COUPLED MAP LATTICE MODEL IDENTIFICATION OF STOCHASTIC DISTRIBUTED PARAMETER SYSTEMS

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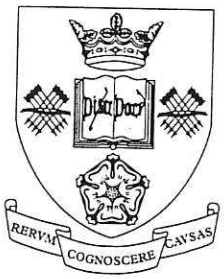

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Coupled Map Lattice Model Identification of Stochastic Distributed Parameter Systems

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Abstract

The identification of Coupled Map Lattice models of linear and nonlinear distributed parameter systems from discrete noisy observations is considered. In the first part of the paper the stochastic CML model is introduced together with some basic tools, the Frobenius-Perron and the equivalent transfer operator, which are used to describe the evolution of densities under the action of the CML transformation. A more general form of the transfer operator that accounts for external stochastic perturbations, which are not necessarily additive or multiplicative, is derived. The identification of the lattice equations which make up the CML model, in the presence of noise, is addressed and some particular implementation issues are discussed. A new identification algorithm for stochastic CML's is introduced and tested using simulated data.

1 Introduction

Nonlinear system identification has generally been associated with the identification of lumped parameter nonlinear dynamical systems. However, many dynamical systems, generically known as Distributed Parameter Systems (DPS), are spatially-extended, infinite-dimensional systems which usually can be described in terms of Partial Differential Equations (PDE's).

Until recently [13] the identification of such systems consisted of estimating just the unknown parameters associated with a structurally known PDE [1], [16], [26]. But the assumption that the form of the model equations is known a priori is in many practical applications unrealistic. The identification of both the structure and parameters in a model of a DPS system is therefore essential if the evolution of many practical spatially extended dynamical systems are to be successfully mapped to a mathematical model.

The Coupled Map Lattice model provides a simple and efficient representation of a wide range of spatially extended systems [21], [22], [23], [14] including myelinated nerve fibers, flows, coupled arrays of chemical reactors and Josephson junctions [38], [17], [9], [12], which usually are described in terms of Partial Differential Equations (PDE's) or as coupled arrays of Ordinary Differential Equations (ODEs).

The identification of Coupled Map Lattice models of deterministic distributed parameter systems directly from data was addressed in a previous paper [5]. Because of the regularity of the lattice model, the identification of the CML can be reduced to the identification of the CML input/output equation corresponding to an arbitrary lattice site. The identification is based on input/output measurements at that particular location as well as input/output

measurements at neighbouring lattice sites. The method can be considered as an extension of the system identification algorithms based on the NARX (Nonlinear AutoRegressive with eXogenous inputs) model for lumped, deterministic nonlinear systems [28].

In the deterministic case all the measurements (inputs and outputs) taken from neighbouring lattice locations can be treated as inputs during identification. This approach however does not work when the data is contaminated by noise. In this paper the identification of stochastic CML's is considered and a new identification algorithm which can accomodate the stochastic variables in the model, is introduced.

The statistical properties of stochastic CML's have been investigated in a number of papers by Losson and Mackey [31], [32], [30]. In the present study a rigorous definition of such systems is introduced in Section 2 together with some basic concepts and tools for the study of such systems, in particular the Frobenius-Perron operator and the analogue transfer operators. These special operators, which govern the evolution of density functions rather than trajectories of a dynamical system, have been studied extensively in the context of lumped dynamical systems [27], [10]. When the dynamical system has no inputs the operator is known as the Frobenius-Perron operator, and this can be used to describe the evolution of densities under an autonomous stochastic CML transformation. This is illustrated in in Section 2.1. A generalisation of this description to systems with lags greater than one is then derived in Section 2.2.

The equivalent operators, derived for stochastically perturbed dynamical systems, are commonly referred to as transfer operators. Exogenous stochastic perturbations are considered in Section 2.3 where a more general form of the transfer operator that describes the evolution of densities for noise perturbed CML transformations, is derived.

The second part of the paper, Section 3, concerns the identification of stochastic CML's from noisy data. The noise is treated as an unobserved stochastic perturbation acting on the system as an additional input variable that has to be estimated during identification. Practical issues regarding the identification of the corresponding lattice equations from noisy data are discussed in Section 3.2. The new identification algorithm, which addresses the particular problems related to the identification of spatially and temporally discrete stochastic systems, is introduced in Section (3.3). Numerical simulations, which demonstrate the applicability of the new identification algorithm, are presented in Section (4).

2 Stochastic CML models

A deterministic CML model [22, 21] is a discrete-time Lattice Dynamical System (LDS), with invariant lattice equations

$$\begin{aligned} x_i(t) &= f_l\left(\mathrm{q}^{n_x} x_i(t), \mathrm{q}^{n_u} u_i(t)\right) + \\ &\quad f_c\left(\mathrm{q}^{n_x} x_i(t), \mathrm{q}^{n_u} u_i(t), \mathrm{s}^m\, \mathrm{q}^{n_x} x_i(t), \mathrm{s}^m \mathrm{q}^{n_u} u_i(t)\right) \end{aligned} \tag{1}$$

where $x_i(t) \in \mathcal{X}_i \subset \mathbb{R}^l$ and $u_i(t) \in \mathcal{U}_i \subset \mathbb{R}^l$ are l-dimensional vectors representing the local state-space and input variables respectively at the ith node of the finite or infinite, d-dimensional lattice of integers $\mathcal{I} \subset \mathbb{Z}^d$ and $f_l, f_c : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}_i$ are piecewise differentiable maps. The CML

model (1) can also be written, in terms of the global state and input variables $x = \{x_i\}_{i\in\mathcal{I}} \in \mathcal{X}$ and $u = \{u_i\}_{i\in\mathcal{I}} \in \mathcal{U}$, as follows

$$x(t) = f(\mathrm{q}^{n_x}x(t), \mathrm{q}^{n_u}u(t)) \tag{2}$$

where $f : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$ is the multivalued function $f = \{f_i\}_{i\in\mathcal{I}}$ with $f_i = f_l + f_c$ and $i = \{i_1, ..., i_d\} \in \mathcal{I}$. In general, equations (2) correspond to an n-dimensional discrete-time dynamical system with $x \in \mathbb{R}^n$, although in principle (2) can be infinite-dimensional.

In equation (1) q is a backward shift operator such that

$$\mathrm{q}^{n_x}x_i(t) = (x_i(t-1) \dots\ x_i(t-n_x)) \tag{3}$$
$$\mathrm{q}^{n_u}u_i(t) = (u_i(t-1) \dots\ u_i(t-n_u)) \tag{4}$$

with n_x and n_u representing the maximum time-lags of x and u and s is a multi-valued spatial shift (translation) operator, invariant over the lattice $\mathcal{I}$

$$\mathrm{s}^m = (s(p_1)\, s(p_2) \dots\ s(p_m)) \tag{5}$$

such that

$$\mathrm{s}^m x_i = (x_{i-p_1} \dots\ x_{i-p_m}) \tag{6}$$

where $\mathrm{p} = \{p_j\}$ with $p_j = (p_j(1) \dots p_j(d)) \in \mathbb{Z}^d$, is a spatial translation index.

In addition, the CML lattice equations (1) are assumed to be symmetrically coupled with neighbouring lattice sites within a neighbourhood N_r of finite radius

$$r = (r(1) \dots r(d)) = (\max_{1\le j\le m} p_j(1) \dots \max_{1\le j\le m} p_j(d)) \tag{7}$$

such that

$$p_j = -p_{m-j+1}, \qquad m = 2m' \quad j = 1, ..., m' \tag{8}$$

and the coupling function f_c is symmetric or anti-symmetric.

If the initial state vector $x_0 = x(0)$ or the input vector u are assumed to be random variables it follows that x is also a vector of random variables and equations (1) define a stochastic CML. Let $x_0 \in \mathcal{X}$ and $u \in \mathcal{U}$ be stochastic variables with probability density functions $\varphi(x)$ and $\psi(u)$ respectively. The result of any nonlinear transformation $y = f(x, u)$ is also a stochastic variable with a probability density function which will depend on both $\varphi(x)$, $\psi(u)$ and the nonlinear transformation f. In this case the study of the evolution of trajectories, usually associated with deterministic dynamical systems, can be replaced with the study of the evolution of densities of the iterates using the Frobenius-Perron operator and related transfer operators introduced in the following sections.

2.1 The Evolution of Densities under an Autonomous CML Transformation

The effect of a nonlinear transformation $f(\cdot)$ on a probability density function can be described using the *Frobenius-Perron* operator [27], [31] associated with the transformation. Consider initially a CML with no inputs with $n_x = 1$ written in the global form (2)

$$x(t) = f(x(t-1)) \tag{9}$$

where $f = \{f_i\}_{i\in\mathcal{I}}$ and $f_i = f_l + f_c$ is invariant over the lattice $\mathcal{I}$. Let the state vector $x = \{x_i\}_{i\in\mathcal{I}} \in A \subset \mathcal{X}$ be a vector of random variables characterised by a probability density function φ. It follows that

$$\text{Prob}\{x \in A\} = \int_A \varphi d\mu \tag{10}$$

where μ is the normalised (probability) measure on A. In this case, the CML system (9) operates on a density function as an initial condition rather than a single point in A.

The probability density function $\varphi_t(x)$ for the state variable $x(t) \in A$, given a nonsingular transformation $f : A \to A$, can be expressed in terms of the probability density function $\varphi_{t-1}(x)$ of $x(t-1) \in A$

$$\int_A \varphi_t(x)d\mu = \int_{f^{-1}(A)} \varphi_{t-1}(x)d\mu \tag{11}$$

where $f^{-1}(A)$ is the counterimage of A under the transformation f.

The *Frobenius-Perron* operator, which describes the evolution of the probability density functions under the CML transformation $f(\cdot)$ as

$$\varphi_t(x) = P_f\varphi_{t-1}(x) \tag{12}$$

is defined as the unique operator $P_f : L^1 \to L^1$ such that $P_f\varphi_t(t)$ is the unique function in L^1 that satisfies the equation

$$\int_A P_f\varphi_{t-1}(x)d\mu = \int_{f^{-1}A} \varphi_{t-1}(x)d\mu \tag{13}$$

The Frobenius-Perron operator can be defined explicitly when f is a piecewise monotone transformation,

$$\varphi_t(x) = P_f\varphi_{t-1}(x) = \sum_{j=1}^{M} \frac{\varphi_{t-1}(f_{\Gamma_j}^{-1}(x))}{|(f_j^{-1}(x))'|}\chi_{f(\Gamma_j)}(x) \tag{14}$$

where f_{Γ_j} is the monotone restriction of f on the subset $\Gamma_j \subset A$, $j = 1, ..., M$ with

$$\bigcup_{j=1}^{M} \Gamma_j = A. \tag{15}$$

In equation (14) $f(\Gamma_j)$ is the image of Γ_j under the transformation f, $\chi_{f(\Gamma_j)}$ is the characteristic function over $f(\Gamma_j)$ and $f_{\Gamma_j}^{-1}(x)'$ is the Jacobian of $f_{\Gamma_j}^{-1}$ evaluated at x.

An important problem concerning the Frobenius-Perron operator induced by a nonsingular transformation is to establish the asymptotic properties of the sequence $P^n\varphi_0(x)$ of iterates of the initial density $\varphi_0(x)$. Stability concepts, similar to those established in the theory of deterministic dynamical systems to characterise the asymptotic behaviour of trajectories, are used to characterise the asymptotic properties of density function iterates. The sequence of density functions could converge to an equilibrium density or to a periodic sequence of densities. The stationary and periodic densities associated with a Frobenius-Perron operator are obvious analogues of the fixed and periodic points defined for deterministic dynamical systems.

