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Enhanced Hybrid Positioning in Wireless Networks I: AoA-ToA

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***Abstract*—Localization in wireless networks presents enormous challenges for scientists and engineers. Some of the most commonly used techniques for localization are based on time of arrival (ToA), received signal strength (RSS) and angle of arrival (AoA) of the signals. In this paper we analyze and propose improvements to the location accuracy of hybrid (AoA-ToA) localization systems. The location coordinates are obtained using a linear least squares (LLS) algorithm. A closed form expression for the mean square error (MSE) of the LLS estimator is derived. Furthermore, the information present in the covariance of the incoming signals is utilized and a novel weighted linear least squares (WLLS) method is proposed. It is shown via simulation that the theoretical MSE accurately predicts the performance of the LLS estimator. It is also shown via simulation that the WLLS algorithm exhibits better performance than the LLS algorithm.**

I. INTRODUCTION

Wireless communication has been one of the most challenging area of research for scientists in the past few decades. Previously the main application of wireless communication was telephony, but with the passage of time and with the emergence of new technologies, the applications of wireless communication have increased significantly. Localization of wireless nodes introduce many new applications. One way of estimating node position is to equip each node with a global positioning system (GPS) chip. However due to the high cost in terms of power consumption and price of the GPS chip and unavailability of direct line of sight (LoS) to the satellite, GPS positioning is not always favored. Hence alternative methods to obtain node location are employed. Various techniques have been developed to localize wireless nodes. These can be classed as receive signal strength (RSS) [1] based positioning that rely on the attenuation of the signal, time of arrival (ToA) [2] that is based on the delay of the incoming signal and angle of arrival (AoA) [3] method capitalizing on the angle of the impinging signal. Most location algorithms use the known location of a certain number of so called anchor nodes (ANs) to locate the target node (TN). Moreover the localization scheme can be either iterative [4] or non-iterative [5], or it can be co-operative [6] or non-cooperative [7]. Positioning of nodes via RSS, ToA and AoA have their own advantages and disadvantages however a new trend is to use all the available information and hence researchers are also working on hybrid systems which combine two or more of these measurements to make one robust and accurate system.

In this paper we study the performance of a linear least squares (LLS) estimator for hybrid AoA-ToA systems [8], we develop a theoretical mean square error (MSE) equation and propose a weighted linear least squares (WLLS) algorithm that shows better performance than the LLS algorithm. The arrival angles can be estimated by a number of methods. One way is by using an array of antennas as in [9] another technique is based on rotating beam of radiation [10]. Similarly a number of techniques are developed for estimating the distance between TNs and ANs via ToA [5], [11]. The techniques used for angle and time estimation are beyond the scope of this paper. This paper deals with optimal utilization of these estimated angles and delays in order to accurately localize the TN.

The rest of the paper is organized as follows. In section II, basic ToA, AoA and the hybrid system are reviewed. Section III deals with the derivation of the covariance matrix and the theoretical MSE expression. The WLLS algorithm is proposed in IV. In Section V we compare the results of LLS with WLLS algorithm and section VI concludes the paper.

II. TOA, AOA AND HYBRID LOCALIZATION

For future use, we define the following notations $\mathbb{R}^n$ is a set of n dimensional real number, $(.)^T$ is the transpose operator, $Tr(M)$ is the trace of matrix M, $E(.)$ refer to the expectation operator.

1) *Time of Arrival.* If only delay estimates are available, at least 3 or 4 ANs are required for 2d or 3d localization respectively. ToA algorithms are based on the delay of the signal from TN to AN to estimated the distance. Each distance estimate can be represented as the radius of a circle, the center of which is the ANs location. Hence we get a number of circles depending upon the number of ANs. The point of intersection of all these circles is the position of the TN.

Let (x, y) be the coordinates of the unknown TN and (x_i, y_i) be the coordinates of i^{th} AN, $i = 1, 2, 3....N$. N being the number of ANs, then the distance between TN and AN is

$$\hat{d}_i = \sqrt{(x - x_i)^2 + (y - y_i)^2} + n_i, \tag{1}$$

where n_i is the noise which is a modeled as zero mean Gaussian i.e. $n_i \sim \left(\mathcal{N}\left(0, \sigma_{n_i}^2\right)\right)$.

