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The Lie Algebra of a Nonlinear Dynamical System and its Application to Control

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ABSTRACT

Using a recently introduced Lie algebra associated with a nonlinear system, several new results in dynamical systems and control theory are obtained. In particular, we show that the solutions carry the structure of the associated Lie group. A number of stability and boundedness results are given and a generalisation of classical modal control is developed.

1. Introduction

In a recent series of papers (Banks and Al-Jurani, 1994, 1996, Banks and McCaffrey, 1998, Banks, 1999) nonlinear systems of the form

$$\dot{x} = A(x)x \tag{1.1}$$

have been considered, the arguments being based on the introduction of a Lie algebra L_A generated by the matrices $\{A(x) : x \in \mathbb{R}^n\}$. Using the classical Levi and Cartan decompositions of a finite-dimensional matrix Lie algebra, we can write

$$L_A = \mathfrak{s} + \mathfrak{g}$$

where $\mathfrak{s}$ is solvable and $\mathfrak{g}$ is semisimple, and

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha \text{ (direct sum)}$$

where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{g}_\alpha$ is a one-dimensional root space. Hence the system (1.1) can be written in the form

$$\dot{x} = S(x)x + H(x)x + \sum_{\alpha\in\Delta} e_\alpha(x)E_\alpha x.$$

Using this decomposition, a number of stability results and a general approach to higher-dimensional chaotic systems have been obtained (Banks and McCaffrey, 1998, Banks, 1999).

However, we have not answered the question of how the Lie algebra $\mathfrak{g} = L_A$ of the system and the corresponding Lie group G of L_A are related to the solutions of the system. The main result of section 3 shows that the solutions of the system (1.1) can be written in the form

$$x(t) = g(t; x_0)x_0$$

where $g(t;x_0) \in G$, so that G acts as a transformation group on $\mathbb{R}^n$ (considered as a set of initial states). Since

$$\exp : \mathfrak{g} = L_A \rightarrow G$$

is the usual exponential map, we have

$$x(t) = e^{A(t;x_0)}x_0.$$

This is useful since the solutions can be seen to carry the structural properties of the group. Thus, for example, if the system

$$\dot{x} = \sum_{\mathbf{i}=0}^{\infty} A_{\mathbf{i}} x^{\mathbf{i}} \cdot x$$

where $\mathbf{i} = (i_1, \cdots, i_n)$ is a multi-index and the matrices $A_{\mathbf{i}}$ all belong to an infinitesimal rotation algebra, then the solutions are all rotations, so that $\|x\|$ is invariant.

In section 4 we consider the notion of equivalent Lie algebras. If we write the system

$$\dot{x} = f(x)\ ,\ f(0) = 0$$

in the form (1.1), there are many different ways of doing this. For example,

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= x_2^2 \end{aligned}$$

has the representations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and also an infinite number of 'degenerate' ones, e.g.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 - x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

It is therefore important to know which represent equivalent Lie algebras. Of course, different representations may be useful, since the solutions then carry the structural properties of several different Lie groups.

In this paper we study various types of Lie algebras and there applications to stability and invariance. In particular, the compact part of a Cartan decomposition is important here. Since a compact Lie algebra generates a compact group, the results of section 3 immediately shows that such systems have bounded (stable) solutions.

In the final section we shall consider the entire approach of using the Lie algebra L_A for control systems as a wdie-ranging generalisation of the classical modal control of a linear dynamical system. Diagonalising the Cartan subalgebra gives a direct extension of the classical method and we obtain a new control technique for nonlinear systems.

In the next section we shall outline the main results of Lie algebras which we shall need. For proofs of the results, see Helagson, 1962, Jacobson, 1962, Sagle and Walde, 1973, Varadarahan, 1976.

2. Lie Algebras

Definition 2.1 A **Lie algebra** $\mathfrak{g}$ is a vector space which has a bilinear product map $[\cdot,\cdot]:$ $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying

(i) $[X,Y]=-[Y,X]$ for all $X,Y\in\mathfrak{g}$.

(ii) $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ for all $X,Y,Z\in\mathfrak{g}$.

(If $[X,Y]=0$ for all $X,Y\in\mathfrak{g}$, we say that $\mathfrak{g}$ is an **abelian** Lie algebra.)

The simplest (nontrivial) example is the space $gl(n)$ of all (complex) $n\times n$ matrices with commutator as Lie algebra product:

$$[X,Y]=XY-YX\text{ ,forall }X,Y\in gl(n).$$

The conditions (i) and (ii) in definition (2.1) are trivial to check in this case. It turns out that every Lie algebra is isomorphic to (i.e. has the same algebraic structure as) a Lie algebra of linear transformations contained in $gl(n)$. Subalgebras of a Lie algebra are defined in the obvious way and subalgebras of $gl(n)$ are called **linear Lie algebras**.

Example 2.2 An important example in physics (because it is the generator of the angular momentum group) is the three-dimensional subalgebra $\mathfrak{g}_3$ of $gl(3)$ generated by the matrices

$$M_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\ M_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\ M_3=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}.$$

Since $[M_i,M_j]=M_k$, where i,j,k is an even permutation of $\{1,2,3\}$ it is clear that this is a three-dimensional Lie algebra and consists of all skew-symmetric (complex) matrices:

$$\begin{pmatrix}0&-x_3&x_2\\x_3&0&-x_1\\-x_2&x_1&0\end{pmatrix}=x_1M_1+x_2M_2+x_3M_3.\ \square \qquad (2.1)$$

Example 2.3 The set A_{n-1} of all trace zero complex matrices in $gl(n)$ is a Lie algebra since $\mathrm{tr}[X,Y]=0$ for all X,Y.

Example 2.4 Another important class of Lie algebras is the class of orthogonal Lie algebras. Let M be a complex $n\times n$ matrix and consider the set of all $n\times n$ matrices X such that $XM=-MX^T$. It is easy to check that this set of matrices is a linear Lie algebra (depending on M). If $n=2m$ is even and $M=\begin{pmatrix}0&I_m\\I_m&0\end{pmatrix}$ we obtain the Lie algebra D_m and if $n=2m+1$ is odd and $M=\begin{pmatrix}1&0&0\\0&0&I_m\\0&-I_m&0\end{pmatrix}$, we obtain the Lie algebra B_m. Finally, if $n=2m$ is even and $M=\begin{pmatrix}0&I_m\\-I_m&0\end{pmatrix}$, we obtain the Lie algebra C_m (the **symplectic Lie algebra**).$\square$

The algebras A_n,B_n,C_n,D_n are called the **classical Lie algebras**. As we shall see, they form (together with the **exceptional Lie algebras** introduced below) essential building blocks for many types of Lie algebras.

Definition 2.5 An **ideal** $\mathfrak{h}$ in a Lie algebra $\mathfrak{g}$ is a (vector) subspace such that $[\mathfrak{h},\mathfrak{g}]\subseteq\mathfrak{h}$ (where $[\mathfrak{h},\mathfrak{g}]$ denotes the subspace spanned by the set of all elements of the form $[X,Y],X\in\mathfrak{h},Y\in\mathfrak{g}$). An ideal $\mathfrak{h}$ in $\mathfrak{g}$ is **minimal** if $\{0\}$ is the only ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$.

We denote by the $\mathfrak{h}_1 + \mathfrak{h}_2$ the subspace spanned by all elements of the form $X + Y, X \in \mathfrak{h}_1, Y \in \mathfrak{h}_2$ for any subsets $\mathfrak{h}_1, \mathfrak{h}_2 \subseteq \mathfrak{g}$. If $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal, then $\mathfrak{g}/\mathfrak{h}$ denotes the quotient Lie algebra which is the quotient of the vector spaces $\mathfrak{g}$ and $\mathfrak{h}$ with bracket

$$[\overline{X}, \overline{Y}] = \overline{[X,Y]}\ ,\ X, Y \in \mathfrak{g}$$

where $\overline{X}$ is the coset of X. Clearly the projection map

$$\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$$

is a homomorphism of Lie algebras with kernel $\mathfrak{h}$. (A **homomorphism** of Lie algebras is a homomorphism of the underlying vector spaces which preserves the bracket.)

Definition 2.6 A Lie algebra $\mathfrak{g}$ is said to be **simple** if $\mathfrak{g}$ and $\{0\}$ are the only ideals of $\mathfrak{g}$.

Theorem 2.7 *The one-dimensional Lie algebra is abelian and simple and any abelian Lie algebra of dimension >1 is not simple. The algebras* A_n, B_n, C_n $(n \geq 1)$ *and* D_n $(n \geq 3)$ *are simple.*□

Definition 2.8 If $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ (vector space direct sum) and each $\mathfrak{g}_i$ is an ideal, then $\mathfrak{g}$ is called the **direct sum** of $\mathfrak{g}_1, \cdots, \mathfrak{g}_k$. Clearly, $\mathfrak{g}_i \cap \mathfrak{g}_j = [\mathfrak{g}_i, \mathfrak{g}_j] = \{0\}$ for $i \neq j$.

