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Multi-periodic Repetitive Control System: A Lyapunov Stability Analysis for MIMO Systems

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Abstract: A multi-input/multi-output (MIMO) repetitive control problem of tracking and disturbance rejection is considered when both reference and disturbance signals are finite linear combinations of periodic but not necessarily sinusoidal signals. Lyapunov stability analyses under a positive real condition (and a natural relaxation) and exponential stability under a strict positive real condition are provided together with bounds on the induced L_2 and RMS gains of the closed loop system. It is shown that similar Lyapunov stability results apply when the plant is a positive real state-delay system. Extension of the analyses to a class of nonlinear systems is discussed and indicates a good degree of robustness in the design.

1. Introduction

Signals associated with repetitive or rotating actions are often of periodic nature or are subjected to forms of periodic disturbances. In order to track exactly or/and reject perfectly this kind of signal, an internal model that generates the corresponding periodic signals should be included in the closed-loop (Francis and Wonham, 1975). This system is called a repetitive control system and has been widely used in control problems with robotics (Kaneko and Horowitz, 1997), motor (Kobayashi et. al., 1999), rolling process (Garimella and Srinivasan, 1996), rotating mechanisms (Fung, et. al., 2000), and many more (Tzou, et. al., 1999; Moon, 1998; Manayathara, et. al., 1996).

If the signal to be tracked or rejected contains only a finite number of sinusoidal frequencies, a finite-dimensional model can serve as internal model (Bai and Wu, 1998). More generally a periodic signal contains infinitely many relevant frequencies i.e. the fundamental frequency and its Fourier harmonics. Within a certain bandwidth, a (higher order) compensator is required but an infinite-dimensional internal model is more appropriate and more accurately reflects the internal model requirements. The one proposed for periodic (and multi-periodic) situations is built on (infinite dimensional) time-delays. Perhaps the first infinite-dimensional repetitive controller was proposed by Inoue et al (1981). The infinitely many poles of the delay system on the imaginary axis are $ik\nu, \mathrm{k} = 0,\pm1,...,$ where $\nu = 2\pi/\tau$ is the fundamental frequency. This pole content makes the system capable of generating signals containing frequency components of any periodic reference/disturbance of period τ. Hara et al (1988) proposed a modified MIMO infinite-dimensional repetitive controller by introducing a low-pass filter aiming at improving the system stability and robustness, at a cost of losing tracking accuracy at high frequencies.

In many practical situations the reference and/or disturbance signals may consist of a superposition of signals of different fundamental periods. The overall signal may not be periodic as the ratio of these frequencies may be irrational. The analysis of this more complex situation leads to the idea of the so-called multi-periodic repetitive control, which has received very little attention (Weiss, 1997; Owens et al., 2001) although there is a known H_∞ stability condition available based on input-output transfer function for linear MIMO single/multi-periodic systems (Weiss, 1997). In contrast, this paper presents a 'Lyapunov condition' providing a state-space characterisation of closed-loop stability. This approach provides insight into dynamics and permits some extensions to the nonlinear case using a quadratic Lyapunov function and clearly defined sector-bounded nonlinearities.

The paper is organized as follows. In section 2, multi-periodic repetitive control is introduced and results for Lyapunov stability for constant filters is proposed based on positive real lemma. In section 3, Lyapunov stability for dynamical filters based on bounded real lemma is studied. Some relaxation of system positive-realness is indicated in section 4 and the possibility of exponential stability is proved under a strict positive real condition in section 5. In section 6, Lyapunov stability analysis is extended to state-delay systems and Section 7 presents a simulation example. Section 8 discusses the possible extension of the Lyapunov analysis to a class of nonlinear systems. Finally in section 9, conclusions are provided.

2. Lyapunov Stability Analysis for MIMO Linear System

The stability of typical single-periodic repetitive control has been studied by many authors (Hara et al, 1988, Weiss, 1997). Weiss also gave a proof of stability for two period repetitive systems. In this section, the basic problem is formulated and Lyapunov's second method is used for stability analysis of arbitrary multi-repetitive multi-input-multi-output (MIMO) systems using a natural form of internal model.

A MIMO multi-periodic repetitive control system considered in this paper is shown in Figure 1. The plant Σ_G to be controlled is *p*-input/*p*-output, finite-dimensional, linear time-invariant, and its transfer function $G(s) = C(sI\text{-}A)^{-1}B$ is a $p \times p$ transfer function matrix. The plant can be assumed to contain other feedback and series control elements but this possibility does not affect the following analysis although it may have impact on control design decisions. The periods of the components of both the external reference r and disturbance d are denoted by $\tau_i, i = 1,\dots,m$. They are assumed known.

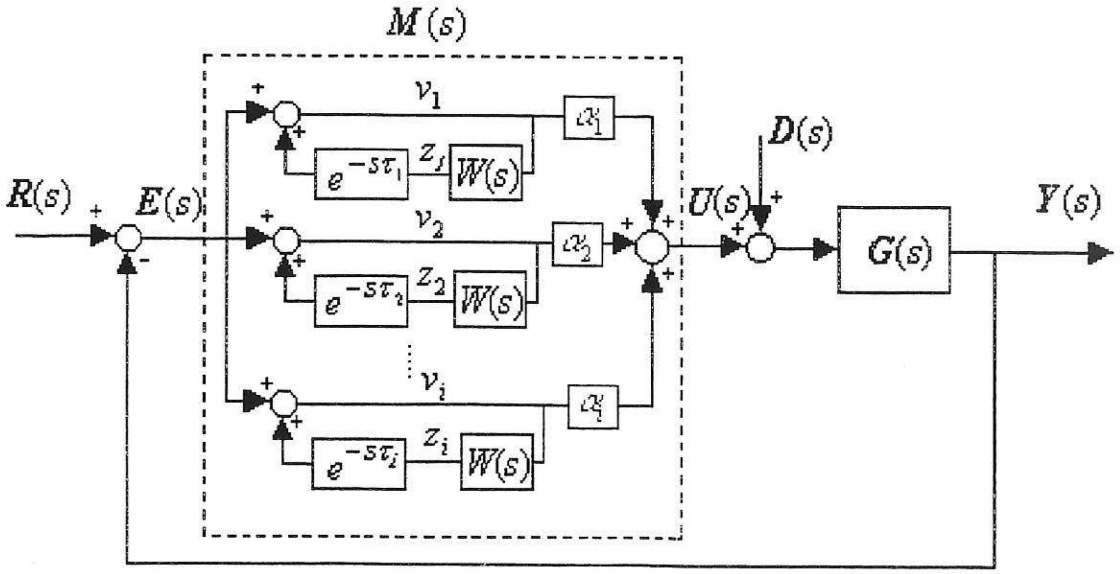


Fig 1. MIMO multi-periodic repetitive control system

The signals R, D, Y, U, E are the reference r, disturbance d, output y, control u and error e respectively.

