}\frac{\eta(\ln\eta - \ln\theta\bar{s}) - (\eta - \theta\bar{s})}{\bar{s}(1-\theta)} = 0$ and $\lim_{\theta\to 1}\phi = \lim_{\theta\to 1}\frac{\hat{s}(\ln\hat{s} - \ln\theta\bar{s}) - (\hat{s} - \theta\bar{s})}{\bar{s}(1-\theta)} = 0$. Then, $\lim_{\theta\to 1}\kappa_1 = \lim_{\theta\to 1}\frac{(\beta+\gamma)\hat{s}}{\gamma + (\alpha+\gamma)\phi} = \frac{\beta+\gamma}{\gamma}\bar{s}$ and $\lim_{\theta\to 1}\kappa_2 = \left(\frac{2}{\alpha+\gamma} + \frac{1}{\beta}\right)\bar{s}$. Substituting them into the relevant equations presented in this section, we get $\tau = \frac{\beta\gamma}{2(\beta+\gamma)}\frac{N}{\bar{s}}$, $t_0' = -\frac{\gamma}{(\beta+\gamma)}\frac{N}{\bar{s}} + \frac{\gamma-\alpha}{(\beta+\gamma)(\alpha+\gamma)}\tau$, $t^+ = t_0' + \frac{\tau}{\beta}$, $t^- = -\frac{\beta}{\gamma}t^+$, and $N_2 = \frac{2\bar{s}\tau}{(\alpha+\gamma)}$. This clearly shows that

$$-t_0'\beta = \frac{\gamma\beta}{(\beta+\gamma)}\frac{N}{\bar{s}} - \frac{(\gamma-\alpha)\beta}{(\beta+\gamma)(\alpha+\gamma)}\tau \le \frac{\gamma\beta}{(\beta+\gamma)}\frac{N}{\bar{s}} = -t_0\beta .$$

Hence, $t_0' > t_0$ when θ approaches to one. This completes the proof. □

Proposition 9 presents another interesting result. That is, in equilibrium, the mean trip cost generated by the optimal single-step coarse toll is less than that generated by no-toll equilibrium, when parameter θ approaches to one. According to Proposition 2, the proposed stochastic bottleneck model becomes the deterministic model when the value of parameter θ approaches to one. Hence, Proposition 9 also holds for the deterministic bottleneck model. It should be noted, however, that this result does not always hold for other θ values. In the next section, we present numerical examples (in Table 3) to illustrate this result.

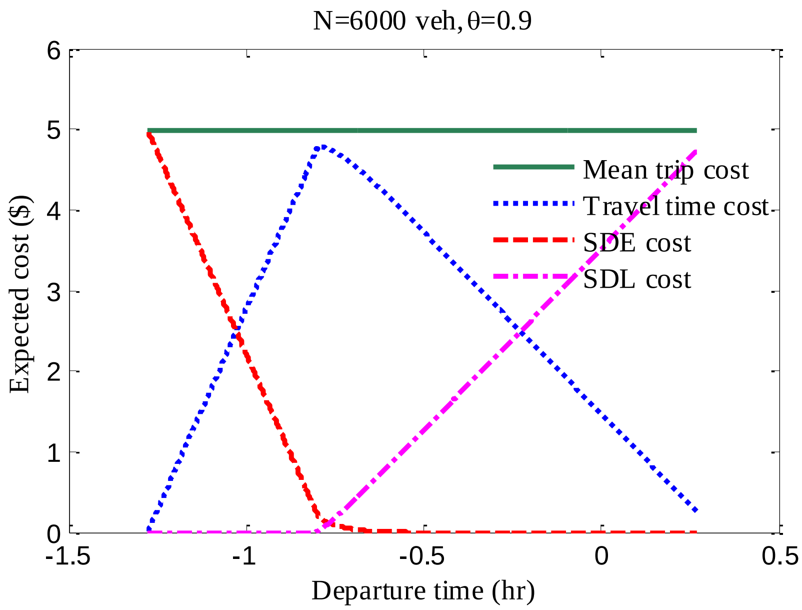

Figure 2 Mean trip cost and its components in no-toll equilibrium

6. Numerical Examples

In this section, we present numerical results for the stochastic bottleneck model without toll, with a time-varying toll, and with a single-step coarse toll. Unless otherwise specified, throughout this section, we adopt the following three parameter values from Arnott et al. (1990b): $\alpha = 6.4$ \$/hr, $\beta = 3.9$ \$/hr, $\gamma = 15.21$ \$/hr, and consider the situation with $\mathrm{N} = 6000\,\mathrm{veh}$, $\bar{s} = 4000\,\mathrm{veh/hr}$ and $\theta = 0.9$.

6.1. No-toll Equilibrium in the Stochastic Bottleneck

The differential equations (11) and (13) are solved by the Euler method with step size equal to 0.005. Figure 2 shows the mean trip cost, mean travel time cost and the mean schedule delay and early costs (SDE and SDL). It can be seen in Figure 2 that the mean trip costs of all commuters are the same and equal to 4.98, but the commuters would endure a trade-off between the cost of travel time and the cost of schedule delay. Note that the SDE and SDL curves cross at a point where their costs are none zero and the travel time cost at the crossing point does not reach the highest point. It is also worth noting that the waiting time cost is none zero at the end of the peak period, which means that the queue still exists.

It is also interesting to investigate the impact of parameter θ on the solution of the stochastic bottleneck model. We change the θ-value from 0.75 to 1.0 and solve the resultant models. Table 1 lists the mean trip costs and the watershed time instants for different θ values. It can be seen that $t_1 = t_2$ and $t_3 = t_e$ when $\theta = 1.0$. This confirms Proposition 2. We can also find from this table that the length of peak period increases as the θ-value decreases. This is consistent with Proposition 3. Since decreasing the θ-value is equivalent to increasing the travel time uncertainty, this means that commuters would leave home earlier to avoid potential losses due to larger uncertainty.

Table 1 Influence of parameter θ on the mean trip cost and the watershed time instants

θ	E[C]	t_0	t_1	t_2	t_3	t_e	$t_e - t_0$
1.00	4.66	-1.19	-0.73	-0.73	0.31	0.31	1.50
0.95	4.81	-1.24	-0.76	-0.64	0.26	0.29	1.53
0.90	4.98	-1.28	-0.80	-0.55	0.21	0.27	1.55
0.85	5.16	-1.32	-0.85	-0.43	0.16	0.25	1.57
0.80	5.36	-1.37	-0.90	-0.31	0.11	0.22	1.59
0.75	5.58	-1.43	-0.95	-0.14	0.05	0.19	1.62

Figure 3 depicts the departure rates for different θ-values. One can observe from the figure that the stochastic bottleneck model follows the deterministic model when the θ-value approaches to one. This is also consistent with Proposition 2. Figure 3 also shows that in equilibrium the departure rate during the peak period is monotonically decreasing with time, which is consistent with Theorem 2.