Definition 2.9 The ideal $\mathfrak{D}\mathfrak{g} \triangleq [\mathfrak{g}, \mathfrak{g}]$ of a Lie algebra $\mathfrak{g}$ is called the **derived algebra** of $\mathfrak{g}$. We have the obvious **derived series** of $\mathfrak{g}$:

$$\mathfrak{g} \supseteq \mathfrak{D}\mathfrak{g} \supseteq \mathfrak{D}^2\mathfrak{g} \supseteq \cdots \supseteq \mathfrak{D}^\ell\mathfrak{g} \supseteq \cdots$$

where $\mathfrak{D}^\ell\mathfrak{g} = \mathfrak{D}(\mathfrak{D}^{\ell-1}\mathfrak{g}\ )$. Each term in the series is clearly an ideal. If $\mathfrak{D}^k\mathfrak{g} = \{0\}$ for some $k > 0$, we say that $\mathfrak{g}$ is a **solvable** Lie algebra.

Definition 2.10 If $\mathfrak{g}$ does not contain any solvable ideal apart from $\{0\}$ (or equivalently does not contain any non-zero abelian ideal), then $\mathfrak{g}$ is said to be **semisimple**. If we exclude the trivial one-dimensional Lie algebra, then all simple Lie algebras are semisimple.

Theorem 2.11 *Every semisimple Lie algebra is the direct sum of all its minimal ideals.* □

Since each minimal ideal of a semisimple Lie algebra is simple, this theorem reduces the study of semisimple Lie algebras to simple ones. The latter can be classified, as we shall see later.□

Theorem 2.12 *Every Lie algebra* $\mathfrak{g}$ *has a unique maximal solvable ideal* $\mathfrak{r}$ *called the radical of* $\mathfrak{g}$. *Then is* $\mathfrak{g}/\mathfrak{r}$ *semisimple.*□

Hence any Lie algebra $\mathfrak{g}$ can be written in the form $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$ where $\mathfrak{r}$ is solvable and $\mathfrak{s}$ is semisimple. This is called a **Levi decomposition** of $\mathfrak{g}$. Note, however, that this decomposition is not a direct sum, so it is not unique. Each Lie algebra has many Levi decompositions.□

Another important class of Lie algebras is the nilpotent class. If $\mathfrak{g}$ is a Lie algebra, let $C^{(0)}\mathfrak{g} = \mathfrak{g}$, $C^{(1)}\mathfrak{g} = [\mathfrak{g}, C^{(0)}\mathfrak{g}], \cdots, C^{(\ell+1)}\mathfrak{g} = [\mathfrak{g}, C^{(\ell)}\mathfrak{g}], \cdots$. Then $C^{(\ell)}\mathfrak{g}$ is an ideal of $\mathfrak{g}$ and we obtain the **descending central series**:

$$C^{(0)}\mathfrak{g} \supseteq C^{(1)}\mathfrak{g} \supseteq \cdots \supseteq C^{(\ell)}\mathfrak{g} \supseteq \cdots.$$

If $C^{(k)}\mathfrak{g} = 0$ for some $k > 0$, then $\mathfrak{g}$ is called **nilpotent**. Note that $\mathfrak{D}^\ell\mathfrak{g} \subseteq C^{(\ell)}\mathfrak{g}$ for each ℓ so that if $\mathfrak{g}$ is nilpotent then it is solvable. In fact, $\mathfrak{g}$ is solvable if and only if $\mathfrak{D}\mathfrak{g}$ is nilpotent.

Example 2.13 The subset of $gl(n)$ consisting of all upper triangular matrices is a solvable Lie algebra. It turns out that any solvable linear Lie algebra is isomorphic to this algebra. (This is based on Lie's theorem, which states that all elements of a solvable linear Lie algebra have a common eigenvector.)

The subset of $gl(n)$ consisting of all upper triangular matrices with equal diagonal terms, i.e. matrices of the form

$$\begin{pmatrix} \lambda & * & * & \cdots & * \\ 0 & \lambda & * & \cdots & * \\ 0 & 0 & \ddots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \ddots & * \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

is a nilpotent Lie algebra. Moreover, any nilpotent linear Lie algebra is isomorphic to a direct sum of such algebras.□

One of the most important linear operators acting on any Lie algebra $\mathfrak{g}$ is the adjoint map *ad*. It is defined for each $X \in \mathfrak{g}$ by

$$(ad\, X)Y = [X, Y].$$

Using this map we can define a geometric structure on a Lie algebra (different from the usual one on the underlying vector space) in terms of a symmetric bilinear form $(\cdot, \cdot)$ called the **Killing form** and defined by

$$(X, Y) = \mathrm{tr}\, ad\, X\, ad\, Y.$$

(i.e. the trace of the operator $ad\, X \circ ad\, Y$).

Example 2.14 Let us calculate the Killing form for the Lie algebra $\mathfrak{g}_3$ defined in example (2.2). For any $X \in \mathfrak{g}_3$ we have $X = x_1M_1 + x_2M_2 + x_3M_3$ (as in (2.1)). Then

$$(ad\, X\,)M_1 = [x_1M_1 + x_2M_2 + x_3M_3, M_1] = -x_2M_3 + x_3M_2$$

and similarly,

$$(ad\, X\,)M_2 = x_1M_3 - x_3M_1\ ,\ (ad\, X\,)M_3 = -x_1M_2 + x_2M_1.$$

Hence,

$$X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

is actually the matrix representation of $ad\, X$ in the basis $\{M_1, M_2, M_3\}$. If $Y = y_1M_1 + y_2M_2 + y_3M_3$ is another such element, then clearly,

$$\begin{aligned} \mathrm{tr}\, ad\, X\, ad\, Y &= \mathrm{tr} \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix} \\ &= -2(x_1y_1 + x_2y_2 + x_3y_3).\square \end{aligned} \tag{2.2}$$

Cartan proved the following important criterion for solvability:

Theorem 2.15 *A Lie algebra $\mathfrak{g}$ is solvable if and only if $(X, X) = 0$ for all $X \in \mathfrak{D}\mathfrak{g}$.*□

Remarkably, semisimplicity is also characterised by the Killing form:

Theorem 2.16 *(Cartan's criterion) A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form of $\mathfrak{g}$ is nondegenerate (i.e. $(X,Y) = 0$ for all $Y \in \mathfrak{g}$ implies that $X = 0$).*□

Just as a vector space can be decomposed into the generalised eigenspaces of any given linear operator (or matrix), any nilpotent linear Lie algebra $\mathfrak{h}$ acting on a vector space V defines a decomposition of V in the following way:

For any given linear function $\alpha : \mathfrak{h} \to \mathbb{C}$ define the set

$$V^\alpha = \{v \in V : (H - \alpha(H)I)^k v = 0 \text{ , for some } k > 0 \text{ and all } H \in \mathfrak{h}\}$$

i.e. V^α is the generalised eigenspace for all $H \in \mathfrak{h}$ with eigenvalue $\alpha(H)$. If $V^\alpha \neq \emptyset$ we say that α is a **weight** or a **root** of $\mathfrak{h}$ in V and V^α is a **weight (root) subspace** of V. Then we have

$$V = \oplus_{\alpha \in \Delta} V^\alpha \tag{2.3}$$

where Δ is the set of all weights of $\mathfrak{h}$ in V.

Now, if $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h}$ is a nilpotent subalgebra, then

$$ad\,\mathfrak{h} \triangleq \{ad\,H : H \in \mathfrak{h}\}$$

is a nilpotent linear Lie algebra acting on $\mathfrak{g}$, so if we apply (2.3) with $V = \mathfrak{g}$ and $\mathfrak{h}$ replaced by $ad\,\mathfrak{h}$, we obtain the following decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \oplus_{\alpha \in \Delta} \mathfrak{g}^\alpha \tag{2.4}$$

where

$$\mathfrak{g}^\alpha = \{G \in \mathfrak{g} : (ad\,H - \alpha(H)I)^k G = 0 \text{ , for some } k > 0 \text{ and all } H \in \mathfrak{h}\}$$

and we have written $\alpha(H)$ for $\alpha(ad\,H)$.

A particularly important summand in (2.4) is the one for which $\alpha(H) = 0$ for all $H \in \mathfrak{h}$, i.e. $\mathfrak{g}^0$, corresponding to the subspace of the zero eigenvalue. Clearly, $\mathfrak{h} \subseteq \mathfrak{g}^0$.

Definition 2.17 If $\mathfrak{h} = \mathfrak{g}^0$, then $\mathfrak{h}$ is called a **Cartan subalgebra** of $\mathfrak{g}$.

It can be shown that every Lie algebra has a Cartan subalgebra and each such subalgebra is a maximal nilpotent subalgebra. Any two Cartan subalgebras are conjugate under a certain group of automorphisms of the algebra.

In the case of a semisimple Lie algebra, the **root space decomposition** (2.4) takes the form

$$\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha$$

where Σ is the set of *nonzero* roots of $\mathfrak{h}$ in $\mathfrak{g}$, and the Cartan subalgebra $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$. The Killing form $(\cdot,\cdot)$ is nondegenerate on $\mathfrak{h}$, each $\mathfrak{g}^\alpha$ is one-dimensional for $\alpha \neq 0$ and there are (dim $\mathfrak{h}$) linearly independent roots. Moreover, if α, β and $\alpha + \beta$ are roots, the $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}$.