The asymptotic properties of the iterates can be investigated using the spectral decomposition of the Frobenius-Perron operator. A stationary (invariant) density function will be an eigenfunction of the Frobenius-Perron operator with eigenvalue 1. An important theoretical result is the theorem of Ionescu-Tulcea and Marinescu (1950), which provides sufficient conditions for an operator to have a finite number of eigenvalues of modulus 1. This result, applied to the operator P_f^k for $k \geq 1$, can be used to establish whether P_f is asymptotically periodic, that is the sequence $P_f^n\varphi_0(x)$ converges to a stable period-k cycle. For $k = 1$ the operator P_f is asymptotically stable and the sequence of density function iterates will converge to an invariant density function. Based on this theorem, some practical conditions that guarantee asymptotic stability or periodicity of the associated Frobenius-Perron operator, have been derived for CML parameters [31].

2.2 Extension to the case $n_x > 1$

In general, the states $x(t) \in A$ of an autonomous CML model at every time instant $t \in \mathbb{Z}_+$, depend on the value of the state vector at previous time instants $x(t-1)$, $x(t-2)$, ..., $x(t-n_x)$ with $n_x > 1$. In turn, the probability density function $\varphi_t(x)$ of $x(t)$ depends on the probability densities of x at previous times φ_{t-1}, φ_{t-2}, ..., φ_{t-n_x}. More precisely, $\varphi_t(x)$ depends on the joint density function $\varphi^*_{t-1}(x^*)$ of the random vector

$$x^*(t-1) = (x(t-1),\, x(t-2),\, ...,\, x(t-n_x)). \tag{16}$$

The evolution of the distribution function $\varphi^*(x^*)$ can be described using the Frobenius-Perron operator associated with the transformation f^*

$$x^*(t) = f^*(x^*(t-1)) \tag{17}$$

defined as

$$f^*(x^*(t)) = \begin{cases} f^{(1)}(x^*(t-1)) \\ f^{(2)}(x^*(t-1)) \\ \vdots \\ f^{(n_x)}(x^*(t-1)) \end{cases} \tag{18}$$

where $f^{(i)}(x(t-1))$, $1 \leq i \leq n_x$ is the ith iterate of the random vector $x^*(t-1)$

$$f^{(i)}(x^*(t-1)) = f(f^{(i-1)}, ..., f^{(1)}, x(t-1), ..., x(t+i-n_x-1)) \tag{19}$$

In equation (17) f^* maps the vector $x^*(t-1)$ given in (16) into $x^*(t) = (x(t+n_x), ..., x(t+1))$.

The joint density function $\varphi_t^*(x^*)$ can be used to calculate the density for any particular variable of the random vector $x^*(t) \in A^{n_x}$ by integrating φ_t^* over the remaining variables [27]. For example, the density function of $\varphi_{t+1}(x)$ is given by

$$\varphi_{t+1}(x) = \underbrace{\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n}}_{n_x-1} \varphi_t^*(x(t+n_x), ..., x(t+1))dx(t+n_x)...dx(t+2) \tag{20}$$

where $x(t) \in A \subset \mathbb{R}^n$.

It is interesting to note that the Frobenius-Perron operator can in principle be applied even if the vector of initial conditions $x(0)$ is fixed rather than random. In this case, which corresponds to a deterministic system, the density function associated with $x(0)$ takes the form of a δ-Dirac distribution supported on a single point which is defined as

$$\delta_{x_0}(x) = \begin{cases} 0 & \text{for } x \neq x_0 \\ \infty & \text{for } x = x_0 \end{cases} \tag{21}$$

where $x^*(0) = x_0 \in A$ is a vector of known initial conditions.

Example

To illustrate this consider a simple example. Let $\mathcal{X} = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be the linear transformation

$$x(t) = f(x(t-1)) = 0.5x(t-1) \tag{22}$$

If $x_0 = x(0) = 2$ is the initial condition, the distribution function $\varphi_0(x)$ is the δ-Dirac distribution function supported on $x = 2$. The Frobenius-Perron operator is given in this case by

$$\varphi_t(x) = P_f\varphi_{t-1}(x) = \frac{\varphi_{t-1}(f^{-1}(x))}{|(f^{-1}(x))'|} \tag{23}$$

The distribution function $\varphi_{t=1}(x)$ can be determined by applying operator P_f on the initial distribution function $\varphi_{t=0}(x) = \delta_2(x)$ hence

$$\varphi_{t=1}(x) = \frac{\varphi_{t=0}(f^{-1}(x))}{|(f^{-1}(x))'|} = \frac{\delta_2(2x)}{2} \tag{24}$$

It is easy to see that the function $\delta_2(2x)$ is in fact the δ-Dirac distribution function $\delta_1(x)$ supported on $x = 1$ so that (24) becomes

$$\varphi_{t=1}(x) = \delta_1(x) \tag{25}$$

which is equivalent to saying that $x(t) = 1$ for $t = 1$. The asymptotic behaviour of the trajectory of the system is also in agreement with the asymptotic behaviour of the density functions which

converges to $\delta_0(x)$. This is the solution of the fixed point equation

$$\delta_{x(t-1)}(x) = P_f \delta_{x(t-1)}(x) \tag{26}$$

This shows that the trajectory of a dynamical system can be generated by iterating the Frobenius-Perron operator associated with the transformation using a δ-Dirac distribution as initial density function.

2.3 The Evolution of Densities under a CML Transformation with Stochastic Perturbations

In addition to the uncertainty introduced by considering the evolution of a random distribution rather than a fixed set of initial conditions, another source of uncertainty in any dynamical system is the existence of an independent stochastic perturbation that affects the system, such as a random variation of a coefficient in the CML for example. Stochastic perturbations are typical in most practical situations and can be viewed as an external input or excitation for the dynamical system considered. These type of dynamical systems have been investigated using densities by several authors [27], [10] including stochastic CML systems [32], [30].

Consider the stochastically perturbed CML system

$$x(t) = f(x(t-1), \xi(t-1)) \tag{27}$$

where $f : \mathcal{X} \times \Xi \rightarrow \mathcal{X}$, $\mathcal{X}, \Xi \subset \mathbb{R}^n$ is a measurable transformation not necessarily nonsingular and $\xi(t-1)$, $\xi(t-2)$,..., are independent random vectors with the same probability density function ψ.

As in the previous sections the task is to derive a relationship between $\varphi_{t-1}(x)$ and ψ the density functions of $x(t-1)$ and $\xi(t)$ and $\varphi_t(x)$, the density associated with $x(t)$. This relationship will take the form of a transfer operator analogous to the Frobenius Perron operator [27].

Transfer operators have been introduced for dynamical systems perturbed by additive and multiplicative stochastic perturbations [27], [10], [32]. In this section, more general stochastic perturbations than multiplicative or additive are considered. This leads to a more general form of the transfer operator associated with a perturbed CML transformation. The new transfer operator is derived following a similar technique to that employed by Lasota and Mackey (1994) for additive and multiplicative perturbations. This technique avoids the introduction of the concept of conditional probabilities required by other approaches.

Let $h : \mathcal{X} \rightarrow \mathbb{R}$ be an arbitrary, bounded measurable function and consider the mathematical expectation of $h(x(t)) = h(y)$

$$\mathcal{E}\{h(y)\} = \int_{\mathcal{X}} h(y)\varphi_t(y)dy \tag{28}$$

where

$$y(t) = f(x(t-1), \xi(t-1)) = x(t) \tag{29}$$

Since x and ξ are independent, the joint density of $(x(t-1), \xi(t-1))$ is $\varphi_{t-1}(x)\psi(\xi)$ so

$$\mathcal{E}\{h(y)\} = \mathcal{E}\{h(f(x(t-1), x(t-1))\} = \int_{\mathcal{X}} \int_{\Xi} h(f(x,\xi))\varphi_{t-1}(x)\psi(\xi)dxd\xi \tag{30}$$

Assume that $y = f(x,\xi)$ is piece-wise invertible with respect to the ξ variable such that

$$\frac{\partial f}{\partial \xi}(x,\xi) := \begin{pmatrix} \frac{\partial f_1}{\partial \xi_1}(x,\xi) & \cdots & \frac{\partial f_1}{\partial \xi_n}(x,\xi) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial \xi_1}(x,\xi) & \cdots & \frac{\partial f_n}{\partial \xi_n}(x,\xi) \end{pmatrix} \tag{31}$$

is nonsingular on every interval $\Gamma_i = \mathcal{X} \times \Gamma_i^{\xi}$ where

$$\bigcup_i \Gamma_i^{\xi} = \Xi \tag{32}$$

This is equivalent to $f(\cdot,\xi)$ being piece-wise monotone with respect to ξ.

It follows that for every $(x,\xi) \in \Gamma_i$ and $y \in \mathcal{X}$ such that $f(x,\xi) = y$ there is a function $\xi(x,y) = f_{\Gamma_i}^{-1}(x,y)$ such that

$$f(x, f_{\Gamma_i}^{-1}(x,y)) - y = 0 \tag{33}$$

Using (32), equation (30) can be written as

$$\mathcal{E}\{h(y(t))\} = \sum_{i=1}^{M_\xi} \int_{\mathcal{X}} \int_{\Gamma_i^{\xi}} h(f(x,\xi))\varphi_{t-1}(x)\psi(\xi)dxd\xi \tag{34}$$

where M_ξ is the total number of subintervals over which $f(\cdot,\xi)$ is monotone with respect to ξ. By a simple change of variable, equation (34) becomes

$$\mathcal{E}\{h(y(t))\} = \sum_{i=1}^{M_\xi} \int_{f(\Gamma_i)} \int_{\mathcal{X}} h(y)\varphi_{t-1}(x)\psi(f_{\Gamma_i}^{-1}(x,y))\frac{\partial f_{\Gamma_i}^{-1}(x,y)}{\partial y}dxdy \tag{35}$$

Assuming that

$$f(\Gamma_i) = \mathcal{X}, \qquad \forall\ \Gamma_i, \quad i = 1, M_\xi \tag{36}$$

it follows that

$$\begin{aligned} \mathcal{E}\{h(x(t))\} &= \sum_{i=1}^{M_\xi} \int_{f(\Gamma_i)} \int_{\mathcal{X}} h(y)\varphi_{t-1}(x)\psi(f_{\Gamma_i}^{-1}(x,y))\frac{\partial f_{\Gamma_i}^{-1}(x,y)}{\partial y}dxdy \\ &= \int_{\mathcal{X}} h(y) \sum_{i=1}^{M_\xi} \int_{\mathcal{X}} \varphi_{t-1}(x)\psi(f_{\Gamma_i}^{-1}(x,y))\frac{\partial f_{\Gamma_i}^{-1}(x,y)}{\partial y}dxdy \end{aligned} \tag{37}$$

Since h was chosen arbitrary, by equating (30) and (37) it follows that the density $\varphi_t(y) =$

$\varphi_t(x(t))$ is given by

$$\varphi_t(y) = \sum_{i=1}^{M_\xi} \int_{\mathcal{X}} \varphi_{t-1}(x)\psi(f_{\Gamma_i}^{-1}(x,y))\frac{\partial f_{\Gamma_i}^{-1}(x,y)}{\partial y}dx \tag{38}$$

Equation (38) defines an operator $P_\xi : L^1 \to L^1$

$$P_\xi\varphi(x) = \sum_{i=1}^{M_\xi} \int_{\mathcal{X}} \varphi(x)\psi(f_{\Gamma_i}^{-1}(x,y))\frac{\partial f_{\Gamma_i}^{-1}(x,y)}{\partial y}dx \tag{39}$$

where $\varphi \in L^1$. This is a more general form of the transfer operator which can describe the CML under the influence of more complex stochastic perturbations than additive or multiplicative.