A multi-periodic repetitive controller containing the required internal model is denoted by the formula

$$M(s)=\sum_{i=1}^{m}\frac{\alpha_i I}{(1-W_i(s)e^{-s\tau_i})} \qquad \textit{(The Internal Model Controller)}$$

It is a linear combination of single-periodic repetitive control elements with each subsystem $W_i(s)$ a stable "filter" (or compensator) introduced to filter out noise and/or trade off tracking accuracy against closed-loop robustness. The "gains" α_i represent the relative weights given to each internal model. It is assumed that $\sum_{i=1}^{m}\alpha_i=\alpha, \alpha_i>0$ but, without loss of generality, α can be set as unity by absorbing the overall "gain" α into the plant gain. If all filters $W_i(s)\equiv 1$, then the poles of $M(s)$ are precisely the infinite set of frequencies of the Fourier components of the reference and disturbance signals. In this case $M(s)$ is an (exact) internal model and marginally stable. In all other cases it can be regarded as an approximate internal model included to achieve a balance between robustness and tracking accuracy (Weiss, 1997).

To illustrate the effect of the filters on the stability of $M(s)$ consider a single filter W with $|W(s)|<1,\ \forall \operatorname{Re} s\geq 0$. As a consequence, for $\operatorname{Re} s\geq 0$, $\left|1-W(s)e^{-s\tau}\right|\geq 1-|W(s)|\left|e^{-s\tau}\right|>0$ so that $1-W(s)e^{-s\tau}\neq 0$ in the closed right-half complex plane. Equivalently, the poles of $M(s)$ lie in the open LHP, as illustrated in Fig 2 and the controller is open loop stable. This stability contributes to robustness but this benefit is obtained (see later results) at the price of tracking accuracy.

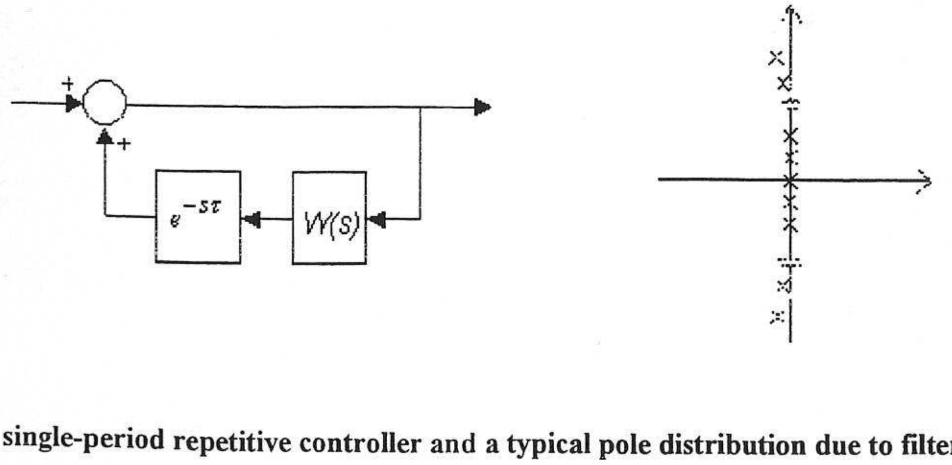


Fig 2. A single-period repetitive controller and a typical pole distribution due to filter action

This paper concentrates on the issue of closed loop stability. Intuitively, the inclusion of delays in the control system will increase the complexity of the stability analysis and may place constraints on the character of system's dynamics. This is supported by the H_∞ condition derived by Weiss (1997) which requires that the plant be positive real or approximately so. To aid the analysis, it is therefore natural to focus attention on positive real systems and constant or strictly bounded real filters. To this end, the following definitions and results are needed.

Definition 1 (Anderson and Vongpanitherd, 1973): The system Σ_G is said to be *positive real* (PR) if

1). All elements of its transfer function matrix $G(s)$ are analytic in $\mathrm{Re}[s] > 0$,

2). $G(s)$ is real for real positive s, and

2). $G(s) + G^*(s) \geq 0$ for $\mathrm{Re}[s] > 0$.

(Note: the superscript $*$ denotes complex conjugate transposition.)

The following are the well-known positive real lemma and bounded real lemma.

Lemma 1(Positive Real Lemma) (Anderson and Vongpanitherd, 1973):
Assume that the state space model

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \quad x(0) = x_0 \end{aligned} \tag{1}$$

is a minimum realisation of Σ_G. *Then* $G(s)$ *is positive real if and only if there exist matrices* $0 < P = P^T \in R^{n\times n}, 0 \leq Q = Q^T \in R^{n\times n}$, $H \in R^{k\times p}$ *and* $L \in R^{p\times q}$ *such that*

$$PA + A^T P = -Q = -LL^T \tag{2}$$

$$D + D^T = H^T H \tag{3}$$

$$PB = C^T - LH \tag{4}$$

(Note: Here the superscript ^{T} denotes transposition and both H and L are assumed to be of full rank.)

It is obvious that if $G(s)$ is strictly proper, that is, $D = 0$, and hence $H = 0$, then (2)~(4) become

$$PA + A^T P = -Q \tag{5}$$

$$PB = C^T \tag{6}$$

This result can be applied to closed loop stability as described in the following result:

Proposition 1 (An equality for the case of constant filters):

Suppose that the plant Σ_G is positive real and strictly proper and that each "filter" $W_i(s) = W_i$ is constant with $|W_i| \le 1$. Suppose that both the reference r and the disturbance d are identically zero. Then the following equality holds for all time:

$$+\infty > V(0) = V(x(t), z(.), t) + \int_0^t x^T(t) Q x(t) dt + \int_0^t \|y(t)\|^2 dt + \sum_{i=1}^{m} \alpha_i (1 - W_i^2) \int_0^t \| v_i(t-\tau_i) \|^2 dt \ge 0$$

where the functional V is defined by the equation

$$V(x(t), z(.), t) = x^T(t) P x(t) + \sum_{i=1}^{m} \alpha_i \int_{t-\tau_i}^{t} \|v_i(\theta)\|^2 d\theta$$

and *$\|.\|$ denotes the Euclidean norm (i.e. $\|f\| = f^T f$).*

Proof: As $r = 0$, $e = -y$ and $v_i = W_i v_i(t-\tau_i) - y(t)$. The system Σ_G has the form

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x(0) = x_0$$

Write $u(t) = \sum_{i=1}^{m} \alpha_i v_i(t)$, $\sum_{i=1}^{m} \alpha_i = 1$ and differentiate V along solutions. Using the positive real lemma yields (where the argument of a function is (t) unless stated otherwise)

$$\dot{V} = -x^T Q x + 2 y^T u + \sum_{i=1}^{m} \alpha_i [v_i^T(t) v_i(t) - v_i^T(t-\tau_i) v_i(t-\tau_i)]$$