3. The Lie Algebra of a Differential Equation

We shall be concerned with nonlinear differential equations which can be written in the form

$$\dot{x} = A(x)x \text{ , } x(0) = x_0 \in \mathbb{R}^n \tag{3.1}$$

where $A : \mathbb{R}^n \to \mathbb{R}^{n^2}$ is a continuously differentiable matrix-valued function (in fact, usually analytic). Such equations are common; for example the analytic system

$$\dot{x} = f(x)$$

for which $f(0) = 0$ can be written in this form. The representation (3.1) is not unique, however. For example, if

$$\tilde{A}(x) = A(x) + \begin{pmatrix} x_2 & -x_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

then the equation

$$\dot{x} = \tilde{A}(x)x \ , \ x(0) = x_0 \tag{3.2}$$

is identical to (3.1). For each representation of the form (3.1) we introduce a Lie algebra associated with the system:

Definition 3.1 For the system of equations (3.1), we denote by L_A the Lie algebra generated by the set

$$\{A(x) : x \in \mathbb{R}^n\},$$

and call it the **Lie algebra associated with the representation (3.1)**.

Remark If $A(x)$ is analytic, we may write

$$A(x) = \sum_{|\mathbf{i}|=0}^{\infty} A_{\mathbf{i}} x^{\mathbf{i}}$$

where $\mathbf{i} = (i_1, \cdots, i_n)$ and $x^{\mathbf{i}} = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$. We could then introduce the Lie algebra $L_{\{A_{\mathbf{i}}\}}$ generated by the matrices $A_{\mathbf{i}}$. However, it is shown in (Banks and McCaffrey, 1998) that $L_A = L_{\{A_{\mathbf{i}}\}}$ and so we may consider either as equivalent definitions of the Lie algebra associated with (3.1). □

Example 3.2 In the following we present some examples to illustrate the kinds of Lie algebras which can arise in the above way.

(1) Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 x_3 - x_1 x_3 \\ \dot{x}_2 &= -x_1 x_3 + x_2 x_3 \\ \dot{x}_3 &= x_1^2 - x_2^2 \, . \end{aligned}$$

It can be written

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & x_3 & -x_1 \\ -x_3 & 0 & x_2 \\ x_1 & -x_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \triangleq A(x) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and so $L_A = so(3)$, the infinitesimal rotation algebra generated by the matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that the solutions are invariant on spheres, i.e. $x_1^2 + x_2^2 + x_3^2 = const.$

(2) The Lie algebra of a system is not uniquely defined as noted above. For example, the trivial linear system

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 \end{aligned}$$

has the representations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 - x_2 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

among infinitely many others. In the first case, the Lie algebra L_A is the trivial one-dimensional (abelian) Lie algebra $\{\lambda I : \lambda \in \mathbb{R}\}$ and in the second case it is the three dimensional Lie algebra

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}.$$

Hence a careful choice of appropriate representation may be necessary. However, in many cases, systems have a 'natural' representation in the form (3.1), as in the first example. The main result of this section is that the solution of (3.1) is given by

$$x(t; x_0) = e^{A(t;x_0)} x_0$$

where $A(t; x_0) \in L_A$ for each t, x_0, i.e. the solution can be regarded as an operation of the Lie group of L_A as a transformation group on $\mathbb{R}^n$, since

$$e^{A(t;x_0)} x_0 = e^{A(t_2; e^{A(t_1;x_0)})} e^{A(t_1;x_0)} x_0 \tag{3.3}$$

for $t = t_1 + t_2$.

Theorem 3.3 *Given a differential equation of the form*

$$\dot{x} = A(x)x \ , \ x(0) = x_0 \in \mathbb{R}^n$$

where $A : \mathbb{R}^n \to \mathbb{R}^{n^2}$ is locally Lipschitz, then the solution for each t (for which the solution exists) can be written in the form

$$x(t; x_0) = e^{A(t;x_0)} x_0 \tag{3.4}$$

where $A(t; x_0) \in L_A$ for each t, x_0, and, moreover, (3.3) holds for all t_1, t_2, t such that $t = t_1 + t_2$.

Proof Since (3.3) is obvious from the group property of solutions, we only need to prove (3.4). To do this we first note that it is sufficient to prove it on some finite interval $[0, T]$. We proceed by introducing a sequence of linear, time-varying approximations:

$$\begin{aligned} \dot{x}^{[0]}(t) &= A(x_0) x^{[0]}(t) \ , \ x^{[0]}(0) = x_0 \\ \dot{x}^{[i]}(t) &= A(x^{[i-1]}(t)) x^{[i]}(t) \ , \ x^{[i]}(0) = x_0 \ , \ i \geq 1. \end{aligned} \tag{3.5}$$

We shall prove that this sequence converges (in $C([0,T];\mathbb{R}^n)$). In fact, note that if $\Phi^{[i-1]}(t,t_0)$ denotes the transition matrix generated by $A(x^{[i-1]}(t))$, then we have (see F.Brauer, 1967) that

$$\left\|\Phi^{[i-1]}(t,t_0)\right\| \leq \exp\left[\int_{t_0}^{t} \mu(A(x^{[i-1]}(\tau)))d\tau\right] \tag{3.6}$$

where $\mu(A)$ is the logarithmic norm of A. The local Lipschitz continuity of A implies that the solutions $x^{[i]}$ of the system (3.5) are bounded, independently of i, for sufficiently small t. In fact, let $B(K,x_0)$ be the ball with centre x_0 and radius K, for any fixed K. Then

$$\|A(x)-A(y)\| \leq \alpha(K)\,\|x-y\|\,,$$

for all $x,y \in B(K,x_0)$, for some $\alpha(K)$, by local Lipschitz continuity. Now,

$$x^{[i]}(t)-x_0 = e^{A(x_0)t}x_0 - x_0 + \int_0^t e^{A(x_0)(t-s)}(A(x^{[i-1]}(s))-A(x_0))ds$$

and so

$$\begin{aligned}\left\|x^{[i]}(t)-x_0\right\| \;\leq\; & \sup_{t\in[0,T]}\left\|e^{A(x_0)t}-I\right\|\cdot\|x_0\| + \sup_{t\in[0,T]} e^{\|A(x_0)t\|}\alpha(K)\cdot \\ & T\sup_{t\in[0,T]}\left\|x^{[i-1]}(t)-x_0\right\|\end{aligned}$$

so if $x^{[i-1]}(t)\in B(K,x_0)$, then $x^{[i]}(t)\in B(K,x_0)$ for $t\in[0,T]$ when T is small enough. Since $x^{[0]}(t)$ clearly belongs to $B(K,x_0)$ for small enough T, we see that all solutions $x^{[i]}(t)$ of (3.5) are bounded for $i\geq 0$, $t\in[0,T]$. Note also that

$$\left\|A(x^{[i-1]}(t))\right\| \leq \alpha(K)\left\|x^{[i-1]}(t)-x_0\right\| + \|A(x_0)\|$$

by local Lipschitz continuity of A and since

$$\mu(A) = \frac{1}{2}\max\sigma(A+A^T)$$

is the standard matrix norm, we have that $\mu(A(x^{[i-1]}(t)))$ is bounded for all i, say

$$\mu(A(x^{[i-1]}(t))) \leq \mu,\ \forall i, t\in[0,T].$$

Hence we have from (3.5) that

$$\begin{aligned}\dot{x}^{[i]}(t)-\dot{x}^{[i-1]}(t) &= A(x^{[i-1]}(t))x^{[i]}(t) - A(x^{[i-2]}(t))x^{[i-1]}(t) \\ &= A(x^{[i-1]}(t))\left(x^{[i]}(t)-x^{[i-1]}(t)\right) \\ &\quad + \left(A(x^{[i-1]}(t))-A(x^{[i-2]}(t))\right)x^{[i-1]}(t)\end{aligned}$$

and so, if

$$\xi^{[i]}(t) = \sup_{s\in[0,T]}\left\|x^{[i]}(s)-x^{[i-1]}(s)\right\|$$

then

$$\xi^{[i]}(t) \leq \int_0^t \left\|\Phi^{[i-1]}(t,s)\right\| \cdot \left\|A(x^{[i-1]}(s)) - A(x^{[i-2]}(s))\right\| \cdot \left\|x^{[i-1]}(s)\right\| ds.$$

Hence,

$$\xi^{[i]}(t) \leq \int_0^t e^{\mu(t-s)}\alpha(K)\xi^{[i-1]}(s)K ds$$

so that

$$\begin{aligned}\xi^{[i]}(T) &\leq \sup_{s\in[0,T]} e^{\mu(T-s)} \cdot \alpha(K)TK\xi^{[i-1]}(T) \\ &\leq \lambda\xi^{[i-1]}(T)\end{aligned}$$

where $\lambda = \sup_{s\in[0,T]} e^{\mu(T-s)} \cdot \alpha(K)TK$. Thus, if T is small enough, $\lambda < 1$ and $\{x^{[i]}(t)\}$ is a Cauchy sequence in $C([0,T];\mathbb{R}^n)$.