Clearly (1) is non-linear and can be solved using iterative techniques, however a linear least squares (LLS) can also be used by first linearizing (1) [12]. Let $\hat{d}_i^2$ be the 2d noisy distance estimates, we have:

$$\hat{d}_i^2 \approx (x - x_i)^2 + (y - y_i)^2, \tag{2}$$

then a reference AN is selected and its distance equation is subtracted from (2) for $i = 1, ...N$ $(i \neq r)$. Let d_r represent the reference distance of this reference AN. The reference AN can be randomly chosen or a special criterion can be developed to choose reference distance as in [13]. Hence we get

$$(x_i - x_r)\, x + (y_i - y_r)\, y = 0.5\Big[\big(x_i^2 + y_i^2\big) - \big(x_r^2 + y_r^2\big) + \hat{d}_r^2 - \hat{d}_i^2\Big]. \tag{3}$$

In matrix form we have,

$$\mathbf{A}_t = \begin{bmatrix} x_1 - x_r & y_1 - y_r \\ x_2 - x_r & y_2 - y_r \\ \vdots & \vdots \\ x_N - x_r & y_N - y_r \end{bmatrix} \in \mathbb{R}^{N\times 2}, \boldsymbol{u}_t = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1},$$

$$\hat{\mathbf{b}}_t = 1/2 \begin{bmatrix} (x_1^2 + y_1^2) - (x_r^2 + y_r^2) + \hat{d}_r^2 - \hat{d}_1^2 \\ (x_2^2 + y_2^2) - (x_r^2 + y_r^2) + \hat{d}_r^2 - \hat{d}_2^2 \\ \vdots \\ \vdots \\ (x_N^2 + y_N^2) - (x_r^2 + y_r^2) + \hat{d}_r^2 - \hat{d}_N^2 \end{bmatrix} \in \mathbb{R}^{N\times 1},$$

then (3) can be rewritten in matrix form as

$$\mathbf{A}_t \boldsymbol{u}_t = 0.5\hat{\mathbf{b}}_t. \tag{4}$$

Moore-Penrose pseudo inverse is taken on both sides to get the location estimates.

$$\hat{\boldsymbol{u}}_t = 0.5(\mathbf{A}_t^T \mathbf{A}_t)^{-1} \mathbf{A}_t^T \hat{\mathbf{b}}_t. \tag{5}$$

2) *Angle of Arrival.* If angle estimates are only available then we need only 2 and 3 ANs for 2d and 3d localization respectively. Each AN forms a line on a 2d plane on which the AN and TN are situated. Hence we get a number of lines depending upon the number of ANs. The point of intersection of these lines is the estimated position of the TN. The AoA system generally shows good results but the estimation error increases significantly as the target moves away from the ANs.

Keeping the same notation as for ToA, we have [14].

$$\hat{\theta}_i \approx \arctan\left[\frac{(y - y_i)}{(x - x_i)}\right] + m_i \tag{6}$$

where m_i represents the zero mean Gaussian noise in the estimate of the i^{th} angle, i.e. $m_i \sim \big(\mathcal{N}\big(0, \sigma_{m_i}^2\big)\big)$.

if

$$\mathbf{A}_a = \begin{bmatrix} \tan\hat{\theta}_1 & -1 \\ \tan\hat{\theta}_2 & -1 \\ \vdots & \vdots \\ \tan\hat{\theta}_N & -1 \end{bmatrix} \in \mathbb{R}^{N\times 2}, \quad \boldsymbol{u}_a = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1},$$

$$\hat{\mathbf{b}}_a = \begin{bmatrix} x_1 \tan\hat{\theta}_1 - y_1 \\ x_2 \tan\hat{\theta}_2 - y_2 \\ \vdots \\ x_N \tan\hat{\theta}_N - y_N \end{bmatrix} \in \mathbb{R}^{N\times 1},$$

then (6) can be written in matrix form as

$$\mathbf{A}_a \boldsymbol{u}_a = \hat{\mathbf{b}}_a. \tag{7}$$

For the standard LLS estimator the solution is given by

$$\hat{\boldsymbol{u}}_a = \big(\mathbf{A}_a^T \mathbf{A}_a\big)^{-1} \mathbf{A}_a^T \hat{\mathbf{b}}_a. \tag{8}$$

3) *Hybrid(*AoA-ToA*).* If both distance and angle estimates are available then localization can be performed with only one AN. In order to enhance results more anchors can be introduced to the system. In this case each AN forms a line, rather than a circle. At one end of the line the AN is situated with known position while at opposite end the TN is situated for which the coordinates are to be estimated. If the slope (AoA) and the magnitude (ToA) of this line is available, the TN coordinates can be easily determined. The error in this case also increases with the increase in distance between AN and TN.