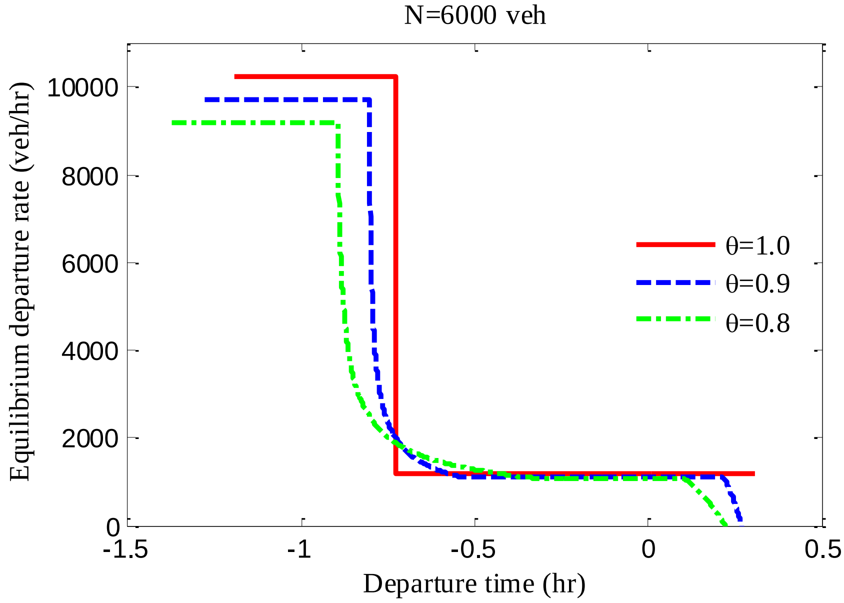

Figure 3 Influence of parameter θ on departure rate

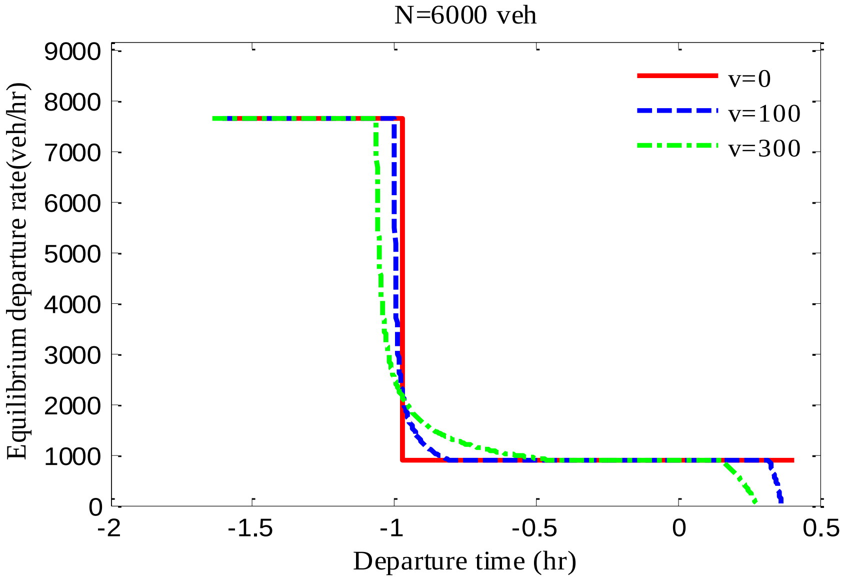

Figure 4 Equilibrium departure rate for different v values

It is interesting to see the influence of the capacity variation on commuters' departure time choice. Consider $s \in [e-v, e+v]$ and let the mean capacity e =3000veh/hr, and we solve the resultant models with different v values. The resulted equilibrium departure rates are shown in Figure 4. It can be seen that the departure rates converge to that of the deterministic bottleneck model when the v-value approaches to zero, which is consistent with Proposition 4. One can also observe from Figure 4 that the equilibrium departure rate during the peak period is monotonically decreasing with time, which is consistent with Theorem 2.

Table 2 presents the mean trip cost and the watershed time instants with different v-values. It can be seen that the length of the peak period decreases with increasing v-value. This result is consistent with Proposition 5, and suggests that commuters would leave home earlier when capacity variation increases.

Table 2 Influence of capacity variation v on the mean trip cost and the watershed time instants

v	$E[C(t)]$	t_0	t_1	t_2	t_3	t_e	$t_e - t_0$
0.00	4.46	-1.19	-0.73	-0.73	0.31	0.31	2.00
100.00	4.70	-1.20	-0.75	-0.63	0.26	0.28	1.48
200.00	4.73	-1.21	-0.76	-0.53	0.21	0.26	1.47
300.00	4.77	-1.22	-0.78	-0.43	0.16	0.23	1.45
400.00	4.81	-1.23	-0.80	-0.32	0.12	0.21	1.44

6.2. Time-varying Toll in the Stochastic Bottleneck

We now consider a time-varying toll in the stochastic bottleneck. Figure 5 depicts the mean travel times before and after implementing the time-varying toll for different θ-values. The no-toll travel times have a concave profile with the travel time peaked in the middle of the departure time period, whilst the travel times under time-varying toll are linearly increasing with departure time. It can also be seen that the travel time with tolling are significantly lower than those without tolling, suggesting that the time-varying toll can significantly reduce the commuters' travel time and the queue behind the bottleneck. Specifically, the queue can be completely eliminated when the θ-value equals one. The mean travel time of the last commuter does not change, no matter whether the toll is applied or not. This is consistent with Proposition 7. The time-varying tolls for four different θ-values are shown in Figure 6. The results confirm the Proposition 6, i.e., the time-varying tolls are nonnegative.

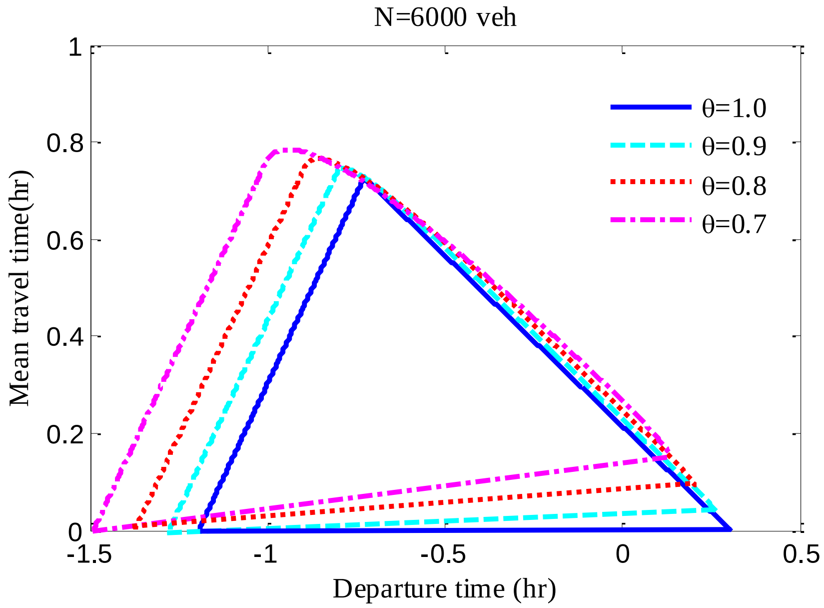

Figure 5 Travel time distributions with departure timeunder no-toll (the upper concave profiles) and time-varying toll (the lower linear distributions)

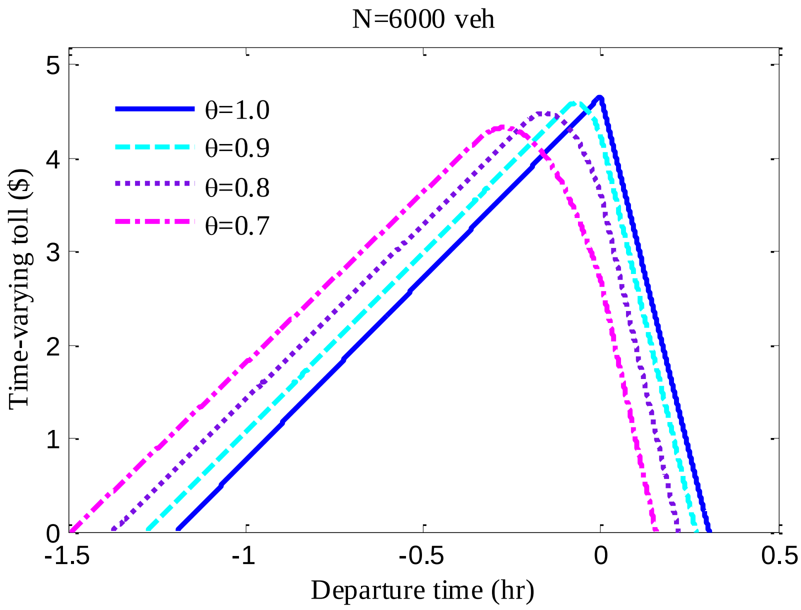

Figure 6 Time-varying toll for different θ values

6.3 Single-step Coarse Toll in the Stochastic Bottleneck

For any given θ-value, the system's total travel costs against parameter η and the minimum are shown in Figure 7(a). The numbers in brackets represent the optimal η-value and the corresponding total system travel cost. We can see that the total travel cost decreases as θ-value increases. When the θ-value is one, the optimal departure rate equals the maximal capacity. This result coincides with that of a single-step coarse toll in the deterministic bottleneck model.