The asymptotic behaviour of the sequence of density iterates can be studied using the spectral decomposition of the transfer operator. Asymptotic periodicity can be established based on the theorem of Ionescu-Tulcea and Marinescu or alternatively by invoking the constrictiveness of the operator [27], [10] which can be checked using a practical criterion reminiscent of the Lyapunov function method in the theory of of deterministic dynamical systems [27].

3 Identification of Stochastic CML

The identification of a CML model from real data normally involves dealing with stochastic perturbations, such as measurement noise, which leads to a stochastic CML.

3.1 The Stochastic input/output CML

Consider the following CML state-space model

$$\begin{aligned} x_i(t) &= f_l\left(\mathrm{q}^{n_x}x_i(t), \mathrm{q}^{n_u}u_i(t)\right) + \\ &\quad f_c\left(\mathrm{q}^{n_x}x_i(t), \mathrm{q}^{n_u}u_i(t), \mathrm{s}^m\,\mathrm{q}^{n_x}x_i(t), \mathrm{s}^m\mathrm{q}^{n_u}u_i(t)\right) \end{aligned} \tag{40}$$

where $i \in \mathcal{I} \subset \mathbb{Z}^d$, the operators q and s are those defined in Section (2). The state-space model is usually complemented with a measurement equation

$$y_i(t) = h_i(x(t), u(t)) \tag{41}$$

where the measurement function is assumed identical for each lattice site $h_i = h$ for $i \in \mathcal{I}$ such that the equivalent input/output equations

$$\begin{aligned} y_i(t) &= F_l\left(\mathrm{q}^{n_y}y_i(t), \mathrm{q}^{n_u}u_i(t)\right) + \\ &\quad F_c\left(\mathrm{q}^{n_y}y_i(t), \mathrm{q}^{n_u}u_i(t), \mathrm{s}^m\,\mathrm{q}^{n_y}y_i(t), \mathrm{s}^m\mathrm{q}^{n_u}u_i(t)\right) \end{aligned} \tag{42}$$

with $i \in \mathcal{I} \subset \mathbb{Z}^d$ is also a CML.

The stochastic perturbations due to unobserved noise can be modelled as additional input

variables for the input/output CML model (42). Since e cannot be measured, the unobserved noise sequence has to be estimated from the identification data using a prediction error approach. This however, requires knowing how the perturbation $e(t)$ acts on the system. Usually $e(t)$ is assumed additive so the resulting stochastic CML model can be written as

$$y(t) = f(\mathrm{q}^{n_y} y(t), \mathrm{q}^{n_u} u(t), \mathrm{q}^{n_e} e(t)) + e(t) \tag{43}$$

where u is the known, deterministic input vector, e represents the unobserved stochastic perturbation and y is the global output vector, which is also a stochastic variable. In equation (43) n_e is the noise lag. The sequence $e(t)$ is assumed to be independent, bounded and uncorrelated with the input u and is characterised by a probability density function $\psi(e)$. The lattice equations corresponding to the ith site of the stochastic CML model can then be written as

$$\begin{aligned} y_i(t) \;=\; & F_l(\mathrm{q}^{n_y} y_i(t), \mathrm{q}^{n_u} u_i(t), \mathrm{q}^{n_e} e_i(t)) + \\ & F_c(\mathrm{q}^{n_y} y_i(t), \mathrm{q}^{n_u} u_i(t), \mathrm{s}^m \mathrm{q}^{n_y} y_i(t), \mathrm{s}^m \mathrm{q}^{n_u} u_i(t), \mathrm{s}^m \mathrm{q}^{n_e} e_i(t)) + e_i(t) \end{aligned} \tag{44}$$

where e_i is the noise vector corresponding to the ith lattice site. Consequently, the prediction error can be calculated using the prediction error equations

$$\varepsilon(t) = y(t) - f(\mathrm{q}^{n_y} y(t), \mathrm{q}^{n_u} u(t), \mathrm{q}^{n_e} \varepsilon(t)) \tag{45}$$

Equation (45) defines a stochastic dynamical system with inputs u and y and outputs the prediction error $\varepsilon(t)$. The identification problem can be formulated in this context as the determination of the nonlinear mapping f such that $\varepsilon(t)$ converges to $e(t)$ as $t \to \infty$. This implies that the transfer operator associated to (45) should be asymptotically stable with ψ the stationary density function. Theoretically, knowing f would allow deriving the transfer operator associated with the transformation (45) to study the asymptotic properties of the iterates.

In practice however, an algorithm to determine f from noisy observation, is required first. This problem is addressed in the following sections.

3.2 Identification of the stochastic lattice equations from data

The identification of deterministic CML's equation has been addressed in a previous paper [13]. The approach exploits the invariance property of the lattice equation with respect to the lattice site, to identify the lattice equations from measurement data recorded just from a small number of spatial locations. If noisy observations are used instead, the algorithm has to be modified accordingly to perform the additional task of estimating the unobserved noise.

Assume that the identification data consists of input/output observations at the ith lattice site and the adjacent sites $\mathrm{s}^{m'} = (s(p'_1), s(p'_2), \ldots, s(p'_{m'}))$ within a neighbourhood $\mathcal{I}_r(i)$ of radius r such that $\{p_1, \ldots, p_m\} \subset \{p'_1, \ldots, p'_{m'}\}$ where $\mathrm{s}^m(i) = (s(p_1)\, s(p_2) \ldots s(p_m))$ is the translation (coupling) operator in (42). Consider the following proposition:

Proposition

If the input/output behaviour,

$$U = (u_i\,,\, u_{i-p'_1}\,,\, \dots\,,\, u_{i-p'_{m'}}),\;\; Y = (y_i\,,\, y_{i-p'_1}\,,\, \dots\,,\, y_{i-p'_{m'}}) \tag{46}$$

corresponding to the state-space model (1) with measurement function

$$y_j(t) = h(x_j(t)), \qquad j = i, i-p'_1, \dots, i-p'_{m'} \tag{47}$$

with $\{p_1\,,\, \dots\,,\, p_m\} \subset \{p'_1\,,\, \dots\,,\, p'_{m'}\}$ has the input/output realisation

$$Y(t) = F(\mathrm{q}^Y(t), \mathrm{q}^U U(t)) \tag{48}$$

where $F = \left(F_i\,,\, F_{i-p'_1} \dots, F_{i-p_{m'}}\right)$ is some $(m'+1)$-dimensional nonlinear function then

$$y_i(t) = F_i(\mathrm{q}^Y(t), \mathrm{q}^U U(t)) \tag{49}$$

$i \in \mathcal{I} \subset \mathbb{Z}^d$ defines the CML in (42).

Proof

If equations (48) exist then these can be derived from the full set of input/output equations (40) by suitable elimination of the output variables not included in $Y = (y_i\,,\, y_{i-p'_1}\,,\, \dots\,,\, y_{i-p'_{m'}})$. Because the ith input/output equation (42) involves only input and output variables in Y and U, it will not be affected by these transformations. Thus (49) is identical to the ith equation in (42).

In the deterministic case the lattice equation (42) was estimated from the input/output data by treating the outputs in the neighbourhood of the ith node as inputs. In the stochastic context however the noise corresponding to neighbouring lattice sites also enter the lattice equations (44). In order to estimate these unobserved stochastic perturbations, the lattice equations corresponding to the noise variables $s^m e_i$, have to be identified as well. Practically by identifying the input/output equations corresponding to

$$U^{\mathrm{s}^m} = (u_i, \mathrm{s}^m u_i),\;\; Y^{\mathrm{s}^m} = (y_i, \mathrm{s}^m y_i) \tag{50}$$

automatically gives (44). The discrete-time realisation of (50) is a (MIMO) NARMAX model [28].

Note that because the inputs cannot usually be eliminated in the same fashion as the outputs, the realisation (48) will involve in general inputs from all lattice sites. This however, does not affect the result stated in Proposition (3.2). In practice if only the input vector U is used in the identification, this will introduce some bias in the estimated residuals $\mathrm{s}^m e_i(t)$ and ultimately some bias in the parameter estimates corresponding to (44). Numerical simulations have shown however that the bias effects on parameter estimates of the final CML model are usually not significant. This can be avoided if the full input vector is used during identification. Whenever the input is invariant over the lattice or the system has no inputs this problem does not occur.

3.3 The identification algorithm

Let $u(t,x)$ and $y(t,x)$ be the input and output of a infinite dimensional system evolving over a compact spatial domain $\Omega \subset \mathbb{R}^d$. As in the previous paper [5], the form of the mathematical representation of the spatially extended dynamical process is assumed unknown. The only information available is a discrete set of input/output data measured from a finite set of sensors located in the spatial domain and the boundary conditions.

Let $\mathcal{I}$ denote the lattice of integers associated with the spatial domain, which corresponds to a uniform sampling of the input and output variables u and y along the spatial coordinate with a sampling step $dx \in \mathbb{R}^d$ such that

$$u_i(t) = u(t, idx) \quad \text{and} \quad y_i(t) = y(t, idx) \quad i \in \mathcal{I} \tag{51}$$

The measurement locations are taken as a subset $\mathcal{I}_r(i) \subset \mathcal{I}$ defined as $\mathcal{I}_r(i) = (i\,,\, i-p'_1\,,\, \ldots\,,\, i-p'_{m'})$, $p'_k \in \mathbb{Z}_+^d$ and $r \in \mathbb{Z}_+$ is a predefined neighbourhood radius. It follows that the identification data will consist of the input and output vectors $U^N = (U(Ndt), ..., U(0)), Y^N = (Y(Ndt), ..., Y(0))$ where

$$U = (u_i\,,\, u_{i-p'_1}\,,\, \ldots\,,\, u_{i-p'_{m'}}) \tag{52}$$

and

$$Y = (y_i\,,\, y_{i-p'_1}\,,\, \ldots\,,\, y_{i-p'_{m'}}) \tag{53}$$

and dt is the sampling time.

The identification task involves estimating a CML model of the dynamics, of the original distributed parameter system, projected on the uniform sampling grid defined over Ω. A particular case, involves the identification of CML models of systems which are inherently discrete in space, which can be addressed in a similar manner.

An essential part of the identification process, involves determining the form or structure of equation (44). This is known as model structure selection, which here has been implemented following a nonparametric regression approach.

Briefly, equations (44) are constructed as a linear expansion in terms of a finite number of known basis functions that are selected from a larger set of candidate regressors postulated a priori. Although the candidate regressors belong to some parametrised function class $\mathcal{F}$, the method is nonparametric because the number of expansion coefficients (parameters) is not known in advance.

A simple and convenient implementation of the regressor set is based on the polynomial expansion. In this case, the regressors belong to the polynomial basis $\mathcal{P}$. In theory $\mathcal{P}$ is infinite dimensional, thus in practice the infinite basis has to be truncated to $\mathcal{P}_l$ which contains polynomial terms up to lth order.

If $\mathcal{P}_l$ is a finite polynomial basis, it follows that equation (44) can be identified as a linear expansion of the polynomial terms

$$y_i(t) = \sum_k \theta_k^i g_k^i(t) + e_i(t) \tag{54}$$

where k is a multi-index and $g_k \in \mathcal{P}_l$.

If the unknown nonlinear function does not belong to the space spanned by the finite polynomial basis only an approximate representation will result. The approximation can be refined by increasing the number and/or the order of the polynomial terms selected in the model (54). If F is continuous, the Stone-Weierstrass density theorem [36] guarantees that in this way the approximation errors can be made as small as desired over any closed interval.

In practice, the model is implemented by selecting only a small set of relevant polynomial terms, hence the number of parameters in equation (54) is usually significantly smaller that the number of candidate regressors in $\mathcal{P}_l$, which sometimes can contain hundreds or thousands of candidate terms.

The selection of the model terms is performed using the the Orthogonal Forward Regression (OFR) algorithm. This least-squares based algorithm involves a stepwise orthogonalisation of the regressors and a forward selection of the relevant terms in (54) based on the Error Reduction Ratio criterion (ERR) [3]. The algorithm also provides the optimum least-squares estimate of the corresponding parameter vector $\Theta = \{\theta_k\}$.