Next note that, for any nonautonomous system

$$\dot{x} = B(t)x \ , \ x(0) = x_0$$

where $B(\cdot) : \mathbb{R} \to \mathbb{R}^{n^2}$ is continuous, we have

$$x(t) = \lim_{n\to\infty} e^{\frac{t}{n}B((n-1)\frac{t}{n})}e^{\frac{t}{n}B((n-2)\frac{t}{n})} \cdots e^{\frac{t}{n}B(\frac{t}{n})}e^{\frac{t}{n}B(0)}x_0$$

(see, for example, Taylor, 1991). Applying this to each term of the sequence $x^{[i]}(t)$ above and taking a diagonal subsequence, the result follows by the Campbell-Hausdorff formula. □

Remark This result says that the solution of equation (3.1) is given by

$$x(t;x_0) = g(t;x_0)x_0$$

where $g(t;x_0) = e^{A(t;x_0)}$ is a smooth curve in the Lie group G_A of L_A. □

Examples 3.4

(1) Any system of the form

$$\dot{x} = \begin{pmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \cdots & \cdots & \cdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{pmatrix} x \ , \ x(0) = x_0 \tag{3.7}$$

where $a_{ij}(x) = -a_{ji}(x)$, i.e. A is skew-symmetric, has a solution of the form

$$x(t;x_0) = O(t;x_0)x_0$$

where $O(t;x_0)$ is an orthogonal matrix for each t. Thus, every system of this form generates rotations of x_0 for each t. In this case, however, the result follows from the elementary fact that $\|x(t;x_0)\|^2$ is invariant; for

$$\frac{d}{dt}\|x(t;x_0)\|^2 = \sum_{i=1}^n x_i\dot{x}_i = \sum_{i=1}^n\sum_{j=1}^n x_i a_{ij}(x)x_j.$$

(2) For any system of the form (3.7) where $\sum_{i=1}^{n} a_{ii}(x) = 0$, i.e. A has trace zero, the solutions are of the form

$$x(t; x_0) = D(t; x_0)x_0$$

where $\det D(t; x_0) = 1$ for each t, since in this case the Lie algebra L_A is $sl(n)$ and G_A is $SL(n)$. For example,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} , \; x(0) = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

has solution

$$\begin{aligned} x_1(t) &= e^t e^{(1-e^{-t})x_{20}} x_{10} \\ x_2(t) &= e^{-t} x_{20} \end{aligned}$$

so that

$$x(t) = \begin{pmatrix} e^t & e^t \left(e^{(1-e^{-t})x_{20}} - 1 \right) x_{10}/x_{20} \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}.$$

Note that $\left(e^{(1-e^{-t})x_{20}} - 1 \right) / x_{20} \to 1 - e^{-t}$ as $x_{20} \to 0$, so this function is well-defined. (Of course, as with $A(x)$, this representation is not unique.)

(3) Let

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

A symplectic matrix A is one which satisfies $A^T J A = J$. Differentiating this gives the Lie algebra $sp(n)$ of infinitesimal symplectic matrices B which satisfy $JB + B^T J = 0$. Thus any differential equation of the form

$$\dot{x} = \begin{pmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & -A_{11}^T(x) \end{pmatrix} x \;, \; x(0) = x_0, \; x \in \mathbb{R}^{2n},$$

where $A_{12}(x) \in \mathbb{R}^{n^2}$, $A_{21}(x) \in \mathbb{R}^{n^2}$ are symmetric, has solutions of the form

$$x(t; x_0) = \begin{pmatrix} B_{11}(t; x_0) & B_{12}(t; x_0) \\ B_{21}(t; x_0) & B_{22}(t; x_0) \end{pmatrix} x_0$$

where

$$\begin{aligned} B_{11}^T B_{21} - B_{21}^T B_{11} &= 0 \\ B_{11}^T B_{22} - B_{21}^T B_{12} &= I \\ -B_{22}^T B_{12} + B_{12}^T B_{22} &= 0. \end{aligned}$$

4. Equivalent Lie Algebras

We have seen that the Lie algebra associated with a system of equations

$$\dot{x} = f(x) \tag{4.1}$$

is not unique and, in fact, depends on the representation of the system in the form

$$\dot{x} = A(x)x. \tag{4.2}$$

In this section we shall consider the Lie algebras generated by the different representations of the nonlinear system (4.1) in the form (4.2). First note the following simple result:

Proposition 4.1 *If we change coordinates to $y = P^{-1}x$ where P is an invertible (real) matrix, then the system (4.2) and the system*

$$\dot{y} = P^{-1}A(Py)Py$$

have isomorphic Lie algebras.

Proof This follows from the facts that $A(Py)$ ranges over the same matrices as $A(x)$ and

$$[P^{-1}AP, P^{-1}BP] = P^{-1}[A, B]P$$

for any A, B. □

Hence we have

Corollary 4.2 *If we have two representations*

$$\dot{x} = A(x)x \ , \ \dot{x} = \bar{A}(x)x$$

of equation (4.1) and we write

$$A(x) = \sum_{\mathbf{i}} A_{\mathbf{i}}x^{\mathbf{i}} \ , \ \bar{A}(x) = \sum_{\mathbf{i}} \bar{A}_{\mathbf{i}}x^{\mathbf{i}},$$

then they generate isomorphic Lie algebras if $\bar{A}_{\mathbf{i}} = P^{-1}A_{\mathbf{i}}P$ for all $\mathbf{i}$ and for some invertible matrix P. □

Example Consider the equations

$$\begin{aligned} \dot{x}_1 &= x_1^2x_2^3 \\ \dot{x}_2 &= x_1x_2^2 \end{aligned}$$

We can write this system in the following forms:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = x_1x_2^3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + x_2^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = x_1^2x_2^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + x_1x_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We see that the change of coordinates

$$x \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x$$

maps one into the other. Moreover, we have

$$L\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\} \cong L\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

since

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note, however, that there are other representations giving nonisomorphic Lie algebras; for example, the representation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = x_1 x_2^3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + x_1 x_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

gives the two-dimensional abelian Lie algebra, while the representation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = x_1^2 x_2^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + x_2^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

gives the three-dimensional Lie algebra with basis

$$\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}. \square$$

We can therefore see that different Lie algebras represent the same system and, conversely, different systems may have the same Lie algebra for some representation. To remove this ambiguity, we will define the **Lie structure** of a system of equations of the form (4.2) to be a pair (L_A, p) consisting of an $n \times n$ matrix Lie algebra L_A together with a map $p : \mathbb{Z}^n \to L_A$ which associates with the monomial $x^{\mathbf{i}}$ an element of L_A.

Definition 4.3 We say that two Lie structures (L_A, p) and $(L_{\bar{A}}, \bar{p})$ are **equivalent** if they generate the same vector field.

Proposition 4.4 *Two Lie structures (L_A, p) and $(L_{\bar{A}}, \bar{p})$ are equivalent if and only if*

$$\sum_{s=1}^{n} A_{\mathbf{i}-1_s}(r, s) = \sum_{s=1}^{n} \bar{A}_{\mathbf{i}-1_s}(r, s)\ ,\ 1 \le r \le n \tag{4.3}$$

where $A_{\mathbf{i}} = p(\mathbf{i})$, $\bar{A}_{\mathbf{i}} = \bar{p}(\mathbf{i})$, $\mathbf{i} = (i_1, \cdots, i_n) \in \mathbb{Z}^n$, $\mathbf{i} - 1_s = (i_1, i_2, \cdots, i_s - 1, i_{s+1}, \cdots, i_n)$ and $B(r, s)$ denotes the (r, s)-element of the matrix B.

