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Nonautonomous Systems, Lie Algebras and Lyapunov Transformations

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Nonautonomous Systems, Lie Algebras and Lyapunov Transformations

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ABSTRACT

An explicit form for the solution of a nonauonomous linear system of differential equations is given by using Duhamel's principle and a generalised Campbell-Haudorff formula. This is applied in the case of a nilpotent generating Lie algebra to Lyapunov transformations.

1. Introduction

In this paper we shall apply Duhamel's principle ([Taylor, 1991]) relating the solution of a nonautonomous linear differential equation to the limit of a repeated sequence of exponentials and a generalised Campbell-Haudorff formula in order to obtain an explicit solution for a nonautonomous linear differential equation. The solution will be given in terms of commutators of the matrices $A(t)$ defining the system. Thus, if the system is

$$\dot{x} = A(t)x. \ x(0) = x_0$$

then the solution will be expressed as an exponential of a matrix which belongs to the Lie algebra generated by the matrices $\{A(t) : t \in \mathbb{R}\}$. As such, we obtain an explicit closed-form solution in the case where this Lie algebra is nilpotent. The use of a system Lie algebra generated by the matrices of a system has also been applied recently to nonlinear systems of the form

$$\dot{x} = A(x)x. \ x(0) = x_0$$

and a number of new results in chaos theory and stability have been obtained (see the series of papers [Banks and Al-Jurani, 1994, 1996, Banks and McCaffrey. 1998. Banks, 1999, Banks, 2000]).

In the case of linear, nonautonomous systems the explicit solution can be applied to a number of problems. We illustrate the application here in the case of Lyapunov transformations, which leads to new stability results.

The general formula developed here requires the computation of a certain set of coefficients (denoted by $\mu(\sigma^{k-1})$ below). These are given by a complicated formula given in theorem 4. The coefficients can be effectively computed by using the symbolic package Maple, and so in the appendix we given a simple Maple program for their computation.

2. The Generalised Campbell-Hausdorff Formula

We begin by stating the well-known Campbell-Haudorff theorem, whose proof can be found in [Miller, 1972]. (For the theory of Lie algebras, see [S.Helagson, 1962, N.Jacobson, 1962, A.A.Sagle and R.E.Walde, 1973, S.Varadarahan, 1976]).

Theorem 1 *If A, B are sufficiently close to 0, then $C = \ln(e^A e^B)$ is given by*

$$C = B + \int_0^1 g[\exp(tAdA)\exp(AdB)](A)dt \tag{2.1}$$

where

$$g(z) = \frac{\ln z}{z-1} = 1 + \frac{1}{2}(1-z) + \frac{1}{3}(1-z)^2 + \cdots = \sum_{\ell=0}^{\infty} \frac{1}{\ell+1}(-1)^\ell (z-1)^\ell.$$ □

Corollary 1 *If A, B are as in the theorem, then*

$$\begin{aligned} C \;=\; & B + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell+1} \sum_{\substack{i_1=0, j_1=0 \\ (i_1, j_1)\neq(0,0)}}^{\infty} \sum_{\substack{i_2=0, j_2=0 \\ (i_2, j_2)\neq(0,0)}}^{\infty} \cdots \sum_{\substack{i_\ell=0, j_\ell=0 \\ (i_\ell, j_\ell)\neq(0,0)}}^{\infty} \frac{1}{i_1! i_2! \cdots i_\ell! j_1! j_2! \cdots j_\ell! (|\mathbf{i}|+1)} \\ & (AdA)^{i_1}(AdB)^{j_1}(AdA)^{i_2}(AdB)^{j_2} \cdots (AdA)^{\ell_2}(AdB)^{\ell_2} \cdot A \end{aligned} \tag{2.2}$$

where $|\mathbf{i}| = i_1 + \cdots + i_\ell$.

Proof From (1) we have

$$C = B + \int_0^1 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell+1} \left(\sum_{\substack{i=0 \\ (i,j)\neq(0,0)}}^{\infty} \sum_{j=0}^{\infty} \frac{t^i}{i!j!}(AdA)^i (AdB)^j \right)^\ell Adt$$

and the result follows. □

We require a generalised Campbell-Haudorff formula for k multiplicands, i.e. $e^{A_1}e^{A_2}\cdots e^{A_k}$. To find this we shall use the argument in [Miller, 1972] which requires the following two lemmas, also proved in [Miller, 1972]:

Lemma 1 *For matrices A, B we have*

$$e^A B e^{-A} = e^{AdA} B = \sum_{j=0}^{\infty} (j!)^{-1}(AdA)^j (B).$$ □

Lemma 2 *If each element of $A(t)$ is analytic and $f(z) = (e^z - 1)/z$ then*

$$e^{A(t)} \frac{d}{dt} e^{-A(t)} = -f(AdA(t))(\dot{A}(t)).$$ □

Theorem 2 *Given k matrices $A_1, \cdots, A_k$ in a sufficiently small neighbourhood of 0, then $C_k = \ln(e^{A_k} e^{A_{k-1}} \cdots e^{A_1})$ is given by*

$$\begin{aligned} C_k &= \int_0^1 g\left[\exp(tAdA_k)\exp(AdA_{k-1})\exp(AdA_{k-2})\cdots\exp(AdA_1)\right](A_k)dt \\ &\quad + C_{k-1} \end{aligned}$$

where $C_{k-1} = \ln(e^{A_{k-1}} e^{A_{k-2}} \cdots e^{A_1})$.