From ToA and AoA equations given by (1) and (6) respectively we obtain

$$\mathbf{A} = \begin{bmatrix} 1_1 & 0 \\ \vdots & \vdots \\ 1_N & 0 \\ 0 & 1_1 \\ \vdots & \vdots \\ 0 & 1_N \end{bmatrix} \in \mathbb{R}^{2N\times 2}, \quad \boldsymbol{u} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1},$$

$$\hat{\mathbf{b}} = \begin{bmatrix} \hat{\mathbf{b}}_x \\ \hat{\mathbf{b}}_y \end{bmatrix} \in \mathbb{R}^{2N\times 1},$$

$$\hat{\mathbf{b}}_x = \begin{bmatrix} x_1 + \hat{d}_1 \cos\hat{\theta}_1 \delta_1 \\ \vdots \\ x_N + \hat{d}_N \cos\hat{\theta}_N \delta_N \end{bmatrix},$$

$$\hat{\mathbf{b}}_y = \begin{bmatrix} x_1 + \hat{d}_1 \sin\hat{\theta}_1 \delta_1 \\ \vdots \\ x_N + \hat{d}_N \sin\hat{\theta}_N \delta_N \end{bmatrix},$$

where $\delta_i = \exp\big(\sigma_{m_i}^2/2\big)$. Then the LLS solution for the hybrid system is given by

$$\hat{\boldsymbol{u}} = \big(\mathbf{A}^T \mathbf{A}\big)^{-1} \mathbf{A}^T \hat{\mathbf{b}}. \tag{9}$$

III. Theoretical MSE of LLS

In this section we derive the theoretical MSE expression for the linear least squares estimator in (9). The MSE for the hybrid system is given by

$$\text{MSE}(\boldsymbol{u}) = Tr\left\{E\left[(\hat{\boldsymbol{u}} - \boldsymbol{u})(\hat{\boldsymbol{u}} - \boldsymbol{u})^T\right]\right\}, \tag{10}$$

where $\hat{\boldsymbol{u}}$ is the erroneous estimated location and $\boldsymbol{u}$ is the location with no error. Putting (9) in (10) we get

$$\mathrm{MSE}\left(\boldsymbol{u}\right)=\mathbf{A}^{\dagger}\mathbf{C}\left(\boldsymbol{u}\right)\mathbf{A}^{\dagger^{T}}, \tag{11}$$

where

$$\mathbf{A}^{\dagger}=\left(\mathbf{A}^{T}\mathbf{A}\right)^{-1}\mathbf{A}^{T},$$

and $\mathbf{C}\left(\boldsymbol{u}\right)=E\left[\left(\hat{\mathbf{b}}-\mathbf{b}\right)\left(\hat{\mathbf{b}}-\mathbf{b}\right)^{T}\right]$, for $\mathbf{b}$ representing the noise free observation. The covariance $\mathbf{C}\left(\boldsymbol{u}\right)$ can be partitioned into separate matrices as follows

$$\mathbf{C}\left(\boldsymbol{u}\right)=\left[\begin{array}{cc}\mathbf{C}\left(x\right) & \mathbf{C}\left(xy\right)\\ \mathbf{C}\left(xy\right) & \mathbf{C}\left(y\right)\end{array}\right], \tag{12}$$

$$\mathbf{C}\left(x\right)=\mathbf{E}\left[\left(\hat{\mathbf{b}}_{x}-\mathbf{b}_{x}\right)\left(\hat{\mathbf{b}}_{x}-\mathbf{b}_{x}\right)^{T}\right]\in\mathbb{R}^{N\times N}. \tag{13}$$

For the diagonal terms i.e. $i=j$,

$$\begin{aligned}\mathbf{C}\left(x\right)_{ii}=&\left(\frac{d_i^2}{2}+\frac{\sigma_{n_i}^2}{2}\right)\exp\left(\sigma_{m_i}^2\right)+\left(\frac{d_i^2}{2}\cos\left(2\theta_i\right)\right.\\&\left.+\frac{\sigma_{n_i}^2}{2}\cos\left(2\theta_i\right)\right)\exp\left(-\sigma_{m_i}^2\right)-\left(d_i\cos\left(\theta_i\right)\right)^2.\end{aligned} \tag{14}$$

