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STATE SPACE RECONSTRUCTION AND SPATIO-TEMPORAL PREDICTION OF LATTICE DYNAMICAL SYSTEMS

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State space reconstruction and spatio-temporal prediction of lattice dynamical systems

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Abstract

This paper addresses the problems of state space reconstruction and spatio-temporal prediction for lattice dynamical systems. It is shown that the state space of any finite lattice dynamical system can be embedded into a reconstruction space for almost every, in the sense of prevalence, smooth measurement mapping as long as the dimension of the reconstruction space is larger than twice the size of the lattice. Based on this result, a spatio-temporal dynamical relation for each site within the lattice is derived and used for sptaio-temporal prediction of the system. In the case of infinite lattice dynamical systems, an approach based on constructing local lattice dynamical systems is proposed. It is shown that the finite dimensional results can be directly applied to the local modelling and spatio-temporal prediction for infinite lattice dynamical systems. Two numerical examples are provided to demonstrate the proposed theory and approach.

1 Introduction

Lattice dynamical systems (LDS) are spatially extended dynamical systems composed of a finite or infinite number of interacting dynamical systems modelled on an underlying spatial lattice with some regular structure, for example, the integer lattice in the plane. Such systems arise as models in many applications, including chemical reactions (Erneux and Nicolis, 1993), material sciences (Cahn, 1960), image processing and pattern recognition (Chua 1998), biology (Keener 1987, 1991), and ecology (Sóle, Valls and Bascompte 1992). A fundamental characteristic of LDS's is the fact that the local state-space variables associated to each lattice node or spatial site are the same over the given lattice, that is, represent the same set of physical quantities such as pressure, temperature, velocity etc., which makes the global states of LDS's distributions on the lattice. This feature distinguishes LDS's from conventional dynamical systems.

As a class of spatially extended dynamical systems, LDS's are able to reproduce complex spatio-temporal patterns and to exhibit surprisingly rich dynamical behaviours, including spatio-temporal chaos, intermittency, traveling waves and pattern formation (Kaneko 1989b, Chua 1998, Hsu 2000, and Bates, Lu, and Wang 2001). Most of the early work relating to the identification and prediction of spatially extended dynamical systems involved time series analysis characterised by correlation dimensions, Lyapunov characteristic exponents and Kolmogorov entropies, but these provide little information about the spatial structure or the underlying spatio-temporal relationships.

Various methods for the identification of local LDS models, in particular Coupled Map Lattice (CML) models, from spatio-temporal observations have already been proposed (Coca and Billings 2001, Mandelj, Grabec and Govekar 2001, Marcos-Nikolaus, Martin-Gonzalez and Sóle 2002, Grabec and Mandeji 1997, Parlitz and Merkwirth 2000). An important step in all of these modelling methods is the proper reconstruction of the local state vectors at some specified site in the lattice from the measured data, that is determining the spatio-temporal region which influences the dynamics of that site. In all previous studies, this spatio-temporal relationship was determined by various heuristic or pre-specified approaches. Then the following question naturally arises, 'does such a reconstruction of the state space exist for any generic observation functions?'

A positive answer to this question is that it is closely related to the embedding problem. The time-series embedding theory for k-dimensional dynamical systems has been established by Packard, Crutchfield, Farmer, and Shaw (1980), Takens (1981), and Sauer, Yorke and Casdagli (1991). These results are dedicated to time-series embedding and prediction of k-dimensional autonomous dynamical systems. Sauer, Yorke and Casdagli (1991) have shown that the delay-coordinate embedding theorem for single-variable measures could be extended to the case of multi-variable measures (cf. Remark 2.9 in Sauer, Yorke, and Casdagli (1991)). The result of Sauer, Yorke and Casdagli (1991) can be directly applied to autonomous LDS's with a finite lattice. For instance, consider a LDS defined on a finite lattice with n sites, then such a spatially extended system can be regarded as a n-dimensional dynamical system when using distributions on the lattice as global states. The result of Sauer, et al. (1991) says that for any spatial region Ω of q sites of the lattice, there generally exists a one-to-one correspondence between the global states and the mixed delay coordinate vectors as long as the dimension of the reconstruction space is larger than $2n$. Then a spatio-temporal dynamical relationship within Ω can be derived which can be used to predict the dynamics in one spatial site knowing the others.

The time-series embedding problem for dynamical systems with external inputs has also been studied by Casdagli (1992). However, Casdagli only gave a heuristic argument for single variable systems. In this paper, the embedding result is extended to the case of finite lattice LDS's with external inputs in a rigorous mathematical manner. It is shown that under similar conditions, for almost every, in the sense of prevalence, set of smooth functions there exists a one-to-one correspondence between the states and the mixed delay coordinate vectors as long as the input sequence is bounded. In the case of an infinite lattice, approaches to local state space reconstruction and spatio-temporal prediction are discussed.

The paper is organised as follows. Section 2 introduces the state space representation of finite lattice LDS's with external inputs and presents the construction of mixed spatio-temporal delay coordinate vectors for such systems. Section 3 provides some fundamental concepts on "prevalence" and "almost every" and then gives the main results about spatio-temporal embedding problems. In section 4 some estimation techniques for the identification and spatio-temporal prediction of LDS's for both finite and infinite lattices are introduced. Section 5 illustrates the proposed approach using two numerical examples. Finally conclusions are given in section 6.

2 Lattice dynamical systems and spatio-temporal delay coordinate vectors

This paper involves a study of a class of LDS's which can be described as follows. Let $d \geq 1$ be an integer and I be the d-dimensional integer lattice or a subset of it. That is, $I \subset Z^d \subset R^d$. The LDS considered herein is a system with discrete time where the spatial variable takes values in the discrete lattice I. The state of the system is represented by the vector $\{x_i\}_{i\in I}$ with $x_i \in R$ for each $i \in I$. In the case $I = Z^d$, the phase space X is an infinite-dimensional Banach space as follows

$$X = \{x = \{x_i\} : i \in I, ||x|| < \infty\} \tag{1}$$

where $x_i \in R$ and the norm $|| \cdot ||$ could be the l^∞ or l^2 norm.

The LDS with discrete time can be expressed by

$$x(t) = \{x_i(t)\}_{i\in I} = f(x(t-1), u(t-1)) = \{f_i(x(t-1), u(t-1))\}_{i\in I} \tag{2}$$

where $u(t) = \{u_i(t)\} \in U$ is the input of the system, and $U = \{u = \{u_i\} : u_i \in R, i \in I, ||u|| < \infty\}$ is the input space. Generally, if $f = \{f_i\}_{i\in I}$ is a diffeomorphism from $X \times U$ to X, and locally Lipschitz in X, then the existence and uniqueness of the initial value problem for LDS holds (Afraimovich and Chow 1994).