This can be written as

$$\dot{V} = -x^T Q x - y^T y + \sum_{i=1}^{m} \alpha_i [y^T y + 2 y^T v_i(t) + v_i^T v_i - W_i^2 v_i^T(t-\tau_i) v_i(t-\tau_i)]$$
$$- \sum_{i=1}^{m} \alpha_i (1 - W_i^2) v_i^T(t-\tau_i) v_i(t-\tau_i)$$

Noting that the third term is identically zero, this reduces to the form

$$\dot{V} = -x^T Q x - \|y\|^2 - \sum_{i=1}^{m} \alpha_i (1 - W_i^2) \|v_i(t-\tau_i)\|^2 \le 0 \tag{14}$$

Integrating and using positivity conditions yields the desired result. ■

The proposition provides a quantity V that is constant over the chosen solution curve of the closed loop system. It is valid for all "t" and hence represents a form of conservation law. By letting $t \to +\infty$ in the above, the following result follows easily by noting that all terms are positive with sum bounded by $V(0)$:

Theorem 1 (Stability with Constant Filters):

With the assumptions of Proposition 1, the closed loop system is asymptotically stable in the sense that $Lx(\cdot) \in L_\infty^n[0,\infty)$ *and the output signal* $y(\cdot) \in L_2^p[0,\infty)$. *If, in addition, the constant filter* W_i *satisfies* $|W_i|<1$ *then the resultant control (sub)signal* $v_i(\cdot) \in L_2^p[0,\infty)$.

It is a simple matter to convert this stability result into a disturbance rejection and tracking stability result although it is useful to distinguish between the two cases of $W_i=1$ and $|W_i|<1$.

1. If all gains $W_i = 1$, consider firstly the case of r=0. It is only necessary to write $d = \sum_{i=1}^{m} \alpha_i d_i$ (where the component d_i has period τ_i). Noting that the map $u \mapsto u+d$ is just the map $v_i \mapsto v_i + d_i$ for all indices "i" and that the this transformation leaves the internal model equations unchanged, it follows that the output y remains $y(\cdot) \in L_2^p[0,\infty)$ and hence the disturbance is rejected asymptotically. Without loss of generality assume therefore that $d=0$ but that $r \neq 0$. The next step is to write $r = \sum_{i=1}^{m} \alpha_i r_i$ and to use Fourier analysis to create locally L_2 periodic inputs $u_{i\infty}$ and corresponding state trajectories that generate the component r_i exactly from some unique initial condition (dependent on r_i). This is essentially a smoothness condition on r_i that has no practical impact. Superposition then trivially yields the existence of an initial condition $x_\infty(0)$, a state trajectory $x_\infty(t)$ and a multi-periodic input $u_\infty(t)$ such that $r(t) \equiv y(t) = Cx_\infty(t)$ i.e. this is the exact multi-periodic solution sought by the feedback scheme! This input has the multi-periodic representation $u_\infty = \sum_{i=1}^{m} \alpha_i u_{i\infty}$ so that writing $u = \sum_{i=1}^{m} \alpha_i v_i$ it can be seen that the map $u \mapsto u - u_\infty$ is equivalent to the maps $v_i \mapsto v_i - u_{i\infty}$ and $y \mapsto y - r = -e$. Noting the invariance of the control equations $v_i(t) = v_i(t-\tau_i) - y(t)$ under these maps and taking differences between actual signals and the signals associated with the exact multi-periodic solution indicates that

the state space mapping relating the input difference u-u_∞ to y-r=-e is equivalent to setting r=0. As a consequence the problem reduces to the case when both d=0 and r=0. Using Theorem 1 then proves that the "pseudo-output" (error) signal $-e(\cdot)\in L_2^p[0,\infty)$.

2. If the W_i are constant but at least one has modulus $|W_i|<1$, then perfect asymptotic tracking of r is not possible. It is still however possible to use the analysis above by constructing a multi-periodic solution of the state equation corresponding to the reference $r(t)$ and reducing the analysis to a stability problem by using differences. This procedure is summarised in the Appendix 1 to the paper for the case of positive real plant and where every $|W_i|<1$.

The stability properties in case 2 can be further underlined by the following Proposition:

Proposition 2 (Gain Bounds and $L_2(0,\infty)$ and RMS BIBO Stability):

If the plant is positive real, the disturbance d=0, $r\in L_2^p(0,T)$ and all filters W_i are constant gains satisfying $|W_i|<1$ then the resultant closed loop system is stable in the bounded-input/bounded-output (BIBO) sense that $M_0\,||r||\geq||y||$ in $L_2^p(0,T)$ (for any value of T including T=∞ and any reference signal $r\in L_2^p(0,T)$). The induced gain of the system is bounded from above by

$$M_0=\left(1+\sum_{j=1}^{m}\alpha_j\frac{W_j^2}{1-W_j^2}\right)^{1/2}$$

This gain also applies to signals measured by their root mean square (RMS) values defined by

$$\langle r\rangle_T=\left(\frac{1}{T}\int_0^T r^T(t)r(t)dt\right)^{1/2}\quad \textit{and}\quad \langle r\rangle_\infty=\lim_{T\to\infty}\left(\langle f\rangle_T\right)^{1/2}$$

(Note: The use of RMS measures permits the application of the bound to Multi-periodic signals even if they do not have the periods used in the construction of the internal model.)

Proof: The proof is deduced from Appendix 2 with the choice of λ=0. ■

The gain bound M_0 is conservative but easy to compute. It does however give useful insight into the effect of changes in reference signals on outputs (from zero initial conditions) via the relation $\langle y_2-y_1\rangle_T\leq M_0\langle r_2-r_1\rangle_T$ for all T>0 bounding the change in output due to the change in reference from r_1 to r_2. The effect of the choice of weights can be illustrated here by examining the case of

$m=1$ and looking at the dependence of M_0 on W. Typical values of the pairs (W,M_0) are as follows: (0.999,22.3), (0.99,7.0), (0.98,5.0), (0.9,2.3), (0.7,1.4).

Note: The result is extended in Proposition 3 in Section 4 with the possibility of improved estimates of gain bounds obtained using measures of the degree to which the plant is positive real.