On the other hand, for the non-diagonal terms i.e. $i\neq j$, we have

$$\mathbf{C}\left(x\right)_{ij}=0. \tag{15}$$

$$\mathbf{C}\left(y\right)=E\left[\left(\hat{\mathbf{b}}_{y}-\mathbf{b}_{y}\right)\left(\hat{\mathbf{b}}_{y}-\mathbf{b}_{y}\right)^{T}\right]\in\mathbb{R}^{N\times N}. \tag{16}$$

For the diagonal terms i.e. $i=j$,

$$\begin{aligned}\mathbf{C}\left(y\right)_{ii}=&\left(\frac{d_i^2}{2}+\frac{\sigma_{n_i}^2}{2}\right)\exp\left(\sigma_{m_i}^2\right)-\left(\frac{d_i^2}{2}\cos 2\theta_i\right.\\&\left.+\frac{\sigma_{n_i}^2}{2}\cos 2\theta_i\right)\exp\left(-\sigma_{m_i}^2\right)-\left(d_i\sin\theta_i\right)^2.\end{aligned} \tag{17}$$

On the other hand, for the non-diagonal terms i.e. $i\neq j$, we have

$$\mathbf{C}\left(y\right)_{ij}=0. \tag{18}$$

Similarly,

$$\mathbf{C}\left(xy\right)=E\left[\left(\hat{\mathbf{b}}_{x}-\mathbf{b}_{x}\right)\left(\hat{\mathbf{b}}_{y}-\mathbf{b}_{y}\right)^{T}\right]\in\mathbb{R}^{N\times N}, \tag{19}$$

for the diagonal terms i.e. $i=j$,

$$\mathbf{C}(xy)_{ii}=\left(d_i^2+\sigma_{n_i}^2\right)\cos\theta_i\sin\theta_i\exp\left(-\sigma_{m_i}^2\right)-d_i^2\cos\theta_i\sin\theta_i. \tag{20}$$

On the other hand, for the non-diagonal terms i.e. $i\neq j$, we have

$$\mathbf{C}\left(xy\right)_{ij}=0. \tag{21}$$

Proof: Appendix. ■

IV. WEIGHTED LINEAR LEAST SQUARES ALGORITHM

The results obtained in (9) show a high error because in (9) the information about the link quality is not utilized. If this information is available at hand we can use it to improve the performance of the system. Thus in this section we propose a weighted linear least squares (WLLS) algorithm by exploiting the covariance matrix The covariance matrix $\mathbf{C}\left(\boldsymbol{u}\right)$ gives us the information about the link quality. Thus the links with high variance are given less weights and vice versa. The WLLS solution is obtained by minimizing the cost function.

$$\varepsilon_{WLLS}\left(\hat{\boldsymbol{u}}\right)=\left(\mathbf{b}-\mathbf{A}\hat{\boldsymbol{u}}\right)^{T}\mathbf{C}\left(\boldsymbol{u}\right)^{-1}\left(\mathbf{b}-\mathbf{A}\hat{\boldsymbol{u}}\right), \tag{22}$$

where $\mathbf{C}\left(\boldsymbol{u}\right)^{-1}$ is the inverse of the covariance matrix defined in (12).

The elements of (12) are dependent on the real values of distance and angles which are not available. Thus we use the estimated values to get the estimated covariance matrix $\mathbf{C}\left(\hat{\boldsymbol{u}}\right)^{-1}$. The WLLS solution is obtained as follows,

$$\hat{\boldsymbol{\theta}}_{WLLS}=\mathbf{A}^{\boldsymbol{\Gamma}}\hat{\mathbf{b}}^{\boldsymbol{\Gamma}}, \tag{23}$$

where
$\mathbf{A}^{\boldsymbol{\Gamma}}=\left[\mathbf{A}^{T}\mathbf{C}\left(\hat{\boldsymbol{u}}\right)\mathbf{A}\right]^{-1}\mathbf{A}^{T}$ and $\hat{\mathbf{b}}^{\boldsymbol{\Gamma}}=\mathbf{C}\left(\hat{\boldsymbol{u}}\right)\hat{\mathbf{b}}$.