In this and the following section, it is assumed that $I \in Z^d$ is a finite subset. That is LDS's with a finite number of interacting dynamical systems are investigated. Let $I = \{i = (i_1, \cdots, i_d), 1 \leq i_j \leq L_d, j = 1, \cdots, d\}$. In this case the state space representation (2) is actually a system of ordinary differential equations (ODEs) on a k-dimensional phase space $X = \{x = \{x_i\} : i \in I, ||x|| < \infty\} \subset R^k$, $k = card(I)$ is the cardinal number of set I

$$x(t) = f(x(t-1), u(t-1)) \tag{3}$$

where $f : X \times U \to X$ is a diffeomorphism, and $U = \{u = \{u_i\} : u_i \in R, i \in I, ||u|| < \infty\} \subset R^k$ is the input space. The existence and uniqueness of the solutions of (3) follows from the regular fundamental theorems of ODE's.

In general, the direct measurement of the state vector x is not possible and only some observable variable y which depends on the state and the input can be measured. Therefore, the state-space model (3) of the LDS is usually complemented with a measurement equation

$$y(t) = h(x(t)) = \{h_i(x(t))\}_{i \in I} \tag{4}$$

where $h = \{h_i\}_{i \in I}$ is a map from X to R^k. For any given measurement map $h = \{h_i\}_{i \in I}$, a spatio-temporal delay coordinate vector can be constructed in the following manner. Consider any subregion $J \subset I$, and any spatial site $i \in J$, iterating state space equation (3) and measurement equation (4) yields

$$\begin{aligned}
y_i(t) &= h_i(x(t)) \\
y_i(t+1) &= h_i(x(t+1)) = h_i(f(x(t), u(t))) \\
y_i(t+2) &= h_i(x(t+2)) = h_i(f(f(x(t), u(t)), u(t+1))) \\
&\vdots \\
y_i(t+n_i-1) &= h_i(x(t+n_i-1)) = h_i(f(\cdots f(f(x(t), u(t)), u(t+1)), \cdots, u(t+n_i-2))
\end{aligned}$$

then a $\sum_{i \in J} n_i$-tuple spatio-temporal delay coordinate vector can be arranged as

$$F(h, f)(x(t), u(t), \cdots, u(t+n-2)) = \{y_i(t), \cdots, y_i(t+n_i-1)\}_{i \in J} \tag{5}$$

where $n = max_{i \in J} n_i$.

In the following section, it will be shown that under some conditions on map f and $n_i, i \in J$, for almost every smooth map h and any given input sequence $u(t), \cdots, u(t+n-2)$, the map F is an immersion with respect to $x(t)$, which means $x(t)$ can be expressed in terms of $\{y_i(t), \cdots, y_i(t+n_i-1)\}_{i \in J}$ and $u(t), \cdots, u(t+n-2)$. The implicit function theorem implies that for any given subset J of the spatial region I a spatio-temporal relationship between inputs and observations at different spatial sites within this spatial region J exits, that is, for any $i \in J$

$$\begin{aligned}
y_i(t+1) &= h_i(x(t+1)) = h_i(f(\cdots f(f(x(t-n_j+1), u(t)), \cdots, u(t-n_j+1)) \\
&= g_i(\{y_j(t), \cdots, y_j(t-n_j+1)\}_{j \in J}, u(t), \cdots, u(t-n+1)) \\
&= g_i(\{y_j(t), \cdots, y_j(t-n_j+1)\}_{j \in I}, \{u_j(t)\}_{j \in I}, \cdots, \{u_j(t-n+1)\}_{j \in I})
\end{aligned} \tag{6}$$

With the establishment of this spatio-temporal relationship, the spatial evolution of the underlying spatio-temporal dynamical systems can be predicted as long as the underlying systems are deterministic.

3 Spatio-temporal embeddings for LDS's

3.1 The finite lattice case

Let V be a complete metric space. Here a measure on V means a nonnegative measure that is defined on the Borel sets of V and is not identically zero. For a subset $S \subset V$, $S + v$ means the translate of S by a vector $v \in V$.

Definition 1 (Hunt, Sauer, and Yorke 1992) *Let V be a complete metric linear space. A measure μ is said to be transverse to a Borel set $S \subset V$ if it has a compact support and $\mu(S+v) = 0$ for any $v \in V$.*

A set $S' \subset V$ is said to be shy if there exists a Borel set S such that $S' \subset S \subset V$ and a Borel measure μ on V which is transverse to S.

A set $S' \subset V$ is said to be prevalent if $V \setminus S'$ is shy.

The motivation of shyness and prevalence comes from the Fubini theorem. Note that any prevalent set $S' \subset V$ is dense in V and that for Euclidean spaces V any $S' \subset V$ is shy if and only if the Lebesque measure of S' vanishes. Therefore for subsets of finite dimensional spaces the term prevalent is synonomous with "almost every", in the sense of outside a set of measure zero.

Definition 2 (Hunt, Sauer, and Yorke 1992) *A finite-dimensional subspace $P \subset V$ is a probe for a set $T \subset V$ if the Lebesque measure supported on P is transverse to a Borel set which contains the complement of T.*

Then if T has a probe, it is prevalent. The detailed theory about prevalence can be found in Hunt, Sauer, and Yorke 1992.

The box-counting dimension of a compact set A in R^k is defined as follows.

Definition 3 *Let A be a compact set in R^k. For a positive number ε, let A_ε be the set of all points within ε of A, that is $A_\varepsilon = \{x \in R^k : ||x - a|| \leq \varepsilon \text{ for some } a \in A\}$. Let $N(\varepsilon)$ be the number of boxes that intersect A. Then the box-counting dimension of A is defined as*

$$boxdim(A) = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon} \tag{7}$$

if the limit exists. If not, the upper or lower box-counting dimension can be defined by replacing the limit by the $\lim inf$ *or* $\lim sup$.

Definition 4 *For a compact differential manifold M, let $T(M) = \{(m, v) : m \in M, v \in T_m M\}$ be the tangent bundle of M, and let $S(M) = \{(m, v) \in T(M) : |v| = 1\}$ denote the unit tangent bundle of M.*

Definition 5 *Let M_1, M_2 and N be manifolds and $f : M_1 \times M_2 \to N$ a C^r map. For $(m_1, m_2) \in M_1 \times M_2$, define $i_{m_1} : M_2 \to M_1 \times M_2$ and $i_{m_2} : M_1 \to M_1 \times M_2$ by*

$$i_{m_1}(y) = (m_1, y), i_{m_2}(x) = (x, m_2) \tag{8}$$

and define $T_1 f(m_1, m_2) : T_{m_1} M_1 \to T_{f(m_1,m_2)} N$ and $T_2 f(m_1, m_2) : T_{m_2} M_2 \to T_{f(m_1,m_2)} N$ by

$$T_1 f(m_1, m_2) = T_{m_1}(f \circ i_{m_2}), T_2 f(m_1, m_2) = T_{m_2}(f \circ i_{m_1}) \tag{9}$$

where $\circ$ denotes the composition.