Although the OFR algorithm is responsible for shaping the model without any prior knowledge of its structure, a number parameters are specified in advance in order to define the candidate regressor set. These are the maximum input, output and noise lags n_u, n_y and n_e and the neighbourhood radius r. A practical approach is to select relatively large values for these parameters and let the selection algorithm sort out the lags and the coupling operator s^m in the final model. Alternatively, a minimum time lag and neighbourhood radius can be selected first. The values are then increased in a stepwise manner until a valid CML model is identified.

An advantage of the OFR algorithm is that it allows the selection of the process terms, involving only input and output variables, to be decoupled from that of the noise terms. Because the noise is not known initially, the deterministic part of the model will be selected first and used to compute a noise estimate. The model can then be augmented with additional terms corresponding to the noise variables. The selection procedure can be terminated when the norm of the residuals is less than a given tolerance

The application of the algorithm in the identification of the stochastic CML model is similar to the identification of multivariable MIMO NARMAX models [3]. Note however that only the identified equation corresponding to y_i will be used to implement the CML model. The neighbouring lattice equations are only needed to estimate the noise variables $\mathrm{s}^m e_i(t)$. For this reason, the identification of the deterministic model structure corresponding to y_i is carried out first. This allows the structure of the coupling operator s^m to be defined, which automatically reduces the number of relevant input/output variables to

$$U_m = (u_i\,,\; u_{i-p_1}\,,\; \cdots\,,\; u_{i-p_m}) \tag{55}$$

and

$$Y_m = (y_{i-p_1}\,,\; \cdots\,,\; y_{i-p_m}\,,\; y_i) \tag{56}$$

The complete identification algorithm can be summarised as follows

- **Step 1:** Select the input, output and noise lags n_u, n_y and n_e, the neighbourhood radius r and the polynomial order l
- **Step 2:** Implement the candidate regressor set using the input and output data U^N, Y^N
- **Step 3:** Use the OFR algorithm to select the process model F_i^p corresponding to y_i and indirectly determine the coupling operator $\mathrm{s}^m = (s(p_1), ..., s(p_m))$
- **Step 4:** For $j = i - p_1, ..., i - p_m$ use the OFR algorithm to select process terms in F_j^p
- **Step 5:** For $j = i, i - p_1, ..., i - p_m$ calculate the initial residuals

$$\varepsilon_j^{(0)}(t) = y_j(t) - F_j^p(t) \tag{57}$$

- **Step 6:** Set $k = 0$. For $j = i, i - p_1, ..., i - p_m$ use the OFR algorithm to augment F_j^p with noise model F_j^n. The combined model $F_j^{(k)}$ is used to estimate new residuals $\varepsilon_j^{(k)}$.
- **Step 7:** Set $k = k + 1$. Recalculate the parameters for each submodel: $\Theta_j^{(k-1)} \rightarrow \Theta_j^{(k)}$ $F_j^{(k-1)} \rightarrow F_j^{(k)}$ and estimate new residuals $\varepsilon_j^{(k)}$
- **Step 8:** Stop if the model parameters converge. Otherwise repeat **Step 7**

Note that the coupling symmetry in equations (44) is enforced in the selection stage by replacing the standard polynomial terms involving coupling variables with suitable symmetric combinations.

3.4 Model Validation

Model validation is an important part of any system identification approach, aimed to ensure that the final model provides a faithful representation of the original system.

If the structure and parameters of the lattice equations (44) have been correctly estimated the predicted noise sequences $e_j(t)$, $j = i, i - p_1, ..., i - p_m$ should be unpredictable that is uncorrelated with all linear and nonlinear combinations of past inputs and outputs.

This hypothesis can be tested for MIMO nonlinear systems, by computing the following correlation functions [3] derived under some mild assumptions [6],

$$\begin{array}{lllll}
\Phi_{\varepsilon_i \varepsilon_j}(\tau) & \tau \neq 0 & i = 1, ..., m+1 & j = i, ..., m+1 & \\
\Phi_{u_i \varepsilon_i}(\tau) & \forall \tau & i = 1, ..., m+1 & j = 1, ..., m+1 & \\
\Phi_{\varepsilon_i^2 (\varepsilon_j u_k)}(\tau) & \tau \geq 0 & i = 1, ..., m+1 & j = i, ..., l & k = 1, ..., m+1 \\
\Phi_{(u_i u_j), \varepsilon_k}(\tau) & \forall \tau & i = 1, ..., m+1 & j = 1, ..., m+1 & k = 1, ..., m+1 \\
\Phi_{(u_i u_j)(\varepsilon_k \varepsilon_l)}(\tau) & \forall \tau & i = 1, ..., m+1 & j = 1, ..., m+1 & k = 1, ..., m+1 \text{ and} \\
 & & l = 1, .., m+1 & &
\end{array} \tag{58}$$

If the above normalised correlations fall within the 95% confidence limits, which for large N are approximately $\pm 1.96/\sqrt{N}$, the model is regarded as adequate.

New correlation tests, which exploit the information in the system outputs, were introduced to enhance the efficiency of the correlation-based validation approach [8].

$$\begin{array}{llll} \Phi_{\varepsilon_i^2(y_j\varepsilon_j)}(\tau) & \tau \neq 0 & i = 1,...,l & j = i,...,l \\ \Phi_{u_i^2\eta_j}(\tau) & \forall\tau & i = 1,...,l & j = i,...,l \end{array} \tag{59}$$

The main problem is that for a MIMO system the number of correlation tests that have to be computed is often very large. For a system with a inputs and b outputs for example the number of individual correlation tests (59) is $ab + b^2 + 2$ and even larger for the first group of tests (58).

To overcome this drawback, a new hierarchical global-to-local validation strategy was introduced in Billings and Zhu (1995). The idea is to test initially the overall model validity using a set of global correlation tests based on a much smaller set of global variables derived from the original ones. For example the following global variables

$$\tilde{\varepsilon}(t) = \sum_{j=1}^{m+1} \varepsilon_j(t) \tag{60}$$

$$\tilde{u}(t) = \sum_{j=1}^{m+1} u_j(t) \tag{61}$$

$$\tilde{y}(t) = \sum_{j=1}^{m+1} y_j(t) \tag{62}$$

$$\tag{63}$$

can be used to compute the correlations (59) and check the global validity of the model. This is equivalent to testing a SISO nonlinear system hence the number of tests that have to be performed is quite small. Ideally if all the subsystems in the MIMO model are valid, the normalised correlations (59) will yield

$$\Phi_{\tilde{\varepsilon}^2(\tilde{y}\tilde{\varepsilon})}(\tau) = \begin{cases} k, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases} \tag{64}$$

$$\Phi_{\tilde{u}^2\tilde{\varepsilon}}(\tau) = 0, \quad \forall\tau \tag{65}$$

where

$$k = \frac{\left(\sum_{t=1}^{N} \tilde{\varepsilon}^2(t) - \overline{\tilde{\varepsilon}^2}\right)^{1/2}}{\left(\sum_{t=1}^{N} \tilde{y}(t)\tilde{\varepsilon}(t) - \overline{\tilde{y}\tilde{\varepsilon}}\right)^{1/2}} \tag{66}$$

is a constant [8]. If the model fails the global model validity tests (60), the individual tests can be performed to detect the offending subsystem or subsystems. Alternative tests, such as the chi-square test introduced by Bohlin (1980), could also be used for model validation.

The correlation tests however, do not reflect whether the dynamics of the CML model,

resulting by mapping the identified equations (44) onto a multi-dimensional lattice, match the original spatially extended dynamical behaviour. From this point to view, much stronger evidence about the adequacy of the model is provided by simulating the CML and comparing the predicted dynamics with all the data available, especially the data not used in the identification.

The (deterministic) model predicted output of the CML model is defined as

$$\begin{aligned} \hat{y}_i(t) &= F_l(\mathbf{q}^{n_y}\hat{y}_i(t), \mathbf{q}^{n_u}u_i(t), 0, ..., 0) + \\ & \quad F_c(\mathbf{q}^{n_y}\hat{y}_i(t), \mathbf{q}^{n_u}u_i(t), \mathbf{s}^m\,\mathbf{q}^{n_y}\hat{y}_i(t), \mathbf{s}^m\mathbf{q}^{n_u}u_i(t), 0, ..., 0) \end{aligned} \tag{67}$$

which leads to the following model prediction error

$$\hat{e}_i(t) = y_i(t) - \hat{y}_i(t) \tag{68}$$

4 Simulation Study

4.1 Example 1; A Linear Diffusion Equation

Consider the diffusion equation

$$\frac{\partial^2 v(t,x)}{\partial t^2} - C\frac{\partial^2 v(t,x)}{\partial x^2} = u(t,x), \quad x \in [0,1] \tag{69}$$

with initial conditions

$$\begin{aligned} v(0,x) &= 0 \\ \frac{dv(0,x)}{dt} &= 4\exp(-x) + \exp(-0.5x) \end{aligned} \tag{70}$$

where

$$u(t,x) = -13\exp(-x)\cos(1.5t) - 9.32\exp(-0.5x)\cos(2.1t) \tag{71}$$

For $C = 1.0$ the exact solution $v(t,x)$ of the initial value problem (69), (70) is

$$\begin{aligned} v(t,x) &= 4\exp(-x)\cos(1.5t) + 2\exp(-0.5)\cos(2.1t) - \\ & \quad 4\exp(-x)\exp(-t) - 2\exp(-x)\exp(-0.5t) \end{aligned} \tag{72}$$

The measurement function is taken as

$$y(t,x) = v(t,x) \tag{73}$$

The reference solution was sampled at 21 equally spaced points over the spatial domain $\Omega = [0,1]$, $x = \{x_1, ..., x_{21}\} = (0, 0.05, ..., 0.95, 1)$. From each location, 1000 input/output data points sampled at $\Delta t = \pi/100$ were generated. The result is plotted in Fig.(1) and Fig.(2) respectively.

The identification data consisted of 1000 data points of input/output data $u_i(t)$, $y_i(t)$

at node $i = 3$ corresponding to $x = x_3 = 0.1$ and input/output data u_{i-1}, $y_{i-1}(t)$ and $u_{i+1}(t)$, $y_{i+1}(t)$ from neighbouring locations $x = x_2 = .05$ and $x = x_4 = .15$.

White noise with standard deviation $\sigma_1 = 0.05$, $\sigma_1 = 0.3258$ was superimposed on the output data Fig.(3).

Based on the noisy input/output observations, the forward regression orthogonal estimator selected a model with 12 terms, which are shown in Table (4.1).

