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Optimal control, Lagrangian manifolds and viscosity solutions
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Abstract

This note proposes a method of synthesis of the optimal control for an infinite time optimal control problem. The control is synthesised from a viscosity solution to the associated Bellman equation. This solution is in turn constructed geometrically from a Lagrangian manifold. This note sets out the argument behind the proposed synthesis in detail and identifies the gaps and conjectures in it.

Optimal control, Lagrangian manifolds and viscosity solutions

An outline of the argument and some conjectures

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Consider the following optimal control problem

$$V(\xi) = \inf_{u(.)\in L_2(0,\infty)} \int_0^\infty \frac{1}{2}\left(x(t)^T q(x(t))x(t) + u(t)^T r(x(t))u(t)\right) dt \qquad (1)$$

subject to

$$\dot{x} = f(x) + g(x)u, \quad x(0) = \xi, \quad \lim_{t\to\infty} x(t) = 0$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and f, g, q and r are analytic functions of the appropriate dimensions. We assume there is an equilibrium at the origin, i.e. $f(0) = 0$. We also assume that $q(x)$ is positive definite for $x \neq 0$ and $r(x)$ is positive definite for all x.

If we assume that the linearisation of (1) at the equilibrium is stabilizable and detectable, then the equilibrium is hyperbolic (Lemma 3 of [4]) . Thus there exists a stable Lagrangian manifold M for the Hamiltonian dynamics corresponding to (1). Assume Hypotheses 1 of [8] hold true and define

$$V(x) = \inf\{S(x,y) : y \text{ such that } (x,y) \in M\} \qquad (2)$$

where $S(x,y)$ is the smooth function defined globally on M and satisfying $dS = -ydx$ on M. The global existence of S follows from Hypotheses 1 of [8]. Assume $\dim M \leq 2$ and that Hypotheses 2 of [8] hold true. Then (by Proposition 3 or Theorem 4 of [8]) V is Lipschitz over Ω, where Ω is the open region of state space covered by M as defined in Hypotheses 1(6) of [8]. Then by Theorem 3 of [3], $V(x)$ is a viscosity solution to the stationary Hamilton-Jacobi-Bellman equation+

$$H(x, -\frac{\partial V}{\partial x}) = \max_u \left\{ -\frac{\partial V}{\partial x}(f(x) + g(x)u) - \frac{1}{2}x^T q(x)x - \frac{1}{2}u^T r(x)u \right\} = 0. \quad (3)$$

In the following note, we investigate the conditions under which V is the value function for the optimal control problem (1). We also investigate how the optimal control can be synthesized from the viscosity solution constructed on M. In the last section of [9] there is a discussion of how to synthesize the optimal control from viscosity solutions to H_∞ control problems. The proof that the 'obvious' control does in fact achieve the infimum of the cost function over all valid controls uses the supersolution property. Due to the generality of the problem the discussion in [9] has to revert to assuming classical supersolutions of the upper Isaacs equation. We indicate in the following how, in the case of infinite time optimal control where the Hamiltonian is convex, the specifics of the construction of the viscosity solution given by (2) allow a proof that the 'obvious' control achieves the infimum when the value function is a viscosity solution rather than a classical solution.

The value function is defined as

$$\hat{V}(x) = \inf_u \sup_t \int_0^t \frac{1}{2}\left(x^T q x + u^T r u\right) dt \tag{4}$$

where the inf is over all controls $u(t)$ such that the solution $x(t)$ to $\dot{x} = f + gu, x(0) = x \in \Omega$ stays in Ω for all t and tends to the origin as $t \to \infty$. By considering the linearisation of (1) at the origin, it can be seen that $S(0,0) = 0$ and so $V(0) = 0$. We first show that $V(x) \leq \hat{V}(x)$ for all $x \in \Omega$.