Proof Let

$$\Gamma(t) = \ln(e^{tA_k} e^{A_{k-1}} \cdots e^{A_1})$$

so that

$$e^{\Gamma(t)} = e^{tA_k} e^{A_{k-1}} \cdots e^{A_1}.$$

Then,

$$\begin{aligned} (\exp[Ad\Gamma(t)])H &= e^{\Gamma(t)} H e^{-\Gamma(t)} \\ &= e^{tA_k} e^{A_{k-1}} \cdots e^{A_1} H e^{-A_1} \cdots e^{-A_{k-1}} e^{-tA_k} \end{aligned}$$

by lemma 1, for any matrix H, and so

$$\exp[Ad\Gamma(t)] = \exp(tAdA_k)\exp(AdA_{k-1})\cdots\exp(AdA_1).$$

Also,

$$\begin{aligned} e^{\Gamma(t)} \frac{d}{dt} e^{-\Gamma(t)} &= e^{tA_k} e^{A_{k-1}} \cdots e^{A_1} \frac{d}{dt}(e^{-A_1} \cdots e^{-A_{k-1}} e^{-tA_k}) \\ &= -A_k \end{aligned}$$

and so, by lemma 2,

$$f(Ad\,\Gamma(t))\dot{\Gamma}(t) = A_k.$$

However,

$$f(\ln z)g(z) = 1 \text{ , for } |1 - z| < 1,$$

and so

$$f(\ln F)g(F) = I \text{ or } g(F) = (f(\ln F))^{-1}$$

for any matrix F with $\|I - F\| < 1$. Setting $F = \exp(Ad\,tA_k)\exp(Ad\,A_{k-1})\cdots\exp(Ad\,A_1)$ gives

$$\begin{aligned} \Gamma(t) &= \int_0^t g\left[\exp(tAdA_k)\exp(AdA_{k-1})\exp(AdA_{k-2})\cdots\exp(AdA_1)\right](A_k)dt \\ &\quad + \text{constant} . \end{aligned}$$

The constant is given by $\Gamma(0) = \ln(e^{A_{k-1}} e^{A_{k-2}} \cdots e^{A_1}) = C_{k-1}$. □

Corollary 2 *If $A_1, \cdots, A_k$ are as in the theorem, then*

$$\begin{aligned} C_k &= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell + 1} \sum_{\substack{\mathbf{i}(1)=0 \\ |\mathbf{i}(1)|\neq 0}}^{\infty} \cdots \sum_{\substack{\mathbf{i}(\ell)=0 \\ |\mathbf{i}(\ell)|\neq 0}}^{\infty} \frac{1}{\mathbf{i}(1)! \cdots \mathbf{i}(\ell)!(i_1(1) + \cdots + i_1(\ell) + 1)} \\ &\quad (Ad\,A_k)^{i_1(1)}(Ad\,A_{k-1})^{i_2(1)} \cdots (Ad\,A_1)^{i_k(1)}(Ad\,A_k)^{i_1(2)}(Ad\,A_{k-1})^{i_2(2)} \cdots (Ad\,A_1)^{i_k(2)} \\ &\quad \cdots (Ad\,A_k)^{i_1(\ell)}(Ad\,A_{k-1})^{i_2(\ell)} \cdots (Ad\,A_1)^{i_k(\ell)} \cdot A_k \end{aligned}$$

where $\mathbf{i}(p) = (i_1(p), \cdots, i_k(p))$, $\mathbf{i}(p)! = i_1(p)! i_2(p)! \cdots i_k(p)!$. □

(If $\ell = 0$ we interpret the value as A_k.)

3. Time-Varying Differential Equations

Next we recall Duhamel's principle ([Taylor, 1991]) for the solution of a linear, time-varying differential system:

Lemma 3 *The system*

$$\dot{x} = A(t)x\ ,\ x(0) = x_0 \tag{3.1}$$

has solution given by

$$x(t) = \lim_{h\to 0} e^{A((m-1)h)h} e^{A((m-2)h)h} \cdots e^{A(2h)h} e^{A(h)h} e^{A(0)h} x_0 \tag{3.2}$$

for any $t > 0$, where $mh = t$. □

From corollary 2 and lemma 3, we have

Lemma 4 *The solution of the system*

$$\dot{x} = A(t)x\ ,\ x(0) = x_0$$

is given by

$$x(t) = \lim_{h\to 0} e^{C_m} x_0$$

where $mh = t$ and

$$\begin{aligned} C_m \;=\; & \sum_{p=2}^{m} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell+1} \sum_{\substack{\mathrm{i}(1)=0 \\ |\mathrm{i}(1)|\neq 0 \\ \mathrm{i}(1)\in\Re^p}}^{\infty} \cdots \sum_{\substack{\mathrm{i}(\ell)=0 \\ |\mathrm{i}(\ell)|\neq 0 \\ \mathrm{i}(\ell)\in\Re^p}}^{\infty} \frac{1}{\mathrm{i}(1)!\cdots \mathrm{i}(\ell)!(i_1(1)+\cdots+i_1(\ell)+1)} \\ & (Ad\,A_p)^{i_1(1)}(Ad\,A_{p-1})^{i_2(1)}\cdots(Ad\,A_1)^{i_p(1)}(Ad\,A_p)^{i_1(2)}(Ad\,A_{p-1})^{i_2(2)}\cdots(Ad\,A_1)^{i_p(2)} \\ & \cdots (Ad\,A_p)^{i_1(\ell)}(Ad\,A_{p-1})^{i_2(\ell)}\cdots(Ad\,A_1)^{i_p(\ell)}\cdot A_p \\ & + A_1 \end{aligned} \tag{3.3}$$

where

$$A_q = A((q-1)h)h. \tag{3.4}$$

□

Combining lemmas 3 and 4 we have

Theorem 3 *The solution of the nonautonomous differential equation (3.1) is given by*

$$\begin{aligned} x(t;x_0) \;=\; & \exp\Bigg(\int_0^t A(\tau)d\tau + \sum_{k=2}^{\infty} \sum_{\sigma^{k-1}\in S_{k-1}} \mu(\sigma^{k-1}) \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_3} \int_0^{\tau_2} \\ & [A(\tau_{\sigma^{k-1}(1)}), [A(\tau_{\sigma^{k-1}(2)}), [\cdots, [A(\tau_{\sigma^{k-1}(k-1)}), A(\tau_k)]\cdots]]] d\tau_1\cdots d\tau_k\Bigg)\, x_0 \end{aligned} \tag{3.5}$$

where S_{k-1} is the set of all permutations of $1, \cdots, k-1$ and $\mu(\sigma^{k-1})$ is a number. depending on k and the permutation, to be found later.