For comparison, we solve the bottleneck model for an increased total number of commuters, i.e., letting $\mathrm{N} = 7000$ veh. The results are presented in Figure 7(b) and they show that, whilst the total system travel costs increased with larger N value, the optimal η-values do not change with N value. This is consistent with Proposition 8.

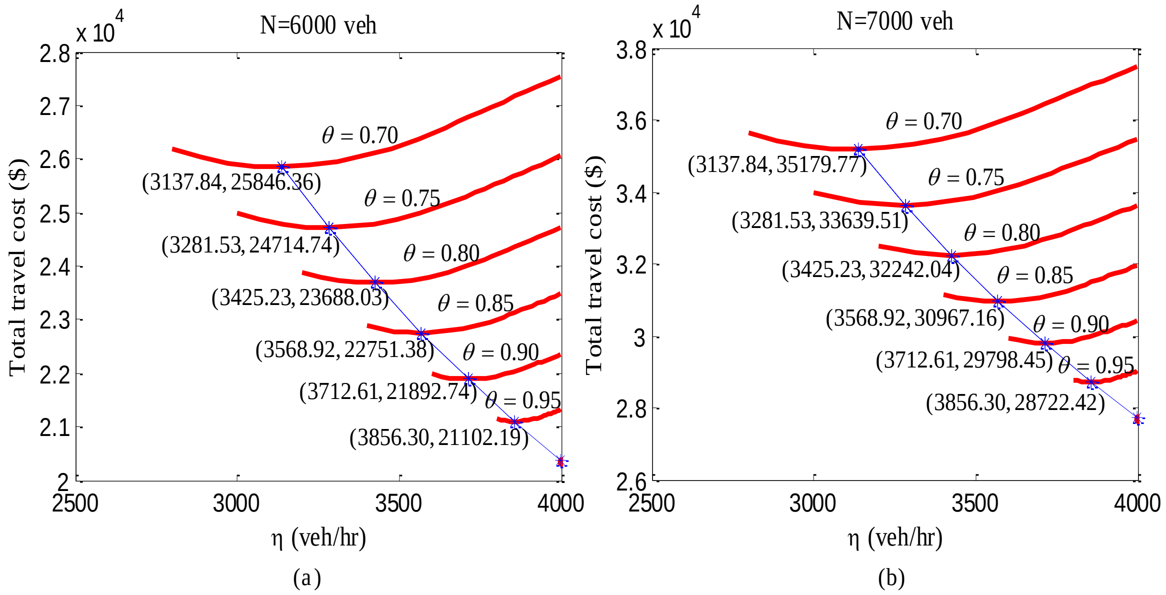

Figure 7 Total travel cost for different θ and N values

We further investigate the difference between the earliest departure time with the single-step coarse toll and that without, as denoted by $t_0' - t_0$. Figure 8 shows the variation of $t_0' - t_0$ with different values of β, γ and θ. Figure 8(a) shows that the value of $t_0' - t_0$ can be either positive or negative when $\theta = 0.7$, implying that the single-step coarse toll can either increase or decrease the commuters' travel cost when capacity variation is large. However, Figure 8(b) shows that the value of $t_0' - t_0$ is always positive when $\theta = 1.0$ (i.e., the deterministic bottleneck model), which is consistent with Proposition 9. Hence, the optimal single-step coarse toll can reduce the commuters' trip cost in the deterministic bottleneck model (as has been reported in Arnott et al. (1990b)), whilst for stochastic bottleneck, the toll can either reduce or increase the trip costs.

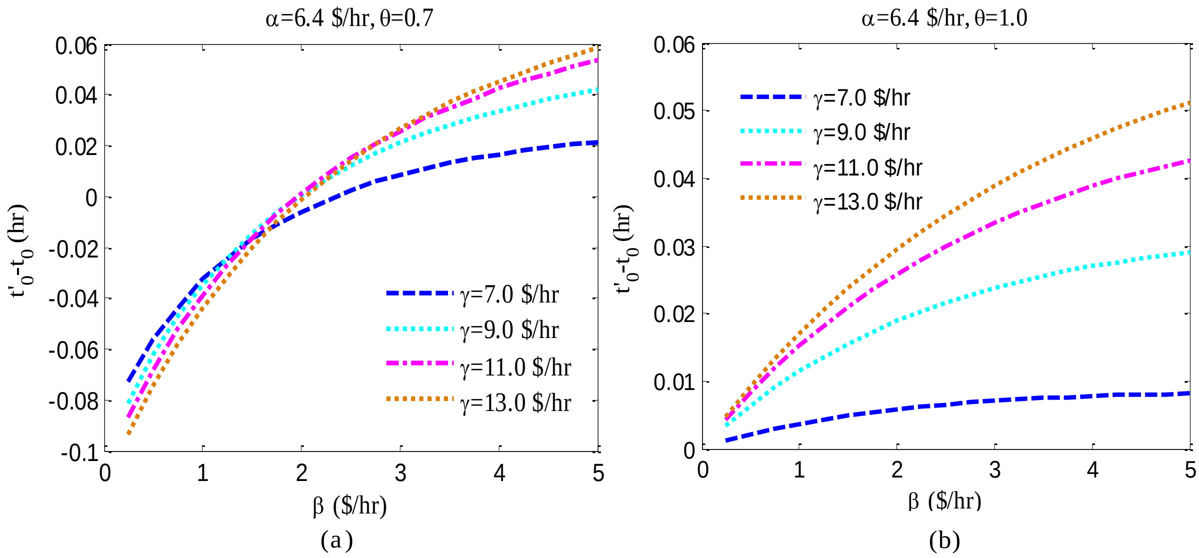

Figure 8 Difference between the earliest departure time with a single step coarse toll and that with no toll

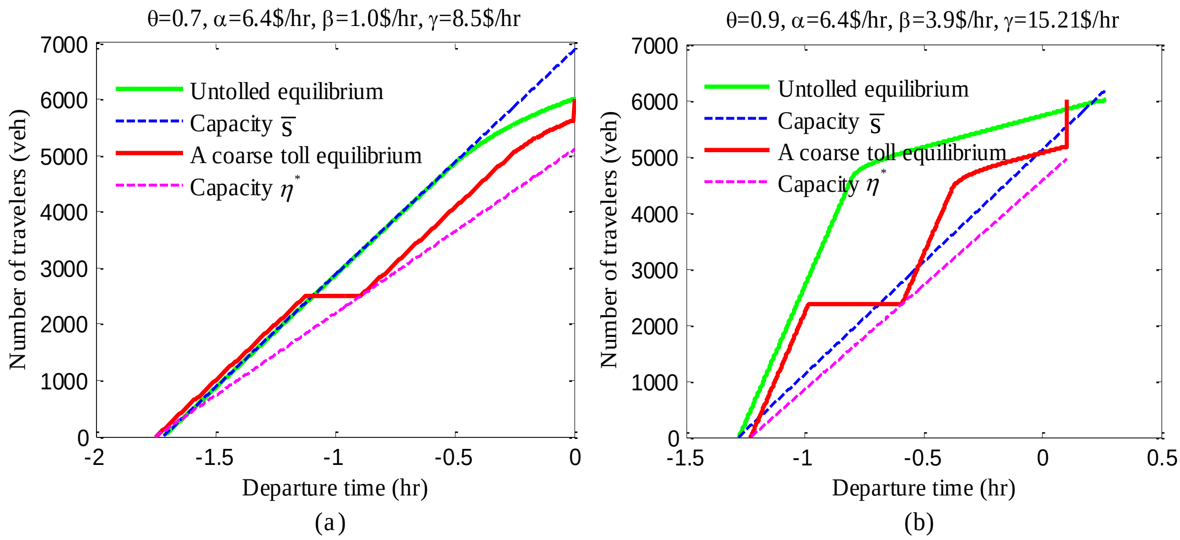

Figure 9 Cumulative departures with different parameters of single-step coarse toll in equilibrium

Figure 9 illustrates two examples in which the single-step coarse toll can either advance or postpone the earliest departure time. In Figure 9(a), with input data of $\alpha = 6.4$ \$/hr, $\beta = 1.0$ \$/hr, $\gamma = 8.5$ \$/hr, $\theta = 0.7$ \$/hr, $\bar{s} = 4000$ veh/hr and $\mathrm{N} = 6000$ veh, it can be seen that the earliest departure time is advanced (i.e., moved earlier) after imposing the single-step coarse toll. Whilst in Figure 9(b), with the input data of $\alpha = 6.4$ \$/hr, $\beta = 3.9$ \$/hr, $\gamma = 15.21$ \$/hr, $\theta = 0.9$, $\bar{s} = 4000$ veh/hr and $\mathrm{N} = 6000$ veh, the earliest departure time is postponed when the single-step coarse toll is implemented.