3. The Inclusion of Dynamical Filters/Compensators

The use of constant filters is of both theoretical and practical value as it demonstrates the success of the exact internal model in achieving perfect tracking accuracy when all $W_i = 1$ and boundedness when $|W_i|<1$. It is recognised however that the use of dynamic filters is of more general practical interest in representing real control elements more faithfully and/or in addressing issues of robustness (particularly with respect to high frequency dynamics). To address this issue for the case of greatest practical interest, it is advantageous to use the following theoretical construct, namely the situation when the gains of the filters are less than unity at all frequencies:

Definition 2 (Anderson and Vongpanitherd, 1973): The system $\sum_W$ is strictly bounded real (SBR) if

1). All elements of its transfer function matrix $W(s)$ are analytic in $\mathrm{Re}[s] > 0$,

2). $I - W^*(j\omega)W(j\omega) > 0 \ \forall \omega \in R$

Note: In classical terms, the transfer function is stable and has a gain that is less than unity at all frequencies. The important background result for this case is as follows:

Lemma 2 (Strictly Bounded Real Lemma) (Anderson and Vongpanitherd, 1973)

Assume that

$$\dot{x}_W = A_W x_W + B_W v$$
$$z = C_W x_W$$

is a minimum realisation of a filter W. Then $W(s)$ is strictly bounded real (SBR) if and only if there exists matrices $0 < P_W = P_W^T$ such that the following Riccatti inequality holds

$$P_W A_W + A_W^T P_W + P_W B_W B_W^T P_W + C_W^T C_W < 0$$

Note: The equivalent linear matrix inequality (*LMI*) formulation requires that the block matrix

$$\Omega = \begin{bmatrix} P_W A_W + A_W^T P_W + C_W^T C_W & P_W B_W \\ (P_W B_W)^{\mathrm{T}} & \text{-I} \end{bmatrix} < 0 \tag{7}$$

It is valuable to introduce a measure γ of how much the filter gains drop below unity. This is achieved as follows by using γ as a measure of the degree to which (7) is negative definite. More precisely:

Corollary 1. If $W(s)$ *is given as in Lemma 2. Then* $W(s)$ *is strictly bounded real if and only if there exist matrices* $0 < P_W = P_W^T$ *and a constant* $\gamma \in (0,1)$ *such that the following LMI holds*

$$\Omega_\alpha = \begin{bmatrix} P_W A_W + A_W^T P_W + C_W^T C_W & P_W B_W \\ (P_W B_W)^{\mathrm{T}} & -\gamma^2 \mathrm{I} \end{bmatrix} \le 0 \tag{8}$$

Proof: From Definition 2, we have $I - W^* W > 0 \ \forall \omega \in R$. As a consequence, $\exists \ \ 0 < \gamma < 1$ such that $I - [\gamma^{-1} W^*][\gamma^{-1} W] > 0 \ \forall \omega \in R$. That is, $\exists \ \ 0 < P_W = P_W^T$ such that

$$\Omega_\alpha' = \begin{bmatrix} P_W A_W + A_W^T P_W + C_W^T C_W & \gamma^{-1} P_W B_W \\ \gamma^{-1} (P_W B_W)^{\mathrm{T}} & \text{-I} \end{bmatrix} < 0$$

As $\Lambda = \begin{pmatrix} I & 0 \\ 0 & \gamma I \end{pmatrix} > 0$, then this implies that

$$\Lambda^T \Omega_\alpha' \Lambda = \Omega_w = \begin{bmatrix} P_W A_W + A_W^T P_W + C_W^T C_W & P_W B_W \\ (P_W B_W)^{\mathrm{T}} & -\gamma^2 \mathrm{I} \end{bmatrix} < 0$$

which proves the result by keeping P_W fixed and reducing γ until semi-definiteness is achieved. ■

The first result of this paper (Theorem 1) examined the case when the "filters" are constant. Extension of the Lyapunov theory to include SBR filters uses the following illustrative construction based on the above corollary.

Let $\eta_W = [x_W^T \ v^T]^T$ be solutions of typical filter dynamics (subscripts omitted for ease of presentation). From (8) it follows that

$$\begin{aligned} 0 &\ge \eta_W^T \Omega \eta_W \\ &= x_W^T (A_W^T P_W + P_W A_W) x_W + v^T B_W^T P_W x_W + x_W^T P_W B_W v + x_W^T C_W^T C_W x_W - \gamma^2 v^T v \\ &= \frac{d}{dt}(x_W^T P_W x_W) + z^T z - \gamma^2 v^T v \end{aligned} \tag{9}$$

which can be expressed as the differential inequality

$$\frac{d}{dt}(x_W^T P_W x_W) \leq \gamma^2 v^T v - z^T z \tag{10}$$

The main result of stability analysis can now be stated below.

Theorem 2 (Stability with SBR filters):

Suppose that the plant Σ_G is positive real and strictly proper and that each filter is SBR. Suppose that both reference r and disturbance d are zero. Then, defining the functional

$$V = x^T Px + \sum_{i=1}^{m} \alpha_i x_{W_i}^T P_{W_i} x_{W_i} + \sum_{i=1}^{m} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta \tag{11}$$

leads to the inequality

$$\infty > V(0) \geq V + \int_0^t x^T Qx dt + \int_0^t \|y\|^2 dt + \sum_{i=1}^{m} \alpha_i (1-\gamma_i^2) \int_0^t \|v_i\|^2 dt \geq 0$$

on all trajectories of the closed-loop systems. The multi-periodic repetitive system in Figure 1 is, as a consequence, globally asymptotically stable.

(Notes:

1. The parameters γ_i seem to play a similar role to the W_i in theorem 2 although the equality is replaced by the inequality above.
2. Using similar arguments to those following theorem 1, the result also implies that the closed loop system is stable in multi-periodic tracking mode and rejects all multi-periodic disturbances of the specified periods.)

Proof: Introducing a positive definite Lyapunov function V for the problem of the form of (11) where the familiar Lyapunov term for the plant is augmented by appropriate quadratic terms for each SBR filter plus an integral term involving variables in the internal model control. Differentiating V along solutions and using the positive real lemma yields (where the argument of a function is (t) unless stated otherwise)

$$\begin{aligned}\dot{V} &\leq -x^T Qx + 2y^T u + \sum_{i=1}^{m} \alpha_i (\gamma_i^2 v_i^T v_i - z_i^T z_i) + \sum_{i=1}^{m} \alpha_i [z_i^T(t) z_i(t) - z_i^T(t-\tau_i) z_i(t-\tau_i)] \\ &= -x^T Qx - y^T y + \sum_{i=1}^{m} \alpha_i [-y^T y + 2y^T z_i(t-\tau_i) + v_i^T v_i - z_i^T(t-\tau_i) z_i(t-\tau_i)] - \sum_{i=1}^{m} \alpha_i (1-\gamma_i^2) v_i^T v_i\end{aligned} \tag{12}$$

The relation $v_i(t)=z_i(t-\tau_i)-y(t)$ is used to prove that the third term in (12) is identically zero, so that

$$\dot{V} \leq -x^T Qx - \|y\|^2 - \sum_{i=1}^{m} \alpha_i (1-\gamma_i^2) \|v_i\|^2 < 0 \tag{13}$$

Integrating and using (12) and the positivity of V yields (where ||.|| denotes the Euclidean norm)

$$\infty > V(0) \geq V + \int_0^t x^T Qxdt + \int_0^t \|y\|^2 dt + \sum_{i=1}^{m} \alpha_i (1-\gamma_i^2) \int_0^t \|v_i\|^2 dt \geq 0 \tag{14}$$

The proof of stability is can now completed in a similar manner to Proposition 2.