Proof This follows from (3.2) and (3.3) since each multiple integral in (3.5) is the limit of typical terms in (3.3) where each $i_j(k) = 1$. The latter condition follows from the fact that, for a sequence

$$(Ad\,A_p)^{i_1(1)}(Ad\,A_{p-1})^{i_2(1)}\cdots(Ad\,A_1)^{i_p(1)}(Ad\,A_p)^{i_1(2)}\cdots A_p$$

of a given total degree $k = \sum_{j=1}^{\ell}(i_1(j) + \cdots + i_p(j))$, any repeated factors will converge to a zero integral since they are multiplied by h^k and there are at most $O(1/(h^{k-1}))$ such terms. □

The only thing remaining, therefore, is to find the multipliers $\mu(\sigma^{k-1})$. This will be done in three steps. Consider first the case of $k = 2$. Clearly for terms with brackets of the form $[A_i, A_j]$ we must have $\ell = 1$ in the expression (3.3); thus we must choose these terms from the expression

$$-\frac{1}{2}\sum_{p=2}^{m}\sum_{\substack{|\mathrm{i}(1)|\neq 0\\ \mathrm{i}(1)\in\mathbb{R}^p}} \frac{1}{\mathrm{i}(1)!(i_1(1)+1)}(Ad\,A_p)^{i_1(1)}(Ad\,A_{p-1})^{i_2(1)}\cdots(Ad\,A_1)^{i_p(1)}A_p.$$

Since we do not have to consider terms of the form $[A_i, A_i] = 0$. we must have $i_1(1) = 0$. and some $i_1(j) \neq 0, j \neq 1$. In this case, all the factors $\frac{1}{\mathrm{i}(1)!(i_1(1)+1)}$ equal 1. so we have

Lemma 5 $\mu(\sigma^1) = -\frac{1}{2}$, *i.e. the second order term in (3.5) is*

$$-\frac{1}{2}\int_0^t\int_0^\tau [A(\rho), A(\tau)]d\rho d\tau.$$

□

Next, terms of order 3 come from (3.3) with $\ell \leq 2$. i.e. from the expressions

$$-\frac{1}{2}\sum_{p=2}^{m}\sum_{\substack{|\mathrm{i}(1)|\leq 2\\ \mathrm{i}(1)\in\mathbb{R}^p}} \frac{1}{\mathrm{i}(1)!(i_1(1)+1)}(Ad\,A_p)^{i_1(1)}(Ad\,A_{p-1})^{i_2(1)}\cdots(Ad\,A_1)^{i_p(1)}A_p$$

$$+\frac{1}{3}\sum_{p=2}^{m}\sum_{\substack{|\mathrm{i}(1)|=1,|\mathrm{i}(2)|=1\\ \mathrm{i}(1)\in\mathbb{R}^p}} \frac{1}{\mathrm{i}(1)!\mathrm{i}(2)!(i_1(1)+i_1(2)+1)}(Ad\,A_p)^{i_1(1)}(Ad\,A_{p-1})^{i_2(1)}\cdots(Ad\,A_1)^{i_p(1)}$$

$$(Ad\,A_p)^{i_1(2)}(Ad\,A_{p-1})^{i_2(2)}\cdots(Ad\,A_1)^{i_p(2)}A_p.$$

We will obtain brackets of the form $[A_i, [A_j, A_k]]$ where (i) $k > i > j$ or (ii) $k > j > i$. Terms of type (i) can come from both the series above and fro any given fixed i, j, k we get a factor of -1/2 from the first and a factor of 1/3 from the second, i.e. a factor of -1/6. Terms of type (ii), however, can only come from the second series because the terms $(Ad\,A_p)^{i_1(1)}(Ad\,A_{p-1})^{i_2(1)}\cdots(Ad\,A_1)^{i_p(1)}A_p$ in the first series are ordered so we must have $k > i > j$. Hence for any term of the second type we have a factor of 1/3, and so we have

Lemma 6 *The third order term in (3.5) is*

$$-\frac{1}{6}\int_0^t\int_0^{\tau_3}\int_0^{\tau_2}[A(\tau_2), [A(\tau_1), A(\tau_3)]]d\tau_1 d\tau_2 d\tau_3 +$$
$$\frac{1}{3}\int_0^t\int_0^{\tau_3}\int_0^{\tau_2}[A(\tau_1), [A(\tau_2), A(\tau_3)]]d\tau_1 d\tau_2 d\tau_3.$$

□

Consider next the case of the k^{th} order terms. As before, each factor $\frac{1}{\mathrm{i}(1)!\cdots\mathrm{i}(\ell)!(i_1(1)+\cdots+i_1(\ell)-1)}$ will reduce to 1 and we will only get k^{th} order terms for $\ell \leq k-1$. Hence we must choose k^{th}

order terms from

$$
\begin{aligned}
& -\frac{1}{2}\sum_{p=k-1}^{m}(Ad\,A_p)(Ad\,A_{p-1})\cdots(Ad\,A_1)A_p \\
& +\frac{1}{3}\sum_{p=k-1}^{m}(Ad\,A_p)\cdots(Ad\,A_1)(Ad\,A_p)\cdots(Ad\,A_1)A_p \\
& -\cdots \\
& +\frac{(-1)^{k-1}}{k}\sum_{p=k-1}^{m}((Ad\,A_p)\cdots(Ad\,A_1))^{k-1}A_p.
\end{aligned}
\tag{3.6}
$$

Consider first the term $B_{i_1}\cdots B_{i_{k-1}}A_{i_k}$ where $i_k > i_1 > i_2 > \cdots > i_{k-1}$ and $B_{i_j} = Ad\,A_v$ for some v depending on i_j. This can be chosen in only one way from the first term in (3.6) and in $k-2$ ways from the second term in (3.6). (We must choose at least one B_v from each group of terms $(Ad\,A_p)\cdots(Ad\,A_1)$, so we could choose the first one, B_{i_1} from the first group and the remaining $k-2$ from the second, or the first two, B_{i_1}, B_{i_2} from the first group and the remaining $k-3$ from the second, etc.) In the r^{th} term in (3.6) we will have r groups $(Ad\,A_p)\cdots(Ad\,A_1)$, i.e.

$$
\underbrace{(Ad\,A_p)\cdots(Ad\,A_1)(Ad\,A_p)\cdots(Ad\,A_1)\cdots(Ad\,A_p)\cdots(Ad\,A_1)}_{r}\,A_p \tag{3.7}
$$

Suppose there are $\rho(s,t)$ ways of selecting terms of the form $(Ad\,A_v)$ from t groups. Then the number of ways of selecting $k-1$ from r, i.e. $\rho(k-1,r)$ is

$$
\rho(k-1,r) = \sum_{i=r-1}^{k-2}\rho(i,r-1).
$$

since we can choose 1 from the first group and $k-2$ from the remaining, i.e. $\rho(k-2,r-1)$ or 2 from the first group and $k-3$ from the remaining, i.e. $\rho(k-3,r-1)$, etc.