To see this, note that V is a subsolution of (3), i.e. for all $p \in D^+V$

$$\max_u \left\{ -p(f + gu) - \frac{1}{2}xqx - \frac{1}{2}uru \right\} \leq 0.$$

So for any valid control $u(t)$ (i.e. such that the solution to $\dot{x} = f + gu(t)$, $x(0) = x \in \Omega$ satisfies $x(t) \in \Omega$ for all t and satisfies $x(t) \to 0$ as $t \to \infty$),

$$-p(f + gu) - \frac{1}{2}xqx - \frac{1}{2}uru \leq 0.$$

Then by Theorem I.14 of [2],

$$-V(x(t)) + V(x) \leq \int_0^t \frac{1}{2}(xqx + uru)\, dt$$

where $x(t)$ is the solution to $\dot{x} = f + gu(t)$, $x(0) = x$. Now $x(t) \to 0$ as $t \to \infty$. So, using the fact that $V(0) = 0$,

$$V(x) \leq \sup_t \int_0^t \frac{1}{2}(xqx + uru)\, dt.$$

Since this holds for all valid controls $u(t)$, we have that

$$V(x) \leq \inf_u \sup_t \int_0^t \frac{1}{2}(xqx + uru)\, dt = \hat{V}(x).$$

To show the other inequality we need to make two assumptions.

Hypotheses 1 *Suppose that $V(x) > 0$ for all $x \in \Omega$, $x \neq 0$. I haven't investigated in any detail the consequences of this failing to be true - at first sight they would appear to be similar to those discussed in the same situation for L_2 gain calculations after Theorem 4 of [3].*

The second assumption involves the synthesis of the optimal control, i.e. the function $\hat{u}(t)$ which achieves the infimum in (4). We first of all state this synthesis as a conjecture. In what follows we give the outline of the argument in support of this conjecture and, hopefully, identify all the remaining holes in the argument.

Conjecture 1 *We conjecture that the optimal control can be obtained as the following feedback for all $x \in \Omega$,*

$$\hat{u}(x) = r^{-1}(x)g^T(x)\hat{y} \tag{5}$$

where $(x, \hat{y})$ is the point on M at which $S(x, y)$ achieves its minimum value over all y such that $(x, y) \in M$. Note that it follows from Hypotheses 1(5) of [8] that the infimum is achieved in the construction (2) for V - see section 2.1.2 of [3]. Note also that the definition of a feedback control $\hat{u}$ on Ω (see e.g. the discussion after Theorem 4.2 of [5]) requires for all $x \in \Omega$ the existence of a solution $\hat{x}(t)$ to the differential equation $\dot{x} = f + g\hat{u}(x)$, $x(0) = x \in \Omega$ such that $\hat{x}(t)$ stays inside Ω for all t and satisfies the terminal conditions - which in this case are $\hat{x}(t) \to 0$ as $t \to \infty$. The next hypothesis addresses this issue. The argument which follows will show that such a solution exists for all initial conditions x lying, not inside Ω, but inside the largest level set of V contained within Ω.

Hypotheses 2 *Suppose that for all $x \in \Omega$ and for all y such that $(x, y) \in M$ the following inequality holds true*

$$-\frac{1}{2}(y-\hat{y})^T g(x) r^{-1}(x) g^T(x)(y-\hat{y}) + \frac{1}{2}\hat{y}^T g(x) r^{-1}(x) g^T(x)\hat{y} + \frac{1}{2}x^T q(x) x > 0$$

where $(x, \hat{y})$ is the point on M at which $S(x, y)$ achieves its minimum value over all y such that $(x, y) \in M$. I have some evidence to indicate that this hypothesis might be true for a sufficiently interesting number of cases. Firstly, it clearly holds when M is single sheeted over state space - i.e. in the classical case. Secondly, in the paper [7], a similar inequality was computed along optimal control trajectories for a simulated inverted pendulum and was found to hold. However, this was a numerical simulation in which the inequality was evaluated at a sample of points which were estimated to lie on the stable manifold M. An analytic expression for M was not available. It would be interesting to test this inequality on an example where M can be identified analytically - for instance Example 4.1.1 of [3].