■

4. Relaxing the Positive Real Condition

The positive real condition is sufficient for stability and can be relaxed to provide more generally applicable results. The idea behind this is to note that $G(s)$ is positive real if and only if its inverse is positive real. It is possible therefore to consider processes where $G^{-1}(s) + \mathrm{K}(s)$ is positive real or equivalently the feedback system $(I + G(s)\mathrm{K}(s))^{-1} G(s)$ is positive real where $G(s)$ is in the form of (1) and K(s) is some minor loop compensation element.

For simplicity of presentation and to underline the importance of the "gain" of K in the analysis, K(s) is now assumed to be of the form K(s)= λI where λ is some scalar gain and I is the unit matrix. Under the conditions explained above, the corresponding relations of (5)-(6) can be stated as the existence of matrices P and Q as before such that

$$PA + A^T P = -Q + 2\lambda C^T \mathrm{C} \tag{15}$$

$$PB = C^T \tag{16}$$

Theorem 3 (A Illustrative Relaxed Positive Real Condition):

Suppose that the plant is Σ_G has the property that $(I + \lambda G)^{-1}$ $G(s)$ is positive real for some λ<1/2 and strictly proper. Suppose that both reference r and disturbance d are multi-periodic with components of period $\tau_i, i = 1,...,m$. Then the multi-periodic repetitive system in Figure 1 is globally asymptotically stable in the sense stated in Theorem 1.

Proof: The proof is similar to that of Theorem 1. Consider the positive definite Lyapunov function

$$V = x^T Px + \sum_{i=1}^{m} \alpha_i x_{W_i}^T P_{W_i} x_{W_i} + \sum_{i=1}^{m} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta \tag{17}$$

By differentiating V along solutions and noting that the analysis of Theorem 2 remains unchanged if Q is replaced by $Q\text{-}2\lambda CC^T$ gives

$$\dot{V} \leq -x^T Qx - (1-2\lambda)\|y\|^2 - \sum_{i=1}^{m} \alpha_i (1 - \gamma^2) v_i^T v_i \leq 0 \tag{18}$$

and all terms are negative if the stated condition $\lambda < \frac{1}{2}$ is used. The proof now follows that of Theorem 1.■

The theorem extends the applicability of the ideas to a broader class of systems although it is still necessary that the underlying state space model satisfies a minimum-phase assumption and CB is non-singular. It also suggests that minor(or inner) loop-compensation may be a useful tool in the control of multi-repetitive systems. This in turn suggest that a combination of output feedback with state feedback may be of great value. These issue aare the subject of current work at Sheffield.

The estimate of gain bounds can be improved in this situation. The details are as follows:

Proposition 3 (Gain Bounds and $L_2(0,\infty)$ and RMS BIBO Stability):

With the assumptions of Theorem 3 but with constant filters $|W_i|<1$, the results of proposition 2 hold true with M_0 replaced by the gain bound

$$M_\lambda = \left(\frac{1}{1-2\lambda} \left[1 + \sum_{j=1}^{m} \alpha_j \frac{W_j^2}{1-W_j^2} \right] \right)^{1/2}$$

This gain also applies to signals measured by their root mean square (RMS) values.

Proof: The proof is provided in Appendix 2. ■

The choice of λ is non-unique but there is an optimal choice: if λ* is the smallest value of λ for which $(I+\lambda G)^{-1}\ G(s)$ is positive real for all 0.5≥λ>λ*, then $M_{\lambda*} < M_0$ if λ*<0. This indicates that the bound provided by Proposition 3 can be better than that of Proposition 2 for strictly positive real systems.

5. Exponential Stability

It has been shown above that a form of asymptotic stability of multi-periodic repetitive control system can be guaranteed if the plant Σ_G is positive real (or satisfies a relaxed positive real condition). This $L_2(0,\infty)$ stability result can be refined when the plant satisfies a strict positive realness condition i.e. $G(s)+G^*(s)\geq 0$ whenever $\mathrm{Re}(s)\geq -\varepsilon, \varepsilon>0$. Under this condition, the results in this section show that the tracking error decays in an exponential manner. For simplicity of presentation, the filtering factor $W(s)$ is chosen as constant, i.e., $|W_i(s)|=|W_i|<1$. The strict inequality is central to the proof of the following result.

Theorem 4 (Exponential Asymptotic Stability):

The Multi-periodic repetitive system in Figure 1, with constant weightings $W_i, i=1,\ldots,m$, is exponentially asymptotically stable if plant Σ_G is strictly positive real and $|W_i|<1, 1\leq i\leq m$.

Proof: The definition of strict positive real can be rephrased as $\mathrm{Re}(G(s'-\varepsilon))\geq 0$ whenever $\mathrm{Re}(s')\geq 0$, by defining $s'=s+\varepsilon$. By denoting $\tilde{f}(t)=e^{\varepsilon t}f(t)$ for any signal f, assuming $r=0$ and $d=0$ and noticing the fact that $C((s'-\varepsilon)I-A)^{-1}B\equiv C(s'I-(\varepsilon I+A))^{-1}B$, equations (3)-(4) become

$$\begin{aligned}\dot{\tilde{x}}&=(A+\varepsilon I)\tilde{x}+B\tilde{u}\\ \tilde{y}&=C\tilde{x}\end{aligned}\tag{19}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ \tilde{u}(t)=\sum_{i=1}^{m}\alpha_i\tilde{v}_i, \qquad \sum_{i=1}^{m}\alpha_i=1$$

$$\tilde{v}_i(t)=(W_i e^{\varepsilon\tau_i})\tilde{z}_i(t-\tau_i)+\tilde{e}(t)\tag{20}$$

v_i

That is the exponential weighting of signals (with $r=0$ and $d=0$) is equivalent to the map $(\{W_i\}_{1\leq i\leq m}, A)\mapsto(\{W_i e^{\varepsilon\tau_i}\}_{1\leq i\leq m}, A+\varepsilon I)$. The transformed system is now positive real. Previous results can now be used as, without loss of generality, we can choose $\varepsilon>0$ so that $\max_i|W_i e^{\varepsilon\tau_i}|<1$. As $\tilde{x}(t)$ is then uniformly bounded there exists an $M>0$ such that $\|\tilde{x}(t)\|\leq M$ i.e. $\|x(t)\|\leq Me^{-\varepsilon t}$ $\forall t>0$. This indicates exponential stability of the state, and hence the tracking error. ∎

6. A Note on Lyapunov Stability Analysis for State-delay System

The above analysis relies heavily on positive real concepts and strict bounded real properties. In this context it can be extended to infinite-dimensional, linear time-invariant, positive real state-delay systems. To illustrate this possibility another known Lemma needs to be introduced.