Finally, we present in Table 3 the commuters' mean trip cost and in Table 4 the system's total travel cost (excluding toll) when schemes of no toll, time-varying toll and single-step coarse toll are separately applied in the stochastic bottleneck model. We can see that both the system total cost and the individual mean trip cost increase with increasing variation of the bottleneck capacity, regardless of the schemes adopted. This is consistent with the theoretical analyses conducted in earlier parts of the paper. For all θ-values analyzed, the time-varying toll scheme leads to the lowest system total cost, followed by the single-step coarse toll scheme, whilst the non toll scheme generates the highest cost. The numerical results in Table 3 also confirm that the time-varying toll does not change the individual's mean trip cost, but the single-step coarse toll can reduce the individual's mean trip cost. With smaller θ-values (i.e. larger capacity variations), the reduction becomes smaller. Table 4 also presents the results for the single-step coarse toll scheme under an average capacity and as expected, the system total travel costs under the optimal capacity are less than those under the average capacity.

Table 3 Individual mean trip cost under three schemes

Individual's mean trip cost ($)				
θ	Non toll	Time-varying toll	Single-step coarse toll	
			Optimal capacity	Average capacity
1.00	4.66	4.66	4.46	4.46
0.95	4.81	4.81	4.62	4.61
0.90	4.98	4.98	4.79	4.77
0.85	5.16	5.16	4.98	4.95
0.80	5.36	5.36	5.18	5.14
0.75	5.58	5.58	5.40	5.34

Table 4 System total travel cost under three schemes and the efficiency of a coarse toll

θ	System's total travel cost (excluding toll) ($)				Efficiency of a single-step coarse toll	
	Non toll	Time-varying toll	Single-step coarse toll		Optimal capacity	Average capacity
			Optimal capacity	Average capacity		
1.00	27936.74	13968.37	20371.48	20371.48	54.2%	54.2%
0.95	28875.50	14738.73	21102.19	21125.88	55.0%	54.8%
0.90	29886.84	15612.44	21892.74	21944.14	56.0%	55.6%
0.85	30980.29	16598.36	22751.38	22835.26	57.2%	56.6%

0.80	32167.25	17718.53	23688.03	23810.14	58.7%	57.8%
0.75	33461.50	19000.82	24714.74	24882.04	60.5%	59.3%

Figure 10 shows the percentage reduction in total travel cost (excluding toll) generated by single-step coarse toll scheme based on optimal capacity and average capacity, as a function of θ. The percentages are computed by using (52). It can be seen that the scheme based on average capacity produces consistently smaller cost reduction than the scheme based on the optimal capacity, and the difference increases with decreasing θ value. At θ =0.75, the difference is 1.2%. As the parameter θ approaches to one, the differences disappears, and the efficiency ω_η value under both schemes is 54.2% (see Table 4).

It is of interest to make a sensitivity analysis of the model parameters on the relative efficiency gains of the different capacity-based coarse tolls. With the default values of $\alpha = 6.4$ \$/hr, $\beta = 3.9$ \$/hr, $\mathrm{s} = 4000$ veh/hr, $\mathrm{N} = 6000$ veh, we let the ratio γ/β vary from 0.5 to 4. Figure 11 plots the relative efficiency of the average capacity-based single-step coarse toll to the optimal capacity-based toll, i.e., $\omega_{\mathrm{a}}/\omega_{\mathrm{o}}$ where the subscripts a and o denote the average capacity and the optimal capacity respectively. ω_η ($\eta = \mathrm{a}$ or o) is defined in (52). It can be seen that the ratio approximates to 1 when γ/β is around 1.5.

Finally, Table 5 summaries the system’s total toll revenues under two tolling schemes in a stochastic bottleneck model (with random capacity following a uniform distribution) vs a deterministic bottleneck model (with a fixed average capacity). It can be seen that the total revenue under both tolling schemes, regardless of the capacity models used, always increases with decreasing value of θ. This suggests that a decline in capacity will increase the revenue collected from tolling. Under a time-varying toll scheme, the revenue collected from a stochastic bottleneck model is less than that from a deterministic bottleneck model with average capacity. However, the contrary is true under a single-step coarse toll scheme.

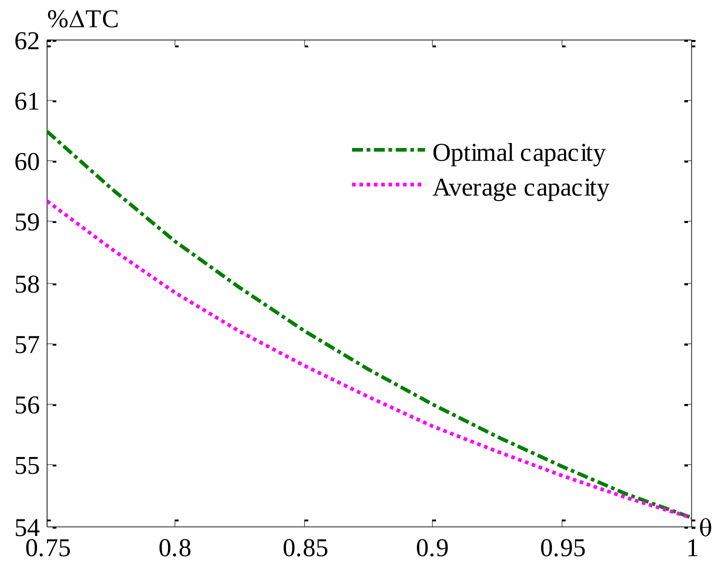

Figure10 Percentage reduction in travel cost (excluding toll) due to coarse toll as a function of θ

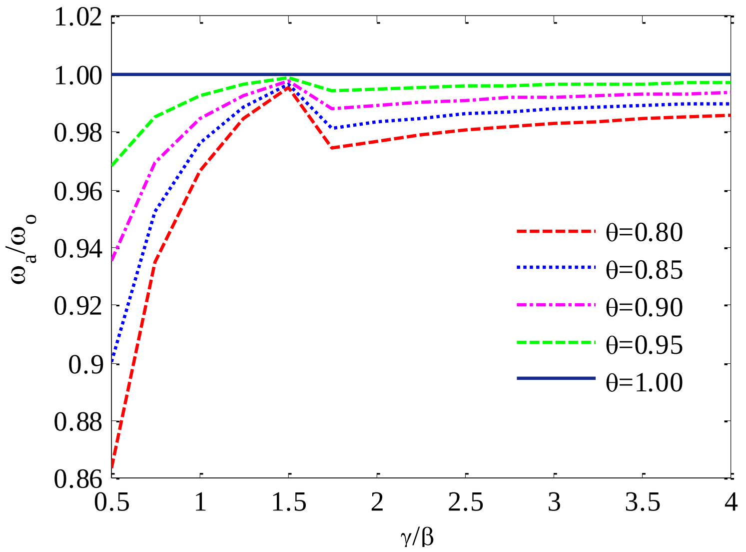

Figure 11 Efficiency gain from a coarse toll upon average capacity against that upon optimal capacity

Table 5 System total toll revenue under two tolling schemes

θ	Random capacity $\hat{s}$	Average capacity $\bar{s}(1+\theta)/2$	System's total toll revenue ($)			
			Time-varying toll		Single-step coarse toll	
			Random	Average	Optimal	Average

			capacity	capacity	capacity	capacity
1.00	4000.00	4000.00	13968.37	13968.37	6403.10	6403.10
0.95	3940.77	3900.00	14136.72	14326.53	6618.98	6567.28
0.90	3881.54	3800.00	14274.40	14703.54	6850.45	6740.10
0.85	3822.30	3700.00	14381.94	15100.94	7099.30	6922.27
0.80	3763.07	3600.00	14448.73	15520.41	7378.83	7114.55
0.75	3703.84	3500.00	14460.68	15963.84	7672.92	7317.83

7. Conclusions

In this paper, we have extended the well-known Vickrey's bottleneck model for studying the commuters' departure time choice behavior with a stochastic bottleneck capacity. We assume the bottleneck capacity follows a uniform distribution and the commuters' departure time choice follows UE principle in terms of their mean trip cost. Analytical solutions of the stochastic bottleneck model have been derived, and numerical results for a range of different scenarios produced. Both analytical and numerical results show that the consideration of capacity uncertainty increases the commuters' mean trip cost and lengthens the peak period.