Note that $T_1 f, T_2 f$ are the analogues on manifolds to partial derivatives of f on Banach spaces.

The following embedding theorem provides a basis for the spatio-temporal prediction for the LDSs composed of a finite number of interacting dynamical systems. Let $I \in Z^d$ be a finite subset and J be any given subset of I. $X = \{x = \{x_i\} : i \in I, ||x|| < \infty\} \subset R^k$ and $U = \{u = \{u_i\} : i \in I, ||u|| < \infty\} \subset R^k$ are open subsets of R^k. For any given sequence of inputs $u_1 = \{u_{1,i}\}_{i \in I}, \cdots, u_p = \{u_{p,i}\}_{i \in I}$, let $f^p(x, u_1, \cdots, u_p) =: f(\cdots f(f(x, u_1), u_2), \cdots, u_p)$.

Theorem 1 *Let $f : X \times U \to X$ be a smooth diffeomorphism, and let A be a smooth compact subset of X with $boxdim(A) = d$ and let $n_i, i \in J$ be integers. Let $u_1 = \{u_{1,i}\}_{i \in I}, u_2 = \{u_{2,i}\}_{i \in I}, \cdots$ be a given sequence of bounded inputs. Assume that for every positive integer $p \leq n = max_{i \in J} n_i$, the set A_p of periodic points of period p under the given input sequence satisfies $boxdim(A_p) < p/2$, and that the $T_1 f^p(x, u_1, \cdots, u_p)$ for any such periodic point x has distinct eigenvalues.*

Then for any smooth map h from X to R^k and mixed spatio-temporal delay coordinate vector $F : X \times U^{n-1} \to R^{\sum_{i \in J} n_i}$, $T_1 F(x, u_1, \cdots, u_{n-1}) : T(X) \to R^{\sum_{i \in J} n_i}$ is an isomorphism on A provided $\sum_{i \in J} n_i > 2k$.

Proof. To prove the theorem, several lemmas are needed.

Lemma 1 (Sauer, Yorke, and Casdagli 1991) *Let n and k be positive integers, $y_1, \cdots, y_n$ distinct points in R^k, and $v_1, \cdots, v_n$ in R^n. Then there exists a polynomial h in k variables of degree at most n such that for $i = 1, \cdots, n$, $\nabla h(y_i) = v_i$.*

Lemma 2 (Sauer, Yorke, and Casdagli 1991) *Let S be a bounded subset of R^k, $boxdim(\bar{S})=d$. and let $G_0, G_1, \cdots, G_l$ be Lipschitz maps from S to R^n. Assume that for each $x \in S$, the rank of the $n \times l$ matrix*

$$\{G_1(x), \cdots, G_l(x)\} \tag{10}$$

is at least r. For each $\alpha \in R^l$, define $G_\alpha = G_0 + \sum_{i=1}^{l} \alpha_i G_i$. Then for almost every $\alpha \in R^l$, the set $G_\alpha^{-1}(0)$ is the nested countable union of sets of lower box-counting dimension at most $d - r$. If $r > d$, then set $G_\alpha^{-1}(0)$ is empty for almost every α.

Remark 1 This lemma can be interpreted in the notations of definition 2. Let $V = Lip(S, R^n)$ be the space of all Lipschitz continuous R^n-valued functions on S. Furthermore, let $T = \{G \in Lip(S, R^n) : G^{-1}(0)\text{is empty}\} \subset V$, and define the l-dimensional subspace $P = span\{G_1, \cdots, G_l\}$ of V, then the above lemma says that under the conditions of Lemma 2 P is a probe space of T so that T is prevalent in V.

Lemma 3 *Let A be a compact subset of a smooth manifold embedded in R^k, and $B \subset R^m$. Let $F_0, F_1, \cdots, F_l : U \times V \to R^n$ be a family of smooth maps from $U \times V$, U is an open neighbourhood of A and $V \subset B$ to R^n. For each positive integer r, let S_r be the set of the unit tangent bundle $S(A)$ such that the $n \times l$ matrix*

$$\{T_1 F_1(u, v)(x), \cdots, T_1 F_l(u, v)(x)\}, (u, v) \in U \times V \tag{11}$$

has rank r and let $d_r = lower\ boxdim(\bar{S}_r)$. Define $F_\alpha = F_0 + \sum_{i=1}^{l} \alpha_i F_i : U \times V \to R^n$. Then if $d_r < r$ for all integers $r \geq 0$, then for almost every $\alpha \in R^l$, the map $T_1 F_\alpha(u, v)$ is an isomorphism on A.

Proof. For $i = 0, \cdots, l$ and for each $(u, v) \in U \times V$, define $G_i : S(A) \to R^n$ by $G_i(u, v, x) = T_1 F_i(u, v)(x)$, and for each $\alpha \in R_l$, define $G_\alpha = G_0 + \sum_{i=1}^{l} \alpha_i G_i$. Because $G_i, i = 0, \cdots, l$ are linear maps they are Lipschitz. Then Lemma 2 applies to show that for almost every $\alpha \in R^l$, $G_\alpha^{-1}(0) \cap S_r$ is the empty set. And since $S(A)$ is the union of all S_r, $G_\alpha^{-1}(0)$ is empty. □

The proof of the theorem can now be developed based on the lemmas above. It is sufficient to prove the case when $J = \{1, 2\}$, that is for two spatial sites. Consider the spatio-temporal delay coordinate vector (5) defined on a subset J of I. For any $h = \{h_i\}_{i \in J}$, any $x \in X$ and $u = (u_1, \cdots, u_{n-1})^T \in U^{n-1}$, rewrite (5) in a general form as follows

$$\begin{aligned} F_{(h,f)}(x, u) &= (h_1(x), h_1(f(x, u_1)), \cdots, h_1(f^{n_1-1}(x, u_1, \cdots, u_{n_1-1}); \\ &\quad h_2(x), h_2(f(x, u_1)), \cdots, h_2(f^{n_2-1}(x, u_1, \cdots, u_{n_2-1}))^T \end{aligned} \tag{12}$$

here the superscript T denotes the transposition. Note that

$$\begin{aligned} T_1 F_{(h,f)}(x, u)(v) &= (\nabla h_1(x)(v), \cdots, \nabla h_1(f^{n_1-1})^T T_1 f^{n_1-1}(x, u_1, \cdots, u_{n_1-1})(v); \\ &\quad \nabla h_2(x)(v), \cdots, \nabla h_2(f^{n_2-1})^T T_1 f^{n_2-1}(x, u_1, \cdots, u_{n_2-1})(v)))^T \end{aligned} \tag{13}$$