Lemma 7 *We have*

$$
\rho(k-1,r) = \frac{1}{(r-1)!}(k-r)(k-r+1)\cdots(k-2)\ ,\ r\geq 2.
$$

Proof Note that $\rho(\nu,1)=1$ for all ν and $\rho(\nu,2)=\nu-1$ for all ν. Hence the formula is correct for $r=2$. Suppose it is true for $r-1$, i.e.

$$
\rho(k-1,r-1) = \frac{1}{(r-2)!}(k-r+1)(k-r+2)\cdots(k-2).
$$

Then,

$$
\begin{aligned}
\rho(k-1,r) &= \sum_{i=r-1}^{k-2}\rho(i,r-1) \\
&= 1+\rho(r,r-1)+\rho(r+1,r-1)+\cdots+\rho(k-2,r-1) \\
&= 1+\frac{1}{(r-2)!}(r+1-r+1)(r+1-r+2)\cdots(r+1-2)+
\end{aligned}
$$

$$\begin{aligned}
&\frac{1}{(r-2)!}(r+2-r+1)(r+2-r+2)\cdots(r+2-2)+\cdots \\
&+\frac{1}{(r-2)!}(k-r)\cdots(k-3) \\
=\;&\frac{1}{(r-2)!}(1.2\cdots(r-2)+2\cdots(r-1)+3\cdots r+\cdots+(k-r)\cdots(k-3)) \\
=\;&\frac{1}{(r-2)!}\sum_{i=1}^{k-r} i(i+1)\cdots(i+(r-2)-1) \\
=\;&\frac{1}{(r-2)!}\frac{(k-r)(k-r+1)\cdots(k-r+r-2)}{(r-2)+1}.
\end{aligned}$$

Corollary *The total number of terms of the form $[B_{i_1},[B_{i_2},[\cdots,[B_{i_{k-1}},A_{i_k}]\cdots]]]$ which can be chosen, where the indices $i_1,i_2,\cdots,i_{k-1}$ are decreasing is given by* $-\frac{1}{k(k-1)}$.

Proof The required number is given by

$$\begin{aligned}
\sum_{\ell=2}^{k}\frac{(-1)^{\ell-1}}{\ell}\rho(k-1,\ell-1) &= \sum_{\ell=2}^{k}\frac{(-1)^{\ell-1}}{\ell}\frac{1}{(\ell-2)!}(k-\ell+1)(k-\ell+2)\cdots(k-2) \\
&= \sum_{\ell=2}^{k}\frac{(-1)^{\ell-1}}{\ell}\frac{1}{(\ell-2)!}\frac{\Gamma(k-1)}{\Gamma(k-\ell+1)} \\
&= -\frac{1}{k(k-1)}.
\end{aligned}$$

(The last sum can be found directly, or by using the symbolic package Maple- see the appendix.)

For the general case, let σ^{k-1} be a permutation of the set $\{1,\cdots,k-1\}$ and write it as $\sigma^{k-1}=(i_1\ \cdots\ i_{k-1})$. We can partition the permutation in the form $(\mathbf{i}^1,\mathbf{i}^2,\cdots,\mathbf{i}^\gamma)$ where

$$\mathbf{i}^1=(i_1,\cdots,i_{v_1})\ ,\ \mathbf{i}^2=(i_{v_1+1},\cdots,i_{v_1+v_2})\ ,\ \cdots$$

such that

$$\begin{aligned}
\mathbf{i}^\alpha \text{ is a decreasing sequence for } \alpha &\in \mathcal{A}\subseteq\{1,\cdots,\gamma\} \\
\mathbf{i}^\beta \text{ is a decreasing sequence for } \beta &\in \{1,\cdots,\gamma\}\backslash\mathcal{A}=\mathcal{B}
\end{aligned}$$

i.e. if $\mathbf{i}^\alpha=(i_{\ell_1},\cdots,i_{\ell_2})$, then $i_{\ell_1}>i_{\ell_1+1}>\cdots>i_{\ell_2}$. Moreover, we choose the partition so that the sets $\mathbf{i}^\alpha$ for $\alpha\in\mathcal{A}$ are maximal. Let

$$\varepsilon=\sum_{\beta\in\mathcal{B}}|\mathbf{i}^\beta|+\aleph(\mathcal{A})$$

where $\aleph(\mathcal{A})$ denotes the cardinality of $\mathcal{A}$ and if $\mathbf{i}^\beta=(i_{k_1},\cdots,i_{k_j})$, then $|\mathbf{i}^\beta|=\sum_{v=1}^{j}i_{k_v}$. If we are selecting from a term of the form ( ) with ℓ repeated strings $(Ad\,A_p)\cdots(Ad\,A_1)$, then we require $\varepsilon\le\ell$. Put $\zeta=\ell-\varepsilon$. If $\zeta>0$, let P_ℓ be the set of distinct partitions of ζ into $\aleph(\mathcal{A})$ pieces, i.e.

$$\zeta=\sum_{\alpha\in\mathcal{A}}\zeta_\alpha.$$

where $\zeta_\alpha \geq 0$. Then the number of possible selections in the ℓ^{th} term is

$$\sum_{\zeta \in P_\ell} \prod_{\alpha \in \mathcal{A}} \rho(|\mathrm{i}^\alpha|, \zeta_\alpha + 1),$$

where we take $\rho(k, r) = 0$ if $r > k$. Hence we have proved
Theorem 4 The number $\mu(\sigma^{k-1})$ is given by

$$\begin{aligned}
\mu(\sigma^{k-1}) &= \sum_{\ell=\varepsilon}^{k-1} \frac{(-1)^\ell}{\ell+1} \sum_{\zeta \in P_\ell} \prod_{\alpha \in \mathcal{A}} \rho(|\mathrm{i}^\alpha|, \zeta_\alpha + 1) \\
&= \sum_{\ell=\varepsilon}^{k-1} \frac{(-1)^\ell}{\ell+1} \sum_{\zeta \in P_\ell} \prod_{\alpha \in \mathcal{A}} \frac{1}{\zeta_\alpha !}(|\mathrm{i}^\alpha| - \zeta_\alpha)(|\mathrm{i}^\alpha| - \zeta_\alpha + 1) \cdots (|\mathrm{i}^\alpha| - 1). \square
\end{aligned}$$