Proof We simply note that each side of (4.3) is the coefficient of $x^{\mathbf{i}}$ in the Taylor expansion of $f_r(x)$. $\square$

5. Solvable Systems

In this section we shall consider systems of the form

$$\dot{x} = A(x)x \tag{5.1}$$

where L_A is a solvable Lie algebra. We can change coordinates so that all the matrices $A(x)$ are upper triangular, so we can write the system (5.1) in the form

$$\dot{x} = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ & a_{21}(x) & \cdots & a_{2n}(x) \\ & & \cdots & \cdots \\ & & & a_{nn}(x) \end{pmatrix} x. \tag{5.2}$$

From the results of section 3, we know that the solution of this system through x_0 at $t = 0$ is given by the limit of the sequence of systems

$$\dot{x}^{[i]}(t) = \begin{pmatrix} a_{11}(x^{[i-1]}(t)) & a_{12}(x^{[i-1]}(t)) & \cdots & a_{1n}(x^{[i-1]}(t)) \\ & a_{21}(x^{[i-1]}(t)) & \cdots & a_{2n}(x^{[i-1]}(t)) \\ & & \cdots & \cdots \\ & & & a_{nn}(x^{[i-1]}(t)) \end{pmatrix} x^{[i]}(t) \ , \ x^{[i]}(0) = x_0. \tag{5.3}$$

We can solve each of these upper triangular time-varying equations explicitly to obtain

$$x^{[i]}(t) = S(x^{[i-1]}(\cdot))(t) \ , \ x^{[0]}(t) = x_0,$$

where

$$S(\xi(t)) = \begin{pmatrix} \sigma_1(\xi(t)) \\ \sigma_2(\xi(t)) \\ \vdots \\ \sigma_n(\xi(t)) \end{pmatrix}$$

and

$$\begin{aligned} \sigma_n(\xi(t)) &= e^{\int_0^t a_{nn}(\xi(s))ds} x_{0n} \\ \sigma_k(\xi(t)) &= e^{\int_0^t a_{kk}(\xi(s))ds} x_{0k} + \int_0^t \sum_{\ell=k+1}^{n} a_{k\ell}(\xi(s))\sigma_\ell(\xi(s)) \times \\ & \quad e^{\int_s^t a_{kk}(\xi(\tau))d\tau} ds \ , \ n-1 \geq k \geq 1. \end{aligned} \tag{5.4}$$

Hence,

$$x^{[i]}(t) = S^i_{\cdot}(x^{[0]})(t).$$

Note also that we can obtain an explicit expression for $A(t; x_0)$ in this case, since

$$\begin{aligned} \sigma_n(\xi(t)) &= e^{\int_0^t a_{nn}(\xi(s))ds} x_{0n} = \alpha_{nn}(\xi; 0, t) x_{0n} \ , \ \text{say} \\ \sigma_{n-1}(\xi(t)) &= \alpha_{n-1n-1}(\xi; 0, t) x_{0n-1} + \int_0^t a_{n-1n}(\xi(s))\alpha_{n-1n-1}(\xi; s, t)\alpha_{nn}(\xi; 0, s) x_{0n} ds \\ &= \alpha_{n-1n-1}(\xi; 0, t) x_{0n-1} + \alpha_{n-1n}(\xi; 0, t) x_{0n} \ , \ \text{say} \\ & \cdots \end{aligned}$$

and so we can write

$$S(\xi(\cdot))(t) = \bar{A}(\xi; t) x_0$$

where

$$\bar{A}(\xi;t) = \begin{pmatrix} \alpha_{11}(\xi;0,t) & \alpha_{12}(\xi;0,t) & \cdots & \alpha_{1n}(\xi;0,t) \\ & \cdots & \cdots & \cdots \\ & & \alpha_{n-1n-1}(\xi;0,t) & \alpha_{n-1n}(\xi;0,t) \\ & & & \alpha_{nn}(\xi;0,t) \end{pmatrix}$$

and so

$$A(t;x_0) = \lim_{i\to\infty} S^i(x^{[1]}(\cdot))\bar{A}(x^{[0]};t)x_0.$$

We can use (5.1) to obtain the following stability result which generalises theorem 4.2 in (Banks and McCaffrey, 1998):

Theorem 5.1 *Let $K > 0, M > 0$ and suppose that*

$$|a_{ii}(x)| \le -\varepsilon_i < 0,\ 1 \le i \le n$$

and put $\delta = \min_i(\varepsilon_i/2)$, $\alpha = \min_i(\varepsilon_i - \delta)$ (>0). Moreover, suppose that

$$|a_{k\ell}(x)| \le L,\ k \ne \ell,\ \|x\| \le K.$$

Then, if

(i) $\frac{nL}{\alpha} < 1$

(ii) $|x_{0k}| \le \left(1 - \frac{nL}{\alpha}\right) M,\ 1 \le k \le n$

(iii) $M \le \frac{K}{\sqrt{n}}$

the system (5.2) is asymptotically stable in the ball $\{x : \|x\| \le K\}$.

Proof Assume that $|\sigma_\ell(t)| \le Me^{-\delta t}$, $k+1 \le \ell \le n$. This is certainly true for $\ell = n$; using the above assumptions. By (5.4) we have

$$\begin{aligned} |\sigma_k(t)| &\le e^{-\varepsilon_k t}|x_{0k}| + \int_0^t \sum_{\ell=k+1}^{n} |a_{k\ell}(\xi)| \cdot |\sigma_\ell(s)| e^{-(t-s)\varepsilon_k} ds \\ &\le e^{-\varepsilon_k t}|x_{0k}| + LM(n-k)e^{-\varepsilon_k t}\int_0^t e^{-\delta s}e^{s\varepsilon_k} ds \\ &\le e^{-\varepsilon_k t}|x_{0k}| + LMne^{-\delta t}e^{-t(\varepsilon_k - \delta)}\int_0^t e^{(-\delta+\varepsilon_k)s} ds \\ &\le e^{-\varepsilon_k t}|x_{0k}| + LMn\frac{e^{-\delta t}}{\alpha} \\ &\le Me^{-\delta t} \end{aligned}$$

by the above conditions. It follows from this that if we have $\left\|x^{[i-1]}(t)\right\| \le K$, then $\left\|x^{[i]}(t)\right\| \le K$ and, in fact,

$$\left\|x^{[i]}(t)\right\| \le Ke^{-\delta t}.$$

However, the same argument as above shows that $\left\|x^{[0]}(t)\right\| \le K$ and so the result follows by induction and the convergence proof for the sequence (5.3). □

6. The Killing Form and Invariant Spaces

Recall that the Killing form of a Lie algebra is defined as the symmetric blinear form

$$(X,Y) = \operatorname{tr}(\operatorname{ad} X, \operatorname{ad} Y).$$

It is important in determining the structure of semisimple Lie algebras and so we shall first see how to find the Killing form of the Lie algebra L_A determined by the differential equation

$$\dot{x} = \left(\sum_{|\mathrm{i}|=0}^{\infty} A_{\mathrm{i}} x^{\mathrm{i}} \right) x,$$

i.e. $L_A = L_{\{A_{\mathrm{i}}\}}$. Suppose that, as a vector space, $\dim L_A = M$. The dimension of the vector space spanned by $\{A_{\mathrm{i}}\}$ is no larger than M; suppose it is $K \leq M$. Then there are K linearly independent matrices in the set $\{A_{\mathrm{i}}\}$; denote them by $C_1, \cdots, C_K$. If $K = M$ we have a basis of L_A- if not we can complete it to a basis of L_A with matrices $C_{K+1}, \cdots, C_M$ where $C_j = [C_{j_1}, C_{j_2}]$ for $K+1 \leq j \leq M$, for some $j_1, j_2 < j$. Let $d^{\gamma}_{\alpha\beta}$ denote the structure constants of L_A with respect to the basis $C_1, \cdots, C_M$; i.e.

$$[C_\alpha, C_\beta] = \sum_{\gamma} d^{\gamma}_{\alpha\beta} C_\gamma.$$

Then we have

Lemma 6.1 *The Killing form is given by*

$$(X,Y) = \sum_i \sum_\ell \sum_k \sum_\gamma y_\ell x_k d^{\gamma}_{\ell i} d^{i}_{k\gamma} \tag{6.1}$$

where $X = \sum x_k C_k, Y = \sum y_\ell C_\ell$.

Proof This is elementary manipulation:

$$[\sum x_k C_k, [\sum y_\ell C_\ell, C_i]] = \sum_j a_{ji} C_j$$

and the left hand side equals

$$\begin{aligned} \sum_\ell \sum_k y_\ell x_k [C_k, [C_\ell, C_i]] &= \sum_\ell \sum_k y_\ell x_k \sum_\gamma d^{\gamma}_{\ell i} [C_k, C_\gamma] \\ &= \sum_\ell \sum_k \sum_\gamma y_\ell x_k d^{\gamma}_{\ell i} \sum_j d^{j}_{k\gamma} C_j \end{aligned}$$

so that

$$a_{ji} = \sum_\ell \sum_k \sum_\gamma y_\ell x_k d^{\gamma}_{\ell i} d^{j}_{k\gamma}$$

and the result follows. □

Corollary 6.2 L_A *is semisimple if and only if the form*

$$((x_1, \cdots, x_M), (y_1, \cdots, y_M)) = \sum_i \sum_\ell \sum_k \sum_\gamma y_\ell x_k d^{\gamma}_{\ell i} d^{j}_{k\gamma}$$

is nondegenerate.