Now, for $i = 1, 2$, let $g_{i,1}, \cdots, g_{i,l_i}$ be a basis for the polynomials in k variables of degree up to n_i. Define the $n_1 + n_2$-dimensional vector-valued function $h_{\alpha,\beta} = h + \sum_{j=1}^{l_1} \alpha_j (g_{1,j}, \mathbf{0})^T + \sum_{j=1}^{l_2} \beta_j (\mathbf{0}, g_{2,j})^T$. Then it follows

$$T_1 F_{(h_{\alpha,\beta},f)}(x,u) = T_1 F_{(h,f)}(x,u) + \sum_{j=1}^{l_1} \alpha_j (T_1 F_{(g_{1,j},f)}(x,u), \mathbf{0})^T + \sum_{j=1}^{l_2} \beta_j (\mathbf{0}, T_1 F_{(g_{2,j},f)}(x,u))^T \quad (14)$$

If x is not a periodic point of period less than $n = max_i\{n_i\}$, then all $x, \cdots, f^{n-1}(x, u_1, u_2, \cdots, u_{n-1})$ are distinct points, which means all $T_1 f(x, u_1)(v), \cdots, T_1 f^{n-1}(x, u_1, u_2, \cdots, u_{n-1})(v)$ are not zeros for any $v \neq 0$ because f is a diffeomorphism. Therefore by Lemma 1, for any given vector $w \in R^{n_1+n_2}$, there exist polynomials g_1 in k variables of degree at most n_1 and g_2 in k variables of degree at most n_2 such that $\nabla g_1(f^{i-1}(x, u_1, u_2, \cdots, u_{i-1})) T_1 f^{i-1}(x, u_1, u_2, \cdots, u_{i-1})(v) = w_i$ for $i = 1, \cdots, n_1$ and $\nabla g_2(f^{i-1}(x, u_1, u_2, \cdots, u_{i-1})) T_1 f^{i-1}(x, u_1, u_2, \cdots, u_{i-1})(v) = w_{i+n_1}$ for $i = 1, \cdots, n_2$, which implies that $\{T_1 F_{(h_{\alpha,\beta},f)} : \alpha \in R^{l_1}, \beta \in R^{l_2}\}$ spans $R^{n_1+n_2}$. Defining a set of maps $G_i : X \times U^{n-1} \to R^{n_1+n_2}, i = 1, \cdots, l_1 + l_2$ as

$$G_i(v) = \begin{cases} (T_1 F_{(g_{1,i},f)}(x,u)(v), \mathbf{0})^T, \text{for } i = 1, \cdots, l_1 \\ (\mathbf{0}, T_1 F_{(g_{2,i-l_1},f)}(x,u)(v))^T, \text{for } i = l_1 + 1, \cdots, l_1 + l_2 \end{cases}$$

then the $(n_1 + n_2) \times (l_1 + l_2)$ matrix $\{G_1, \cdots, G_{l_1+l_2}\}$ has rank $n_1 + n_2$. It follows that if f has no periodic points of period less than $n = max\{n_1, n_2\}$, then S_r is empty for all $r \geq 0$ but $r = n_1 + n_2$ and because by hypothesis $n_1 + n_2 > 2k > lowerboxdim(\bar{S}_{n_1+n_2})$, the proof is completed by Lemma 3.

If x is a periodic point of period $p < max\{n_1, n_2\}$, then in order to apply Lemma 3, we need to show that the rank of $\{G_1, G_2, \cdots, G_{l_1+l_2}\}$ is strictly larger than $boxdim(\bar{S}_r)$. There are two cases.

Case 1: $p < min\{n_1, n_2\}$.

In this case, following the discussion of Theorem 4.14 in Sauer, Yorke and Casdagli (1991), the $n_1 + n_2$-dimensional vector $G_i(x, u, v), i = 1, \cdots, l_1$ can be written as equation (4.2) of Sauer, Yorke and Casdagli (1991) with H_j there replaced by $H_{i,j} =: \nabla g_{1,i}(x_j)$ and the last n_2 rows by zeros, and $G_i(x, u, v), i = l_1 + 1, \cdots, l_1 + l_2$ can be similarly written with H_j there replaced by $H_{i,j} =: \nabla g_{2,i}(x_j)$ and the first n_1 rows by zeros.

Then following the arguments of Theorem 4.14 in Sauer, Yorke and Casdagli (1991), and the hypothesis here, Lemma 3 applies.

Case 2: $n_1 \leq p < n_2$.

In this case, it is sufficient to consider the $n_1 + n_2$-dimensional vector $G_i(x, u, v), i = l_1 + 1, \cdots, l_1 + l_2$. Again this vector can be written as equation (4.2) in Sauer, Yorke and Casdagli (1991) with

H_j there replaced by $H_{i,j} =: \nabla g_{2,i}(x_j)$ and the first n_1 rows by zeros. The conclusion then follows. $\square$.

Remark 2 According to Theorem 1, for a LDS defined on a finite lattice $I = \{1, \cdots, n\}$, and for any given subset $J \subset I$ with m nodes, there generally exists a one-to-one correspondence between the global states and the spatio-temporal delay coordinate vector with a lag n_i for each node $i = 1, \cdots, m$ as long as $\sum_{i=1}^{m} n_i > 2n$. This means that the smaller the region J is, the more lags are needed. In practice, this provides an indication about the determination of the spatial neighbourhoods and time lags when identifying such a spatio-temporal dynamical system at a given lattice site. From the system prediction point of view, this theorem implies that the dynamics of a LDS at one site or a spatial subregion can be predicted based on the dynamics at any other site or spatial subregion.

3.2 The infinite lattice case

In this section, the problem of local state space reconstruction for infinite lattice LDS's will be discussed.

Let $I \subset Z^d \subset R^d$ be the d-dimensional integer lattice. A LDS defined on I has a phase space $X = \{x = \{x_i\} : i \in I, ||x|| < \infty\}$ which is an infinite-dimensional Banach space. The systems of interest herein are the infinite lattice LDS's having the property of finite-range interaction. Specifically, let $G \subset I \subset Z^d$ be a fixed finite subset of the lattice, which represents the finite-range interaction. Then the infinite lattice LDS (2) can be written as

$$x_i(t) = f(\{x_{i+j}(t-1)\}_{j\in G}, \{u_{i+j}(t-1)\}_{j\in G}), i \in I \tag{15}$$

where $f : \{\{x'_j\}_{j\in G}\} \times \{\{u'_j\}_{j\in G}\} \to R$ is a smooth function.