Further more, we have developed two tolling schemes for the stochastic bottleneck model, namely the time-varying toll and the single-step coarse toll. We have shown that the proposed piecewise time-varying toll is non-negative and can effectively reduce, even eliminate when the capacity is constant, the queues behind the bottleneck. We have also found that the single-step coarse toll may either advance or postpone the earliest departure time. The numerical results demonstrate that the system total travel costs (excluding toll) decrease under both pricing schemes, suggesting that the schemes improve the efficiency of the stochastic bottleneck.

In our future work, we plan to further extend the stochastic bottleneck model to consider commuter heterogeneity, risk preference, demand uncertainty, multiple transport modes and flexible work start times.

Acknowledgments

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Appendix A: Derivation of the departure rates in four time intervals

There are four departure time intervals to consider. In $[t_0, t_1]$, there is no schedule delay late, the expected travel cost is

$$E\left[C(t)\right] = \alpha \int_{\theta \bar{s}}^{\bar{s}} \left(\frac{R(t)}{s} + t_0 - t \right) f(s)ds + \beta \int_{\theta \bar{s}}^{\bar{s}} -\left(\frac{R(t)}{s} + t_0 \right) f(s)ds,$$

where $f(s)$ is the probability density function of the stochastic capacity, and $f(s) = 1/\left(\bar{s} - \theta \bar{s}\right)$. At equilibrium, $E\left[C(t)\right]$ must be constant, i.e., $dE\left[C(t)\right]/dt = 0$ which directly leads to

$$r(t) = \frac{\alpha}{\alpha - \beta} \frac{\bar{s}\left(1 - \theta\right)}{\ln \theta^{-1}}, \quad t_0 \le t \le t_1.$$

In $(t_1, t_2]$, both schedule delay early and late may occur, we then have

$$E\left[C(t)\right] = \alpha \int_{\theta \bar{s}}^{\bar{s}} \left(\frac{R(t)}{s} + t_0 - t \right) f(s)ds + \beta \int_{-R(t)/t_0}^{\bar{s}} -\left(\frac{R(t)}{s} + t_0 \right) f(s)ds$$
$$+ \gamma \int_{\theta \bar{s}}^{-R(t)/t_0} \left(\frac{R(t)}{s} + t_0 \right) f(s)ds.$$

At equilibrium, i.e., $dE\left[C(t)\right]/dt = 0$, the above leads to

$$r(t) = \frac{\alpha}{A + B\left(\ln R(t) + 1\right)}, \quad t_1 < t \le t_2,$$

where $A = -\left(\alpha \ln \theta + \beta \ln\left(-t_0 \bar{s}\right) + \gamma \ln\left(-t_0 \theta \bar{s}\right) + \left(\beta + \gamma\right)\right) / (\bar{s} - \bar{s}\theta)$ and $B = \left(\beta + \gamma\right) / \left(\bar{s} - \bar{s}\theta\right)$.

In $(t_2, t_3]$ with no schedule early, we have

$$E\left[C(t)\right] = \alpha \int_{\theta \bar{s}}^{\bar{s}} \left(\frac{R(t)}{s} + t_0 - t \right) f(s)ds + \gamma \int_{\theta \bar{s}}^{\bar{s}} \left(\frac{R(t)}{s} + t_0 \right) f(s)ds.$$

At equilibrium, we get

$$\mathrm{r}(\mathrm{t})=\frac{\alpha}{\alpha+\gamma}\frac{\overline{\mathrm{s}}(1-\theta)}{\ln\theta^{-1}},\quad \mathrm{t}_2<\mathrm{t}\le\mathrm{t}_3.$$

In $(\mathrm{t}_3,\mathrm{t}_\mathrm{e}]$, there is no schedule early but a queue may exist. We can find a watershed capacity of the bottleneck such that the queuing length equals zero, i.e. $\mathrm{R}(\mathrm{t})=\mathrm{s}(\mathrm{t}-\mathrm{t}_0)$, and hence the watershed capacity is $\mathrm{R}(\mathrm{t})/(\mathrm{t}-\mathrm{t}_0)$. We then have

$$\mathrm{E}[\mathrm{C}(\mathrm{t})]=\alpha\int_{\theta\overline{\mathrm{s}}}^{\frac{\mathrm{R}(\mathrm{t})}{\mathrm{t}-\mathrm{t}_0}}\left(\frac{\mathrm{R}(\mathrm{t})}{\mathrm{s}}+\mathrm{t}_0-\mathrm{t}\right)\mathrm{f}(\mathrm{s})\mathrm{d}\mathrm{s}+\gamma\int_{\theta\overline{\mathrm{s}}}^{\frac{\mathrm{R}(\mathrm{t})}{\mathrm{t}-\mathrm{t}_0}}\left(\frac{\mathrm{R}(\mathrm{t})}{\mathrm{s}}+\mathrm{t}_0\right)\mathrm{f}(\mathrm{s})\mathrm{d}\mathrm{s}+\gamma\int_{\frac{\mathrm{R}(\mathrm{t})}{\mathrm{t}-\mathrm{t}_0}}^{\overline{\mathrm{s}}}\mathrm{t}\mathrm{f}(\mathrm{s})\mathrm{d}\mathrm{s}.$$

The equilibrium condition leads to

$$\mathrm{r}(\mathrm{t})=\frac{(\alpha+\gamma)\mathrm{R}(\mathrm{t})/(\mathrm{t}-\mathrm{t}_0)-(\alpha\theta+\gamma)\overline{\mathrm{s}}}{(\alpha+\gamma)\left(\ln\mathrm{R}(\mathrm{t})-\ln\left(\theta\overline{\mathrm{s}}(\mathrm{t}-\mathrm{t}_0)\right)\right)},\quad \mathrm{t}_3<\mathrm{t}\le\mathrm{t}_\mathrm{e}.$$

Appendix B: Analytical results for light traffic demand

Following the method for analyzing the case of heavy traffic demand, here we investigate the four possible situations for light traffic demand during the peak period. We use t_1, t_2 and t_3 to denote the watershed lines of these four situations. The corresponding departure rates in these four time intervals are directly given as follows.

In $[\mathrm{t}_0,\mathrm{t}_1]$ with no schedule late, we have $\mathrm{r}(\mathrm{t})=(\alpha/(\alpha-\beta))\left(\overline{\mathrm{s}}(1-\theta)/\ln\theta^{-1}\right)$ and the boundary condition $\overline{\mathrm{SDE}}(\mathrm{t}_1)=0$.