V. SIMULATION RESULTS

In this section, results obtained by Monte Carlo simulation are compared to the theoretical MSE derived in the previous section. Also performance of LLS and WLLS algorithms are compared. Furthermore system performance is analyzed by changing variables like angle noise variance, distance noise variance and number of ANs. All the simulations are run independently η number of times and the noise variance of all communication links is considered same.

In Fig. 1 the average root mean square error (Avg. RMSE) in the location estimates of TNs is plotted against the angle and distance noise variance. Fig. 1 compares the results obtained by theoretical MSE with Monte Carlo simulation for LLS. Five TNs are taken at random locations and the performance of the system is plotted for 4, 6, 8 and 10 ANs. From Fig. 1 it is seen that the theoretical analysis accurately predicts the system performance.

In Fig. 2 the Avg. RMSE is obtained for LLS and WLLS algorithms. The performance of the system is plotted for 4, 6, 8 and 10 ANs while 5 TNs are taken at random locations. As expected the performance is improved by increasing the number of ANs. Also expected is the improved performance of WLLS compared to LLS.

In Fig. 3 the Avg. RMSE is plotted against different number of ANs, while keeping the angle and distance noise variance fixed. The figure shows the improvement in system performance with the addition of more ANs to the network. Again with utilization of weights the system shows improved performance.

In order to analyze the effect of angle noise variance and distance noise variance on location estimate, a 3D plot is generated in Fig. 4 where the Avg. RMSE for LLS and

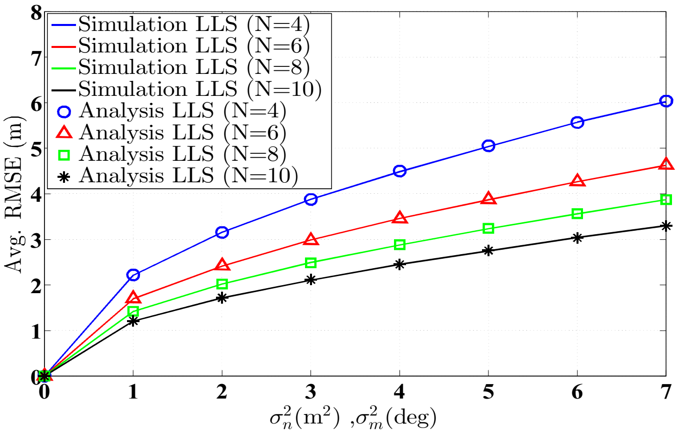

Figure 1. Avg. RMSE for analytical LLS and simulation LLS., $\eta = 2000$.

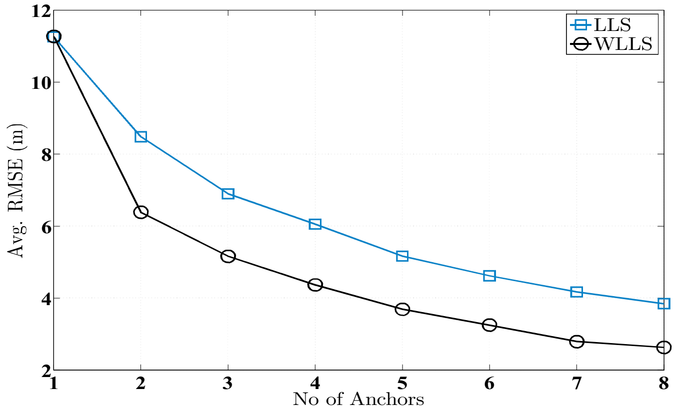

Figure 3. Performance comparison between LLS and WLLS $\sigma_n^2 = 7$m, $\sigma_m^2 = 7$degrees, Network 40 x 40, Number of Anchors=[2,3,4,5,6,7,8], $\eta = 2000$.

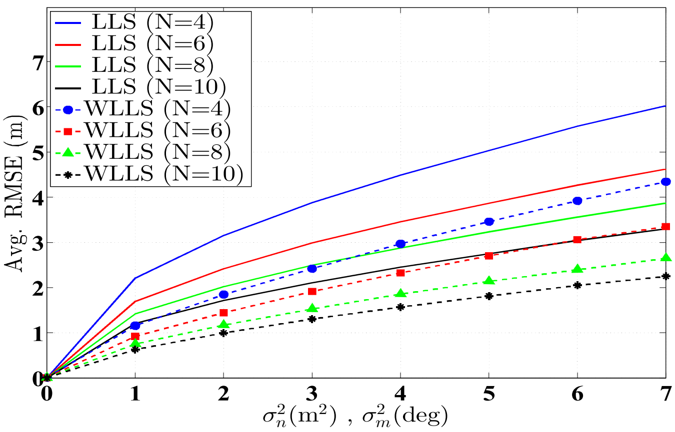

Figure 2. Performance comparison between LLS and WLLS $\sigma_n^2 = [0, 1, 2, 3, 4, 5, 6, 7]$metres, $\sigma_m^2 = [0, 1, 2, 3, 4, 5, 6, 7]$degrees, Network 40 x 40, $\eta = 2000$.