Proof This follows from the above result and Cartan's criterion theorem 2.16. □

Example 6.3 (See example 2.14.) Let

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; M_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These are linearly independent and involutive in the sense that $[M_i, M_j] = M_k$ where i, j, k is an even permutation of {1,2,3}. Hence $\dim L_{\{M_1,M_2,M_3\}} = 3$ and the structure constants d_{ij}^k are nonzero only if i, j, k are distinct and hence a permutation of {1,2,3}. Clearly, then

$$(X, Y) = -2(x_1y_1 + x_2y_2 + x_3y_3)$$

and so $L_{\{M_1,M_2,M_3\}}$ is semisimple (in fact, simple). Consider the system

$$\dot{x} = f_1(x)M_1x + f_2(x)M_2x + f_3(x)M_3x.$$

By theorem 3.1, this has a solution of the form

$$x(t) = e^{A(t,x_0)}x_0$$

where

$$A(t, x_0) = \sum_{i=1}^{3} \alpha_i(t, x_0)M_i \in L_{\{M_1,M_2,M_3\}}$$

for some functions $\alpha_1, \alpha_2, \alpha_3$. In fact, the system generates rotations as can be seen directly:

$$\begin{aligned} \frac{d}{dt}\|x(t)\|^2 &= 2\sum_{i=1}^{3} x_i\dot{x}_i \\ &= 2x_1(f_2x_3 - f_3x_2) + 2x_2(-f_1x_3 + f_3x_1) + 2x_3(f_1x_2 - f_2x_1) \\ &= 0. \end{aligned}$$

Hence, spheres are invariant for this dynamical system. Now suppose that $f = (f_1, f_2, f_3)$ is of the form $f = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3}\right) = \operatorname{grad} V$ for some function V. Then the equation is

$$\dot{x} = \sum_{i=1}^{3} \frac{\partial V}{\partial x_i} M_i x$$

and the level curves of V are also invariant, i.e. $\frac{dV}{dt} = 0$ as can easily be checked. Hence for this system, the trajectory starting at x_0 lies in $\{x : \|x\| = \|x_0\|, V(x) = V(x_0)\}$.

Consider now any system of the form

$$\dot{y} = \sum_{i=1}^{3} h_i(y)E_iy \tag{6.2}$$

where $[E_i, E_j] = E_k$ for any even permutation i, j, k of 1,2,3. Then $L_{\{E_1,E_2,E_3\}} \cong L_{\{M_1,M_2,M_3\}}$ and so there exists P such that

$$E_i = P^{-1} M_i P, \; 1 \leq i \leq 3. \tag{6.3}$$

Hence, if we put $x = Py$, we have

$$\dot{x} = \sum_{i=1}^{3} h_i(P^{-1}x) M_i x$$

and if there exists a function $V(x)$ such that

$$(h_1(P^{-1}x), h_2(P^{-1}x), h_3(P^{-1}x)) = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3} \right)$$

the system will be invariant on the level curves $V(x) = V(x_0)$. For this we require

$$\frac{\partial h_i(P^{-1}x)}{\partial x_j} = \frac{\partial h_j(P^{-1}x)}{\partial x_i} \;, \; i \neq j$$

i.e.

$$\frac{\partial h_i(y)}{\partial y} P'_j = \frac{\partial h_j(y)}{\partial y} P'_i \;, \; i \neq j \tag{6.4}$$

where P'_i is the i^{th} column of P^{-1}. Hence we have proved

Theorem 6.4 *Given a system of the form (6.2) where $[E_i, E_j] = E_k$ for an even permutation i, j, k of 1,2,3, if the functions h_i satisfy (6.4) for each y, where P is given by (6.3), then there exists a function $W(y)$ such that the trajectories of the system with initial state y_0 lie on* $\{y : \|Py\| = \|Py_0\|, W(y) = W(y_0)\}$. □

Example 6.5 Consider a system of the form

$$\begin{aligned} \dot{y} &= \left[h_1(y) \begin{pmatrix} 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & -1 & 1 \\ 0 & -2 & 1 \end{pmatrix} + h_2(y) \begin{pmatrix} -1 & 0 & \frac{2}{3} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ -3 & 0 & 1 \end{pmatrix} + h_3(y) \begin{pmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \right] y \\ &\triangleq (h_1(y)E_1 + h_2(y)E_2 + h_3(y)E_3)\, y. \end{aligned}$$

It is easy to find a matrix P such that

$$PE_iP^{-1} = M_i \;, \; 1 \leq i \leq 3.$$

In fact,

$$P = \begin{pmatrix} 3 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \; P^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Conditions (6.4) become

$$\begin{aligned} -\frac{1}{2}\frac{\partial h_1}{\partial y_2} &= \frac{1}{3}\frac{\partial h_2}{\partial y_1} \\ \frac{1}{3}\frac{\partial h_1}{\partial y_1} + \frac{1}{2}\frac{\partial h_1}{\partial y_2} + \frac{\partial h_1}{\partial y_3} &= \frac{1}{3}\frac{\partial h_3}{\partial y_1} \\ \frac{1}{3}\frac{\partial h_2}{\partial y_1} + \frac{1}{2}\frac{\partial h_2}{\partial y_2} + \frac{\partial h_2}{\partial y_3} &= -\frac{1}{2}\frac{\partial h_3}{\partial y_2}. \end{aligned}$$

These equations are satisfied by the functions

$$h_1(y) = 6y_1 - 2y_3 \ , \ h_2(y) = -12y_2 + 6y_3 \ , \ h_3(y) = 6y_3^2,$$

and substituting in $x = Py$ we get

$$\frac{\partial V}{\partial x_1} = 2x_1 \ , \ \frac{\partial V}{\partial x_2} = 6x_2 \ , \ \frac{\partial V}{\partial x_3} = 6x_3^2$$

from which we conclude that

$$V(x) = x_1^2 + 3x_2^2 + 2x_3^3$$

i.e.

$$W(y) = V(x) = V(Py) = (3y_1 - y_3)^2 + 2y_3^3 + 3(-2y_2 + y_3)^2$$

and so for the system

$$\dot{y} = \left[(6y_1 - 2y_3) \begin{pmatrix} 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & -1 & 1 \\ 0 & -2 & 1 \end{pmatrix} + (-12y_2 + 6y_3) \begin{pmatrix} -1 & 0 & \frac{2}{3} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ -3 & 0 & 1 \end{pmatrix} + 6y_3^2 \begin{pmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \right] y$$

i.e.

$$\begin{aligned} \dot{y}_1 &= 8y_1y_2 - 4y_1y_3 - (20/3)y_2y_3 + 4y_2y_3^2 + (10/3)y_3^2 - 2y_3^3 \\ \dot{y}_2 &= 12y_1y_2 - 3y_1y_3 - 9y_1y_3^2 - 4y_2y_3 + y_3^2 + 3y_3^3 \\ \dot{y}_3 &= 24y_1y_2 - 12y_1y_3 - 8y_2y_3 + 4y_3^2 \end{aligned}$$

the sets

$$\{y : (3y_1 - y_3)^2 + y_3^2 + (-4 + y_3)^2 = \text{const.}\}$$

and

$$\{y : (3y_1 - y_3)^2 + 2y_3^3 + 3(-2y_2 + y_3)^2 = \text{const.}\}$$

are invariant.

7. Compact Lie Algebras

In this section we shall study systems which define compact Lie algebras. First we recall the general theory of such algebras (see Helgason, 1962).

Definition 7.1 A real Lie algebra $\mathfrak{g}_0$ is **compact** if one can define a symmetric, negative-definite blinear form $B(X, Y)$ on it for which

$$B(\text{ad}\, A \cdot X, Y) + B(X, \text{ad}\, A \cdot Y) = 0 \ , \ \forall \ A \in \mathfrak{g}_0.$$

It can be shown that the Lie group of a compact Lie algebra is compact in the topological sense.

If $\mathfrak{g}$ is a complex Lie algebra, we let $\mathfrak{g}^{\mathbb{R}}$ denote the real Lie algebra obtained from the vector space $\mathfrak{g}$ by restricting to real scalars and with bracket which satisfies

$$[X, iY] = i[X, Y].$$

Note that, on $\mathfrak{g}^{\mathbb{R}}$, $J = i$ is an isomorphism for which $J^2 = i^2 = -I$. Such a map is called a complex structure. Then $\mathfrak{g}$ is said to have a **real form** $\mathfrak{g}_0$ if

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus J\mathfrak{g}_0$$

(vector space direct sum). (Of course, $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{R}}$ are identical as sets.) Every element $Z \in \mathfrak{g}$ can be written

$$Z = X + JY = X + iY \ , \ X, Y \in \mathfrak{g}_0,$$

and so $\mathfrak{g}$ is isomorphic to the complexification of $\mathfrak{g}_0$. The map

$$\sigma : X + iY \to X - iY \ , \ X, Y \in \mathfrak{g}_0$$

is called the **conjugation** of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. A direct decomposition

$$\mathfrak{g}_0 = \mathfrak{t}_0 + \mathfrak{p}_0$$

where $\mathfrak{t}_0$ is a subalgebra and $\mathfrak{p}_0$ is a vector subspace is called a **Cartan decomposition** if the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ has a compact real form $\mathfrak{g}_k$ such that

$$\sigma \cdot \mathfrak{g}_k \subseteq \mathfrak{g}_k \ , \ \mathfrak{t}_0 = \mathfrak{g}_0 \cap \mathfrak{g}_k \ , \ \mathfrak{p}_0 = \mathfrak{g}_0 \cap (i\mathfrak{g}_k).$$

Cartan decompositions are conjugate under inner automorphisms of $\mathfrak{g}_0$. The two main results we require concerning compact Lie algebras and Cartan decompositions are the following:

Lemma 7.2 *(i) A real Lie algebra is compact if and only if its Killing form is strictly negative definite (and so the Lie algebra is necessarily semisimple).*

(ii) Every compact Lie algebra $\mathfrak{g}$ *is a direct sum*

$$\mathfrak{g} = \mathfrak{z} + [\mathfrak{g}, \mathfrak{g}]$$

where $\mathfrak{z}$ *is the centre* $\mathfrak{g}$ *of and the ideal* $[\mathfrak{g}, \mathfrak{g}]$ *is compact (and semisimple).* □

Lemma 7.3 *Let* $\mathfrak{g}_0$ *be a real semisimple Lie algebra which is a direct sum* $\mathfrak{t}_0 + \mathfrak{p}_0$ *where* $\mathfrak{t}_0$ *is a subalgebra and* $\mathfrak{p}_0$ *is a vector subspace. Then the following statements are equivalent:*

(i) $\mathfrak{g}_0 = \mathfrak{t}_0 + \mathfrak{p}_0$ *is a Cartan decomposition of* $\mathfrak{g}_0$.