In what follows, the local dynamics of the LDS will be constructed. To this end, let $I_k \subset I, k = (k_1, k_2, \cdots, k_d)$ be the following finite lattice

$$I_k = \{(i_1, i_2, \cdots, i_d) \in I : 1 \leq i_j \leq k_j\} \tag{16}$$

Then the local finite lattice LDS defined on I_k can be constructed according to (15) with some boundary conditions. The width of the frame of boundary sites is usually equal to the neighbourhood size r. For instance, if $d = 2$, $G = (i, j), i, j = -1, 0, 1$, then $r = 1$ and the boundary sites are

$$\{(k_1+1, j), (0, j), (i, k_2+1), (i, 0), 0 \leq i \leq k_1, 0 \leq j \leq k_2\} \tag{17}$$

The three most commonly chosen boundary conditions are: Neumann, periodic and Dirichlet boundary conditions. The Neumann boundary condition is the zero flux boundary condition, that is the states of the boundary sites are set equal to the states at the corresponding neighbourhood sites in I_k. For the example above, then $0 \leq i \leq k_1+1, 0 \leq j \leq k_2+1$, and

$$x_{k_1+1,j} = x_{k_1,j}, x_{i,k_2+1} = x_{i,k_2} x_{0,j} = x_{1,j}, x_{i,0} = x_{i,1} \tag{18}$$

The periodic boundary condition identifies the first and the last rows (respectively, columns) of the array I_k, namely, for $0 \leq i \leq k_1+1, 0 \leq j \leq k_2+1$,

$$x_{1,j} = x_{k_1,j}, x_{0,j} = x_{k_1-1,j}, x_{2,j} = x_{k_1+1,j}; x_{i,1} = x_{i,k_2}, x_{i,0} = x_{i,k_2-1}, x_{i,2} = x_{i,k_2+1}; \tag{19}$$

The Dirichlet boundary condition involves fixed boundary values which are prescribed on the boundary sites.

If the finite subset I_k is large enough ($d \geq r$), then with any specified boundary condition, the LDS defined on I_k forms a system of ODE's on a $k = \prod_{i=1}^{d} k_i$-dimensional phase space. Therefore, Theorem 1 can be directly applied to such a finite dimensional system to obtain a local spatio-temporal input-output relationship between the sites with I_k for the measurement functions defined on the same lattice I_k. Discussions on the effects of these boundary conditions on the global spatio-temporal patterns can be found in Shin(2000).

4 System identification and spatio-temporal prediction

The results of previous sections suggest how a good choice of time lags for the spatio-temporal model depends on the underlying global state space dimension for any given neighbourhood. In the case of finite LDS's, it is a straightforward task, that is, it can be determined in such a way that the summation of the numbers of the sites in the given neighbourhood and the time lags should be larger than twice the global state space dimension. In the case of infinite LDS's, it depends on the dimension of the local lattice LDS's.

In what follows, the identification problem of the LDS's will be considered. The system identification of LDS's involves the construction of the spatio-temporal input-output relationship from a set of inputs and observations. From (6), the task of the identification is to reproduce the dynamical relation g from the measured data along time and space. This problem has been extensively studied (Coca and Billing 2001, 2002). In this paper, for a specific site i, the identification procedure can be outlined as below

i) Determine the spatial neighbourhood sites for the ith site;

ii) Determine the time lags;

iii) Apply the Orthogonal Least Squares (OLS) algorithm to obtain the parameters of the LDS model (polynomials as regressors).

Note that in the classical identification procedure, the spatial neighbourhood sites of the identified site and the time lags were considered as fixed. In other words, the neighbourhood of the identified site was physically considered as a region around that site which directly influences the dynamics of that site in the spatial domain, and the time lags were considered as the direct influences from the history of the system evolution. But from the spatio-temporal embedding theorem 1, it is realised that the idea behind the state space reconstruction method is that it should be sufficient to use a number of 'proper' independent spatial and temporal quantities to specify the state of a LDS at any given time and any given site. This means that a LDS can be modelled in many different ways depending on the selection of independent variables. Example 1 in the next section is an illustration of this point where three spatio-temporal models are obtained for a single LDS.

In practical system identification, a general rule for the selection of the neighbourhood sites and the time lags can be outlined. Due to the spatio-temporal embedding theorem, for finite lattice LDS's, the set J of the neighbourhood sites for site i can be chosen as any subset including the site i, of the lattice I. In this case a one-to-one correspondence between the system states and the mixed spatio-temporal delay coordinate vectors exits as long as $\sum_{j\in J} n_j > 2 \times card(J)$. In the case of infinite lattice LDS's discussed in the previous section, if the finite interaction range G is known, then this can be chosen to be equal to I_k sites while the time lags can be chosen as satisfying $\sum_{j\in I_k} n_j > 2 \times card(I_k)$. If G is unknown, the same rule applies as long as I_k is chosen large enough.

5 Numerical examples

5.1 Example 1 - Linear Diffusion Equation

Consider the following diffusion equation

$$\frac{\partial^2 v(t,x)}{\partial t^2} - C\frac{\partial^2 v(t,x)}{\partial x^2} = u(t,x), x \in [0,1] \tag{20}$$

with initial conditions

$$\begin{aligned} v(0,x) &= 0 \\ \frac{dv(0,x)}{dt} &= 4\exp(-x) + \exp(-0.5x) \end{aligned} \tag{21}$$

where

$$u(t,x) = -13\exp(-x)\cos(1.5t) - 9.32\exp(-0.5x)\cos(2.1t) \tag{22}$$

For $C = 1.0$ the exact solution $v(t,x)$ of the initial value problem (20), (22) is

$$\begin{aligned} v(t,x) &= 4\exp(-x)\cos(1.5t) + 2\exp(-0.5x)\cos(2.1t) \\ &\quad -4\exp(-x)\exp(-t) - 2\exp(-0.5x)\exp(-0.5t) \end{aligned} \tag{23}$$

Such a spatially extended dynamical system can be considered as a lattice dynamical system when both the space and time are properly discretised.

In the simulation, the measurement function was taken as

$$y(t,x) = v(t,x) \tag{24}$$

and the spatial domain was sampled at 21 equally spaced points over $[0,1]$. In this way, it is regarded as a lattice dynamical system with a finite lattice $I = \{1, \cdots, 21\}$ where $\{x_1, \cdots, x_{21}\} = \{0, 0.05, \cdots, 0.95, 1\}$. Therefore, the global state space is R^{21}.

According to the spatio-temporal embedding theorem, it is generic that any other sites can be chosen as a neighbourhood of any site for the local state space reconstruction. Therefore, in the simulation, for a given site i, three sets of neighbourhood were studied, that is, case 1: $i+1$; case 2: $i-1, i+1$; case 3: $i-2, i-1, i+1, i+2$. In all three cases it is theoretically required that the time delay for each site should be larger than 42, 21, and 10 on the average. However, the simulation results show that a small time delay is enough for this special system.

In the numerical simulation, for each site, 100 input/output data points sampled at $\Delta t = 0.1$ were generated. The data are plotted in Fig.(1).

The identification data consisted of 100 data points of input/output data $u_i(t), y_i(t)$ at site $i = 11$ corresponding to $x = x_{11} = 0.5$. In addition, 100 input and output data from neighbouring locations acted as regressors during the identification. The three identified models are listed in Table (1), where ERR denotes the Error Reduction Ratio and STD denotes the standard deviations.

The model predicted output and the model predicted error are plotted in Fig.(2) which show very good agreement between the exact solution and the CML model output. Furthermore, the simulation results also show that such a LDS can be embedded into different reconstruction spaces, which justifies the developed spatio-temporal embedding theorem.