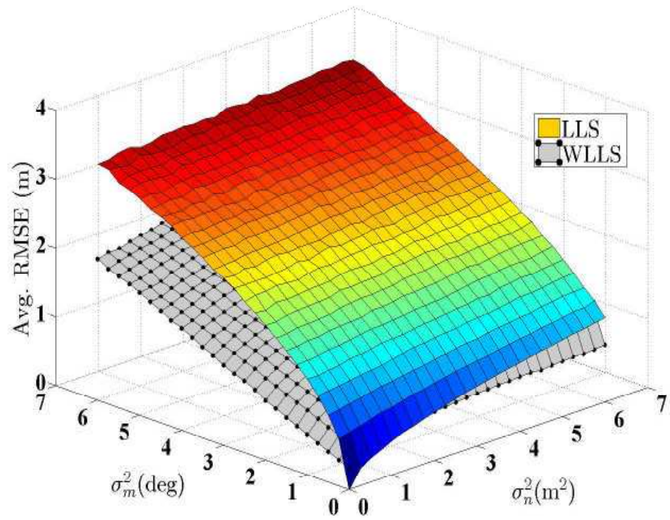

Figure 4. Performance comparison between LLS and WLLS against range error and angle error on x-axis and y-axis receptively, $\eta = 3000$.

WLLS is plotted against distance noise variance along x-axis and angle noise variance along y-axis. From Fig. 4 it is seen that the angle noise variance has a bigger impact on performance degradation as compared to distance noise variance. This is because the angle noise variance is distance dependent and introduces a large error if the TNs are located at large distance from the ANs. The figure also shows the improved performance of WLLS as compared to LLS.

VI. Conclusion

In this paper a hybrid AoA-ToA localization system was analyzed. TN's locations were estimated using a least square approach. To predict the systems performance a covariance matrix for the LS model was developed and an analytical MSE equation was derived. Performance was improved by introducing weights to the communication links, hence a WLLS algorithm which exploits the covariance matrix was proposed. Via simulation it was shown that the analytical MSE accurately predicts the system performance. It was also shown through simulation that the proposed WLLS algorithm shows better performance than its LLS counterpart. The derived algorithm will be analyzed for AoA-RSS hybrid model which will reduce the systems complexity. In order to achieve better accuracy a combined PLE/location estimator will also be developed using AoA-RSS model in the future.

Appendix

Proof of (14) and (15): Putting values of $\hat{\mathbf{b}}_x$ and $\mathbf{b}_x$ in (13) for diagonal terms i.e $i = j$.

$$\begin{aligned}\mathbf{C}\left(x\right)_{ii} &= E_{m_i,n_i}\left\{\left[\left(d_i + n_i\right)\cos\left(\theta_i + m_i\right)\delta_i - d_i\cos\theta_i\right]^2\right\} \\ &= E_{m_i,n_i}\left\{\left[\left(d_i + n_i\right)^2\cos^2\left(\theta_i + m_i\right)\right]\delta_i^2 + \left(d_i\cos\theta_i\right)^2\right. \\ &\qquad \left. - 2\delta_i\left(d_i\cos\theta_i\right)\left(d_i + n_i\right)\cos\left(\theta_i + m_i\right)\right\},\end{aligned} \tag{24}$$

$$=E_{m_i,n_i}\Bigg\{\Big(d_i^2+n_i^2+2d_in_i\Big)\Big(0.5+0.5\Big[\cos\big(2\theta_i\big)\cos\big(2m_i\big)$$
$$-\sin\big(2\theta_i\big)\sin\big(2m_i\big)\Big]\Big)\delta_i^2+\Big(d_i\cos\theta_i\Big)^2-2\delta_i\Big(d_i^2+d_in_i\Big)$$
$$\times\Big(\cos^2\big(\theta_i\big)\cos\big(m_i\big)-\cos\big(\theta_i\big)\sin\big(\theta_i\big)\sin\big(m_i\big)\Big)\Bigg\}. \tag{25}$$