(ii) $B(T, T) < 0$, *for* $T \neq 0$ *in* $\mathfrak{t}_0$,

$B(X, X) < 0$, *for* $X \neq 0$ *in* $\mathfrak{p}_0$

and the mapping $\mathfrak{s}_0 : T + X \to T - X, T \in \mathfrak{t}_0, X \in \mathfrak{p}_0$ *is an automorphism.* □

We can obtain a simple stability result from the assumption of compactness of the system Lie algebra:

Lemma 7.4 *If the system*

$$\dot{x} = A(x)x \tag{7.1}$$

generates a compact Lie algebra L_A, *then it is a stable system. Morever, the system*

$$\dot{y} = -\alpha y + \bar{A}(y)y \tag{7.2}$$

is asymptotically stable, if again $\bar{A}$ *generates a compact Lie algebra* $L_{\bar{A}}$.

Proof The Lie group G_A generated by L_A is compact and the solution of (7.1) is of the form

$$x(t) = e^{A(t;x_0)}x_0$$

by theorem 3.1, where $e^{A(t;x_0)} \in G_A$ (and $A(t;x_0) \in L_A$). Since G_A is compact,

$$\|x(t)\| \le K\,\|x_0\|$$

for some K independent of x_0, and hence we have stability.

In (7.2), put

$$z = e^{\alpha t} y.$$

Then,

$$\dot{z} = \bar{A}(e^{-\alpha t} z) z$$

and $\bar{A}$ generates a compact Lie algebra, so the system is stable, by the first part. Hence, $y = e^{-\alpha t} z$ is asymptotically stable. □

We shall need the following simple result:

Lemma 7.5 *Let A and B be two square matrices and let $K = \operatorname{Ker} B$. Suppose that $K_1 \subseteq K$ is the largest invariant subspace of K under A, i.e. $AK_1 \subseteq K_1$. Then*

$$\exp(A+B)x = (\exp A)x$$

for all $x \in K_1$.

Proof Use the power series expansion of $\exp(A+B)$ and note that for any term in the expansion containing at least one B must be zero when operating on $x \in K_1$ since K_1 is the invariant under A. □

We can now state the main result of this section.

Theorem 7.6 *Consider the nonlinear differential equation*

$$\dot{x} = A(x)x \tag{7.3}$$

and suppose that $\{A(x)\}$ generates a semisimple Lie algebra L_A which has a Cartan decomposition

$$L_A = \mathfrak{k}_0 + \mathfrak{p}_0. \tag{7.4}$$

Let $\operatorname{Ker} \mathfrak{p}_0$ denote the set

$$\operatorname{Ker} \mathfrak{p}_0 = \cap\{\operatorname{Ker} B : B \in \mathfrak{p}_0\}$$

and let K be the largest invariant subspace of $\operatorname{Ker} \mathfrak{p}_0$ under $\mathfrak{k}_0$, i.e.

$$AK \subseteq K$$

for all $A \in \mathfrak{k}_0$. Then the solutions of (7.3) are stable on K and we can choose coordinates so that the system can be written in the form

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} A_1(y) & A_2(y) \\ 0 & A_3(y) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{7.5}$$

where $\{A_1(y)\}$ generates a compact Lie algebra and $\begin{pmatrix} y_1 \\ 0 \end{pmatrix}$ parameterises K.

Proof By theorem 3.3 and the decomposition (7.4) we may write the solution in the form

$$x(t) = \left(e^{A_1(t;x_0)+A_2(t;x_0)}\right) x_0$$

where $A_1 \in \mathfrak{t}_0$ and $A_2 \in \mathfrak{p}_0$. If $x_0 \in K$ then, by lemma 7.5, we have

$$x(t) = e^{A_1(t;x_0)}x_0$$

and so stability follows from lemma 7.4. The decomposition (7.5) now follows by standard linear algebra. □

Remark We can obtain a similar result if L_A is not semisimple, by writing $L_A = \mathfrak{g} + \mathfrak{s}$ where $\mathfrak{g}$ is semisimple and $\mathfrak{s}$ is solvable. If $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{t}_0 + \mathfrak{p}_0$ then we replace $\mathfrak{p}_0$ by $\mathfrak{p}_0 + \mathfrak{s}$ in the theorem. □

Examples 7.7 (1) For the system in example 6.5, $\mathfrak{p}_0 = \{0\}$ and the space K is $\mathbb{R}^3$ so A_2, A_3 are zero in this case.

(2) Consider the system

$$\begin{aligned}
\dot{x} &= x_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} x + x_1^2 x_4 \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} x + x_2^3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} x + \\
&\quad x_2 x_5 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix} + x_1 x_3 x_4 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} x \\
&= (x_1 A_{10000} + x_1^2 x_4 A_{20010} + x_2^3 A_{03000} + x_2 x_5 A_{01001} + x_1 x_3 x_4 A_{10110})x \text{ , say.} \qquad (7.6)
\end{aligned}$$

Then

$$L_A = L\{A_{10000}, A_{20010}, A_{03000}, A_{01001}, A_{10110}\}$$

which has the basis

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Killing form is

$$K(\xi, \eta) = -6\xi_1\eta_1 - 6\xi_2\eta_2 - 6\xi_6\eta_6 - 56\xi_{10}\eta_{10} + 14\xi_4\eta_4 + 14\xi_5\eta_5$$

as can easily be checked using, for example, Maple. This is degenerate so we shall use the remark after theorem 7.6. We note that, by examining the structure constants, we have a compact Lie subalgebra generated by the basis elements

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which is isomorphic to the Lie algebra $\{M_1, M_2, M_3\}$ in example 2.2. By a change of coordinates we can write these matrices in the form

$$\begin{pmatrix} 0 & 1 & 0 & \\ -1 & 0 & 0 & * \\ 0 & 0 & 0 & \\ & * & & * \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & \\ 0 & 0 & 0 & * \\ -1 & 0 & 0 & \\ & * & & * \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \\ 0 & 0 & 1 & * \\ 0 & -1 & 0 & \\ & * & & * \end{pmatrix}.$$

Such a map is given by $y = P^{-1}x$ where

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The system (7.6) becomes

$$\begin{aligned} \dot{y} &= x_1 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} x + x_1^2 x_4 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} x + x_2^3 \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} x + \\ & x_2 x_5 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + x_1 x_3 x_4 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} y \\ &= \begin{pmatrix} 0 & y_3 & 0 & (y_1+y_3)^3 & 0 \\ -y_3 & 0 & y_3^2 y_4 & 0 & (y_1+y_3)(y_1+y_5) \\ 0 & -y_3^2 y_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(y_1+y_3)(y_1+y_5) & y_2 y_3 y_4 \\ 0 & 0 & 0 & y_2 y_3 y_4 & (y_1+y_3)(y_1+y_5) \end{pmatrix} y \end{aligned}$$

and so the sphere $y_1^2 + y_2^2 + y_3^2 = \text{const}$, $y_4 = y_5 = 0$ is invariant under the dynamics of this equation. Hence the sets

$$\{(x_1, x_2, x_3, x_4, x_5) : 2x_1^2 - 2x_1x_2 + x_2^2 + x_3^2 = \text{const} \ , \ x_4 = 0, x_1 - x_2 + x_5 = 0\}$$

are invariant for the system (7.6), i.e. the system

$$\begin{aligned}
\dot{x}_1 &= -x_1^2 x_3 x_4 \\
\dot{x}_2 &= x_1 x_3 - x_1^2 x_3 x_4 + x_2^3 x_4 \\
\dot{x}_3 &= x_1^2 - x_1 x_2 + x_1^3 x_4 + x_1 x_2 x_5 - x_2^2 x_5 + x_2 x_5^2 \\
\dot{x}_4 &= -x_2 x_4 x_5 + x_1^2 x_3 x_4 - x_1 x_2 x_3 x_4 + x_1 x_3 x_4 x_5 \\
\dot{x}_5 &= x_1 x_3 + x_2^3 x_4 + x_1 x_2 x_5 - x_2^2 x_5 + x_2 x_5^2 + x_1 x_3 x_4^2
\end{aligned}$$

It is therefore seen that studying the Lie algebra generated by a differential equation and looking for Cartan decompositions of this algebra can give some insight into the invariant subspaces of the system dynamics.