Consider the following state delay system:

$$\Sigma G : \begin{cases} \dot{x} = Ax + A_d x(t - \tau_d) + Bu \\ y = Cx \end{cases} \tag{21}$$

where $x \in R^n, y \in R^p, u \in R^p$ are the state, the output and the input of the system, τ_d is the state time delay.

Lemma 3(Niculescu and Lozano, 2000): Assuming that $\lambda(A) \in C^-$, *then* Σ_G *is strictly positive real if and only if there exist matrices* $0 < P = P^T \in R^{n \times n}$ and $0 < Q = Q^T \in R^{n \times n}$ *such that*

$$PA + A^T P + PA_d Q^{-1} A_d^T P + Q < 0 \tag{22}$$

$$C = B^T P \tag{23}$$

Remark: The Algebraic Riccati Inequality (ARI) (22) can be equivalently expressed as LMI

$$\Delta = \begin{bmatrix} PA + A^T P + Q & PA_d \\ A_d^T P & -Q \end{bmatrix} < 0 \tag{24}$$

Theorem 5 (Stability of Multi-periodic State Delay Systems):

Suppose that he same conditions as given in Theorem 2 hold except that ΣG *is a state delay system as in (21). Suppose that* ΣG *is positive real. Then the multi-periodic repetitive system in Figure 1 is globally asymptotically stable.*

Proof: Introduce a positive definite Lyapunov-like functional V of the form

$$V = x^T Px + \int_{t-t_d}^{t} x^T Qx + \sum_{i=1}^{m} \alpha_i x_{W_i}^T P_{W_i} x_{W_i} + \sum_{i=1}^{m} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta \tag{25}$$

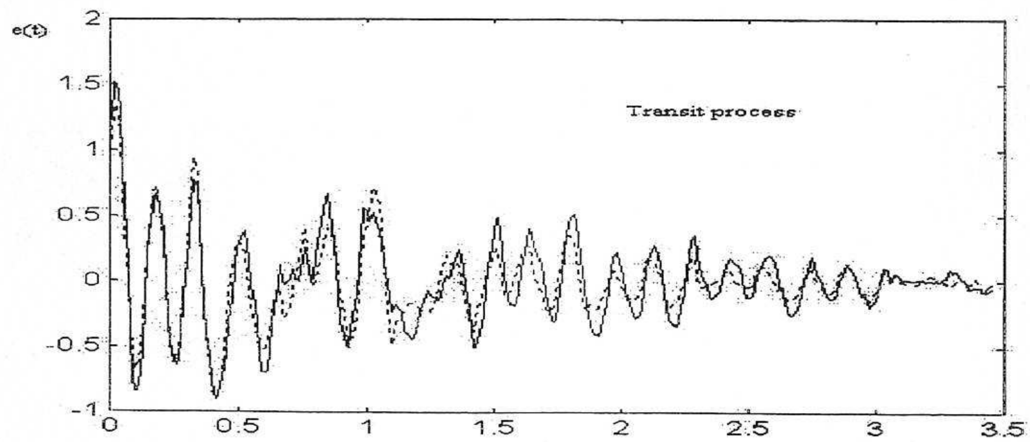
By differentiating V along solutions and by using (10) and (22)-(24),

$$\begin{aligned} \dot{V} &< x^T (PA + A^T P)x + x^T (t - \tau_d) A_d^T Px + x^T PA_d x(t - \tau_d) x^T Qx \\ &\quad - x^T (t - \tau_d) Qx(t - \tau_d) + 2y^T u + \sum_{i=1}^{m} \alpha_i (\gamma^2 v_i^T v_i - z_i^T z_i) \\ &\quad + \sum_{i=1}^{m} \alpha_i [z_i^T z_i - z_i^T (t - \tau_i) z_i (t - \tau_i)] \\ &= -\xi^T \Delta \xi - y^T y + \sum_{i=1}^{m} \alpha_i [y^T y + 2y^T v_i + v_i^T v_i - z_i^T (t - \tau_i) z_i (t - \tau_i)] - \sum_{i=1}^{m} \alpha_i (1 - \gamma_i^2) v_i^T v_i \\ &= -\xi^T \Delta \xi - \|y\|^2 - \sum_{i=1}^{m} \alpha_i (1 - \gamma_i^2) v_i^T v_i \\ &< 0 \end{aligned} \tag{26}$$

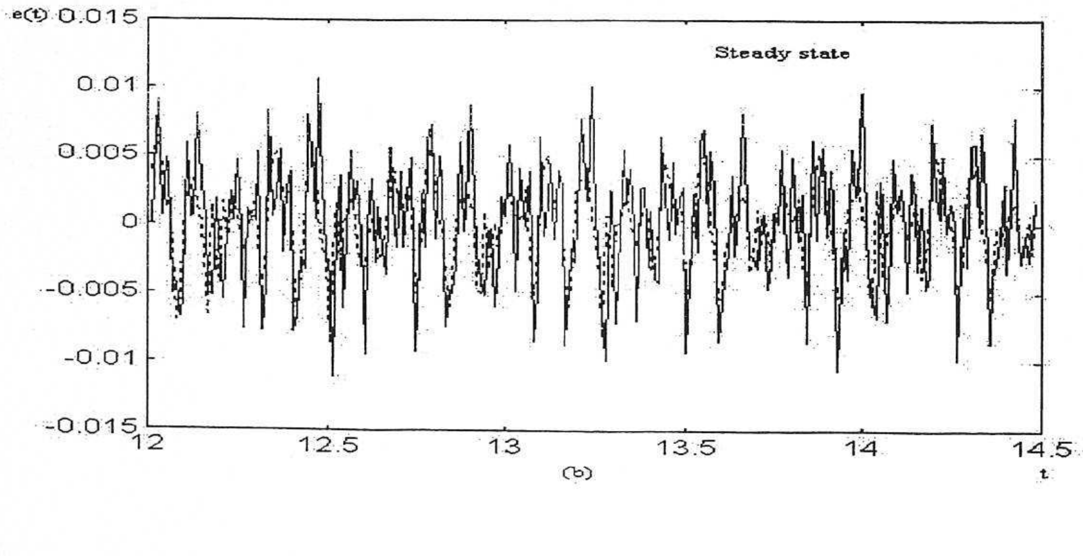
where $\xi^T = [x^T \; x^T(t-\tau_d)]$. The methods used earlier in this paper complete the proof of the result. ∎