(25) is obtained from (24) by using the identity $\cos^2(t) = 0.5{+}0.5\cos(2t)$. Also using the expectation $E_{m_i}[\sin(m_i)] = 0$ and $E_{m_i}[\sin(2m_i)] = 0$, (28) is obtained.

$$\mathbf{C}(x)_{ii}=\Bigg\{\Big(\frac{d_i^2}{2}\Big)+\Big(\frac{d_i^2}{2}\Big)\cos 2\theta_i E_{m_i}\Big[\cos\Big(2m_i\Big)\Big]+E_{n_i}\Big[\frac{n_i^2}{2}\Big]$$
$$+E_{n_i}\Big[\frac{n_i^2}{2}\Big]E_{m_i}\Big[\cos\Big(2m_i\Big)\Big]\Bigg\}\delta_i^2+\Big(d_i\cos\Big(\theta_i\Big)\Big)^2$$
$$-2\delta_i\Big(d_i\cos\Big(\theta_i\Big)\Big)^2E_{m_i}\Big[\cos\Big(m_i\Big)\Big]. \tag{28}$$

Finally, using expectations

$$E_{m_i}[\cos(m_i)] = \exp\left(-\frac{\sigma_{m_i}^2}{2}\right), \tag{29}$$
$$E_{m_i}[\cos(2m_i)] = \exp\left(-2\sigma_{m_i}^2\right), \tag{30}$$

we conclude the proof by obtaining (14).

For non-diagonal terms i.e. $i \neq j$, putting values of $\hat{\mathbf{b}}_x$ and $\mathbf{b}_x$ in (13).

$$\mathbf{C}(x)_{ij}=E_{m_{ij},n_{ij}}\Bigg\{\Big[\Big(d_i+n_i\Big)\cos\Big(\theta_i+m_i\Big)\delta_i-d_i\cos\theta_i\Big]$$
$$\times\Big[\Big(d_j+n_j\Big)\cos\Big(\theta_j+m_j\Big)\delta_j-d_j\cos\theta_j\Big]\Bigg\},$$

which yields (26) and then (27) given at the top of next page.

The derivation of $\mathbf{C}(y)$ is similar to $\mathbf{C}(x)$ other than that x coordinates are replaced by y.

Proof of (20) and (21): Putting values of $\hat{\mathbf{b}}_x$, $\mathbf{b}_x$, $\hat{\mathbf{b}}_y$ and $\mathbf{b}_y$ in (19) for diagonal terms i.e. $i = j$.

$$\mathbf{C}(\mathrm{xy})_{ii}=E_{m_i,n_i}\Bigg\{\Big[\Big(d_i+n_i\Big)\cos\Big(\theta_i+m_i\Big)\delta_i-d_i\cos\theta_i\Big]$$
$$\times\Big[\Big(d_i+n_i\Big)\sin\Big(\theta_i+m_i\Big)\delta_i-d_i\sin\theta_i\Big]\Bigg\},$$

$$=E_{m_i,n_i}\Bigg\{\Big[\Big(d_i+n_i\Big)^2\cos\Big(\theta_i+m_i\Big)\sin\Big(\theta_i+m_i\Big)\Big]\delta_i^2$$
$$-\Big[d_i\Big(d_i+n_i\Big)\cos\Big(\theta_i+m_i\Big)\sin\theta_i\Big]\delta_i-\Big[d_i\Big(d_i+n_i\Big)$$
$$\sin\Big(\theta_i+m_i\Big)\cos\theta_i\Big]\delta_i+d_i^2\cos\theta_i\sin\theta_i\Bigg\}. \tag{31}$$

Using the identities $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$, $\cos^2 t - \sin^2 t = \cos 2t$ and then taking expectation $E_{m_i}[\sin(m_i)] = 0$, (32) is obtained.

$$\mathbf{C}(\mathrm{xy})_{ii}=\Bigg\{\Big(d_i^2+E_{n_i}\Big[n_i^2\Big]\Big)\cos\Big(\theta_i\Big)\sin\Big(\theta_i\Big)E_{m_i}\Big[\cos\Big(2m_i\Big)\Big]\Bigg\}\delta_i^2$$
$$-\Bigg\{d_i^2\cos\Big(\theta_i\Big)E_{m_i}\Big[\cos\Big(m_i\Big)\Big]\sin\Big(\theta_i\Big)\Bigg\}\delta_i-\Bigg\{d_i^2\sin\Big(\theta_i\Big)$$
$$E_{m_i}\Big[\cos\Big(m_i\Big)\Big]\cos\Big(\theta_i\Big)\Bigg\}\delta_i+d_i^2\cos\theta_i\sin\theta_i. \tag{32}$$

Finally, using expectation (29) and (30) we conclude the proof by obtaining (20).