Example Consider, for example, the permutation of $\{1,2,3,4,5\}$ given by $\sigma^5 = (5\ 2\ 3\ 4\ 1)$. Here we have

$$(5\ 2\ 3\ 4\ 1) = (\mathrm{i}^1, \mathrm{i}^2, \mathrm{i}^3)$$

where

$$\mathrm{i}^1 = (5, 2)\ ,\ \mathrm{i}^2 = (3)\ ,\ \mathrm{i}^3 = (4, 1),$$

so $\mathcal{A} = \{1, 2, 3\}, \mathcal{B} = \emptyset$ and $\varepsilon = 3$. For $\ell = 3$ there is only one choice, so the contribution to $\mu(\sigma^5)$ is -1/4 in this case. For $\ell = 4$ we have $\zeta = 1$ and the partitions are (0,1) and (1,0), so the contribution from this term is

$$\frac{1}{5}(\rho(2, 1) \cdot \rho(2, 2) + \rho(2, 2) \cdot \rho(2, 1)) = 2/5.$$

Finally, for $\ell = 5$ we have $\zeta = 2$ and the partitions are (2,0),(0,2) and (1,1). Hence the contribution here is

$$-\frac{1}{6}(\rho(2, 3) \cdot \rho(2, 1) + \rho(2, 1) \cdot \rho(2, 3) + \rho(2, 2) \cdot \rho(2, 2)) = -1/6$$

since $\rho(2, 3) = 0$. Hence we have $\mu(\sigma^5) = -\frac{1}{4} + \frac{2}{5} - \frac{1}{6} = -\frac{1}{60}$.
Remark We obtain the same answer if we regard the singleton $\mathrm{i}^2 = (3)$ as increasing or decreasing. We have regarded it as increasing in the example.

4. Structure Constants and Nilpotent Systems

The explicit formula (3.5) in theorem 3 for the solution of a general non-autonomous differential equation of the form

$$\dot{x} = A(t)x\ ,\ x(0) = x_0 \tag{4.1}$$

will now be applied to obtain some general results about such systems. First, let L_A denote the Lie algebra generated by the matrices $\{A(t) : t \in \mathbb{R}\}$. It has been shown ([S.P.Banks and D.McCaffrey, 1998]) that, if $A(t)$ is analytic, so that we can write $A(t) = \sum_{i=0}^{\infty} t^i A_i$ for some matrices A_i, then L_A is equal to the Lie algebra generated by the matrices $\{A_i : 0 \leq i < \infty\}$. Suppose that $\{E_k : 1 \leq k \leq r\}$ is a basis of L_A, so that

$$A(t) = \sum_{k=1}^{r} g_k(t) E_k \tag{4.2}$$

for some functions g_k, $1 \leq k \leq r$. Let c_{ij}^k be the structure constants of L_A, so that

$$[E_i, E_j] = \sum_{k=1}^{r} c_{ij}^k E_k$$

and so

$$[A(t), A(\tau)] = \sum_i \sum_j \sum_k c_{ij}^k g_i(t) g_j(\tau) E_k.$$

Then from theorem we have

Theorem 5 *If $A(t)$ is given by (4.2) then the solution of equation (4.1) is given by*

$$\begin{aligned}
x(t;x_0) &= \exp\{\sum_{k=1}^{r} \int_0^t g_k(\tau) d\tau E_k + \sum_{k=2}^{\infty} \sum_{\sigma^{k-1} \in S_{k-1}} \mu(\sigma^{k-1}) \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_3} \int_0^{\tau_2} \\
&\quad \sum_{i_1} \cdots \sum_{i_k} \sum_{w} \sum_{v_{k-2}} \cdots \sum_{v_1} c_{i_1 v_{k-2}}^{w} c_{i_2 v_{k-3}}^{v_{k-2}} \cdots c_{i_{k-3} v_2}^{v_3} c_{i_{k-2} v_1}^{v_2} c_{i_{k-1} i_k}^{v_1} \\
&\quad g_{i_1}(\tau_{\sigma^{k-1}(1)}) g_{i_2}(\tau_{\sigma^{k-1}(2)}) \cdots g_{i_{k-1}}(\tau_{\sigma^{k-1}(k-1)}) g_{i_k}(\tau_k) E_w d\tau_1 \cdots d\tau_k\} x_0 \\
&= \exp\{\sum_{k=1}^{r} \int_0^t g_k(\tau) d\tau E_k + \sum_{k=2}^{\infty} \sum_{\sigma^{k-1} \in S_{k-1}} \mu(\sigma^{k-1}) \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_3} \int_0^{\tau_2} \\
&\quad \sum_{i_1} \cdots \sum_{i_k} \sum_{w} C(w, i_1, \cdots, i_k) \\
&\quad g_{i_1}(\tau_{\sigma^{k-1}(1)}) g_{i_2}(\tau_{\sigma^{k-1}(2)}) \cdots g_{i_{k-1}}(\tau_{\sigma^{k-1}(k-1)}) g_{i_k}(\tau_k) E_w d\tau_1 \cdots d\tau_k\} x_0
\end{aligned} \tag{4.3}$$

where

$$C(w, i_1, \cdots, i_k) = \sum_{v_{k-2}} \cdots \sum_{v_1} c_{i_1 v_{k-2}}^{w} c_{i_2 v_{k-3}}^{v_{k-2}} \cdots c_{i_{k-3} v_2}^{v_3} c_{i_{k-2} v_1}^{v_2} c_{i_{k-1} i_k}^{v_1}.$$

□

As a specific example, consider the system with $so(3)$ as its Lie algebra:

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 0 & -g_3(t) & -g_2(t) \\ g_3(t) & 0 & -g_1(t) \\ g_2(t) & g_1(t) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= (g_1(t) M_1 + g_2(t) M_2 + g_3(t) M_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\end{aligned}$$

where

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here,

$$[M_1, M_2] = M_3 \ , \ [M_2, M_3] = M_1 \ , \ [M_3, M_1] = M_2$$

and we have the structure constants

$$\begin{aligned} c_{12}^3 &= c_{23}^1 = c_{31}^2 = -c_{21}^3 = -c_{32}^1 = -c_{13}^2 = 1, \\ c_{ij}^k &= 0 \text{ if } \{i, j, k\} \text{ is not a permutation of 1.2.3.} \end{aligned}$$