Terms	Estimates	$[ERR]_i$	Std. Dev.
$y^*(t-1)$ [a]	$-0.52530E-1$	$0.99423E+0$	$0.24558E+0$
$u_i(t-4)$	$0.73558E+2$	$0.84751E-3$	$0.36561E+2$
$u_i(t-1)$	$-0.75663E+2$	$0.15307E-2$	$0.37360E+2$
$y^*(t-2)$ [b]	$0.58299E-1$	$0.29037E-3$	$0.25173E+0$
$y_i(t-1)$	$0.40114E+0$	$0.49428E-4$	$0.54737E+0$
$u_i(t-2)$	$0.22418E+3$	$0.21004E-4$	$0.11096E+3$
$u_i(t-3)$	$-0.22209E+3$	$0.74501E-4$	$0.11016E+3$
$y_i(t-2)$	$0.54881E+0$	$0.32791E-4$	$0.49716E+0$
$e_i(t-2)$	$-0.53762E+0$	$0.26781E-3$	$0.48343E+0$
$e_i(t-1)$	$-0.38105E+0$	$0.35140E-5$	$0.54845E+0$
$e_{i-1}(t-1)$	$0.56219E-1$	$0.67460E-7$	$0.24621E+0$
$e_{i+1}(t-1)$	$0.57649E-1$	$0.74653E-7$	$0.24317E+0$
$e_{i-1}(t-2)$	$-0.51340E-1$	$0.12020E-6$	$0.24495E+0$
$e_{i+1}(t-2)$	$-0.53230E-1$	$0.14210E-6$	$0.24234E+0$

Table 4.1 Estimated CML Model for Example 4.1

[a] $y^*(t-1) = y_{i-1}(t-1) + y_{i+1}(t-1)$
[b] $y^*(t-2) = y_{i-1}(t-2) + y_{i+1}(t-2)$

To ensure the symmetry of the equations with respect to the coupling variables, the coupling topology was chosen to be symmetric and the estimated coefficients of $y_{i-1}(t-1)$ and $y_{i+1}(t-1)$ as well as of $y_{i-1}(t-2)$ and $y_{i+1}(t-2)$ were forced to be identical. The global model validity tests are shown in Figs. (4).

Based on the model (4.1), a one-dimensional CML model with 21 lattice nodes was implemented and simulated using the reference solutions computed at $x = -0.05$ and $x = 1.05$, $y_0(t)$ and $y_{22}(t)$ respectively, as boundary conditions.

The model simulated output is plotted in Fig.(5) for comparison. The model predicted errors eqn. (68) (not the fitting error) in Fig. (6) show excellent agreement between the exact solution and the CML model output.

4.2 Example 2: A Nonlinear Reaction-Diffusion Equation

Consider the nonlinear FitzHugh-Nagumo reaction-diffusion equation

$$\frac{\partial u_1}{\partial t} = d_1 \frac{\partial^2 u_1}{\partial x^2} + g(u_1) - u_2 \tag{74}$$

$$\frac{\partial u_2}{\partial t} = d_2 \frac{\partial^2 u_2}{\partial x^2} + \delta u_1 - \gamma u_2 \tag{75}$$

with $g(u_1) = -u_1(u_1 - \alpha)(u_1 - 1)$, $x \in \Omega = [0, 1]$ and Dirichlet boundary conditions (i.e. $u_1(t, 0) = u_1(t, 1) = u_2(t, 0) = u_2(t, 1) = 0$).

Equations (74), which normally describe the conduction of nerve impulses along the axon [15], [33], are used to illustrate further the applicability of the proposed identification method for a stochastically perturbed nonlinear CML.

The nonlinear PDE's (74) with parameter values $d_1 = d_2 = 6.188e - 4$, $\delta = 40$ and $\gamma = -0.2$, were numerically integrated with high accuracy, using initial conditions

$$\begin{aligned} u_1(0, x) &= \sin(\pi x/2) \\ u_2(0, x) &= \sin(\pi x/2) \end{aligned} \tag{76}$$

The solution was uniformly sampled along the spatial coordinate x with a sampling step $\Delta x = 0.02$. The objective was to identify a CML representation of the original DPS dynamics restricted on the discrete 1×49 lattice corresponding to the spatial coordinates $\{x_k\}_{k=1}^{49} = (0.02\,,\ 0.04,\ ,\ldots,\ 0.98)$. The outputs were defined as

$$\begin{aligned} y_1(t, x) &= u_1(t, x) \\ y_2(t, x) &= u_2(t, x) \end{aligned} \tag{77}$$

The identification data consisted of 6×1000 samples corresponding to the output vector $(y_{i-1}\,,\ y_i\,,\ y_{i+1})$ with $y_i = (y_i^1\,,\ y_i^2)$, recorded with the sampling time $\Delta t = 0.01$ at three spatial locations $(x_{i-1}, x_i, x_{i+1}) = (0.04\,,\ 0.06\,,\ 0.08)$, $i = 2, 3, 4$. White noise with standard deviation $\sigma^1 = (\sigma_{i-1}^1\,,\ \sigma_i^1\,,\ \sigma_{i+1}^1) = (0.0347\,,\ 0.0417\,,\ 0.0452)$ and $\sigma^2 = (\sigma_{i-1}^2\,,\ \sigma_i^2\,,\ \sigma_{i+1}^2) = (0.2675\,,\ 0.2907\,,\ 0.3027)$ respectively was added to the data as shown in Fig.(8).

The noise-corrupted data was used to identify a multivariable NARMAX model with 6 outputs. A polynomial model set was defined for each subsystem. The polynomial orders and maximum lags were $m_k^1 = 3$, $m_k^2 = 1$ and $n_{y_k^j} = n_{e_k^j} = 1$ respectively where $k = i - 1, i, i + 1$ and $j = 1, 2$. The full cubic polynomial model set consisted of 1330 terms.

The forward-regression orthogonal estimator selected 12 and 6 terms respectively, shown in Table (4.2), for the input/output equations corresponding to y_i^1 and y_i^2, $i = 3$. Note that the model set for the y_i^1 and y_i^2 outputs was modified to ensure symmetry of the coupling function.

Output	Terms	Estimates	$[ERR]_i$	Std. Dev.
	$y^*_{1,i}(t-1)^a$	$0.16947E+0$	$0.99254E+0$	$0.34240E-1$
	$y_{2,i}(t-1)$	$-0.10348E-1$	$0.33566E-2$	$0.10522E-2$
	$y_{1,i}(t-1)$	$0.65900E+0$	$0.31275E-3$	$0.67522E-1$
	${y^*_{1,i}}^2(t-1)$	$-0.27817E-2$	$0.29609E-3$	$0.34292E-2$
	${y^*_{1,i}}^3(t-1)$	$-0.14856E-2$	$0.75699E-4$	$0.54862E-3$
	${y_{2,i}}^3(t-1)$	$0.61035E-5$	$0.30733E-5$	$0.15208E-4$
	${y_{2,i}}^2(t-1)$	$-0.28524E-4$	$0.12151E-4$	$0.75556E-4$
$y_{i,1}(t)$	$y_{1,i}(t-1)\,y_{2,i}(t-1)$	$0.13950E-1$	$0.61931E-5$	$0.42710E-2$
	$y_{2,i}(t-1)y^*_{1,i}(t-1)$	$-0.70271E-2$	$0.27895E-4$	$0.21396E-2$
	${y_{1,i}}^2(t-1)$	$0.24272E-1$	$0.96026E-5$	$0.13713E-1$
	$e_{1,i}(t-1)$	$-0.68983E+0$	$0.71053E-3$	$0.72069E-1$
	${y^*_{1,i}}^2(t-1)e_{1,i-1}(t-1)$	$0.20361E-1$	$0.24118E-4$	$0.67551E-2$
	${y_{1,i}}^2(t-1)^* e_{1,i+1}(t-1)$	$0.21033E-1$	$0.23448E-4$	$0.63340E-2$
	$y^*_{2,i}(t-1)^b$	$0.42113E-1$	$0.99341E+0$	$0.21397E-1$
	$y_{1,i}(t-1)$	$0.39993E+0$	$0.29571E-2$	$0.11111E-1$
$y_{2,i}(t)$	$y_{2,i}(t-1)$	$0.93425E+0$	$0.34800E-3$	$0.42765E-1$
	const.	$0.30430E-2$	$0.32643E-5$	$0.11199E-1$
	$e_{2,i}(t-1)$	$-0.93938E+0$	$0.83749E-3$	$0.51229E-1$
	$e_{1,i}(t-1)$	$-0.45389E+0$	$0.11989E-4$	$0.20501E+0$

Table 4.2 Estimated CML Model for Example 4.2

[a] $y^*_{1,i}(t-1) = y_{1,i-1}(t-1) + y_{1,i+1}(t-1)$
[b] $y^*_{2,i}(t-1) = y_{2,i-1}(t-1) + y_{2,i+1}(t-1)$

The global model validity test computed for this model is shown in Fig. (9). The identified input/output equations corresponding to the $i = 3$ site (Table 4.2) were subsequently used to implement a 2×49 sites CML as a finite dimensional model for the original distributed parameter system. The resulting CML model, with Dirichlet boundary conditions, was simulated for 20000 steps to check stability. The interpolated reference solution and model simulated output over a $5s$ interval (500 iterations) are plotted in Figs.(7a,b) and Figs.(10a,b) for comparison. The interpolated error surfaces shown in Figs.(11) illustrate good agreement between the original solution and the simulated CML output.

5 Conclusions

A stochastic formulation of the input/output CML model has been introduced and studied using the Frobenius-Perron and transfer operators associated with the CML transformation. The operator description allows the dynamics of the stochastic lattice equations to be analysed in terms of densities rather than trajectories. A formulation of the Frobenius-Perron operator for CML transformations with multiple time-lags has been introduced and a more general transfer operator, which describes the evolution of probability density functions under a CML transformation perturbed by noise, not necessarily additive or multiplicative, was derived.

In the second part, the identification of stochastic Coupled Map Lattices was addressed for the first time. A new identification method, which works for linear and nonlinear spatially-extended dynamical systems including systems of PDE's, was introduced and tested using simulated noisy data.

6 Acknowledgment

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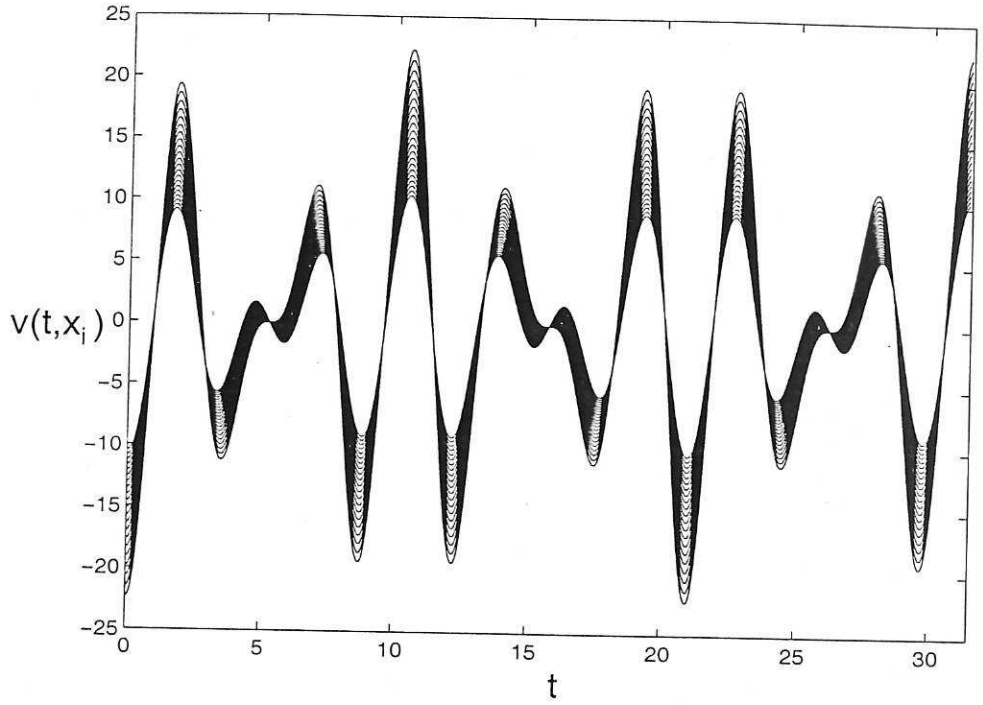


Figure 1: **Example 1**,Simulated input data $v(t,x)$: $dt = \pi/100$, $x = (0.05\,,\ 0.01\,,\ \ldots\,,\ 1)$