5.2 Example 2 - Sine-Gordon Equation

Consider the two-dimensional Sine-Gordon Equation (Hirota 1973)

	Terms	Estimates	ERR	STD
Case 1	$y_i(t-1)$	0.51010E+00	0.97473E+00	0.11133E+00
	$y_{i+1}(t-2)$	0.29727E+00	0.25102E-01	0.13878E+00
	$u_i(t-5)$	0.64905E-01	0.13236E-03	0.57504E-01
	$y_i(t-2)$	0.45267E+00	0.12739E-04	0.13546E+00
	$u_i(t-1)$	-0.26402E+00	0.21892E-04	0.61936E-01
	$y_{i+1}(t-4)$	-0.31134E+00	0.24077E-06	0.13837E+00
	constant	0.12748E-02	0.54424E-08	0.29894E-03
	$y_i(t-4)$	0.28483E-01	0.47670E-08	0.13295E+00
	$y_{i+1}(t-5)$	-0.33505E+00	0.20063E-07	0.13160E+00
	$y_i(t-3)$	0.14801E+00	0.64642E-08	0.13479E+00
	$y_{i+1}(t-3)$	0.13662E+00	0.34670E-08	0.13880E+00
	$u_i(t-4)$	-0.23632E+00	0.43983E-08	0.10595E+00
	$u_i(t-2)$	0.46216E+00	0.27615E-07	0.10946E+00
	$y_i(t-5)$	-0.63146E-01	0.31759E-08	0.11490E+00
	$u_i(t-5)^2$	-0.95824E-04	0.30765E-09	0.20913E-04
	$u_i(t-5)y_{i+1}(t-5)$	-0.77616E-03	0.24184E-08	0.16436E-03
	$y_i(t-4)y_i(t-5)$	-0.13775E-02	0.28060E-08	0.36203E-03
	$y_i(t-1)u_i(t-5)$	-0.22775E-03	0.19617E-07	0.54374E-04
	$y_i(t-4)u_i(t-4)$	0.22760E-03	0.78590E-08	0.89990E-04
	$y_{i+1}(t-1)$	0.15953E+00	0.20172E-08	0.13461E+00
Case 2	$y_i(t-1)$	0.44886E+00	0.97450E+00	0.14608E+00
	$y_{i+1}(t-2)$	-0.12130E+00	0.25229E-01	0.11473E+00
	$u_i(t-2)$	0.62552E-02	0.21500E-03	0.26842E-02
	$y_{i-1}(t-2)$	-0.46516E+00	0.56718E-04	0.96212E-01
	$y_{i-1}(t-1)$	0.10690E+01	0.35351E-05	0.97783E-01
	$y_{i+1}(t-1)$	0.43696E+00	0.30027E-07	0.12168E+00
	$y_i(t-2)$	-0.37547E+00	0.12109E-07	0.14645E+00
	$u_i(t-1)$	0.32525E-02	0.40834E-08	0.28909E-02
	constant	-0.10198E-03	0.13912E-08	0.12100E-03
Case 3	$y_i(t-1)$	0.39445E+00	0.97450E+00	0.13572E+00
	$y_{i+1}(t-1)$	-0.12496E+00	0.25232E-01	0.12018E+00
	$u_i(t-2)$	0.38259E-02	0.21183E-03	0.43191E-02
	$y_{i-1}(t-2)$	-0.18572E+00	0.56175E-04	0.13305E+00
	$y_{i-1}(t-1)$	0.59432E+00	0.36235E-05	0.13273E+00
	$y_{i+1}(t-1)$	0.25854E+00	0.24180E-07	0.12267E+00
	$y_{i-2}(t-1)$	0.37694E+00	0.20224E-07	0.11270E+00
	$y_{i-2}(t-2)$	-0.23501E+00	0.13585E-07	0.11128E+00
	$y_{i+2}(t-1)$	0.34798E+00	0.11117E-07	0.10910E+00
	$y_{i+2}(t-2)$	-0.19187E+00	0.98894E-08	0.10551E+00
	constant	-0.12259E-03	0.39321E-08	0.11837E-03
	$y_i(t-2)$	-0.24013E+00	0.34177E-08	0.13700E+00
	$u_i(t-1)$	0.58602E-02	0.24592E-08	0.46488E-02

Table 1: Example 1: The terms and parameters of the final LDS models for three cases

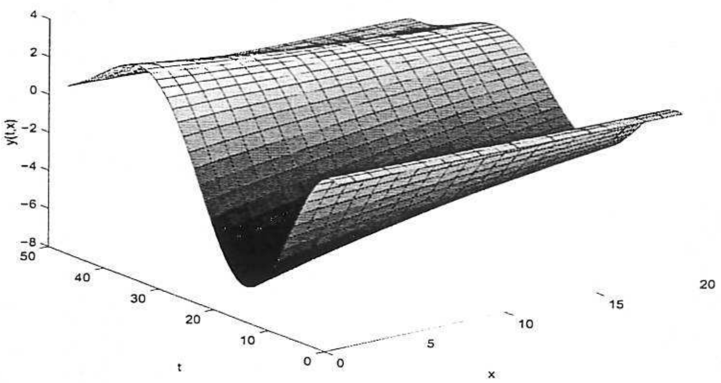

Figure 1: Example 1: System output

$$\frac{\partial^2 u(t,x,y)}{\partial x^2} + \frac{\partial^2 u(t,x,y)}{\partial y^2} - \frac{\partial^2 u(t,x,y)}{\partial t^2} = \sin(u(t,x,y)) \tag{25}$$

which describes the motion of the magnetic flux quanta on a Josephson junction transmission line.

The exact three-soliton solution of (25) can be expressed in the following form

$$u(t,x,y) = 4\tan^{-1}(\frac{g(t,x,y)}{f(t,x,y)}) \tag{26}$$

where

$$\begin{aligned} f(t,x,y) &= 1 + a(1,2)\exp(\eta_1+\eta_2) + a(1,3)\exp(\eta_1+\eta_3) + a(2,3)\exp(\eta_2+\eta_3) \\ g(t,x,y) &= \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + a(1,2)a(1,3)a(2,3)\exp(\eta_1+\eta_2+\eta_3)) \end{aligned} \tag{27}$$

in which

$$a(i,j) = \frac{(P_i - P_j)^2 + (q_i - q_j)^2 - (w_i - w_j)^2}{(P_i + P_j)^2 + (q_i + q_j)^2 - (w_i + w_j)^2}$$
$$\eta_i = P_i x + q_i y - w_i t - \eta_i^0, (\eta_i^0 \text{ is constant})$$
$$P_i^2 + q_i^2 - w_i^2 = 1, \text{for } i,j = 1,2,3$$