For non-diagonal terms i.e. $i \neq j$, putting values of $\hat{\mathbf{b}}_x$, $\mathbf{b}_x$, $\hat{\mathbf{b}}_y$ and $\mathbf{b}_y$ in (19).

$$\mathbf{C}(xy)_{ij}=E_{m_{ij},n_{ij}}\Bigg\{\Big[\Big(d_i+n_i\Big)\cos\Big(\theta_i+m_i\Big)\delta_i-d_i\cos\theta_i\Big]$$
$$\times\Big[\Big(d_j+n_j\Big)\sin\Big(\theta_j+m_j\Big)\delta_j-d_j\sin\theta_j\Big]\Bigg\},$$

which yields (33) and then (34) given at the top of next page.

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$$\mathbf{C}(x)_{ij} = E_{m_{ij},n_{ij}}\Bigg\{\Big[\Big(d_i d_j + d_i n_j + d_j n_i + n_i n_j\Big)\Big(\cos(\theta_i)\cos(m_i) + \sin(\theta_i)\sin(m_i)\Big)\Big(\cos(\theta_j)\cos(m_j) + \sin(\theta_j)\sin(m_j)\Big)\delta_{ij}\Big]$$

$$\Big[\Big(d_i d_j + d_j n_i\Big)\Big(\cos(\theta_i)\cos(\theta_j)\cos(m_i) + \sin(\theta_i)\cos(\theta_j)\sin(m_i)\Big)\delta_i\Big]$$

$$- \Big[\Big(d_i d_j + d_i n_j\Big)\Big(\cos(\theta_i)\cos(\theta_j)\cos(m_j) + \sin(\theta_i)\cos(\theta_j)\sin(m_j)\Big)\delta_j\Big] + d_i d_j \cos(\theta_i)\cos(\theta_j)\Bigg\}, \quad (26)$$

where $\delta_{ij} = \delta_i \delta_j$.
Taking expectations in (26), we obtain

$$\mathbf{C}(x)_{ij} = d_i d_j \cos(\theta_i)\cos(\theta_j) - d_i d_j \cos(\theta_i)\cos(\theta_j) - d_i d_j \cos(\theta_i)\cos(\theta_j) + d_i d_j \cos(\theta_i)\cos(\theta_j) = 0. \quad (27)$$

$$\mathbf{C}(xy)_{ij} = E_{m_{ij},n_{ij}}\Bigg\{\Big[\Big(d_i d_j + d_i n_j + d_j n_i + n_i n_j\Big)\Big(\cos(\theta_i)\cos(m_i) - \sin(\theta_i)\sin(m_i)\Big)\Big(\sin(\theta_j)\cos(m_j) + \cos(\theta_j)\sin(m_j)\Big)\delta_{ij}\Big]$$

$$- \Big[\Big(d_i d_j + d_j n_i\Big)\Big(\cos(\theta_i)\sin(\theta_j)\cos(m_i) + \sin(\theta_i)\sin(\theta_j)\sin(m_i)\Big)\delta_i\Big]$$

$$- \Big[\Big(d_i d_j + d_i n_j\Big)\Big(\cos(\theta_i)\sin(\theta_j)\cos(m_j) + \cos(\theta_i)\cos(\theta_j)\sin(m_j)\Big)\delta_j\Big] + d_i d_j \cos(\theta_i)\sin(\theta_j)\Bigg\}. \quad (33)$$

Taking expectations in (33), we get

$$\mathbf{C}(xy)_{ij} = d_i d_j \cos(\theta_i)\sin(\theta_j) - d_i d_j \cos(\theta_i)\sin(\theta_j) - d_i d_j \cos(\theta_i)\sin(\theta_j) + d_i d_j \cos(\theta_i)\sin(\theta_j) = 0. \quad (34)$$

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