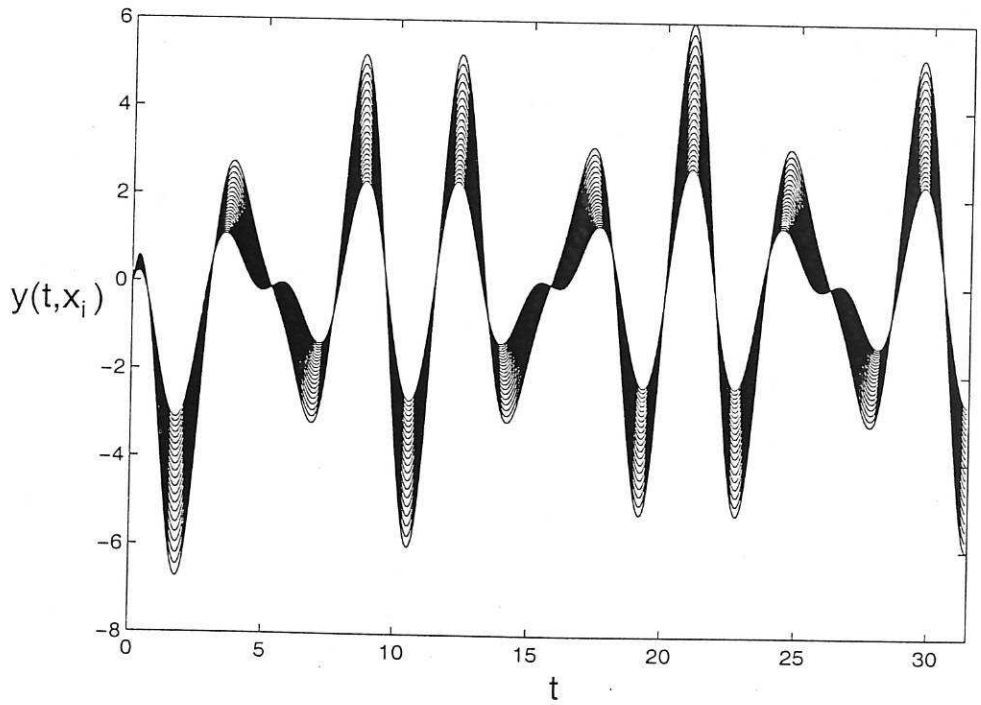


Figure 2: **Example 1**, Simulated output data $y(t,x)$: $dt = \pi/100$, $x = (0.05\,,\ 0.1\,,\ \ldots\,,\ 1)$

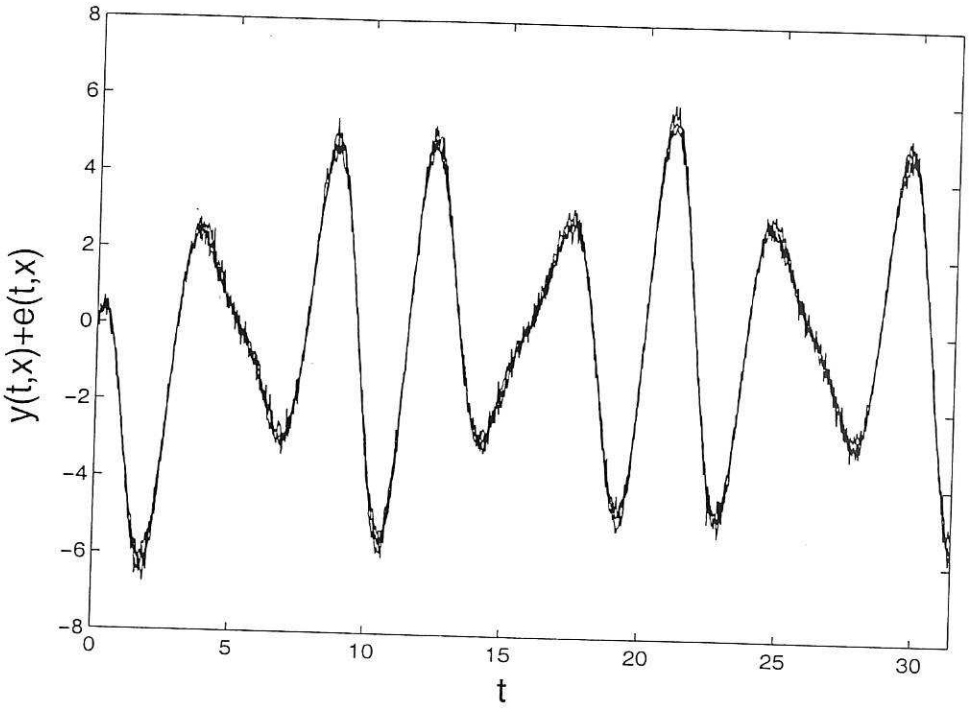
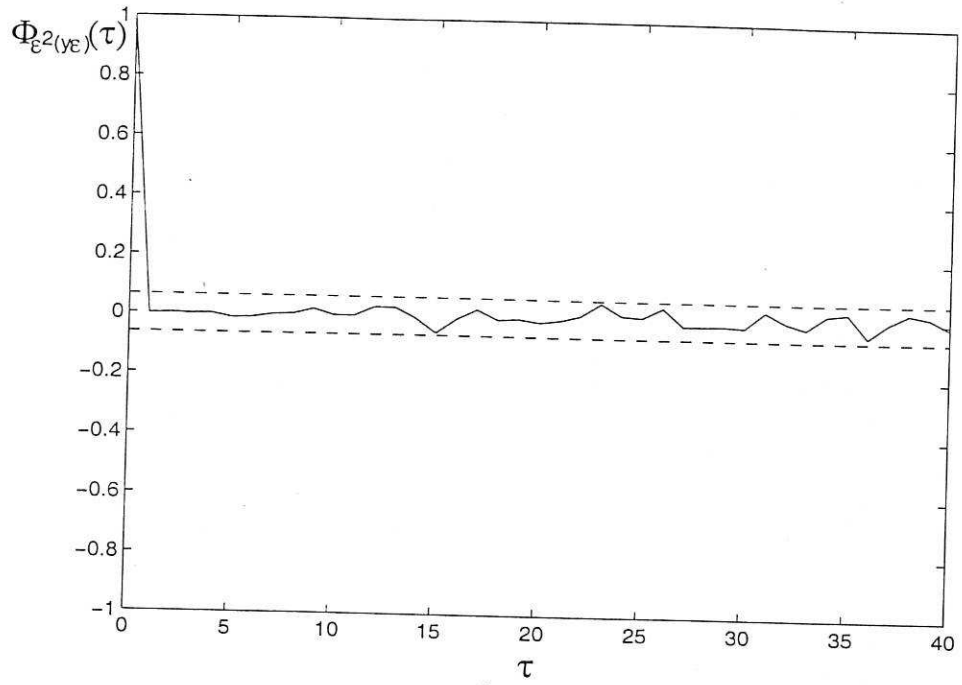
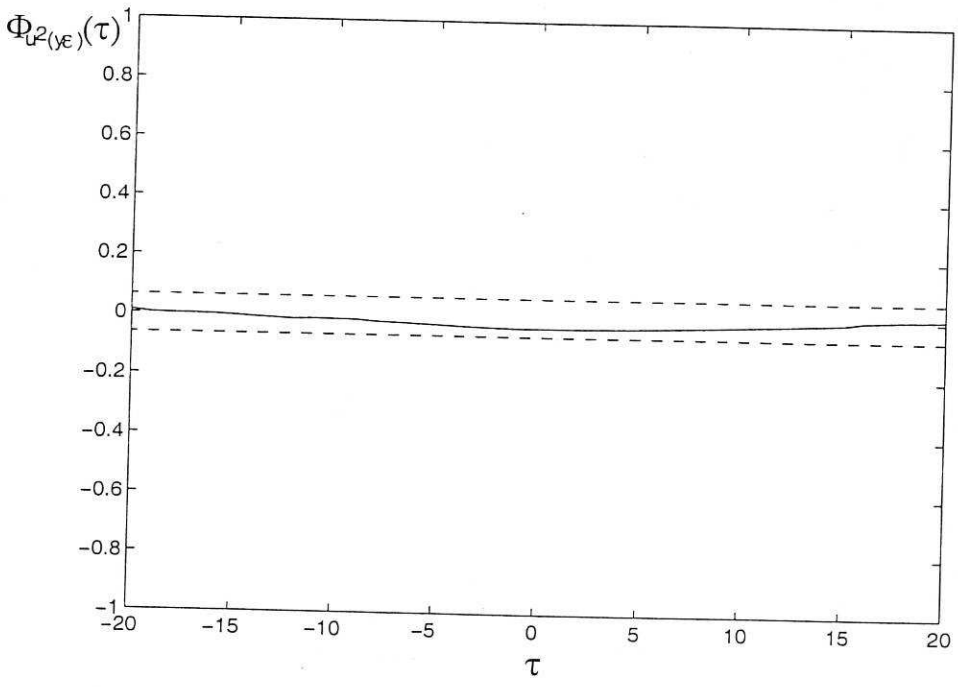


Figure 3: **Example 1**, Noisy identification data $y(t, x_i)$: $dt = \pi/100$, $x_i = (0.05, 0.1, 0.15)$



a



b

Figure 4: **Example 1**, Global model validity tests

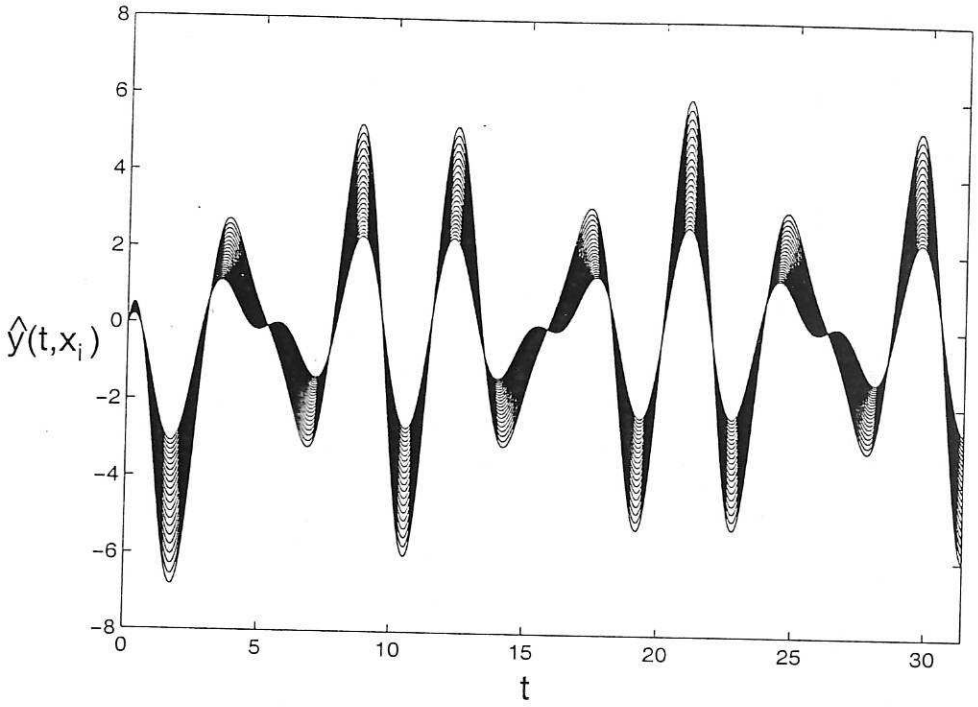


Figure 5: **Example 1**, Model predicted PDE solution $\hat{y}(t, x_i)$ for $i = 1, ..., 21$