Given these two assumptions we now outline the argument to show that $V(x) \geq \hat{V}(x)$ and that the feedback defined by (5) is the optimal control. As promised in the conjecture above, we start by showing that the feedback control (5) gives an asymptotically stable solution which stays inside the largest level set of V contained within Ω.

Let $p \in D^-V$, the subdifferential of V, and consider the following expression

$$pf + pgr^{-1}g^T\hat{y}. \tag{6}$$

By Proposition 3 or Theorem 4 of [8], V is Lipschitz. It follows from [6] that, $D^-V \subset \partial V$, the generalised gradient of V. Then, it is shown in the proof of

Theorem 3 of [3] that for a Lipschitz V defined by (2), $\partial V \subset \text{co}\{-y : (x,y) \in M\}$ where co denotes the convex hull. Since the expression (6) is convex in p,

$$pf + pgr^{-1}g^T\hat{y} \leq \max_{y:(x,y)\in M} \{-yf - ygr^{-1}g^T\hat{y}\}$$

i.e.

$$-pf - pgr^{-1}g^T\hat{y} \geq \min_{y:(x,y)\in M} \{yf + ygr^{-1}g^T\hat{y}\}. \tag{7}$$

Now, by Hypothesis 1(2) of [8], $H(x,y) = 0$ for all $(x,y) \in M$. In this case,

$$\begin{aligned} H(x,y) &= \max_u \left\{ yf + ygu - \frac{1}{2}xqx - \frac{1}{2}uru \right\} \\ &= yf + \frac{1}{2}ygr^{-1}g^Ty - \frac{1}{2}xqx \\ &= 0. \end{aligned}$$

Then from (7),

$$\begin{aligned} -pf - pgr^{-1}g^T\hat{y} &\geq \min_{y:(x,y)\in M} \left\{ -\frac{1}{2}ygr^{-1}g^Ty + ygr^{-1}g^T\hat{y} + \frac{1}{2}xqx \right\} \\ &= \min_{y:(x,y)\in M} \left\{ -\frac{1}{2}(y-\hat{y})^Tgr^{-1}g^T(y-\hat{y}) + \frac{1}{2}\hat{y}^Tgr^{-1}g^T\hat{y} + \frac{1}{2}x^Tqx \right\} \\ &> 0 \end{aligned}$$

by Hypothesis 2. Now let $\hat{x}(t)$ denote the solution to $\dot{x} = f + gr^{-1}g^T\hat{y}$, $x(0) = x \in \Omega$. Then by Theorem I.14 of [2],

$$-V(\hat{x}(t)) + V(x) > 0$$

i.e. V is strictly decreasing along trajectories of $\dot{x} = f + gr^{-1}g^T\hat{y}$ which start in Ω. Since, by Hypothesis 1, $V(x) > 0$ for all $x \neq 0$ in Ω, it follows that $\hat{u} = r^{-1}g^T\hat{y}$ gives an asymptotically stable feedback control within the largest sublevel set of V contained within Ω. In other words, denote the set $\{x \in \mathbb{R}^n : V(x) \leq c\}$ by V_c and let $c^* = \sup\{c \geq 0 : V_c \subseteq \Omega\}$. By considering the linearisation of (1) at the origin, it can be seen that $c^* > 0$. Then $\hat{u} = r^{-1}g^T\hat{y}$ gives an asymptotically stable feedback control for all initial points $x = x(0)$ lying within the set $V_{c^*} \subseteq \Omega$.