In $(\mathrm{t}_1,\mathrm{t}_2]$ with possible schedule either early or late, we have $\mathrm{r}(\mathrm{t})=\alpha/\left(\mathrm{A}+\mathrm{B}(\ln\mathrm{R}(\mathrm{t})+1)\right)$ and the boundary condition $\mathrm{R}(\mathrm{t}_2)=(\mathrm{t}_2-\mathrm{t}_0)\overline{\mathrm{s}}$ when $\mathrm{s}=\overline{\mathrm{s}}$;

In $(\mathrm{t}_2,\mathrm{t}_3]$ with possible schedule either early or late and a possible queue, we have

$$\mathrm{r}(\mathrm{t})=\frac{(\alpha-\beta)\mathrm{R}(\mathrm{t})/(\mathrm{t}-\mathrm{t}_0)-(\alpha\theta-\beta)\overline{\mathrm{s}}}{(\alpha+\gamma)\ln(\mathrm{R}(\mathrm{t})/\theta\overline{\mathrm{s}})-(\alpha-\beta)\ln(\mathrm{t}-\mathrm{t}_0)-(\gamma+\beta)\ln(-\mathrm{t}_0)}.$$

The boundary condition is $\mathrm{r}(\mathrm{t})=0$, if $\mathrm{t}\ge0$. Correspondingly, we have $\mathrm{r}(\mathrm{t}_3)=0$ and $\mathrm{R}(\mathrm{t}_3)=\tilde{\mathrm{s}}(\mathrm{t}_3-\mathrm{t}_0)$, where $\tilde{\mathrm{s}}=\overline{\mathrm{s}}(\alpha\theta-\beta)/(\alpha-\beta)$ and hence $\mathrm{t}_3=\mathrm{t}_\mathrm{e}$. If $\mathrm{r}(\mathrm{t})=0$, $\mathrm{t}<0$, the boundary condition is then $\mathrm{t}_3=0$.

In $(t_3, t_e]$, there is no schedule early but a queue may exist. The departure rate in this interval is

$$r(t) = \frac{(\alpha+\gamma) R(t) / (t - t_0) - (\alpha\theta + \gamma) \bar{s}}{(\alpha+\gamma)\left(\ln R(t) - \ln\left(\theta \bar{s}\,(t - t_0)\right)\right)}.$$

The boundary condition is $r(t_e) = 0$. Similarly, we have $R(t_e) = \hat{s}\left(t_e - t_0\right)$, where, $\hat{s} = \bar{s}\left(\alpha\theta + \gamma\right)/(\alpha+\gamma)$.

Comparing the above results with those for heavy traffic demand, we can see that the difference only occurs in the third possible situation. This suggests that commuters who are confronted with no schedule early and a queue would switch to the interval with possible schedule delay and possible queue because the system is not very congested.

Appendix C: Impact of demand elasticity

To simplify the analysis, the following linear demand function is adopted:

$$N = N_0 + b\left(C_0 - C\right),$$

where N_0 and C_0 are the reference demand and the reference trip cost respectively, b is a positive parameter which reflects the sensitivity of traffic demand to trip cost, and C is the mean trip cost. When b approaches zero, it becomes the case of fixed demand, and when b tends to infinity, it becomes the case of perfectly elastic demand.

Substituting equation (18) into the above demand function and rearranging the equation, we obtain the total number of commuters as:

$$N = \frac{N_0 + bC_0}{1 + b\beta / \left(\hat{s}(1 - k_0)\right)}.$$

Substituting the above result into equation (23), we get the length of peak period,

$$t_e - t_0 = \frac{N}{\hat{s}} = \frac{N_0 + bC_0}{\hat{s} + b\beta/(1 - k_0)}.$$

Define $u(\theta) = \hat{s} + b\beta / \left(1 - k_0\right)$. Substituting k_0 given in equation (16) and $\hat{s} = \bar{s}\left(\alpha\theta + \gamma\right)/\left(\alpha + \gamma\right)$ into $u(\theta)$, we can obtain the first order derivative of $u(\theta)$ with respect to θ,

$$u'(\theta)=\frac{\alpha\bar{s}}{(\alpha+\gamma)}+\frac{b\beta}{(\beta+\gamma)}\frac{(\alpha+\gamma)\left(\ln\hat{s}-\ln(\theta\bar{s})\right)-\gamma(1-\theta)/\theta}{(1-\theta)^2}.$$

The first term of the above equation's right hand side is positive. In order to understand the second term, we denote $v(\theta)=(\alpha+\gamma)\left(\ln\hat{s}-\ln(\theta\bar{s})\right)-\gamma(1-\theta)/\theta$ and derive its first order derivative with respect to θ,

$$v'(\theta)=\left(1-\frac{\theta\bar{s}}{\hat{s}}\right)\frac{1}{\theta^2}\gamma.$$

Clearly $v'(\theta)\geq 0$ holds because $\theta\bar{s}\leq\hat{s}$. Therefore, $v(\theta)$ is an increasing function with respect to θ. Since $v(1)=0$, then $v(\theta)\leq 0$ for all $\theta\in(0,1]$.

If $0\leq b<\dfrac{(1-\theta)^2(\beta+\gamma)}{-v(\theta)\beta}\dfrac{\alpha\bar{s}}{\alpha+\gamma}$ holds, then $u'(\theta)>0$. This implies t_e-t_0 is a monotonically decreasing function of variable θ. Otherwise, $u'(\theta)\leq 0$ and it implies that t_e-t_0 is a monotonically increasing function of θ. Intuitively, whilst both the elastic demand N and the random capacity $\hat{s}$ are both increasing with θ, depending on the demand sensitivity b, the amount of demand increase with θ may be greater or lesser than that of capacity increase. Therefore, the length of peak period t_e-t_0 under elastic demand may also increase or decrease with respect to the value of θ, depending of the demand sensitivity to cost.

Appendix D: Proof of Theorem 2

According to equations (10), (11), (12) and (13), the departure rate $r(t)$ is continuous within each of intervals $[t_0,t_1)$, (t_1,t_2), (t_2,t_3) and $(t_3,t_e]$. From equations (11), (12) and (13), we have

$$\lim_{t\to t_1^+}r(t)=\lim_{t\to t_1^+}\frac{\alpha}{A+B\left(\ln R(t)+1\right)}=\frac{\alpha}{\alpha-\beta}\frac{\bar{s}(1-\theta)}{\ln\theta^{-1}}=\lim_{t\to t_1^-}r(t),$$

$$\lim_{t\to t_2^-}r(t)=\frac{\alpha}{A+B\left(\ln R(t_2)+1\right)}=\frac{\alpha}{\alpha+\gamma}\frac{\bar{s}(1-\theta)}{\ln\theta^{-1}}=\lim_{t\to t_2^+}r(t),$$

and

$$\lim_{t\to t_3^+} r(t) = \frac{(\alpha+\gamma)\,R(t_3)/(t_3-t_0)-(\alpha\theta+\gamma)\bar{s}}{(\alpha+\gamma)\ln\left(R(t_3)/\left(\theta\bar{s}\,(t_3-t_0)\right)\right)} = \frac{\alpha}{\alpha+\gamma}\frac{\bar{s}\,(1-\theta)}{\ln\theta^{-1}} = \lim_{t\to t_3^-} r(t)\,.$$

Hence, $r(t)$ is continuous within the interval $[t_0, t_e]$.

Equations (10) and (12) state that the departure rate $r(t)$ is constant during periods $t_0 \le t \le t_1$ and $t_2 \le t \le t_3$. Since departure rate $r(t)$ is non-negative and α is positive, the denominator of the right hand side in (11) must be positive. Furthermore, since $\theta \in [0,1)$ and $0 < \beta < \gamma$, then $B = (\beta+\gamma)/(\bar{s}-\bar{s}\theta) > 0$. By definition, the cumulative departure flow $R(t)$ is non-decreasing with respect to time t, thus, the denominator of the right hand side of (11) is non-decreasing with respect to time t. Therefore, the right hand side of (11) is non-increasing with respect to time t, i.e., the departure rate $r(t)$ is monotonically decreasing within $[t_1, t_2]$.