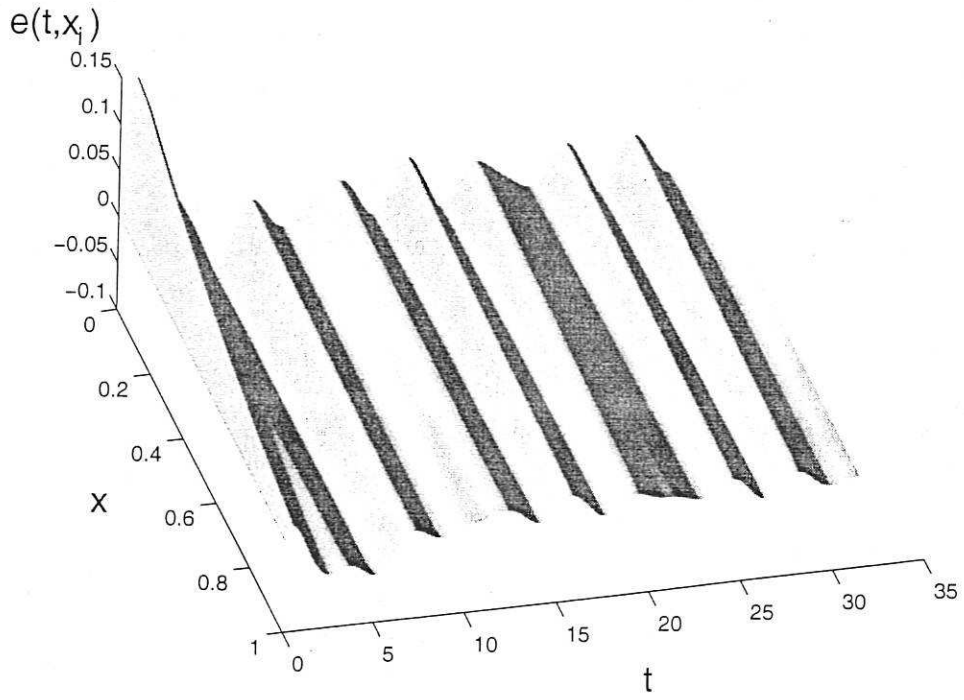
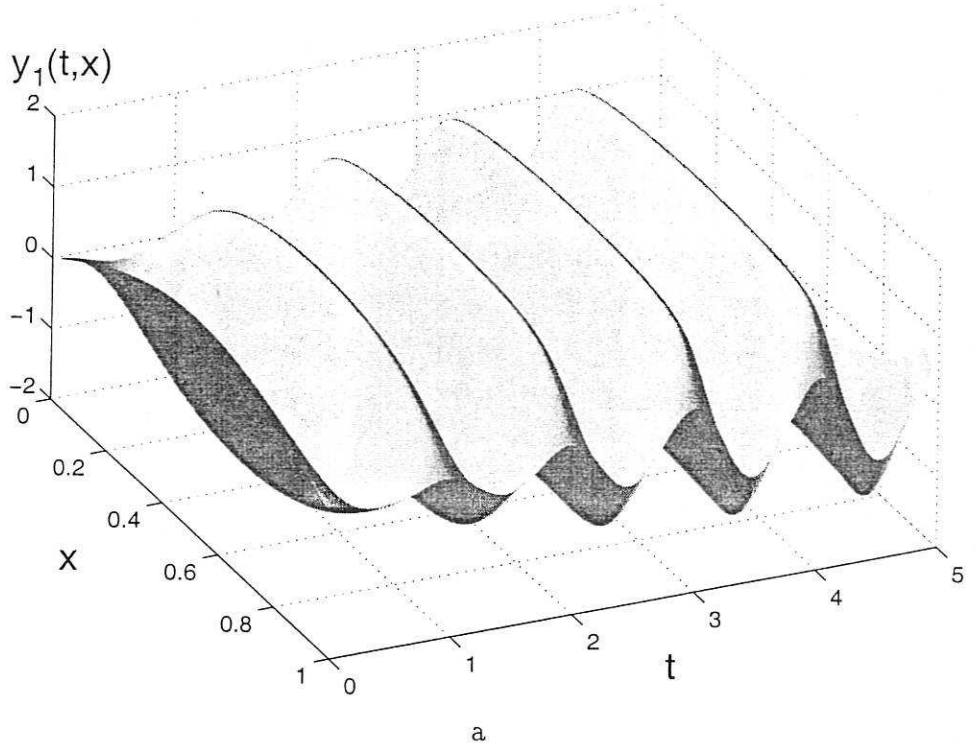
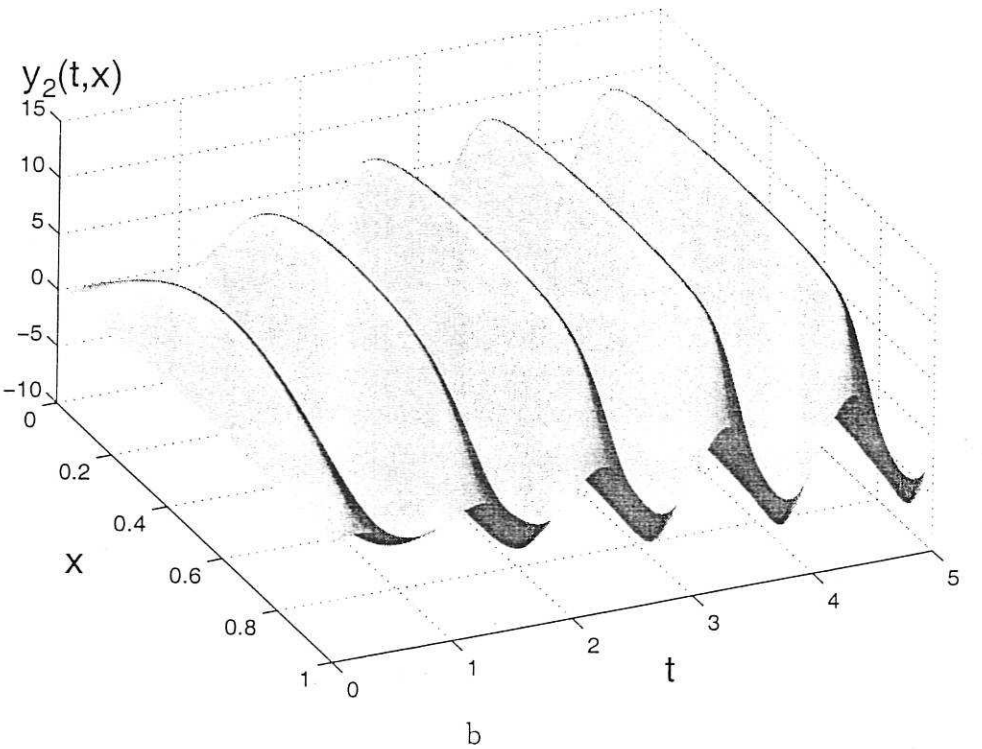


Figure 6: **Example 1**, Model predicted error $e(t, x_i) = y(t, x_i) - \hat{y}(t, x_i)$

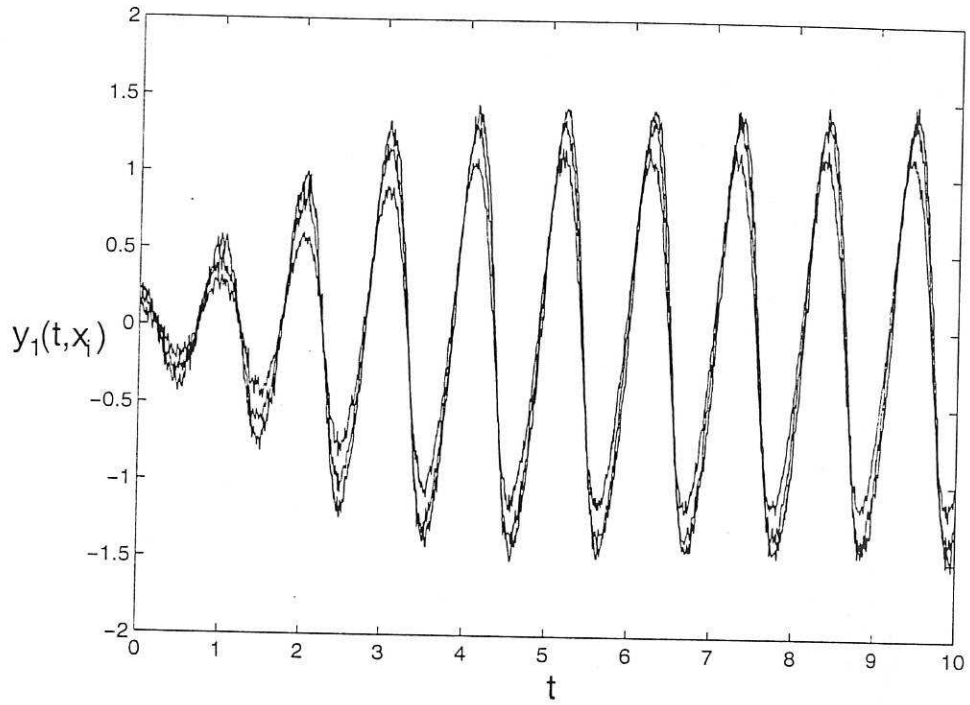


a

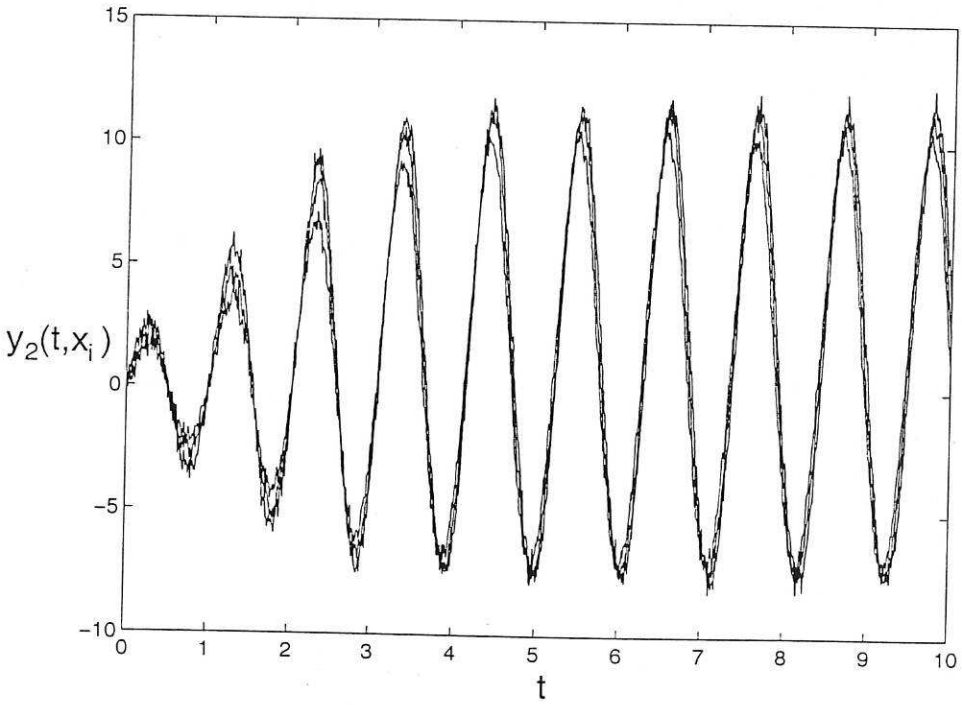


b

Figure 7: **Example 2**, Simulated data for the FitzHugh-Nagumo Equations: a) $y_1(t,x)$ b) $y_2(t,x)$



a



b

Figure 8: **Example 2**, Noisy identification data for the FitzHugh-Nagumo Equations: a) $y_1(t, x_i)$ b) $y_2(t, x_i)$ where $x_i = (0.04\,,\, 0.06\,,\, 0.08)$