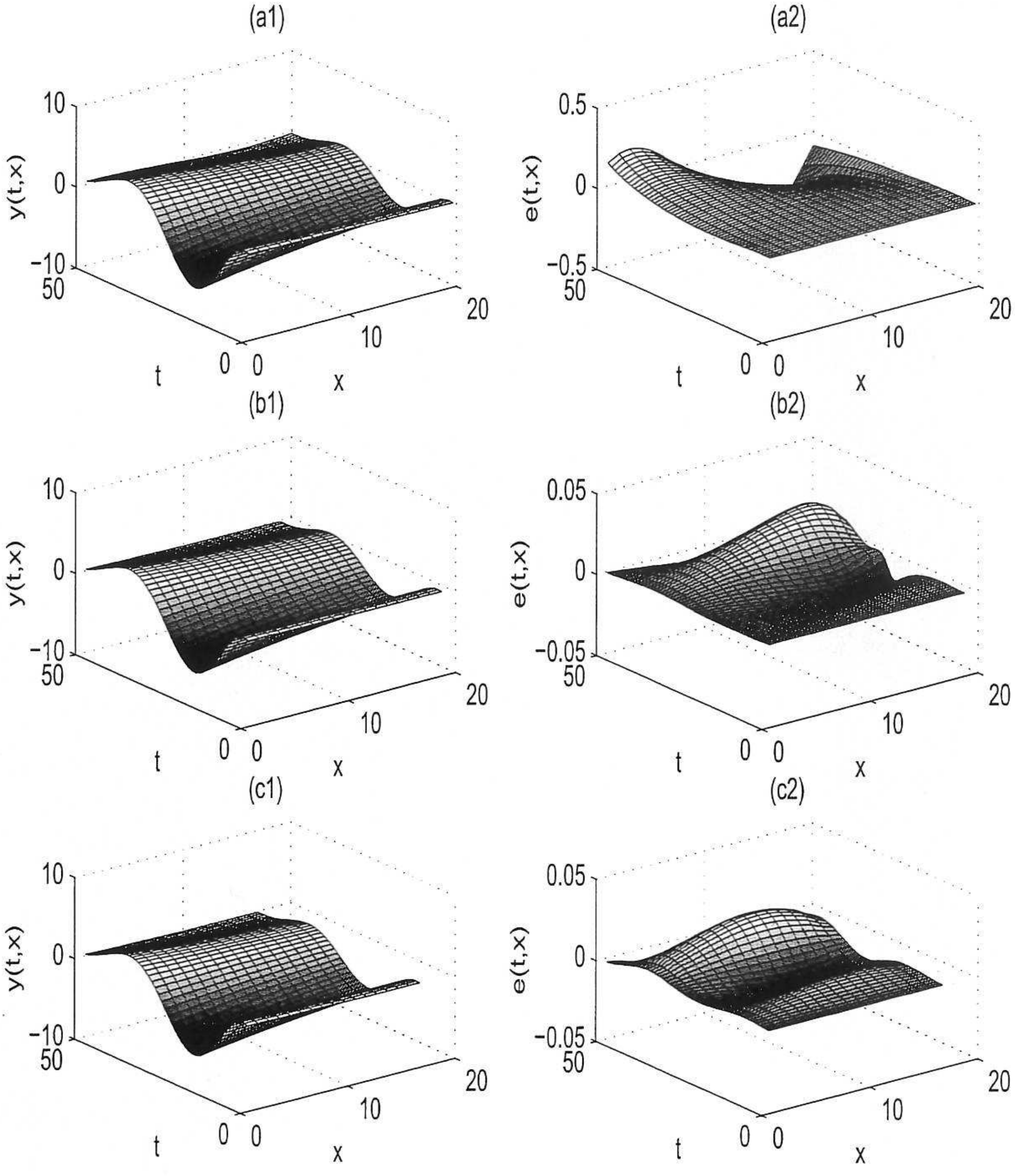

Figure 2: Example 1: Model predicted output and error (a1) and (a2): case 1, (b1) and (b2): case 2, (c1) and (c2): case 3

provided that the parameters P_i, q_i, and $w_i, i = 1, 2, 3$ satisfy the condition

$$det \begin{pmatrix} P_1 & q_1 & w_1 \\ P_2 & q_2 & w_2 \\ P_3 & q_3 & w_3 \end{pmatrix} = 0$$

By approximating $\nabla^2 u(t,x,y) = \frac{\partial^2 u(t,x,y)}{\partial x^2} + \frac{\partial^2 u(t,x,y)}{\partial y^2}$ as

$$\nabla^2 u(t,x,y) \approx \frac{1}{h^2}(u(t,x+h,y) + u(t,x-h,y) + u(t,x,y+h) + u(t,x,y-h) - 4u(t,x,y)) \quad (28)$$

and letting $v(t,x,y) = du(t,x,y)/dt$ then the system (25) can be considered as the following lattice dynamical system

$$\begin{aligned} \frac{du_{(i,j)}(t)}{dt} &= v_{(i,j)}(t) \\ \frac{dv_{(i,j)}(t)}{dt} &= -\sin(u_{(i,j)}(t) + \frac{1}{h^2}(u_{(i+1,j)}(t) + u_{(i-1,j)}(t) + u_{(i,j+1)}(t) + u_{(i,j-1)}(t) - 4u_{(i,j)}(t))) \end{aligned} \quad (29)$$

where $u_{(i,j)}(t) = u(t, x_i, y_j)$, $v_{(i,j)}(t) = \dot{u}_{(i,j)}(t)$, $h = x_i - x_{i-1} = y_j - y_{j-1}$, $i = 1, \cdots, M, j = 1, \cdots, N$.

The simulation data was generated with $M = N = 30$ and $h = 0.1$ with the following parameter values, $P_1 = 1.1, P_2 = P_3 = 0.3; q_1 = 0.0, q_2 = q_3 = 1.2$ and $w_1 = 0.4583, w_2 = w_3 = 0.6633$, and initial conditions

$$u(0, x_i, y_j) = 4\tan^{-1}(\frac{g(0, x_i, y_j)}{f(0, x_i, y_j)}) \quad (30)$$

The measurement function was taken as

$$y(t,x,y) = u(t,x,y) \quad (31)$$

From each location, 50 input/output data points sampled at $T = 0.5$ were generated. Fig.(3) shows four snapshots of $y(t,x,y)$ at $t = 0.5 \times 1 = 0.5$ and $t = 0.5 \times 5 = 2.5$, respectively.

In this simulation, the neighbourhood for the site i was chosen as $(i-1, j)$ and $(i, j-2)$. The identification data consisted of 15 data points of input/output data $u_i(t), y_i(t)$ at the node $(i,j) = (25, 25)$. The identified model is listed in Table (2)

The model predicted outputs are plotted in Fig.(4), which show that the identified LDS model can reproduce the spatio-temporal patterns of the original system very well.