Let $p(\theta) = \hat{s}/\bar{s} - 1 + \ln\hat{s} - \ln(\theta\bar{s})$. Substituting $\hat{s} = \bar{s}\,(\alpha\theta+\gamma)/(\alpha+\gamma)$ into $p(\theta)$, we obtain the first order derivative of $p(\theta)$ with respect to θ,

$$p'(\theta) = \frac{\alpha}{\alpha+\gamma} - \frac{\gamma}{\theta(\alpha\theta+\gamma)}\,.$$

It is clear that $\alpha+\gamma \ge (\alpha\theta+\gamma)\theta$, for $0 < \theta \le 1$. Since $\gamma > \alpha$, we get $p'(\theta) < 0$. So, $p(\theta)$ is a monotonically decreasing function of the parameter θ. Due to $p(1) = 0$ and $p(\theta) \ge 0$ for all $\theta \in (0,1]$, we then have

$$\frac{\hat{s}}{\bar{s}} \ge 1 - \ln\frac{\hat{s}}{\theta\bar{s}}\,.$$

Multiplying both hand sides of the above equation by $t - t_0$, we get

$$\hat{s}\,(t-t_0) \ge \bar{s}\,(t-t_0)\left(1-\ln\frac{\hat{s}}{\theta\bar{s}}\right).$$

Substituting the second inequality of equation (14), i.e., $R(t) \le \bar{s}\,(t-t_0)$, $t \in [t_3, t_e]$, into the above, we obtain

$$\hat{s}\,(t-t_0) \ge R(t)\left(1-\ln\frac{\hat{s}}{\theta\bar{s}}\right).$$

From the first inequality of equation (14), i.e., $\hat{s}(t-t_0) \le R(t)$, $t \in [t_3, t_e]$, we have

$$\ln \frac{\hat{s}}{\theta \bar{s}} = \ln \frac{\hat{s}(t-t_0)}{\theta \bar{s}(t-t_0)} \le \ln \frac{R(t)}{\theta \bar{s}(t-t_0)}.$$

Combining the above equation with the one before it leads to

$$\ln \frac{R(t)}{\theta \bar{s}(t-t_0)} + \frac{\hat{s}(t-t_0)}{R(t)} - 1 \ge 0,$$

which can be rewritten as

$$R(t) \ge \frac{R(t)/(t-t_0) - \hat{s}}{\ln R(t) - \ln(\theta \bar{s}(t-t_0))}(t-t_0) = r(t)(t-t_0), \quad t \in [t_3, t_e].$$

Note that the above equality employs the definition $\hat{s} = \bar{s}(\alpha\theta + \gamma)/(\alpha + \gamma)$ and (13) for $r(t)$.

The first order derivative of (13) is

$$\frac{dr(t)}{dt} = \frac{\ln R(t) - \ln \theta \bar{s}(t-t_0) + \hat{s}(t-t_0)(R(t))^{-1} - 1}{\left(\ln R(t) - \ln(\theta \bar{s}(t-t_0))\right)^2}\left(\frac{r(t)(t-t_0) - R(t)}{(t-t_0)^2}\right), \quad t \in [t_3, t_e].$$

Thus, we can conclude $dr(t)/dt \le 0$ for all $t \in (t_3, t_e]$.

In summary, the departure rate $r(t)$ is monotonically decreasing within all four intervals and at their boundaries. Considering the continuity of $r(t)$ for all $t \in [t_0, t_e]$, we conclude that $r(t)$ is monotonically decreasing within $[t_0, t_e]$. This completes the proof. □

Appendix E: Derivation of the time-varying toll

There are three departure time intervals to be considered. In $[t_0, t_1]$, there is no schedule late, commuters departing at time t must arrive at destination early and may experience queue, then the expected trip cost can be formulated as

$$E[c(t)] = \alpha \int_{\theta\bar{s}}^{\hat{s}} \frac{\hat{s}-s}{s}(t-t_0)\ f(s)ds - \beta \int_{\theta\bar{s}}^{\hat{s}} \left\{ \frac{\hat{s}(t-t_0)}{s} + t_0 \right\} f(s)ds - \beta \int_{\hat{s}}^{\bar{s}} t\ f(s)ds$$

$$= (\alpha - \beta)(t-t_0)\frac{\hat{s}\left(\ln \hat{s} - \ln(\theta\bar{s}) - 1\right) + \theta\bar{s}}{\bar{s}(1-\theta)} - \beta t$$

The boundary condition for this situation is $\overline{SDE}(t_1) = 0$, $t_1 = t_0(\hat{s} - \theta\bar{s})/\hat{s}$ when $s = \theta\bar{s}$, we then have $R(t_1) = -t_0\theta\bar{s}$.

In $(t_1, t_2]$, commuters departing home and traversing the bottleneck for work will endure both schedule delay early and late. If the capacity of the bottleneck is large enough, only schedule delay early will occur. On the other hand, schedule delay late occurs when the capacity is very small. The watershed capacity satisfies $s = -\hat{s}(t - t_0)/t_0$, therefore, the trip cost in the absence of toll can be reformulated as

$$E[c(t)] = \alpha\int_{\theta\bar{s}}^{\hat{s}} \frac{\hat{s} - s}{s}(t - t_0) f(s)ds - \beta\left(\int_{\frac{\hat{s}(t-t_0)}{-t_0}}^{\hat{s}} \left(\frac{\hat{s}}{s}(t - t_0) + t_0\right) f(s)ds + \int_{\hat{s}}^{\bar{s}} tf(s)ds\right)$$

$$+\gamma\int_{\theta\bar{s}}^{\frac{\hat{s}(t-t_0)}{-t_0}} \left(\frac{\hat{s}}{s}(t - t_0) + t_0\right) f(s)ds$$

$$= \alpha(t - t_0)\frac{\hat{s}\left(\ln\hat{s} - \ln(\theta\bar{s}) - 1\right) + \theta\bar{s}}{\bar{s}(1-\theta)} - \beta\frac{\hat{s}(t - t_0)\left(\ln t_0 - \ln(t_0 - t)\right) + t\bar{s}}{\bar{s}(1-\theta)}$$

$$+\gamma\frac{\hat{s}(t - t_0)\left(\ln\left(\hat{s}(t_0 - t)\right) - \ln\left(t_0\theta\bar{s}\right) - 1\right) - t_0\theta\bar{s}}{\bar{s}(1-\theta)}.$$

The boundary condition for this case is $\overline{SDE}(t_2) = \overline{SDL}(t_2) = 0$ when $s = \hat{s}$, we then have $R(t_2) = -t_0\hat{s}$, which means $t_2 = 0$.

Commuters departing at time $[t_2, t_e]$ only experience schedule delay late. Thus, the mean trip cost without toll can be formulated as

$$E[c(t)] = \alpha\int_{\theta\bar{s}}^{\hat{s}} \frac{(\hat{s} - s)}{s}(t - t_0)\ f(s)ds + \gamma\ f(s)\left(\int_{\theta\bar{s}}^{\hat{s}}\left(\frac{\hat{s}}{s}(t - t_0) + t_0\right)ds + \int_{\hat{s}}^{\bar{s}} tds\right)$$

$$= \frac{(t - t_0)(\alpha\theta + \gamma)\left(\ln\hat{s} - \ln(\theta\bar{s})\right)}{(1-\theta)} + \gamma t_0.$$

The boundary condition for this case is $R(t_e) = \hat{s}(t_e - t_0)$, i.e., the queuing length at time t_e equals zero when $s = \hat{s}$.

Finally, by letting $E[C(t)] = -t_0\beta$ and using (24), we can obtain the piecewise time-varying toll as represented in (27).