Finally we outline the steps required to show that $V(x) \geq \hat{V}(x)$ and that $\hat{u} = r^{-1}g^T\hat{y}$ achieves the infimum in (4). To start with, let $x_0 \in \Omega$ and consider the point $(x_0, \hat{y}_0) \in M$ at which $S(x_0, y)$ achieves its minimum over all y such that $(x_0, y) \in M$. Recall from above that we are assuming $\text{dim}M \leq 2$. If $\text{dim}M = 1$ then by Proposition 3 of [8], the projection π from M onto state space is non-singular at $(x_0, \hat{y}_0)$. If $\text{dim}M = 2$ then by Theorems 4 and 5 of [8], π is either non-singular at $(x_0, \hat{y}_0)$ or $(x_0, \hat{y}_0)$ is a singular point of type A_3 - see Section 4 of [8] for a discussion of the classification of singularities of projections of low dimensional Lagrangian manifolds onto state space.

If π is non-singular at $(x_0, \hat{y}_0)$ then the vector of state variables x_0 forms a canonical coordinate chart for M in a neighbourhood of $(x_0, \hat{y}_0)$ - i.e. if $\text{dim}M = 1$ then x_0 is the coordinate on M and if $\text{dim}M = 2$ then (x_{0_1}, x_{0_2}) are the coordinates on M. Then there exists a smooth generating function $G(x)$ for M defined in a ball $B_\delta(x_0)$ such that the set $W = \{(x, \hat{y}) : x \in B_\delta(x_0), \hat{y} = \partial G(x)/\partial x\}$ defines a neighbourhood of $(x_0, \hat{y}_0)$ on M. Furthermore, in W, $S(x, \hat{y}) = G(x)$. Now suppose that by taking W small enough, we can assume that each $(x, \hat{y}) \in W$ is the point at which $S(x, y)$ achieves its minimum over all y such that $(x, y) \in M$. Then, using the fact that S is defined on M as the solution to $dS = -ydx$, we get that for $x \in B_\delta(x_0)$,

$$V(x) = S(x, \hat{y}) = G(x) = -\int_0^t \hat{y}dx + V(x_0)$$

where the integral is taken along any trajectory in $B_\delta(x_0)$ connecting x_0 and $x = x(t)$ and where $(x(t), \hat{y}(x(t))) \in W$ is the corresponding trajectory on M.

Suppose, on the other hand, that no matter how small we take W, it is not true that each $(x, \hat{y}) \in W$ is the point at which $S(x, y)$ achieves its minimum over all y such that $(x, y) \in M$. Then $(x_0, \hat{y}_0)$ is a point at which the minimising y jumps from one branch of M to another. So there exists a set of points $(x_0, \hat{y}_0^i) \in M$, $i \in I$ which all project onto x_0 and at each of which S achieves its minimum value over x_0. Again, it is shown in the proof of Theorem 4 of [8] that π must be non-singular at each of these points. Then, as above, for each $(x_0, \hat{y}_0^i)$ we have a neighbourhood $B_{\delta^i}(x_0)$ in state space, a neighbourhood W^i on the corresponding branch of M and an associated generating function G^i for that branch of M.

GAP IN THE ARGUMENT (1): we assume that an appropriate analytical argument can be constructed to show the existence of a ball $B_\delta(x_0) \subseteq B_{\delta^i}(x_0)$ for all i such that for any $x \in B_\delta(x_0)$, we can chose the appropriate branch of M on which

$$V(x) = S(x, \hat{y}^i) = G^i(x) = -\int_0^t \hat{y}^i dx + V(x_0)$$

holds true where again the integral is taken along any trajectory in $B_\delta(x_0)$ connecting x_0 and $x = x(t)$ and where $(x(t), \hat{y}^i(x(t))) \in W^i$ is the corresponding trajectory of minimising points for S on M.

Lastly, if $(x_0, \hat{y}_0)$ is a singular point of type A_3, then by the proof of Theorem 4 of [8], for some $\delta > 0$ the minimising point for S over all $x \in B_\delta(x_0)$ lies on the same branch of M as $(x_0, \hat{y}_0)$. Note, the singularity in M at $(x_0, \hat{y}_0)$ is a tuck. So, given two points x_1 and x_2 in $B_\delta(x_0)\backslash\{x_0\}$, it is possible that each of the corresponding minimising points for S can lie on the same branch of M as $(x_0, \hat{y}_0)$, while they lie on different branches with respect to one another. Also, by inspection of the canonical generating function for the branch of M corresponding to a singularity of type A_3, we see that V is Lipschitz in $B_\delta(x_0)$.