Hence,

$$c_{jk}^i = \varepsilon_{ijk}$$

where

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ is an even permutation of 1.2.3} \\ -1 & \text{if } i, j, k \text{ is an odd permutation of 1.2.3} \\ 0 & \text{otherwise} \end{cases}$$

(the standard tensorial ε-function), and so from theorem , we have

$$\begin{aligned} x(t; x_0) &= \exp\{\sum_{k=1}^{r} \int_0^t g_k(\tau) d\tau E_k + \sum_{k=2}^{\infty} \sum_{\sigma^{k-1} \in S_{k-1}} \mu(\sigma^{k-1}) \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_3} \int_0^{\tau_2} \\ & \sum_{w=1}^{3} \sum_{i_1=1}^{3} \cdots \sum_{i_k=1}^{3} \sum_{v_{k-2}=1}^{3} \cdots \sum_{v_1=1}^{3} \varepsilon_{w i_1 v_{k-2}} \varepsilon_{v_{k-2} i_2 v_{k-3}} \cdots \varepsilon_{v_3 i_{k-3} v_2} \varepsilon_{v_2 i_{k-2} v_1} \varepsilon_{v_1 i_{k-1} i_k} \\ & g_{i_1}(\tau_{\sigma^{k-1}(1)}) g_{i_2}(\tau_{\sigma^{k-1}(2)}) \cdots g_{i_{k-1}}(\tau_{\sigma^{k-1}(k-1)}) g_{i_k}(\tau_k) E_w d\tau_1 \cdots d\tau_k\} x_0 \\ &= \exp\{\sum_{k=1}^{r} \int_0^t g_k(\tau) d\tau E_k + \sum_{k=2}^{\infty} \sum_{\sigma^{k-1} \in S_{k-1}} \mu(\sigma^{k-1}) \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_3} \int_0^{\tau_2} \\ & \sum_{w=1}^{3} \sum_{i_1=1}^{3} \cdots \sum_{i_k=1}^{3} \sum_{v_{k-2}=1}^{3} \cdots \sum_{v_1=1}^{3} \Xi^w(i, v) \\ & g_{i_1}(\tau_{\sigma^{k-1}(1)}) g_{i_2}(\tau_{\sigma^{k-1}(2)}) \cdots g_{i_{k-1}}(\tau_{\sigma^{k-1}(k-1)}) g_{i_k}(\tau_k) d\tau_1 \cdots d\tau_k\} E_w x_0 \end{aligned}$$

where

$$\begin{aligned} \Xi^w(i, v) &= \varepsilon_{w i_1 v_{k-2}} \varepsilon_{v_{k-2} i_2 v_{k-3}} \cdots \varepsilon_{v_3 i_{k-3} v_2} \varepsilon_{v_2 i_{k-2} v_1} \varepsilon_{v_1 i_{k-1} i_k} \\ &= \pm 1 \end{aligned}$$

In the case of systems with nilpotent Lie algebra, we get an explicit closed form

$$\begin{aligned} x(t; x_0) &= \exp\{\sum_{k=1}^{r} \int_0^t g_k(\tau) d\tau E_k + \sum_{k=2}^{K} \sum_{\sigma^{k-1} \in S_{k-1}} \mu(\sigma^{k-1}) \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_3} \int_0^{\tau_2} \\ & \sum_{i_1} \cdots \sum_{i_k} \sum_{w} \sum_{v_{k-2}} \cdots \sum_{v_1} c_{i_1 v_{k-2}}^{w} c_{i_2 v_{k-3}}^{v_{k-2}} \cdots c_{i_{k-3} v_2}^{v_3} c_{i_{k-2} v_1}^{v_2} c_{i_{k-1} i_k}^{v_1} \\ & g_{i_1}(\tau_{\sigma^{k-1}(1)}) g_{i_2}(\tau_{\sigma^{k-1}(2)}) \cdots g_{i_{k-1}}(\tau_{\sigma^{k-1}(k-1)}) g_{i_k}(\tau_k) E_w d\tau_1 \cdots d\tau_k\} x_0 \end{aligned}$$

where K is the degree of nilpotency. For example, consider the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left[\begin{pmatrix} -4 & -3 & 2 \\ 12 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix}\cos t + \begin{pmatrix} -2 & -1 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\sin t + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} t^2\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x(0) = x_0.$$

Put

$$F_1 = \begin{pmatrix} -4 & -3 & 2 \\ 12 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad F_2 = \begin{pmatrix} -2 & -1 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then F_1, F_2, F_3 form a basis of a nilpotent Lie algebra with

$$[F_1, F_2] = -2F_3$$

and all other commutators zero. Hence

$$c_{12}^3 = c_{21}^3 = -2$$

are the only nonzero structure constants. It follows that the solution of the system is

$$\begin{aligned} x(t; x_0) &= \exp\left(\int_0^t \{F_1\cos\tau + F_2\sin\tau + F_3\tau^2\}d\tau \right. \\ &\quad \left. -\frac{1}{2}\int_0^t\int_0^\tau(-2\cos\rho\sin\tau - 2\sin\rho\cos\tau)F_3 d\rho d\tau\right) \\ &= \exp\left(\sin t F_1 + (1-\cos t)F_2 + \frac{t^3}{3}F_3 + \sin t(1-\cos t)F_3\right) x_0. \end{aligned}$$

5. Application to Lyapunov Transformations

Recall the basic properties of the Lyapunov transformation (see [Vincent & Grantham, 1997]). Consider the linear, nonautonomous system

$$\dot{x} = A(t)x, \quad x(0) = x_0$$

and let

$$y(t) = P(t)x(t)$$

for some invertible matrix-valued function $P(t)$. Then,

$$\begin{aligned} \dot{y} &= \dot{P}x + P\dot{x} \\ &= \dot{P}x + PA(t)x \\ &= \dot{P}P^{-1}y + PA(t)P^{-1}y \\ &= By \end{aligned}$$

where

$$B = \dot{P}P^{-1} + PA(t)P^{-1}.$$

Hence

$$\dot{P} = -PA(t) + BP\ ,\ P(0) = I.$$

Then the basic result is

Lemma 8 *If $\|P(\cdot)\|_{L^\infty(0,\infty;\mathbb{R}^{n^2})}$ and $\|P^{-1}(\cdot)\|_{L^\infty(0,\infty;\mathbb{R}^{n^2})}$ exist, then $y(t)$ is asymptotically stable if and only if $x(t)$ is asymptotically stable.* □