8. Modal Control

In the classical theory of control, an effective control design method for linear systems of the form

$$\dot{x} = Ax + bu$$

is to diagonalise A (or reduce it to Jordan form if it is not diagonalisable) by changing coordinates

$$y = P^{-1}x$$

and

$$\dot{y} = \Lambda + (P^{-1}b)u$$

where

$$\Lambda = P^{-1}AP.$$

We then choose the control u in a simple way to stabilise the system (assuming this is possible). In this section we shall generalise this approach to nonlinear systems of the form

$$\dot{x} = A(x)x + b(x)u \tag{8.1}$$

by introducing the Lie algebra $\mathcal{L}_{\{A\}}$ of the system as in the previous sections. Let

$$\mathcal{L}_{\{A(x)\}} = \mathfrak{s} + \mathfrak{g}$$

be a Levi decomposition of $\mathcal{L}_{\{A\}}$ into a solvable part ($\mathfrak{s}$) and a semisimple part ($\mathfrak{g}$); this is not a direct sum so the decomposition is not unique. Next let

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Sigma} \mathfrak{g}^{\alpha}$$

be a Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{h}$ is a Cartan subalgebra and the root spaces $\mathfrak{g}^{\alpha}$ are one dimensional (Σ is the set of nonzero roots). Then we can write the system (8.1) in the form

$$\dot{x} = S(x)x + H(x)x + \sum_{\alpha\in\Sigma} e_{\alpha}(x)E_{\alpha}x + b(x)u \tag{8.2}$$

where $S(x)$ is upper triangularisable and $H(x)$ is diagonalisable (simultaneously, of course, independently of x). Let $y = P^{-1}x$ be a change of coordinates which diagonalises the $H(x)$ matrices, i.e.

$$\begin{aligned} P^{-1}H(x)P &= \Lambda(x) \\ &= \operatorname{diag}(\lambda_1(x), \cdots, \lambda_n(x)). \end{aligned}$$

Then (8.2) becomes

$$\dot{y} = \Lambda(Py)y + R(y)y + P^{-1}b(Py)u \tag{8.3}$$

where

$$R(y) = P^{-1}\left(S(Py) + \sum_{\alpha\in\Sigma} e_\alpha(Py)E_\alpha\right)P.$$

Suppose that we can find a feedback control $u = u(y)$ such that

$$\sum_{i=1}^{n} \lambda_i(Py)y_i^2 + y^T P^{-1}b(Py)u(y) \le -\mu\,\|y\|^2 \tag{8.4}$$

for some $\mu > 0$. From (8.3) we have

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 = y^T\dot{y} = -\mu\,\|y\|^2 + y^T R(y)y$$

so

$$\begin{aligned} \|y\|^2 &= e^{-2\mu t}\|y_0\|^2 + \int_0^t 2e^{-2\mu(t-s)}y^T R(y)y\,ds \\ &\le e^{-2\mu t}\|y_0\|^2 + \int_0^t 2e^{-2\mu(t-s)}\|y(s)\|^2\,\|R(y)\|\,ds. \end{aligned}$$

Assume that

$$\|R(y)\| \le \lambda$$

for $y \in B_{0,\Delta} = \{x : \|x\| \le \Delta\}$, and so, by Gronwall's lemma we have

$$\|y\|^2 \le e^{-2(\mu-\lambda)t}\|y_0\|^2$$

and we have stability if $\lambda < \mu$. The original system (8.2) will be stable in the set

$$\{x : x^T(P^T)^{-1}P^{-1}x \le \Delta\}.$$

Hence we have proved
Theorem 8.1 *Consider the control system*

$$\dot{x} = A(x)x + b(x)u$$

and suppose that $\mathcal{L}_{\{A\}} = \mathfrak{s} + \mathfrak{g} = \mathfrak{s} + \mathfrak{h} + \sum_{\alpha\in\Sigma}\mathfrak{g}^\alpha$. *Then we can write*

$$\dot{x} = H(x)x + R(x)x + b(x)u$$

where $H(x) \in \mathfrak{h}$, $R(x) \in \mathfrak{s} + \sum_{\alpha \in \Sigma} \mathfrak{g}^\alpha$. Suppose that P is a nonsingular matrix which diagonalises $H(x)$ (independently of x) and that the pair $(P^{-1}H(x)P, P^{-1}b(x))$ is exponentially stabilisable in the sense that (8.4) holds for some $\mu > 0$. Moreover, if

$$\left\| P^{-1}R(Py)P \right\| \leq \lambda$$

for $y \in B_{0,\Delta}$ and $\lambda < \mu$, then the system is exponentially stabilisable in $\{x : x^T(P^T)^{-1}P^{-1}x \leq \Delta\}$ and

$$\|x\|^2 \leq \rho^2 e^{-2(\mu-\lambda)t} \|P\|^2 \|x_0\|^2$$

where $\|Px\|^2 \geq \rho^2 \|x\|^2$. □

Example 8.2 Consider the system

$$\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} &= \begin{pmatrix} 3 - x_1^2 & -4 + x_2^2 & x_2^2 \\ x_2^2 & -1 - x_1^2 & x_2^2 \\ -5 - x_2^2 & 4 - x_2^2 & -2 - x_1^2 - x_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + (1 + x_1^2 \sin^2 x_3) \begin{pmatrix} 1/3 \\ -1/6 \\ -2/3 \end{pmatrix} u \\
&= \begin{pmatrix} 3 - x_1^2 & -4 & 0 \\ 0 & -1 - x_1^2 & 0 \\ -5 & 4 & -2 - x_1^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + x_2^2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&\quad + (1 + x_1^2 \sin^2 x_3) \begin{pmatrix} 1/3 \\ -1/6 \\ -2/3 \end{pmatrix} u
\end{aligned}$$

The matrix

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix}$$

diagonalises the first matrix in the right hand side of the equation, so that, if $y = P^{-1}x$ we have

$$\dot{y} = \begin{pmatrix} -1 - x_1^2 & 0 & 0 \\ 0 & -2 - x_1^2 & 0 \\ 0 & 0 & 3 - x_1^2 \end{pmatrix} y + x_2^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} y + (1 + x_1^2 \sin^2 x_3) P^{-1} \begin{pmatrix} 1/3 \\ -1/6 \\ -2/3 \end{pmatrix} u.$$

The Lie algebra of this system is $\mathcal{L}_{\{A\}} = \mathfrak{s} + \mathfrak{g}$ where $\mathfrak{g} = A_2$ and $\mathfrak{s}$ is generated by $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} x_1^2$.

However, in this case it is better to combine this part with the Cartan subalgebra generated by $\begin{pmatrix} -1 - x_1^2 & 0 & 0 \\ 0 & -2 - x_1^2 & 0 \\ 0 & 0 & 3 + 2x_1^2 \end{pmatrix}$ since the combined system is stable everywhere. If we choose the control

$$\begin{aligned}
u &= \frac{1}{1 + x_1^2 \sin^2 x_3} \begin{pmatrix} -12 & 12 & 0 \end{pmatrix} x \\
&= \frac{12}{1 + x_1^2 \sin^2 x_3} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} Py
\end{aligned}$$

then we obtain the equation

$$\dot{y} = \begin{pmatrix} -1-x_1^2 & 0 & 0 \\ 0 & -2-x_1^2 & 0 \\ 0 & 0 & -3-x_1^2 \end{pmatrix} y + x_2^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} y.$$

In this case we have $\mu = 1$ and

$$\left\| x_2^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\| = x_2^2$$

and we have stability for x satisfying $\|x\|^2 \le \delta < 1$ (for some δ). Of course, we can do better by choosing a control to move the poles of the Cartan subalgebra part further to the left.

9. Conculsions

In this paper we have studied in some detail the Lie algebra generated by a differential equation, which was introduced in an earlier paper. It was first shown that the Lie algebra generates a finite-dimensional Lie group which acts as a transformation group on the state space, giving the solutions of the system as a continuous curve in the Lie group operating on the initial value. This showed that the solutions carry the properties of the Lie group, generalising the linear theory. Next the nonuniqueness of the Lie algebra was discussed and conditions for different Lie algebras to generate the same system were derived. The stability theory of solvable systems was given in the following section, using the fact that the system could be triangularised and then applying the convergence theory for an approximating system of triangular time-varying systems which can be solved explicitly. The Killing form and invariance were discussed and applied in the case of compact Lie algebras using the Cartan decomposition. This allowed us to isolate certain invariant subsets of the dynamics by splitting off the compact part of the Lie algebra and transforming to suitable coordinates. Finally, the whole theory of the classical root space decomposition of semisimple Lie algebras was shown to generalise directly the classical modal control theory to nonlinear systems.

It is hoped that this theory demonstrates that the Lie algebra associated with a differential equation in the above way has many applications to dynamical systems theory. Other aspects of the classical theory of Lie algebras and Lie groups should also have applications in this field. For example, the Iwasawa decomposition and the theory of representations of the Lie algebras and Lie groups invovled (see, for example, Helgason, 1962, Knapp and Vogan, 1995). These issues will be examined in a future paper.

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