GAP IN THE ARGUMENT (2): we assume that an appropriate analytical argument can be constructed to show that, for any $x \in B_\delta(x_0)$,

$$V(x) = S(x, \hat{y}^i) = G^i(x) = -\int_0^t \hat{y}^i dx + V(x_0)$$

holds true where again the integral is taken along any trajectory in $B_\delta(x_0)$ connecting x_0 and $x = x(t)$ and where $(x(t), \hat{y}^i(x(t))) \in W^i$ is the corresponding trajectory of minimising points for S on M. As mentioned above, given that $(x_0, \hat{y}_0)$ is a type A_3 singularity, then a minimising point for S over $x \in$

$B_\delta(x_0)\backslash\{x_0\}$ can jump from one branch of M to another. So, as with the previous case, the argument involves piecing together trajectories of minimising points for S on different branches of M. However, here things are simpler because we know from the particular form of the generating function for an A_3 singularity that there that are only two branches to consider.

So, in all three cases we get the same local relationship for $V(x)$. Now, apply this relationship along the particular trajectory which solves $\dot{x} = f + gr^{-1}g^T\hat{y}$, $x(0) = x_0$. Take t small enough to ensure that $x(t) \in B_\delta(x_0)$. Then, using the fact that

$$H(x,y) = yf + \frac{1}{2}ygr^{-1}g^Ty - \frac{1}{2}xqx = 0$$

for all $(x,y) \in M$, we get that

$$\begin{aligned}
V(x(t)) &= -\int_0^t \hat{y}\dot{x}dt + V(x_0) \\
&= -\int_0^t \left(\hat{y}f + \hat{y}gr^{-1}g^T\hat{y}\right) dt + V(x_0) \\
&= -\int_0^t \frac{1}{2}\left(xqx + \hat{y}gr^{-1}g^T\hat{y}\right) dt + V(x_0) \\
&= -\int_0^t \frac{1}{2}\left(xqx + \hat{u}r\hat{u}\right) dt + V(x_0)
\end{aligned}$$

GAP IN THE ARGUMENT (3): we know from above that the solution to $\dot{x} = f + g\hat{u}$ for all initial conitions $x(0) = x_0 \in V_{c^*} \subseteq \Omega$ stays in V_{c^*} and tends to 0 as $t \to \infty$. We assume that an analytical argument can be constructed to prove that, along the solution $x(t)$ to $\dot{x} = f + g\hat{u}$, $x(0) = x_0$, the above local neighbourhood expressions for $V(x(t))$ can be pieced together to give

$$-V(x(t)) + V(x_0) = \int_0^t \frac{1}{2}\left(xqx + \hat{u}r\hat{u}\right) dt$$

for all $t > 0$. Then, since $x(t) \to 0$ as $t \to \infty$, it follows that

$$V(x_0) = \sup_t \int_0^t \frac{1}{2}\left(xqx + \hat{u}r\hat{u}\right) dt.$$

Hence, for all $x \in V_{c^*} \subseteq \Omega$,

$$V(x) = \hat{V}(x) = \inf_u \sup_t \int_0^t \frac{1}{2}\left(xqx + uru\right) dt$$

and the infimum is achieved by the feedback control $\hat{u}(x) = r^{-1}(x)g^T(x)\hat{y}$, where $(x, \hat{y})$ is the point on M at which $S(x,y)$ achieves its minimum value over all y such that $(x,y) \in M$.

In other words, on the subset V_{c^*} of Ω, $\hat{u} = r^{-1}g^T\hat{y}$ is the optimal feedback control for the problem defined in (1) - subject to Hypotheses 1 and 2 being satisfied and the above three gaps in the argument being successfully filled. This is the conjecture put forward in this note.

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