7. Simulation Example

The following examples are chosen to illustrate the theory and the effect of the choice of parameters. For the sake of simplicity, a SISO system is examined to illustrate the control system performance. $W_i(s)$ is set to be a constant close to 1(i.e., 0.999) throughout the simulations. The positive real plant is a state delay system where $A=\begin{bmatrix}-15 & 0\\ 0 & -20\end{bmatrix}$, $A_d=\begin{bmatrix}-4 & 0\\ -5 & -5\end{bmatrix}$, $B=\begin{bmatrix}15\\ 7.5\end{bmatrix}$, $C=[1 \;\; 0]$. The reference signal is chosen to be a mix of harmonics of two fundamental frequencies, $\omega_1 = 2.6\pi \; rad/sec$, $\omega_1 = 6\pi \; rad/sec$, combined in the form $r = r_1 + r_2$, where $r_1 = 2\sin\omega_1 t + 1.5\sin 3\omega_1 t + 0.7\sin 5\omega_1 t$ and $r_2 = \sin\omega_2 t + 3\sin 2\omega_2 t$. The two fundamental periods are $\tau_1 = 1/1.3$, $\tau_2 = 1/3$. For simplicity, equal weights $a_1 = 0.5$ and $a_2 = 0.5$ are chosen, which means equal weighting in the controller on the two components of reference signal. A gain $K = 10$ is also put into the feedforward path with the intention of improving response speeds. This does not affect the positive-real nature of the process.



(a)



(b)

Fig 3. Error $e(t)$ when (1). Disturbance $d(t)=0$ (solid) and (2). Disturbance $d(t)\neq 0$ (dotted)

Simulations show that this repetitive controller is also capable of effectively rejecting a periodic disturbance. More precisely, suppose in the followings that the disturbance is a square wave at a frequency of $14\pi\ rad/sec$ and at peak value ± 2. A square wave is chosen to indicate that the scheme can cope with signals with infinite frequency content. A third repetitive sub-controller is added to $M(s)$ at a fundamental period of $\tau_3 = 1/7$, with weightings replaced by $\alpha_1 = 0.4, \alpha_2 = 0.4$ and $\alpha_3 = 0.2$. The asymptotic part of the tracking error (for t>12) is also demonstrated in Figure 3 (dotted) and is almost identical to that of disturbance-free case (Figure 3, solid). The result suggests that the influence of the disturbance signal is negligible and ultimately rejected.

8. Effect of Nonlinear Perturbations to Plant Dynamics

In this section, extension of the above Lyapunov stability to a certain class of nonlinear system is briefly discussed. The plant Σ_G is now a nonlinear system, described by

$$\begin{aligned} \dot{x} &= Ax + Bu + f(x) \\ y &= Cx, \quad x(0) = x_0 \end{aligned} \tag{27}$$

where $f(0)=0$ and the linear part of the system (A,B,C) is positive real. Let $P=P^T>0$ and $Q=Q^T\geq 0$ be the associated Lyapunov matrices. Suppose also that, for all x_1 and x_2, the nonlinearity satisfies the inequality

$$(x_2-x_1)^T P(f(x_2)-f(x_1)) \leq 0 \tag{28}$$

A simple example of this could be $f(x)=Bg(y)$ when (28) reduces to $0\geq (y_2-y_1)^T(g(y_2)-g(y_1))$ for all output values y_1 and y_2. This is just a familiar sector constraint used, for example, in absolute stability problems.

The stability analysis is stated below. For simplicity, the filtering factor $W(s)$ is chosen again as constant, i.e, $|W_i(s)|=|W_i|<1$.

Theorem 6: (Stability in the Presence of Nonlinear Perturbations)

Suppose that both reference r and disturbance d are identically zero and that $|W_i|<1, 1\leq i\leq m$. Then the multi-periodic nonlinear repetitive system in Figure 1 with Σ_G a nonlinear system governed by eqn (27)-(28) is globally asymptotically stable. In addition, Propositions 2 and 3 remain true.

Proof: The proof is similar to that in section 2. Let $r=d=0$. Introduce a positive definite function $V = x^T Px + \sum_{i=1}^{m} \alpha_i \int_{t-\tau_i}^{t} \|v_i(\theta)\|^2 d\theta$. Differentiating V along solutions and using the condition in (28)-(30) yields

$$\begin{aligned}\dot{V} &= -x^T Qx - \|y\|^2 - \sum_{i=1}^{m} \alpha_i (1 - W_i^2) \|v_i(t-\tau_i)\|^2 + 2x^T Pf(x) \\ &\leq -x^T Qx - \|y\|^2 - \sum_{i=1}^{m} \alpha_i (1 - W_i^2) \|v_i(t-\tau_i)\|^2\end{aligned} \tag{29}$$

which, replaces the equality if Proposition 1 with the inequality

$$+\infty > V(0) \geq V(x(t), z(.), t) + \int_0^t x^T(t) Qx(t)dt + \int_0^t \|y(t)\|^2 dt + \sum_{i=1}^{m} \alpha_i (1 - W_i^2) \int_0^t \| v_i(t - \tau_i) \|^2 dt \geq 0 \tag{3}$$

Replacing equality with inequality does not negate the conclusions obtained by using arguments similar to those for the linear case of $f=0$.. The proofs that Propositions 2 and 3 remain valid follow a similar argument. ■

The feedback scheme hence has a degree of robustness to nonlinear perturbations. The proof of the above result indicates that the ideas also apply to the analysis of section 3 where dynamic filters are included. It also indicates that Propositions 2 and 3 remain valid so that the gain estimates also apply to this class of nonlinear systems.

9. Conclusions

MIMO multi-periodic repetitive control systems have been studied and a number of stability conditions derived using Lyapunov-type analyses. The potential power of the approach has been illustrated. In particular it has been shown that asymptotic $L_2(0,\infty)$ stability and perfect disturbance rejection is guaranteed if the plant is positive real and that this can be extended to prove exponential stability if it is strictly positive real. The positive real condition seems to be central to both frequency domain (Weiss, 1997) and the analyses in this paper. It can however be partially relaxed by demanding that a feedback version of the plant is positive real. In all cases, the theory provides simple estimates of L_2 and RMS induced gains of the closed loop system.

It has also been shown that positive real conditions make it possible to extend the ideas to a class of nonlinear systems and a class of linear differential delay systems. A feature of the analysis is the possibility of including strictly bounded real filters in the delay loops to improve robustness.