Appendix F: Derivation of the end time of tolling

For $\eta \le \hat{s}$, according to the definitions made for $E[c(t^+,\eta)]$ and $E[c(t^-,\eta)]$, we have

$$E[c(t^+,\eta)] = \alpha E[T(t^+,\eta)] - \beta E[T(t^+,\eta)+t^+]$$

$$= (\alpha-\beta)\int_{\theta\bar{s}}^{\eta}(\frac{N_0}{s} - t^+ + t_0')\, f(s)ds - \beta\int_{\theta\bar{s}}^{\bar{s}} t^+ f(s)ds$$

$$= (\alpha-\beta)\left(t^+ - t_0'\right)\psi - \beta t^+,$$

$$E[c(t^-,\eta)] = \alpha E[T(t^-,\eta)] + \gamma E[T(t^-,\eta)+t^-]$$

$$= (\alpha+\gamma)\left(\int_{\theta\bar{s}}^{\eta}\frac{N_1+N_0-s\left(t^- - t_0'\right)}{s} f(s)ds + \int_{\eta}^{\hat{s}}\frac{N_1 - s\left(t^- - t^+\right)}{s} f(s)ds\right) + \gamma\int_{\theta\bar{s}}^{\bar{s}} t^- f(s)ds$$

$$= (\alpha+\gamma)(t^+ - t_0')\psi + (\alpha+\gamma)(t^- - t^+)\phi + \gamma t^-,$$

where $\psi = \dfrac{\eta\left(\ln\eta - \ln(\theta\bar{s})\right) - \left(\eta - \theta\bar{s}\right)}{\bar{s}\left(1-\theta\right)}$. Substituting (31) into the above two equations and equalizing them, we get

$$t^- = \frac{\dfrac{\tau\left(\beta+\gamma\right)\psi}{\left(\alpha-\beta\right)\psi-\beta} + \left(\left(\alpha+\gamma\right)\phi-\beta\right)t^+}{\gamma+\left(\alpha+\gamma\right)\phi},$$

where $\phi = \dfrac{\hat{s}\left(\ln\hat{s} - \ln(\theta\bar{s})\right) - \left(\hat{s} - \theta\bar{s}\right)}{\bar{s}\left(1-\theta\right)}$.

For $\eta > \hat{s}$, similarly, we have

$$E[c(t^+,\eta)] = (\alpha-\beta)\left(t^+ - t_0'\right)\psi - \beta t^+,$$

$$E[c(t^-,\eta)] = (\alpha+\gamma)\int_{\theta\bar{s}}^{\hat{s}}\left(\frac{R(t^-)}{s} - \left(t^- - t_0'\right)\right) f(s)ds + \gamma\int_{\theta\bar{s}}^{\bar{s}} t^- f(s)ds$$

$$= (\alpha+\gamma)\left(t^- - t_0'\right)\phi + \gamma t^-,$$

$$t^- = \frac{\dfrac{\left(\alpha+\gamma\right)\phi-\left(\alpha-\beta\right)\psi}{\left(\alpha-\beta\right)\psi-\beta}\tau + \left(\left(\alpha+\gamma\right)\phi-\beta\right)t^+}{\gamma+\left(\alpha+\gamma\right)\phi}.$$

Appendix G: Single-step coarse toll when $\gamma \le \alpha$

Lindsey et al. (2012) showed that in the deterministic model, if $\gamma \le \alpha$ and assuming the queue from the mass departure has not yet dissipated, a commupter who departs at t'_d after the mass must experience the cost equal to the expected generalized cost of being in the mass, i.e., $\gamma t'_d + (\alpha+\gamma)\left(N_2 - s\left(t'_d - t^-\right)\right)/\bar{s} = \gamma t^- + (\alpha+\gamma) N_2/(2\bar{s})$. Similarly, in the stochastic model, we have

$$\gamma t'_d + (\alpha+\gamma)\int_{\theta\bar{s}}^{\bar{s}} \frac{N_2 - s\left(t'_d - t^-\right)}{s} f(s)\,ds = \gamma t^- + (\alpha+\gamma)\int_{\theta\bar{s}}^{\bar{s}} \frac{N_2}{2s} f(s)\,ds .$$

Then,

$$t'_d = \frac{(\alpha+\gamma)}{\alpha}\frac{N_2 \ln\theta^{-1}}{2\bar{s}(1-\theta)} + t^- .$$

Since the single-step coarse toll does not affect the departure rate of commuters before the mass for either condition $\alpha < \gamma$ and $\alpha \ge \gamma$, we have the same departure rates as shown in (32), (33), (34) and (35) with respect to the corresponding time intervals. In stochastic model, there are two possible situations for commuters who depart after the mass. We use t'_d and t'_s to denote the watershed lines of departure times, N_3 and N_4 the number of commuters in these two situations, respectively.

Situation I. Users always arrive late in $\left[t'_d, t'_s\right]$. Similar to the no-toll equilibrium, the departure rate in this interval is

$$r(t) = \frac{\alpha}{\alpha+\gamma}\frac{\bar{s}(1-\theta)}{\ln\theta^{-1}}, \quad t'_d < t \le t'_s .$$

The boundary condition for this situation is $N_2 + N_3 = \bar{s}\left(t'_s - t^-\right)$, i.e. the queue length at time t'_s equals zero when $s = \bar{s}$.

Situation II. Users always arrive late and may or may not incur a queuing delay in $\left[t'_s, t'_e\right]$. In this interval, the queue length may fall to zero and the departure rate becomes

$$r(t)=\begin{cases}\left(\dfrac{R(t)-(N_0+N_1)}{(t-t^-)}-\hat{s}\right)\Big/\ln\dfrac{R(t)-(N_0+N_1)}{(t-t^-)\theta\bar{s}}, & \text{for } t\in[t_s',t_e'],\ \text{if } \eta\le\hat{s}\\ \left(\dfrac{R(t)}{(t-t_0)}-\hat{s}\right)\Big/\ln\dfrac{R(t)}{(t-t_0)\theta\bar{s}}, & \text{for } t\in[t_s',t_e'],\ \text{otherwise}\end{cases}$$

The boundary condition for this situation is $r(t_e')=0$, i.e., $N_2+N_3+N_4=\hat{s}\left(t_e'-t^-\right)$.

Using the boundary condition $N_3+N_2=\bar{s}\left(t_s'-t^-\right)$ and the constant departure rate $r(t)$ in time interval $[t_d',t_s']$, we can get

$$t_s'=\frac{N_2-r(t)t_d'+\bar{s}t^-}{\bar{s}-r(t)}.$$

Finally, combining t_d' and t_s' with (31), (36), (39), (40), (41) and substituting them into (44), and letting the first-order partial derivative of (44) with respect to t^+ be zero, we obtain

$$t^+=g_1(\eta)N,$$

where

$$g_1(\eta)=\begin{cases}\dfrac{-\beta(\hat{s}\sigma_3-\eta\sigma_1)/(\hat{s}\sigma_2-\eta\sigma_1)}{2\sigma_4\hat{s}(1-k_0)\left(\psi\hat{s}(\beta+\gamma)+\hat{s}\sigma_2-\eta\sigma_1\right)}-\dfrac{\sigma_1\left(2\psi\hat{s}+(\hat{s}\sigma_2-\eta\sigma_1)/(\beta+\gamma)\right)}{2\left(\psi\hat{s}(\beta+\gamma)+\hat{s}\sigma_2-\eta\sigma_1\right)}, & \text{if } \eta\le\hat{s}\\ \dfrac{1}{\hat{s}(k_0-1)}-\dfrac{1/2}{\hat{s}\sigma_4/\sigma_2-\eta}, & \text{otherwise}\end{cases}$$

$\sigma_1=(\alpha+\gamma)\psi-\beta$, $\sigma_2=(\alpha+\gamma)\phi+\gamma$, $\sigma_3=(\alpha+\gamma)\phi-\beta$ and $\sigma_4=(\alpha-\beta)\psi-\beta$.

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Figure and Table captions:

Figure 1 Equilibrium departures with stochastic capacity in a single bottleneck

Figure 2 Mean trip cost and its components in no-toll equilibrium

Figure 3 Influence of parameter value θ on departure rate

Figure 4 Equilibrium departure rate for different v values

Figure 5 Travel time distributions with departure timeunder no-toll (the upper concave profiles) and time-varying toll (the lower linear distributions). Figure 6 Time-varying toll for different θ values

Figure 7 Total travel cost for different θ and N values

Figure 8 Difference between the earliest departure time with a single step coarse toll and that with no toll

Figure 9 Cumulative departures with different parameters of single-step coarse toll in equilibrium

Figure 10 Percentage reduction in travel cost (excluding toll) as a function of θ

Figure 11 Relative efficiency from a coarse toll upon average capacity against that upon optimal capacity

Table 1 Influence of parameter θ on the mean trip cost and the watershed time instants

Table 2 Influence of capacity variation v on the mean trip cost and the watershed time instants

Table 3 Individuals mean trip cost under three schemes

Table 4 System total travel cost under three schemes and the efficiency of a coarse toll

Table 5 System total toll revenue under two pricing schemes