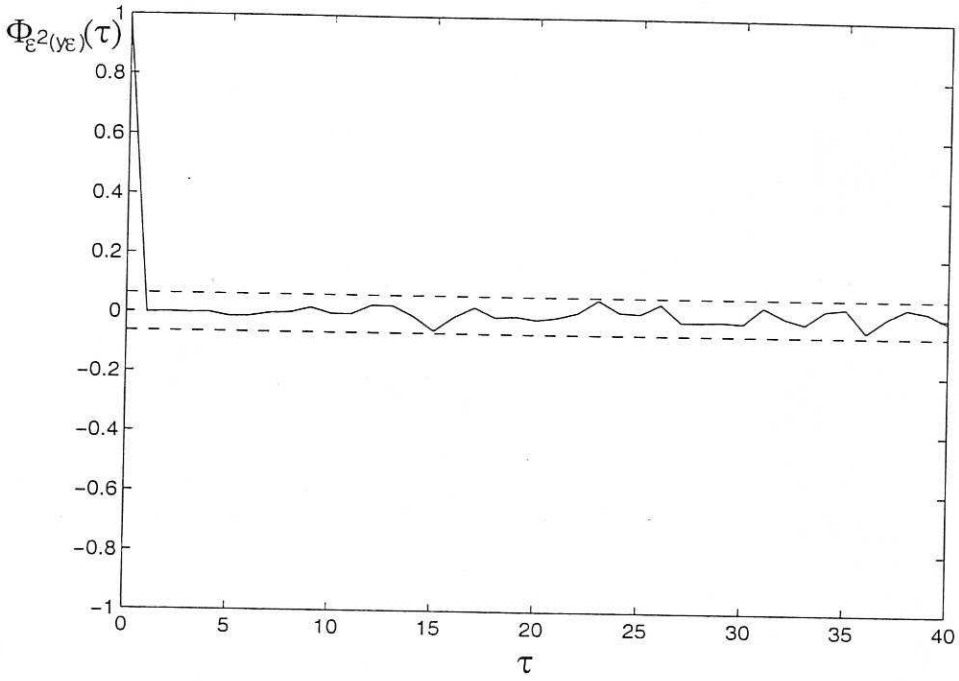
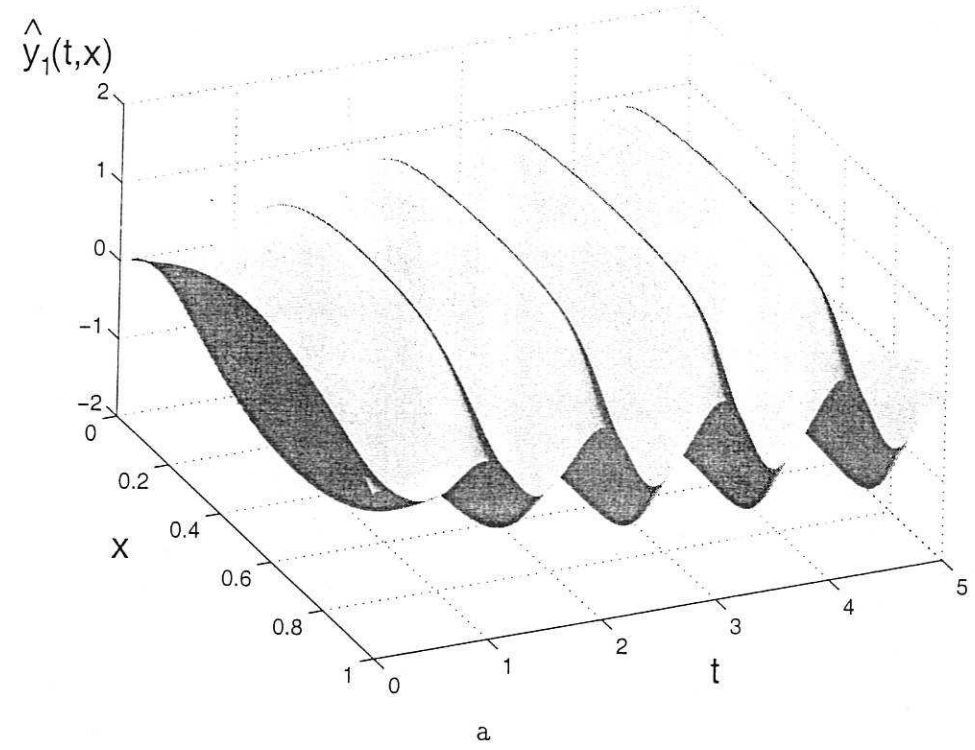
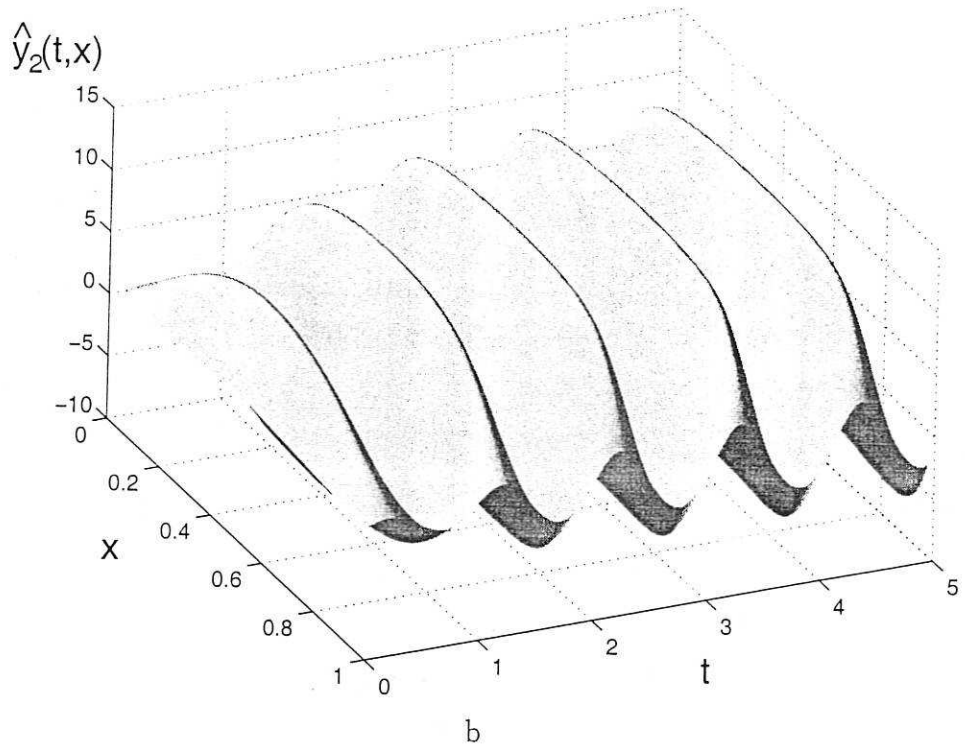


Figure 9: **Example 2**, Global model validity test

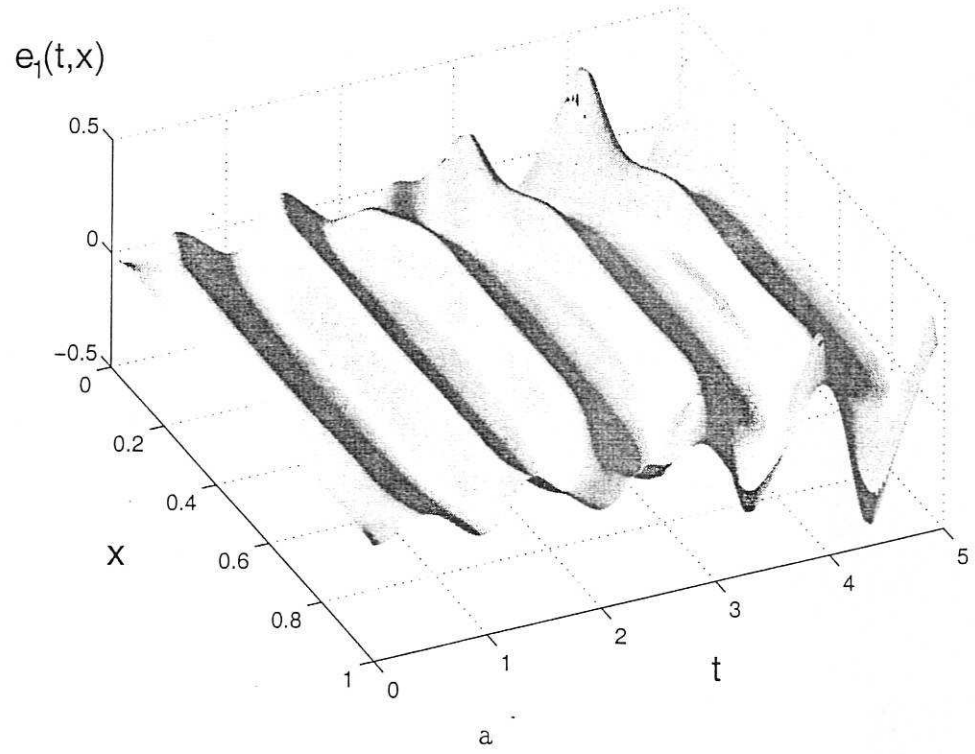


a



b

Figure 10: **Example 2**, Model predicted PDE solutions: a) $\hat{y}^1(t, x_k)$, b) $\hat{y}^2(t, x_k)$



a

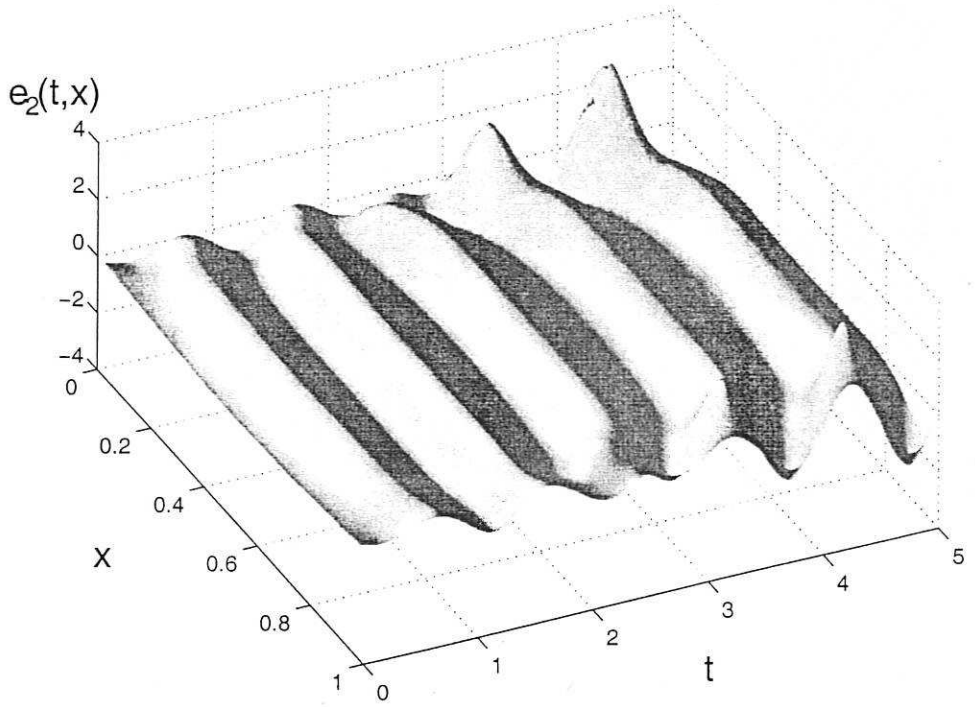


Figure 11: **Example 2**, Model prediction errors: a) $e_1(t,x) = y_1(t,x) - \hat{y}_1(t,x)$ b) $e_2(t,x) = y_2(t,x) - \hat{y}_2(t,x)$

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