The usual application is to choose B to be time invariant and Hurwitz. so that y stability immediately implies x stability. Unfortunately, choosing P so that lemma 8 holds is difficult. in general. In this paper we shall split $A(t)$ into two pieces, i.e. $A(t) = A_N(t) + A_S(t)$. using Lie algebra theory and select P in accordance with the first part $A_N(t)$. Using the explicit formula for the solution of a time-varying system we shall obtain a new stability result for these systems. Consider the system

$$\dot{x} = A(t)x$$

and split the Lie algebra L_A generated by $A(t)$ into its solvable and semisimple parts : $A(t) = A_S(t) + A_\Sigma(t)$. We then obtain the system

$$\dot{x} = (A_S(t) + A_\Sigma(t))x.$$

We shall introduce a Lyapunov transformation for the system

$$\dot{\xi} = A_S(t)\xi,$$

so that $y = P\xi$, where

$$\dot{P} = -PA_S(t) + BP. \tag{5.1}$$

Consider the operator $\mathfrak{A}_B^S$ defined by

$$\mathfrak{A}_B^S P = -PA_S(t) + BP.$$

Lemma 9 $L_{\mathfrak{A}_B^S} \cong L_{A_S}$, *i.e. the Lie algebra generated by the operators $\mathfrak{A}_B^S(t)$ is isomorphic to a subalgebra of that generated by $A_S(t)$.*

Proof Consider the map $A_S(t) \to \mathfrak{A}_B^S(t)$. We have, for $t_1 \neq t_2$,

$$\begin{aligned}
[\mathfrak{A}_B^S(t_1), \mathfrak{A}_B^S(t_2)]P &= \mathfrak{A}_B^S(t_1)\mathfrak{A}_B^S(t_2)P - \mathfrak{A}_B^S(t_2)\mathfrak{A}_B^S(t_1)P \\
&= \mathfrak{A}_B^S(t_1)(-PA_S(t_2) + BP) - \mathfrak{A}_B^S(t_2)(-PA_S(t_1) + BP) \\
&= -(-PA_S(t_2) + BP)A_S(t_1) + B(-PA_S(t_2) + BP) \\
&\quad -(-(-PA_S(t_1) + BP)A_S(t_2) + B(-PA_S(t_1) + BP)) \\
&= PA_S(t_2)A_S(t_1) - PA_S(t_1)A_S(t_2) \\
&= P[A_S(t_2), A_S(t_1)].
\end{aligned}$$

Since P is invertible, the result follows.□

Remark If we simply attempt to insert and remove a 'stabilising' matrix B into the equation, i.e.

$$\dot{x} = (B + A_S(t) - B + A_\Sigma(t))x$$

then the Lie algebra generated by $\{A_S(t), t \in \mathbb{R}\} \cup \{B\}$ is bigger than L_{A_S}, since

$$[A_S(t_1) + B, A_S(t_2) + B] = [A_S(t_1), A_S(t_2)] + [B, A_S(t_2)] + [A_S(t_1), B]$$

and so any special properties of L_{A_S} are lost.

Now write

$$A_S(t) = A_N(t) + A_D(t)$$

where L_{A_N} is nilpotent and $A_D(t)$ is diagonalisable. (This is always possible since $A_S(t)$ belongs to a solvable algebra.) Note, however, that this splitting is not unique. Hence, the equation becomes

$$\dot{x} = (A_N(t) + A_D(t) + A_\Sigma(t))x.$$

From lemma we have

Lemma 10 $L_{\mathfrak{A}_B^N}$ *, the Lie algebra generated by the operators*

$$\mathfrak{A}_B^N P = -PA_N(t) + BP,$$

is nilpotent. □

The equation for P^{-1} is easy to obtain:

Lemma 11 *If P satisfies the equation*

$$\dot{P} = \mathfrak{A}_B^N(t)P = -PA_N(t) + BP$$

then

$$\dot{P}^{-1} = A_N(t)P^{-1} - P^{-1}B.$$

Proof $PP^{-1} = I$ so $\dot{P}P^{-1} + P\dot{P}^{-1} = 0$ and so

$$\begin{aligned} \dot{P}^{-1} &= -P^{-1}\dot{P}P^{-1} \\ &= -P^{-1}(-PA_N(t) + BP)P^{-1} \\ &= A_N(t)P^{-1} - P^{-1}B. \square \end{aligned}$$

If $\overline{\mathfrak{A}_B^N}$ is defined by

$$\overline{\mathfrak{A}_B^N}Q = A_N(t)Q - QB$$

then $L_{\overline{\mathfrak{A}_B^N}}$ is nilpotent (just as in lemma 10).

Our main result is

Theorem 6 *Suppose that B is a Hurwitz matrix so that $\|e^{Bt}\| \le Me^{-\omega t}$ for some $M > 0$ and $\omega > 0$. If we have*

$$\|P(A_D(t) + A_\Sigma(t))P^{-1}\| < \omega$$

where P satisfies

$$\dot{P} = -PA_N(t) + BP$$

and $\|P^{-1}(t)\|$ is bounded for all t, then the system

$$\dot{x} = (A_N(t) + A_D(t) + A_\Sigma(t))x$$

is asymptotically stable.

Proof Put $z = P(t)$. Then

$$\begin{aligned} \dot{z} &= \dot{P}x + P\dot{x} \\ &= \dot{P}P^{-1}z + P(A_N(t) + A_D(t) + A_\Sigma(t))P^{-1}z \\ &= (\dot{P}P^{-1} + PA_N(t)P^{-1})z + P(A_D(t) + A_\Sigma(t))P^{-1}z \\ &= Bz + P(A_D(t) + A_\Sigma(t))P^{-1}z. \end{aligned}$$