The paper has made a clear statement about the potential of Lyapunov analysis for multi-periodic systems in the presence of positive real or "almost positive real" processes. Although some systems are positive real in nature, like large space structures (Benhabib, et. al., 1981), typical plants met in practice are not necessarily positive real. This problem, together with issues of design of compensators and inner loop feedback will be addressed in future papers as will issues associated with systematic design of stable high performance loops.

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APPENDIX 1:

The purpose of this appendix is to indicate the proof of the assertion that stability analysis can be transformed into a tracking theory. Suppose that the "filters" W_i are constant. If $|W_i|<1,\ 1\le i\le m$, write the multi-periodic reference signal as a absolutely convergent complex Fourier series:

$$r(t) = R_0 + \sum_{r=1}^{m}\sum_{k\neq 0} R_{rk} e^{jk\omega_r t}$$

and search for a steady state solution

$$y(t) = Y_0 + \sum_{r=1}^{m}\sum_{k\neq 0} Y_{rk} e^{jk\omega_r t}$$

A simple calculation then yields the formula, $1\le i\le m$,

$$z_i(t) = \frac{R_0 - Y_0}{1-W_i} + \sum_{r=1}^{m}\sum_{k\neq 0}\frac{1}{1-W_i e^{-jk\omega_r\tau_i}}(R_{rk} - Y_{rk})e^{jk\omega_r t}$$

with a corresponding formula for $v(t)$. This then gives the corresponding multi-periodic output

$$y(t) = \sum_{i=1}^{m}\frac{\alpha_i}{1-W_i}G(0)(R_0 - Y_0) + \sum_{i=1}^{m}\alpha_i\sum_{r=1}^{m}\sum_{k\neq 0}\frac{e^{ik\omega_r t}}{1-W_i e^{-jk\omega_r\tau_i}}G(jk\omega_r)(R_{rk} - Y_{rk})$$

For consistency with the assumed construction of $y(t)$, choose

$$Y_0 = (I + \sum_{i=1}^{m}\frac{\alpha_i}{1-W_i}G(0))^{-1}\sum_{i=1}^{m}\frac{\alpha_i}{1-W_i}G(0)R_0$$

and $Y_{rk} = (I + P_{rk})^{-1} P_{rk} R_{rk}$ where $P_{rk} = \sum_{i=1}^{m} \frac{\alpha_i}{1 - W_i e^{-jk\omega_r \tau_i}} G(jk\omega_r)$.

Note that Y_{rk} is well-defined if $|W_i| < 1$ and $G(s)$ is positive real (a sufficient condition). A simple calculation then indicates that all signals lie in $L_2(0,T)$ in any interval $[0,T]$.

A similar analysis yields an infinite multi-periodic series for the state vector and hence the initial condition that generates the multi-periodic solution.

Finally, if the subscript r now denotes the multi-periodic solution constructed above, it is easily seen that, by construction,

$$\frac{d}{dt}[x_r(t) - x(t)] = A[x_r(t) - x(t)] + B[v_r(t) - v(t)]$$
$$x_r(t) - x(t)\big|_{t=0} = x_{r0} - x_0$$
$$y_r(t) - y(t) = C[x_r(t) - x(t)]$$
$$v_r(t) - v(t) = \sum_{i=1}^{M} \alpha_i [z_{ri}(t) - z_i(t)]$$
$$z_{ri}(t) - z_i(t) = W_j [z_{ri}(t - \tau_i) - z_i(t - \tau_i)] - [y_r(t) - y(t)]$$

These equations have the same structure as the equations used in the proof of Proposition 1 and Theorem 1. As a consequence, the stability proof can be applied directly to prove stability properties with

$$V = (x_r(t) - \mathrm{x(t)})^{\mathrm{T}} P (x_r(t) - \mathrm{x(t)}) + \sum_{i=1}^{m} \alpha_i \int_{t-\tau_i}^{t} \|z_{ri}(\theta) - z_i(\theta)\|^2 d\theta$$

APPENDIX 2:

The proof of Propositions 2 and 3 are given below, the proof of Proposition 2 being obtained by setting λ=0. From Proposition 1, examine the functional

$$V(x(t), z(.), t) = x^T(t) P x(t) + \sum_{i=1}^{m} \alpha_i \int_{t-\tau_i}^{t} \|v_i(\theta)\|^2 d\theta$$

in the general case where $r \neq 0$ but $d=0$. and differentiate V along solutions of the closed-loop system from zero initial conditions on x(0) and the delayed variables $v_i(t), -\tau_i \le t < 0, 1 \le i \le m$. Using the positive real lemma and the construction of Theorem 3 yields (where the argument of a function is (t) unless stated otherwise)

$$\dot{V} = -x^T Qx + 2\lambda y^T y + 2y^T u + \sum_{i=1}^{m} \alpha_i [v_i^T(t)v_i(t) - v_i^T(t-\tau_i)v_i(t-\tau_i)]$$

Write $u(t) = \sum_{i=1}^{m} \alpha_i v_i(t)$, $\sum_{i=1}^{m} \alpha_i = 1$ and use the identity $v_j(t) + y(t) \equiv r(t) + W_j v_j(t-\tau_j)$ to give

$$\dot{V} = -x^T Qx - (1-2\lambda)y^T y + \sum_{i=1}^{m} \alpha_i [y^T y + 2y^T v_i(t) + v_i^T v_i - v_i^T(t-\tau_i)v_i(t-\tau_i)]$$

$$= -x^T Qx - (1-2\lambda)y^T y + \sum_{i=1}^{m} \alpha_i [\| r + W_j v_j(t-\tau_j)\|^2 - \| v_i(t-\tau_i)\|^2]$$

$$= -x^T Qx - (1-2\lambda)y^T y + \sum_{i=1}^{m} \alpha_i [\| r + W_j v_j(t-\tau_j)\|^2 - \| v_i(t-\tau_i)\|^2]$$

$$= -x^T Qx - (1-2\lambda)y^T y + \sum_{i=1}^{m} \alpha_i [\| r\|^2 + 2W_j r^T v_j(t-\tau_j) - (1-W_j^2)\| v_i(t-\tau_i)\|^2]$$

$$\leq -(1-2\lambda)y^T y + \| r\|^2 + \sum_{i=1}^{m} \alpha_i \frac{W_i^2}{1-W_i^2}\| r\|^2$$

where the terms in the summation have been replaced by their maximum values over all possible values of delayed variables. Integrating and using the zero initial conditions on the state and delayed variables, then gives, for all t>0,

$$0 \leq -(1-2\lambda)y^T(t)y(t) + r^T(t)r(t)\left(1 + \sum_{i=1}^{m} \alpha_i \frac{W_i^2}{1-W_i^2}\right)$$

Integrating yields the desired inequality, which proves the proposition. ■

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