Terms	Estimates	ERR	STD
$y_{i,j-2}(t-1)$	0.31071E+00	0.99487E+00	0.41674E+00
$y_{i,j-2}(t-2)$	0.13599E+00	0.50331E-02	0.54055E+00
constant	0.18561E+00	0.85853E-04	0.58803E-01
$y_{i-1,j}(t-1)$	0.11221E+01	0.69613E-05	0.87463E+00
$y_{i,j}(t-2)$	-0.15402E+01	0.33817E-05	0.11698E+01
$y_{i-1,j}(t-2)$	0.12513E+01	0.12143E-05	0.14617E+01
$y_{i,j}(t-1)$	-0.38742E+00	0.18126E-06	0.93930E+00

Table 2: Example 2: The terms and parameters of the final LDS model

6 Conclusions

A spatio-temporal embedding theorem for finite lattice LDS's with external inputs has been proved. Although this result is similar to that of Sauer, Yorke, and Casdagli (1991) who studied the simpler time series case, it is believed that the insight the new embedding result brings for the spatio-temporal case is an important step in the development of spatially extended dynamical systems. With this result, the spatio-temporal prediction of LDS's can be realised. It has also been demonstrated that the new embedding theorem can be directly applied to the local modelling of LDS's defined on an infinite lattice. Furthermore, from a practical point of view, the new results provide a guide to determining the neighbourhood and time lags for the problem of system identification of LDS's.

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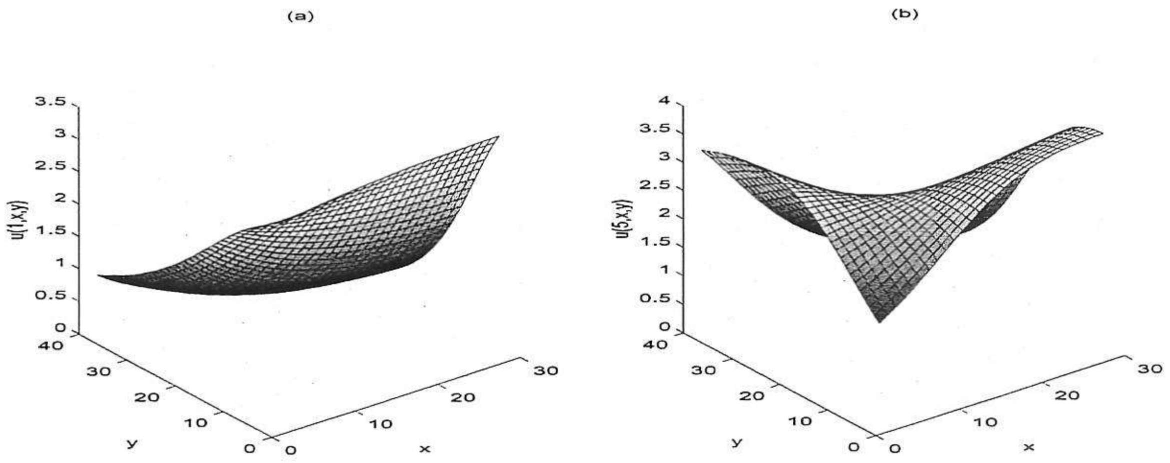

Figure 3: Example 2: Sytem outputs at (a) $t = 0.5 \times 1 = 0.5$, (b) $t = 0.5 \times 5 = 2.5$

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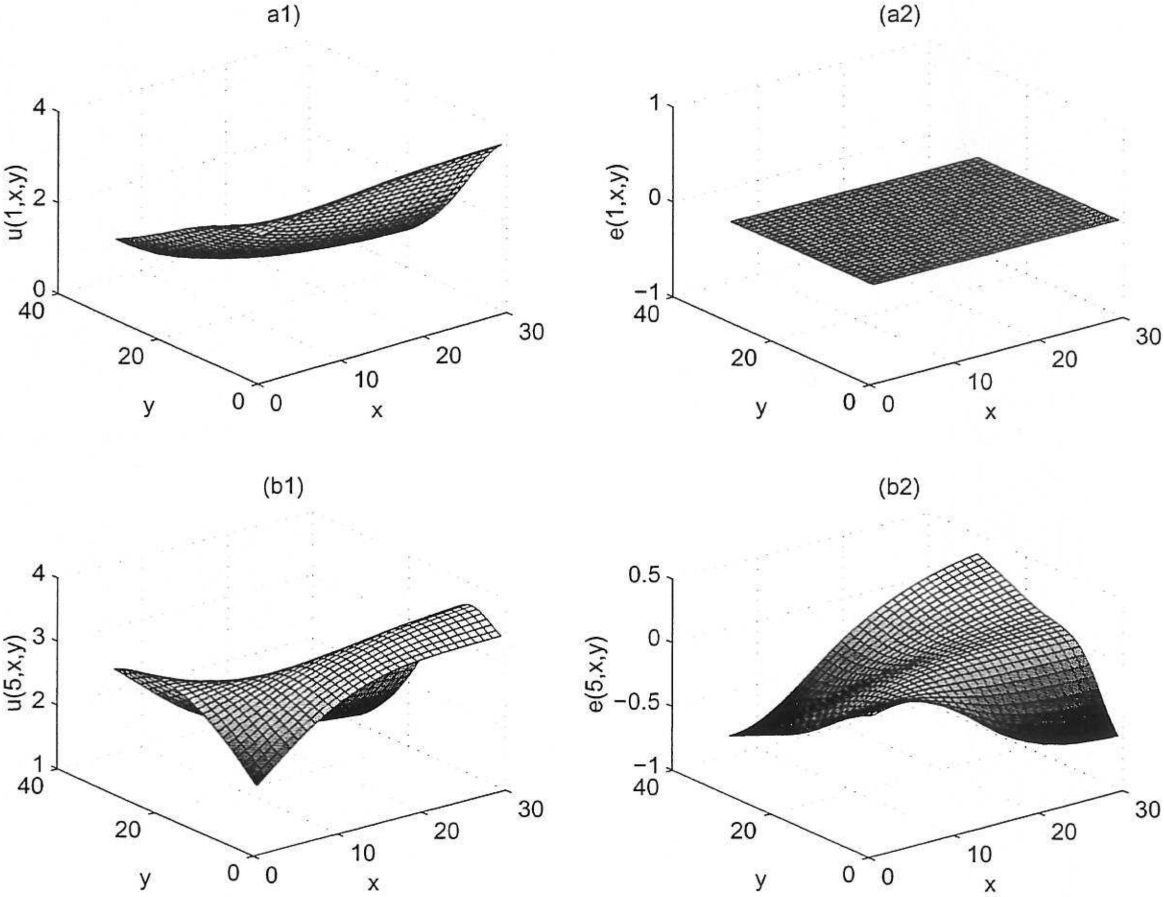

Figure 4: Example 2: Model predicted output and error at $t = 0.5 \times 1 = 0.5$: (a1) output and (a2) error, and at $t = 0.5 \times 5 = 2.5$: (b1) output and (b2) error