Now use Gronwall's inequality to give the stability of z: then the boundedness of $\|P^{-1}(t)\|$ gives the stability of x since

$$\|x(t)\| \le \left\|P^{-1}(t)\right\| \cdot \|z(t)\| \, .$$

□

Example Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2/3 & \frac{1}{30}\frac{1-t^2}{1+t^2} & \frac{1}{40}\frac{\sin t}{1+t^4} \\ -2\cos t + \frac{1}{1+t^2} & -1 & -\cos t \\ \cos t - \frac{1}{34}\frac{t}{1+t^2} & 0 & -59/60 \end{pmatrix} x \\ &= (A_N(t) + A_\Sigma(t))x \end{aligned}$$

where

$$\begin{aligned} A_N(t) &= \begin{pmatrix} -1 & 0 & 0 \\ -2\cos t + \frac{1}{1+t^2} & -1 & -\cos t \\ \cos t & 0 & -1 \end{pmatrix}. \\ A_\Sigma(t) &= \begin{pmatrix} \frac{1}{3} & \frac{1}{30}\frac{1-t^2}{1+t^2} & \frac{1}{40}\frac{\sin t}{1+t^4} \\ -2\cos t + \frac{1}{1+t^2} & 0 & 0 \\ -\frac{1}{34}\frac{t}{1-t^2} & 0 & 1/60 \end{pmatrix}. \end{aligned}$$

Then, L_{A_N} has basis

$$\left\{ A_N^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, A_N^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_N^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_N^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Note that L_{A_N} is nilpotent; in fact,

$$[A_N^1, A_N^2] = A_N^3 \, , \; [A_N^1, A_N^3] = [A_N^2, A_N^3] = 0 \, , \; [A_N^4, A_N^i] = 0$$

for $1 \le i \le 3$. If $B = -I$, then $P(t)$ and $P^{-1}(t)$ are given by

$$\begin{aligned} P(t) = \exp \Bigg[&- \int_0^t \begin{pmatrix} 0 & 0 & 0 \\ -2\cos t + \frac{1}{1+t^2} & 0 & -\cos t \\ \cos t & 0 & 0 \end{pmatrix} dt + \\ &\frac{1}{2}\int_0^t \int_0^\tau (\cos\rho\cos\tau - \cos\rho\cos\tau) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Bigg] d\rho d\tau \end{aligned}$$

$$= \exp\left[-\int_0^t \begin{pmatrix} 0 & 0 & 0 \\ -2\cos t + \frac{1}{1+t^2} & 0 & --\cos t \\ \cos t & 0 & 0 \end{pmatrix} dt\right]$$

$$P^{-1}(t) = \exp\left[\int_0^t \begin{pmatrix} 0 & 0 & 0 \\ -2\cos t + \frac{1}{1+t^2} & 0 & -\cos t \\ \cos t & 0 & 0 \end{pmatrix} dt\right].$$

It is easy to check that

$$\|P(t)\|, \left\|P^{-1}(t)\right\| \leq 5.2.$$

The theorem now shows that the system is asmptotically stable.

6. Conclusions

In this paper we have studied nonautonomous linear differential equations and. using the theory of Lie algebras. we have obtained an explicit expression for the solution in terms of an exponential of an infinite series of integrals of commutators of the matrix of the equation. In the case where the system matrix $A(t)$ generates a nilpotent Lie algebra. we obtain an explicit closed-form solution of the equation. The formula for the solution depends on a combinatorial coefficient specified in theorem 4. This coefficient can be evaluated effectively by using Maple and a simple program which performs this task is given in the appendix.

Using the formula for nilpotent systems. we have applied the theory to Lyapunov transformations and stability. Further applications will be given in a future paper.

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8. Appendix

In this appendix we shall give a Maple program for the computation of the coefficients $\mu(\sigma^{k-1})$. The first procedure, *noughts*, simply produces a sequence of zeros. The procedure *partit* produces a list of all partitions of a given type by a recursive insertion of new numbers for the next higher-order partitions, based on the procedure *insert_number*. The procedure *_rho* simply computes the function ρ in lemma 7. Finally the procedure *coeff_mu* calculates $\mu(\sigma^{k-1})$ from the expression in theorem 4. Here is the program:

```
noughts:=proc(k)
local L,i;
L:=0;
if k>1 then for i from 1 to k-1 do L:=L,0;od:fi:
L;
end:
#
#
insert_number:=proc(L,n)
local LL,i;
LL:=NULL;
for i from 1 to nops(L) do
LL:=LL,[n,op(L[i])];
od;
[LL];
end:
#
#
partit:=proc(n,m)
local L,LL,i;
L:=NULL;
if m=1 then RETURN([[n]]);fi;
for i from 0 to n do
if n-i=0 then LL:=[[i,noughts(m-1)]] else LL:=insert_number(partit(n-i,m-1),i);fi;
L:=L,op(LL);
od;
[L];
end:
#
#
_rho:=proc(k,r)
local i;
if r>k then RETURN(0);fi;
if r=1 then RETURN(1);fi;
product(k+1-r+i,i=0..r-2)/(r-1)!;
end:
#
```

```
#
decreasing_subsequences:=proc(L)
local i,LL,LLL;
LL:=NULL;
for i from 1 to nops(L) do
if i=1 then LLL:=L[1]
else
if L[i]<L[i-1] then LLL:=LLL,L[i]
else
LL:=LL,[LLL];
LLL:=L[i];
fi;
fi;
od;
[LL,[LLL]];
end:
#
#
reduced_decreasing_subsequences:=proc(L)
local i,LL,LLL;
LLL:=NULL;
LL:=decreasing_subsequences(L);
for i from 1 to nops(LL) do
if nops(LL[i])>1 then LLL:=LLL,LL[i]; fi;
od;
[LLL];
end:
#
#
coeff_mu:=proc(L)
local LL,i,k,myepsilon,LLL,mymu,p,card_a,temp_sum,j,temp_prod;
# L is a permutation in the form of a list, e.g.[2,5,4,3,1]
LL:=reduced_decreasing_subsequences(L);
k:=0;
card_a:=nops(LL);
for i from 1 to nops(LL) do k:=k+nops(LL[i]);od;
myepsilon:=nops(L)-k+nops(LL);
# find the lengths of the decreasing sequences:
LLL:=NULL;
for i from 1 to nops(LL) do LLL:=LLL,nops(LL[i]);od;
LLL:=[LLL];
# compute mu
mymu:=0;
for i from myepsilon to nops(L) do
p:=partit(i-myepsilon,card_a);
```

```
temp_sum:=0;
for j from 1 to nops(p) do
temp_prod:=1;
for k from 1 to card_a do
temp_prod:=temp_prod*_rho(LLL[k],p[j][k]+1);
od;
temp_sum:=temp_sum+temp_prod;
od;
mymu:=mymu+temp_sum*((-1)^i)/(i+1);
od;
mymu;
end:
```